Special Relativity and Fields

Short questions, due 6th October

1.1. What are the assumptions made to derive the Lorentz transformations? **[2]**
1.2. Explain briefly the graphic interpretation of the Lorentz transformations that was developed in the course. **[2]**
1.3. Why is the matrix

$$\Lambda^{-1} = \begin{pmatrix} \cosh\xi & -\sinh\xi \\ -\sinh\xi & \cosh\xi \end{pmatrix} \text{ the matrix inverse of } \Lambda = \begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix}?$$

Show that

$$\begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^{-\xi} & 0 \\ 0 & e^{\xi} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

[2]
1.4. Consider a Lorentz transformation of the space-time coordinates (ct', x', y', z') from a moving frame to the rest frame with coordinates (ct, x, y, z). How would you describe this transformation if the moving frame has the velocity $\vec{u} = (u_x, u_y, u_z) = (u\cos\phi, u\sin\phi, 0)$? **[2]**
1.5. Show that

$$1 - \frac{v^2}{c^2} = \frac{\left(1 - \frac{u^2}{c^2}\right)\left(1 - \frac{v'^2}{c^2}\right)}{\left(1 + \frac{\vec{u}\cdot\vec{v}'}{c^2}\right)^2}.$$

[2]
1.6. Consider the velocity $\vec{v}$ as seen from an inertial frame moving in the x direction (velocity addition theorem). It is understandable that the x-component of $\vec{v}$ is transformed, but why are the y- and z-components transformed as well? What do you think is the physical reason for this? **[2]**
1.7 What is the space-time metric and the metric tensor? **[2]**
1.8 What do you think is the most important property of the metric? **[2]**
1.9 What is proper time and how is it related to the metric? **[2]**
1.10 Explain briefly the twin paradox. **[2]**

Short questions, due 13th October

2.1. What is the motivation to introduce the four-velocity $u^\alpha = \mathrm{d}x^\alpha/\mathrm{d}s$? **[2]**
2.2. What are contravariant and covariant four-vectors? **[2]**
2.3. Why is ∂_α a covariant four-vector? **[2]**
2.4. Why is $A_\alpha A^\alpha = A_\beta A^\beta$? Why is $g_{\alpha\beta}$ the matrix inverse of itself? (Here $g_{\alpha\beta}$ is the metric tensor of inertial frames in special relativity.) **[2]**
2.5. Consider a covariant four-vector A_α with $A_0 = \cosh\eta$, $A_1 = \sinh\eta\,\sin\theta\,\cos\phi$, $A_2 = \sinh\eta\,\sin\theta\,\sin\phi$, $A_3 = \sinh\eta\,\cos\theta$. Calculate A^α and $A_\alpha A^\alpha$. **[2]**
2.6. Explain briefly the principle of least action in classical mechanics. **[2]**
2.7. How to deduce the action of a free relativistic particle? **[2]**
2.8. Why is the Lagrangian $L = -mc^2\sqrt{1 - v^2/c^2}$? **[2]**
2.9. Use this Lagrangian and the formulas $\vec{p} = \partial L/\partial\vec{v}, E = \vec{p}\cdot\vec{v} - L$ from classical mechanics to show that $\vec{p} = m(v)\vec{v}$, $E = m(v)c^2$, with $m(v) = (1 - v^2/c^2)^{-1/2}$. Use these explicit expressions and the expressions for the four-velocity u^α to show that $p^\alpha = mcu^\alpha$, with $p^\alpha = (E/c, \vec{p})$. **[2]**
2.10. Why does the energy and the canonical momentum constitute the covariant four-vector p_α? **[2]**

Short questions, due 20th October

3.1. Use $p_\alpha = -\partial_\alpha S$ to derive $p^\alpha = mc\,u^\alpha$. What does the relation mean? **[2]**
3.2. What is the meaning of $\delta\Omega^{\alpha\beta}$? Why is $\delta\Omega^{\alpha\beta} = -\delta\Omega^{\beta\alpha}$? **[2]**
3.3. How is the covariant tensor $\delta\Omega_{\alpha\beta}$ defined? Show that from 3.2. follows $\delta\Omega_{\alpha\beta} = -\delta\Omega_{\beta\alpha}$. **[2]**
3.4. Show that $\vec{L} = \Sigma_a \vec{r}_a \times \vec{p}_a$ and $\vec{R} = \Sigma_a (t\vec{p}_a - (E_a/c^2)\vec{r}_a)$ for the explicit matrix structure of the angular momentum tensor $L^{\alpha\beta}$ in terms of the L's and R's that was given in the course. **[3]**
3.5. We have shown that $\vec{L}$ and $\vec{R}$ are conserved. What is the meaning of $\vec{L}$? What is the non-relativistic limit of $\vec{R}$? What is the meaning of $\vec{R}$? **[3]**
3.6. Write down the action S of a charged particle in an electromagnetic field with four-potential A_α and explain the motivations used to deduce each term of the action. **[3]**
3.7. Why does the Hamiltonian depend on the vector potential, when the energy does not? **[2]**
3.8. What is gauge invariance? **[3]**

Short questions, due 27th October

4.1. As was shown in the course, the Hamiltonian of a charged particle in the electromagnetic field is

$$H = c\sqrt{m^2c^2 + (\vec{p} - q\vec{A})^2} + qU.$$

Use Hamilton's equations

$$\frac{d\vec{x}}{dt} = \frac{\partial H}{\partial \vec{p}}\,, \quad \frac{d\vec{p}}{dt} = -\frac{\partial H}{\partial \vec{x}}$$

to derive the equation of motion

$$\frac{dm(v)\vec{v}}{dt} = q\left(\vec{E} + \vec{v} \times \vec{B}\right).$$

[4]

4.2. Why does the electric field change the kinetic energy, whereas the magnetic field does not? **[2]**

4.3 Consider a charged particle in a uniform electric field. Explain briefly the characteristic features of the relativistic trajectory. **[2]**

4.4. Explain briefly how a charged particle moves in a magnetic field and why this is the case. **[2]**

4.5. For particles in crossed E and B fields (or in the B field alone) we described the trajectories in the plane orthogonal to the B field using complex numbers. Deduce the equation $\ddot{z} = i\omega(v - \dot{z})$ from the Lorentz force in the non-relativistic limit, where ω denotes the cyclotron frequency qB/m and v the drift velocity E/B. **[2]**

4.6. Why does a charged particle drift in crossed magnetic and electric fields? It sounds paradoxical that the particle moves on average in an orthogonal direction to the electric field that is accelerating it. Give a qualitative explanation. **[2]**

4.7. Show that $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ takes the form

$$F_{\alpha\beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -cB_z & cB_y \\ -E_y & cB_z & 0 & -cB_x \\ -E_z & -cB_y & cB_x & 0 \end{pmatrix}$$

and that

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix}.$$

[6]

Short questions, due 3rd November

5.1. Show that

$$F_{\alpha\beta}u^{\beta} = \frac{1}{\sqrt{1-v^2/c^2}} = \left(\frac{\vec{E}\cdot\vec{v}}{c}, -\vec{E}-\vec{v}\times\vec{B}\right)^T .$$

[2]

5.2. Show that

$$mc\frac{du_{\alpha}}{ds} = \frac{1}{c\sqrt{1-v^2/c^2}}\left(\frac{\vec{F}\cdot\vec{v}}{c}, -\vec{F}\right)^T \quad \text{with} \quad \vec{F} = \frac{dm(v)\vec{v}}{dt}.$$

[2]

5.3. What is the advantage of expressing the electromagnetic field in tensor form? **[2]**

5.4. Write down the 3 rotations and the 3 Lorentz boosts of the Riemann-Silberstein vector in matrix form. **[2]**

5.5. Why is the square of the Riemann-Silberstein vector invariant under rotations and Lorentz transformations? **[2]**

5.6. Why is $F_{\alpha\beta}F^{\alpha\beta} = -2(E^2 - c^2B^2)$? Give a short argument, if possible. **[2]**

5.7. Show that the relations $\vec{E} = -\partial\vec{A}/\partial t - \nabla U$ and $\vec{B} = \nabla\times\vec{A}$ imply the first group of Maxwell's equations, $\nabla\times\vec{E} = -\partial\vec{B}/\partial t$ and $\nabla\cdot\vec{B} = 0$. Explain briefly the meaning of these Maxwell equations. **[2]**

5.8. Show that $\partial_{\mu}F^{*\mu\nu} = 0$ is equivalent to the first group of Maxwell's equations in the traditional 3D form. ($F^{*\mu\nu}$ denotes the dual tensor given in the course.) **[2]**

5.9. Explain the Principle of Least Action for fields. **[2]**

5.10. What are the assumptions made for deducing the Lagrangian of the free electromagnetic field? **[2]**

Short questions, due 17th November

6.1. In the course we derived the Euler-Lagrange equations for 4D vector fields A_α. Consider the simpler case of a scalar field ϕ with Lagrangian density $\mathscr{L}(\phi, \partial_\alpha \phi)$. Derive the Euler-Lagrange equations. **[3]**

6.2. Consider the field ϕ in one-dimensional space (1+1 dimensional space-time). Show that you can write the Euler-Lagrange equations as

$$\frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{\partial \dot{\phi}} + \frac{\partial}{\partial x}\frac{\partial \mathscr{L}}{\partial \phi'} = \frac{\partial \mathscr{L}}{\partial \phi},$$

where the dot denotes the time derivative and the prime the space derivative. Explain why this is a generalization of the Euler-Lagrange equations from mechanical objects to fields. **[2]**

6.3. Consider a linear chain of identical point masses interacting with one another via springs with identical spring constants k. The Lagrangian for the departures q_a of the masses from their equilibrium positions is

$$L = \sum_a \left[\frac{m}{2}\left(\frac{dq_a}{dt}\right)^2 - \frac{k}{2}\left(q_a - q_{a-1}\right)^2\right].$$

Show that in the continuum limit $q_a(t) \sim q(t, x)$, you get

$$L = \int \mathscr{L} dx, \quad \mathscr{L} = \frac{\rho}{2}\left[\left(\frac{\partial q}{\partial t}\right)^2 - c^2\left(\frac{\partial q}{\partial x}\right)^2\right], \quad \rho = \frac{m}{\delta}, \quad c^2 = \frac{k}{m}\delta^2,$$

where δ denotes the spacing between the masses. **[3]**

6.4. Derive the Euler-Lagrange equations for the linear chain of discrete point masses and in the continuum limit. Which physical phenomenon do the equations describe? **[3]**

6.5. Give a physical interpretation for the 4D current of point charges that was derived in the course. **[2]**

6.6. Derive the second group of Maxwell's equations in tensor form from the Euler-Lagrange equations generated by the Lagrangian $\mathscr{L} = -(\epsilon_0/4)F_{\alpha\beta}F^{\alpha\beta} - A_\alpha j^\alpha$. **[3]**

6.7. Show that the charge-conservation law, $\partial_\alpha j^\alpha = 0$, follows from the second group of Maxwell's equations in tensor form. **[2]**

6.8. Why does the Lagrangian $-(\epsilon_0/4)F_{\alpha\beta}F^{\alpha\beta} - A_\alpha j^\alpha$ lead to a gauge-invariant action? **[2]**

Short questions, due 24th November

7.1. Give a physical interpretation for the retarded potential

$$A^\alpha = \int \frac{j^\alpha(t - |\vec{r} - \vec{r}'|/c, \vec{r}')}{4\pi\varepsilon_0 \, |\vec{r} - \vec{r}'|} \, d^3r' \, .$$

[2]

7.2. What is the Lorentz gauge and what is its advantage? **[2]**

7.3. Why is $f(x,t) = f_1(t - x/c) + f_2(t + x/c)$ with the two arbitrary functions $f_1(t)$ and $f_2(t)$ the solution of $\partial_\alpha \partial^\alpha f = 0$ in one spatial dimension (1+1 space-time dimensions)? **[2]**

7.4. Consider the electromagnetic radiation emitted from a localized source (dipole radiation). Use the expressions for $\vec{A}$ and U derived in the course and the underlying approximations to show that

$$\vec{E} \sim \frac{\vec{r} \times (\vec{r} \times \ddot{\vec{d}})}{4\pi\varepsilon_0 r^3} \, .$$

[3]

7.5. Give a physical interpretation for the electric and the magnetic field of dipole radiation (mostly in words, not formulas): Why does the radiation at radius r and time t depend on the source at time $t - r/c$? How are $\vec{E}$ and $\vec{B}$ related to each other, why is the radiation transversal and why do $\vec{E}$ and $\vec{B}$ fall like $1/r$? Why does the radiation depend on the second time derivative of the dipole moment? **[2]**

7.6. Give a physical interpretation for the structure of the symmetric energy-momentum tensor $T^{\mu\nu}$ of the electromagnetic field: What is the meaning of the $T^{0\nu}$ and of the $T^{k\nu}$ components? Why do you need a tensor to describe the conservation of energy and momentum for fields? **[2]**

7.7. Show that $T^{\mu\nu} = \varepsilon_0 F^{\mu\alpha} F_{\alpha\beta} g^{\beta\nu} + (\varepsilon_0/4) F^{\alpha\beta} F_{\alpha\beta} \, g^{\mu\nu}$ is a symmetric tensor such that $T^{\mu\nu} = T^{\nu\mu}$. What is the advantage of this symmetry? **[2]**

7.8. Use the traditional 3D form of Maxwell's equations to prove that

$$\frac{\partial \mathscr{H}}{\partial t} + \nabla \cdot \vec{S} = 0$$

for the energy density $\mathscr{H}$ and the Poynting vector $\vec{S}$. **[2]**

7.9. Show that for dipole radiation with

$$\vec{E} = -c \, \frac{\vec{r}}{r} \times \vec{B} \, , \quad \vec{B} = \frac{\ddot{\vec{d}} \times \vec{r}}{4\pi\varepsilon_0 c r^2} \, , \quad \ddot{\vec{d}} \cdot \vec{r} = |\ddot{\vec{d}}| \, r \, \cos\theta$$

the Poynting vector is

$$\vec{S} = \frac{c}{(4\pi)^2 \varepsilon_0} \, \frac{\vec{r}}{r^2} \, \ddot{\vec{d}}^2 \sin^2\theta \, .$$

Give a physical interpretation for this result. In the case of a mobile phone, $\ddot{\vec{d}}$ points in the direction of the antenna. In which direction does most of the radiation go? **[3]**

Short questions, due 1st December

8.1. Why can you regard the quantum-mechanical wave function $\psi(t, \vec{x})$ as a field? What is the traditional physical interpretation for this field? (If you know a non-standard interpretation, such as Bohm's pilot waves, write it down as well if you like.) **[2]**
8.2. Why is the Schrödinger equation a first-order differential equation in time? **[2]**
8.3. Assuming that the relativistic version of the Schrödinger equation is first-order in time, why does it have to be of first order in the spatial derivatives as well? **[2]**
8.4. What are the assumptions made for deducing the Dirac equation? **[2]**
8.5. Why do you need matrices and not just numbers to satisfy the crucial relation $\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2g^{\alpha\beta}$? **[2]**
8.6. Why does the relation $\Lambda^{-1}\gamma^\alpha\Lambda = \Lambda'^{\alpha}_{\beta}\gamma^\beta$ guarantee the Lorentz invariance of the Dirac equation? **[2]**
8.7. Show that $\cosh(\xi/2)\mathbb{1}_4 + \sinh(\xi/2)\sigma_L$ is the inverse of $\cosh(\xi/2)\mathbb{1}_4 - \sinh(\xi/2)\sigma_L$ and that $\cos(\phi/2)\mathbb{1}_4 - i\sin(\phi/2)\sigma_R$ is the inverse of $\cos(\phi/2)\mathbb{1}_4 + i\sin(\phi/2)\sigma_R$. **[1]**
8.8. Prove that the two expressions for ψ_0 given in section 5.3 are solutions of the Dirac equation. Give interpretations for the two ψ_0. **[2]**
8.9. Transform the two ψ_0 to an inertial frame moving in the x-direction. Give interpretations for the resulting wave functions. **[2]**
8.10. Explain the central ideas of local phase invariance and the connection to gauge fields. **[3]**

Short questions, due 8th December

9.6. Consider instead of the Dirac spinor a scalar complex field ψ with the Lagrangian density

$$\mathscr{L} = \frac{\hbar^2}{2m} g^{\alpha\beta} \left(D_\alpha \psi\right)^* \left(D_\beta \psi\right) - \frac{mc^2}{2} \psi^* \psi \,, \quad D_\alpha = \partial_\alpha + i \frac{q}{\hbar c} A_\alpha \,.$$

The complex conjugation sign applies both to ψ and to the operator D_α, such that $D_\alpha^* = \partial_\alpha - i(q/\hbar c)A_\alpha$. Show that $\mathscr{L}$ is invariant under local phase shifts of the field ψ if A_α is transformed by appropriate gauge transformations. **[2]**

9.7. In the Lagrangian density of 9.6., ψ and ψ^* should be regarded as independent fields. (The real and imaginary parts of ψ constitute two real-valued fields. Alternatively, you can regard ψ and ψ^* as two independent entities.) Write down the Euler-Lagrange equations for ψ and ψ^* and show that both are equivalent to

$$D^\alpha D_\alpha \psi + \frac{m^2 c^2}{\hbar^2} \psi = 0 \,.$$

This equation is known as the Klein-Gordon equation. However, Schrödinger formulated it first, before he discovered the non-relativistic Schrödinger equation. Schrödinger realized that the scalar relativistic equation does not give the Rydberg spectrum of the hydrogen atom and abandoned this approach. Months later, he found the Schrödinger equation. However, the Dirac equation describes the hydrogen atom very well, apart from QED corrections (Lamb shift) and corrections due to the finite size of the nucleus (QCD effects). **[3]**

9.8. Return to Dirac's theory with the Lagrangian density

$$\mathscr{L} = \bar{\psi} \left(\gamma^\alpha \left(i\hbar c \partial_\alpha - q A_\alpha\right) - mc^2\right) \psi \,.$$

Regard ψ and $\bar{\psi}$ as independent fields and show that the Euler-Lagrange equations for ψ and $\bar{\psi}$ lead to the Dirac equation for ψ in the presence of the electromagnetic field. **[2]**

9.9. What is the numerical value of $\mathscr{L}$ when ψ obeys the Dirac equation? Does it matter? And if not, why? **[2]**

9.* What do you think about the course? What is the best and what is the worst thing? Did you learn something? Please list the topics that you would like to have discussed in the tutorial sessions during the last week. **[1]**