Classical Mechanics homework due 23rd February

A Motion in polar and spherical coordinates

Physicists often solve theoretical problems by taking advantage of symmetries. Usually, this involves expressing the equations of motion in coordinates that are suited to the specific symmetry of the problem. Finding the suitable equations is particularly easy and elegant using Lagrangian mechanics, because the Lagrangian formalism is independent of any coordinate system (all coordinates q are general coordinates to begin with). In this homework sheet you will consider two important cases of symmetry, circular and spherical symmetry.

A.1 Consider the mechanical motion in the (x, y) plane under the influence of a potential $U(r)$ that depends only on the radius $r = \sqrt{x^2 + y^2}$ (circular symmetry). In this case the best coordinates usually are polar coordinates r and φ with $x = r\cos\varphi$ and $y = r\sin\varphi$. Draw a picture that illustrates the meaning if r and φ. **[1]**

A.2 Express the velocity $(\dot{x}, \dot{y})$ in terms of r and φ and derive a simple expression for $v^2 = \dot{x}^2 + \dot{y}^2$. **[2]**

A.3 Use the result of A.2 to express the Lagrangian $L = mv^2/2 - U$ in polar coordinates. Calculate the derivatives $\partial L/\partial\dot{r}$, $\partial L/\partial r$, $\partial L/\partial\dot{\varphi}$ and $\partial L/\partial\varphi$ and write down the Euler-Lagrange equations. **[3]**

A.4 Express the z-component of the angular momentum, $L_z = m(x\dot{y} - y\dot{x})$ in terms of r and φ. Use the Euler-Lagrange equation for φ to draw an interesting conclusion about L_z. **[2]**

A.5 Spherical coordinates r, φ, θ are often useful to describe the motion of a particle in three-dimensional space in the presence of a spherically-symmetric potential $U(r)$ where $r = \sqrt{x^2 + y^2 + z^2}$. The Cartesian coordinates (x, y, z) are $(r\cos\varphi\ \sin\theta$, $r\sin\varphi\ \sin\theta$, $r\cos\theta)$. Draw a picture that illustrates the meaning of r, φ, θ. **[1]**

A.6 Similar to A.2, express the velocity $(\dot{x}, \dot{y}, \dot{z})$ in terms of r, φ, θ and show that $v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{r}^2 + r^2(\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2)$. **[4]**

A.7 Derive the Euler-Lagrange equations in spherical coordinates from the Lagrangian $L = mv^2/2 - U$. **[3]**

A.8 For a trajectory with given initial position and velocity, one can always rotate the system of coordinates such that initially $d\theta/dt = 0$ and $\theta = \pi/2$ for, say, $t = 0$. Use the Euler-Lagrange equations to show that θ remains at $\pi/2$ for all times t. What is the geometrical meaning of this fact? **[2]**

A.9 Express the z-component of the angular momentum, $m(x\dot{y} - y\dot{x})$ in spherical coordinates r, φ, θ. Show that your result is consistent with A.4 for the situation considered in A.8. **[2]**

Classical Mechanics solutions (sketched)

A.2 $\dot{x} = \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi$, $\dot{y} = \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi$, $v^2 = (\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi)^2 + (\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi)^2 = \dot{r}^2 + r^2\dot{\varphi}^2$ and $L = (m/2)\,(\dot{r}^2 + r^2\dot{\varphi}^2) - U$.
A.3

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{r}} &= m\dot{r}\,, \quad \frac{\partial L}{\partial r} = mr\dot{\varphi}^2 - \frac{dU}{dr}\,, \quad \frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi}\,, \quad \frac{\partial L}{\partial \varphi} = 0\,, \\
0 &= \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = m\left(\ddot{r} - r\dot{\varphi}^2\right) + \frac{dU}{dr}\,, \quad 0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = m\,\frac{dr^2\dot{\varphi}}{dt}\,.
\end{aligned}
$$

A.4 $x\dot{y} - y\dot{x} = r\cos\varphi(\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi) - r\sin\varphi(r\cos\varphi - r\dot{\varphi}\sin\varphi) = r^2\dot{\varphi}$. From equation of motion follows $L_z = \text{const.}$ Conservation of angular momentum.
A.6

$$
\begin{aligned}
\dot{x} &= \dot{r}\cos\varphi\sin\theta - r\dot{\varphi}\sin\varphi\sin\theta + r\dot{\theta}\cos\varphi\cos\theta\,, \\
\dot{y} &= \dot{r}\sin\varphi\sin\theta + r\dot{\varphi}\cos\varphi\sin\theta + r\dot{\theta}\sin\varphi\cos\theta\,, \\
\dot{z} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta\,, \\
v^2 &= \Big((\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi)\sin\theta + r\dot{\theta}\cos\varphi\cos\theta\Big)^2 \\
&\quad + \Big((\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi)\sin\theta + r\dot{\theta}\sin\varphi\cos\theta\Big)^2 + \Big(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\Big)^2 \\
&= \Big((\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi)^2 + (\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi)^2\Big)\sin^2\theta + r^2\dot{\theta}^2\cos^2\theta \\
&\quad +2\Big((\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi)\cos\varphi + (\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi)\sin\varphi\Big)\sin\theta\, r\dot{\theta}\,\cos\theta \\
&\quad + \Big(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\Big)^2 \\
&= \Big(\dot{r}^2 + r^2\dot{\varphi}^2\Big)\sin^2\theta + 2\dot{r}r\dot{\theta}\cos\theta\sin\theta + \Big(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\Big)^2 + r^2\dot{\theta}^2\cos^2\theta \\
&= \dot{r}^2 + r^2(\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2)\,.
\end{aligned}
$$

A.7

$$
\begin{aligned}
L &= \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\sin^2\theta + r^2\dot{\theta}^2\right) - U \\
\frac{\partial L}{\partial \dot{r}} &= m\dot{r}\,, \quad \frac{\partial L}{\partial r} = mr(\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2) - \frac{dU}{dr}\,, \\
\frac{\partial L}{\partial \dot{\varphi}} &= mr^2\sin^2\theta\,\dot{\varphi}\,, \quad \frac{\partial L}{\partial \varphi} = 0\,, \quad \frac{\partial L}{\partial \dot{\theta}} = mr^2\,\dot{\theta}\,, \quad \frac{\partial L}{\partial \theta} = mr^2\dot{\varphi}^2\sin\theta\,\cos\theta\,, \\
0 &= m\left(\ddot{r} - r(\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2)\right) + \frac{dU}{dr}\,, \\
0 &= m\,\frac{d(r^2\dot{\varphi}\sin^2\theta)}{dt}\,, \quad m\,\frac{d(r^2\dot{\theta})}{dt} = mr^2\dot{\varphi}^2\sin\theta\,\cos\theta\,.
\end{aligned}
$$

A.8 Assume $\theta(t) = \pi/2$. This solves the differential equation $d(r^2\dot{\theta})/dt = r^2\dot{\varphi}^2\sin\theta\cos\theta$, because $\cos(\pi/2) = 0$ and $\dot{\theta} = 0$, and it fits the given initial conditions $\theta(0) = \pi/2$ and $\dot{\theta}(0) = 0$. Geometric meaning: motion in (x, y) plane.
A.9 $x\dot{y} - y\dot{x} = r^2\dot{\varphi}\sin^2\theta$. Consistent with A.4 for $\theta = \pi/2$, because $\sin(\pi/2) = 1$.

Classical Mechanics homework due 8th March

B Evacuation slide

Calculate the optimal shape of an aircraft evacuation slide. The (friction-less) slide should be designed such that it takes the shortest possible time to slide from an exit at height h down to the ground some distance x away. The figure illustrates the notation used.

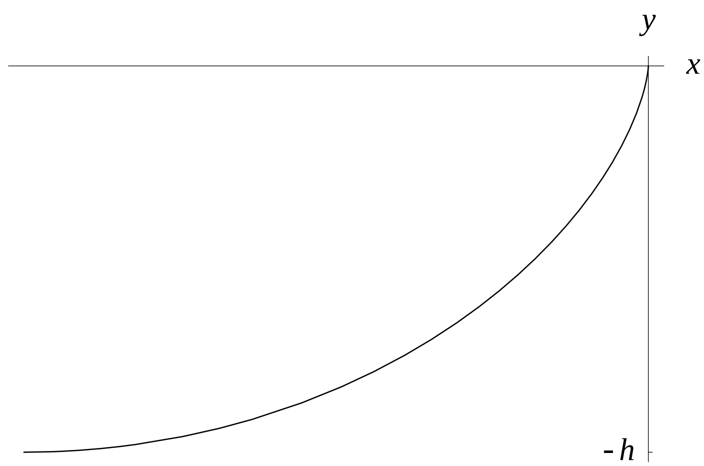

Figure 1: Evacuation slide from height $y = 0$ to $y = -h$ the distance $-x$ away from the aircraft.

To address the problem use the conservation law of energy. For each passenger with mass m the sum of the kinetic energy $m[(dx/dt)^2 + (dy/dt)^2]/2$ and the potential energy gmy is conserved. For simplicity we put the potential energy at the top of the slide ($y = 0$) to zero. Assume that the passengers enter the slide with zero velocity. In this case the initial total energy is zero. Energy conservation implies that the total energy remains zero. Therefore,

$$\frac{m}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] = -mgy\,.$$

The total time T a passenger takes to slide down is the integral of the time increments dt expressed as a function of one of the coordinates, say the height y. Since we wish to optimize the total time, T plays the role of the action in this problem. The solution of the Euler-Lagrange equation with respect to y will describe the optimal shape.

B.1 Express the total time $T = \int dt$ in terms of y and $x' = dx/dy$. **[2]**
B.2 In order to optimize T, write down the equivalent of the Euler-Lagrange equation. Show that you get

$$\frac{\partial}{\partial y} \frac{x'}{\sqrt{-y(1+x'^2)}} = 0\,,$$

which implies

$$\frac{x'}{\sqrt{-y(1+x'^2)}} = \text{const} \equiv \frac{1}{\sqrt{\eta}}\,.$$

[3]
B.3 Solve this differential equation for $x(y)$. The function $x(y)$ describes the shape of the slide. **[5]**
B.4 The solution depends on the parameter η. This means that you have found a family of optimal shapes. In the interest of the passengers, the slide should have zero slope dy/dx at the end when $y = -h$. Show that this implies $\eta = h$. **[2]**
B.5 Calculate the time T it takes to slide down from height zero to $-h$ for the optimal shape with $\eta = h$. Hint: It may be advantageous to use the formula given in B.2 in the expression for T that you derived in B.1. **[3]**
B.6 Compare the shortest time T to slide away from the aircraft with the time of a free fall from the exit. **[2]**
B.7 Give a qualitative explanation for the shape of the evacuation slide illustrated in the figure. In particular, the figure shows that the lower y is the lower is the slope of the slide dy/dx. Why would you expect this from an optimal slide? **[3]**

Classical Mechanics solutions

This problem is known as the brachistochrone problem. The term derives from the Greek $\beta\rho\alpha\chi\iota\sigma\tau\omega\varsigma$ (brachistos) "the shortest" and $\chi\rho\omega\nu\omega\varsigma$ (chronos) "time, delay." See also `http://mathworld.wolfram.com/BrachistochroneProblem.html`.

B.1 We solve $m[(dx/dt)^2 + (dy/dt)^2]/2 = -mgy$ for dt and get

$$T = \int dt = \int \sqrt{-\frac{1+x'^2}{2gy}}\, dy\,.$$

B.2 The Euler-Lagrange equation is

$$\frac{\partial}{\partial y}\frac{\partial}{\partial x'}\sqrt{-\frac{1+x'^2}{2gy}} - \frac{\partial}{\partial x}\sqrt{-\frac{1+x'^2}{2gy}} = 0\,.$$

We obtain the result given in B.2 from

$$\frac{\partial}{\partial x}\sqrt{-\frac{1+x'^2}{2gy}} = 0 \quad \text{and} \quad \frac{\partial}{\partial x'}\sqrt{-\frac{1+x'^2}{2gy}} = \frac{x'}{\sqrt{-2gy(1+x'^2)}}\,.$$

B.3 We solve the equation for x' and get

$$x' = \sqrt{\frac{-y/\eta}{1+y/\eta}}\,, \quad \text{from which follows} \quad x = \int \sqrt{\frac{-y/\eta}{1+y/\eta}}\, dy\,.$$

Substituting $\sin\varphi = \sqrt{-y/\eta}$ with $2d\varphi = -dy/\sqrt{-\eta y(1+y/\eta)}$ we obtain

$$\begin{aligned} x &= -2\int y\, d\varphi = -2\eta \int \sin^2\varphi\, d\varphi = -\eta(\varphi - \cos\varphi\, \sin\varphi) \\ &= \eta\left[\sqrt{-\frac{y}{\eta}\left(1+\frac{y}{\eta}\right)} - \arcsin\left(\sqrt{-\frac{y}{\eta}}\right)\right]. \end{aligned}$$

B.4 The slope dy/dx is zero when $x' = dx/dy$ is infinite. We see from the starting point in the solution of B.3 that this is the case for $y = -\eta$. Therefore, zero slope at $y = -h$ requires $\eta = h$.

B.5 We get from B.2 the relation $1 + x'^2 = 1/(1 + y/h)$ and from B.1

$$T = \int_{-h}^{0} \frac{dy}{\sqrt{-2gy(1+y/h)}} = \sqrt{\frac{2h}{g}}\int d\varphi = \sqrt{\frac{2h}{g}}\,\frac{\pi}{2}\,.$$

B.6 For a free fall from $y = 0$ to $y = -h$ we get $-h = -gt^2/2$, and hence $t = \sqrt{2h/g}$. The optimal time of sliding sideways differs from the time of the free fall by merely a factor of $\pi/2$.

B.7 The lower y is (the more far down the slide the passengers have gone) the larger is the kinetic energy and hence the velocity. To save time, sliding sideways should be done when the speed is high. Therefore the slope should decrease when the speed gets higher, *i.e.* when y gets lower.

Classical Mechanics homework due 22nd March

C Spherical oscillator

The spherical oscillator is an important physical model. For example, it serves as a simple model for quark confinement (Nobel Prize 2004), describing the fact that the effective force between quarks grows linearly with increasing distance. Such oscillators also model atom traps where ultracold atoms are confined by optical or magnetic forces (Nobel Prize 2001). From a theoretical point of view, the spherical oscillator and the Kepler problem are the only cases where central forces lead to closed trajectories for all energies of bound trajectories. In this homework problem you should explore the physics of the spherical oscillator. The Lagrangian of the spherical oscillator is

$$L = \frac{\mu v^2}{2} - \frac{k r^2}{2}$$

where $\vec{v}$ is the relative velocity and $\vec{r}$ the relative position of the two interacting mass points, μ denotes the reduced mass and k the Hooke constant.

C.1 It turns out that the total Lagrangian is the sum of three independent Lagrangians of one-dimensional harmonic oscillators with identical frequencies $\omega = \sqrt{k/\mu}$. Why is this the case? **[1]**

C.2 From C.1 immediately follows the solution of the equations of motion in Cartesian coordinates,

$$x(t) = x_0 \sin(\omega t + \varphi_x)\,, \quad y(t) = y_0 \sin(\omega t + \varphi_y)\,, \quad z(t) = z_0 \sin(\omega t + \varphi_z)\,.$$

Calculate the angular-momentum vector $\vec{L}$. You should see that $\vec{L}$ is indeed a constant vector that depends on the initial conditions. **[2]**

C.3 Imagine that you rotate your system of coordinates such that $\vec{L}$ points in z direction. Show from your result of C.2 that this implies that $z_0 = 0$, *i.e.* the oscillation occurs in the x, y plane, unless $\vec{L}$ vanishes altogether. How does the oscillator behave in the latter case? **[3]**

C.4 Assume in addition that the coordinates in the x, y plane are rotated such that $x(t) = x_0 \cos(\omega t)$ and $y(t) = y_0 \sin(\omega t)$. Show that the trajectory of the oscillator forms an ellipse with axes x_0 and y_0. Express the excentricity ϵ and the parameter p of this ellipse in terms of the energy E and the z-component L_z of the angular momentum. [Hint: the ellipse differs from the one of the Kepler problem.] **[3]**

C.5 Represent the trajectory in polar coordinates with $x = r\cos\varphi$ and $y = r\sin\varphi$. Show that you get the simple relation

$$\cos(2\varphi) = \frac{1 - \nu^2/(2\xi^2)}{\sqrt{1-\nu^2}}\,, \quad r = \sqrt{\frac{2E}{k}}\,\xi\,, \quad \nu = \frac{L_z \omega}{E}\,.$$

[6]

C.6 Now consider the spherical oscillator using the theory of the two-body problem, section 3.2 of the course,

$$\varphi - \varphi_0 = \pm L_z \int \frac{dr}{r^2\sqrt{2\mu(E - U_{\text{eff}})}}\,.$$

Show that the result of C.5 is consistent with this formula **[5]**

Classical Mechanics homework due 12th April

D Electronic transmission lines

Canonical Mechanics is much more universal than the traditional subject of mechanics. In this homework problem you will discuss an application of the canonical formalism in electronics.

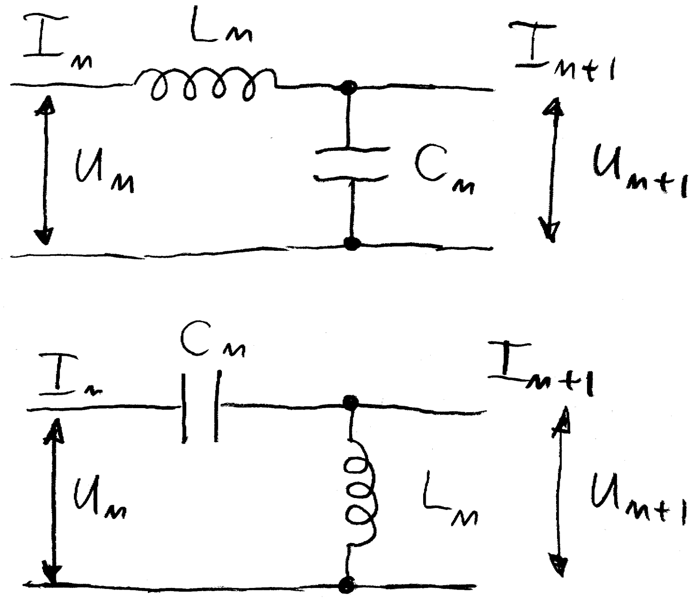

Figure 2: Two types of electronic transmission lines. The figures shows their elementary cells. The cells are connected to each other to form lines.

D.1 Consider an electromagnetic transmission line that consists of many of the cells shown in the top part of the figure connected to each other. To find the "equations of motion" for the voltages U_n and currents I_n we note: The charge of the capacitor equals the voltage after the n-th cell, U_{n+1}, times the capacitance C_n. The charge is the integral of the charging current, $I_n - I_{n+1}$. This gives one equation of motion. Another one follows from Faraday's law of electromagnetic induction, which implies that any change of the current, $\dot{I}_n$, generates the voltage difference $U_n - U_{n+1}$ divided by the inductance L_n. Consequently, we get

$$C_n \dot{U}_{n+1} = I_n - I_{n+1}\,, \quad L_n \dot{I}_n = U_n - U_{n+1}\,.$$

Deduce a Lagrangian that generates these equations of motion. Hint: It may be wise to consider the charge $Q_n = -\int I_n dt$, because one equation of motion could be already satisfied by putting $C_n U_{n+1} = Q_{n+1} - Q_n$ *per definitionem.* All you need to do is to infer a Lagrangian that generates the other equation expressed in terms of the Q_n. **[4]**

D.2 A Lagrangian is determined up to a prefactor and an integration constant, in general. Show that you can chose the Lagrangian of D.1 such that the energy E and the Hamiltonian H of the transmission line are

$$E = \sum_n \left(\frac{L_n \dot{Q}_n^2}{2} + \frac{C_n U_{n+1}^2}{2} \right), \quad H = \frac{1}{2} \sum_n \left(\frac{P_n^2}{L_n} + \frac{(Q_{n+1} - Q_n)^2}{C_n} \right),$$

where the P_n are the canonical momenta of Q_n. **[2]**

D.3 Consider the other type of transmission line shown in the lower part of the figure. Here the equations of motion are

$$C_n(\dot{U}_n - \dot{U}_{n+1}) = I_n\,, \quad L_n(\dot{I}_n - \dot{I}_{n+1}) = U_{n+1}\,.$$

Introduce, like in D.1, the charge $Q_n = -\int I_n dt$ and show that the Lagrangian is

$$L = \frac{1}{2} \sum_n \left(L_n(\dot{Q}_{n+1} - \dot{Q}_n)^2 - \frac{Q_n^2}{C_n} \right).$$

[5]

D.4 Use the Lagrangian of D.3 to calculate the energy. Give an interpretation for the energy. **[3]**

D.5 Consider the transmission lines in the continuum limit. For this, assume that the voltages and charges vary little over the scale of a few elementary cells of the lines. You approximate the $U_n(t)$ and $Q_n(t)$ by continuous functions $U(x,t)$ and $Q(x,t)$ where x indicates the position of the n-th cell. Assume, for simplicity, that the distance between the cells is a constant δ and that the C_n and L_n are constant. For the transmission line introduced in D.1 you get from $C_n\dot{U}_{n+1} = I_n - I_{n+1}$ the relation $U = C^{-1}\delta\,(\partial Q/\partial x)$ and from $L_n\dot{I}_n = U_n - U_{n+1}$ you infer $\ddot{Q} = -\dot{I} = L^{-1}\delta\,(\partial U/\partial x)$. Consequently, you obtain the usual linear wave equation

$$\frac{\partial^2 Q}{\partial t^2} = c^2 \frac{\partial^2 Q}{\partial x^2} \quad \text{with the phase velocity} \quad c = \frac{\delta}{\sqrt{LC}}\,.$$

Similar to geometrical optics, regard Q as the real part of $\mathcal{Q}\exp(i\int k\,dx - i\omega t)$ where $\mathcal{Q}$ is a slowly varying envelope. You get the dispersion relation $\omega = ck$. From one of Hamilton's equations follows that c is also the group velocity,

$$v = \frac{\partial \omega}{\partial k} = c\,.$$

Now consider the other transmission line introduced in D.3 and follow a similar procedure in this case. First, deduce the wave equation. (You will find a less usual wave equation). Then establish the dispersion relation. Show that

$$v = -\frac{\omega}{k}\,,$$

which implies that, for a positive frequency ω, the group velocity is positive only when k is negative. Any ideas what this means? **[6]**