Special Relativity and Fields

Homework problem, due 13th October

A. Relativistic rocket science. A rocket is initially at rest (with respect to the inertial frame K). Then the engines are fired and the rocket starts moving. Assume that the exhaust gases are ejected at a constant rate with the fixed speed v_e in opposite direction with respect to the rocket. Let m_1 be the rest mass of the rocket that will, of course, change when the exhaust material is being ejected. In this exercise you should calculate the velocity of the rocket as a function of the remaining mass m_1.

A.1. As a first step, note that the system rocket plus exhaust is isolated; the total energy and the total momentum are conserved. Consider what is happening from one infinitesimal step of rocket acceleration to the next. Let $\mathrm{d}m_2$ be the mass of the small chunk of exhaust material ejected within each infinitesimal step. Let v_1 be the velocity of the rocket and v_2 the velocity of the exhaust gas, as seen in the inertial frame K. Both m_1, v_1, and v_2 vary when $\mathrm{d}m_2$ is ejected. The conservation laws are

$$\mathrm{d}\left(\frac{m_1c^2}{\sqrt{1-v_1^2/c^2}}\right)+\frac{c^2\mathrm{dm}_2}{\sqrt{1-v_2^2/c^2}}=0\,,\quad \mathrm{d}\left(\frac{m_1v_1}{\sqrt{1-v_1^2/c^2}}\right)+\frac{v_2\mathrm{dm}_2}{\sqrt{1-v_2^2/c^2}}=0\,.$$

In each conservation law you need to consider the differential of the entire first term, but in the second term only $\mathrm{d}m_2$. Why? **[2]**

A.2. Calculate $\mathrm{d}m_1$ and $\mathrm{d}m_2$ as functions of m_1, v_1, $\mathrm{d}v_1$, and v_2. **[3]**

A.3. Consider the gained velocity of the rocket in relation to the lost rest mass, $\mathrm{d}v_1/\mathrm{d}m_1$, using your result of A.2. Apply the addition theorem of velocities in order to represent v_2 as a function of v_1 and of the constant exhaust speed relative to the rocket, v_e. Show that you get the simple result

$$\frac{\mathrm{d}v_1}{\mathrm{d}m_1}=\frac{v_e}{m_1}\left(1-\frac{v_1^2}{c^2}\right)\,.$$

[4]

A.4. Solve this differential equation and represent the solution $v_1(m_1)$ as a function of the ratio R between the present rest mass m_1 and the initial mass m_0 of the rocket. **[4]**

A.5. What is the required mass ratio R in order to reach a given velocity v_1? Why is it an advantage to use the fastest exhaust gas possible (in the best case, light)? **[3]**

A.6. Given enough mass, the rocket can reach any velocity $|v_1|<c$. In particular, it can exceed the speed of the exhaust gas that is propelling the rocket, as long as $|v_e|<c$. Why is this possible? Give a physical reason. **[2]**

A.7. To illustrate your findings, put in some numbers. Suppose that you want to reach half the speed of light with the rocket. How many tons of rocket fuel plus rocket do you need for each ton of rocket, if you use a chemical rocket engine with an exhaust speed $|v_e|=10^4m/s$? How many tons do you need for a photon drive with $|v_e|=c$? **[2]**

Homework problem, due 20th October

B. Oscillating Force. Suppose that an atom is illuminated by highly intense laser light. The outer electrons of the atom may experience an oscillating electromagnetic field that is much stronger than the electric field of the nucleus and of the other electrons, binding the atom together. The electrons oscillate with the light, rapidly reaching relativistic velocities. Consider the following simple model for each of the electrons. Suppose that the electron is a classical particle, subject to the electric force of the electromagnetic field. We ignore the Lorentz force and we assume the electric field to be uniform along the trajectory of the electron, for simplicity. The electron is initially at rest and is then accelerated in the direction of the electric field, *i.e.* in the polarization direction of light. Assume linear polarization in the x-direction. The equation of motion is, for light with frequency ω,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{v}{\sqrt{1-v^2/c^2}} = a\cos\omega t\,.$$

B.1. Solve the equation of motion for $v(t)$ and express the result in terms of the dimensionless parameter $\alpha = a/(\omega c)$. When is v^2 maximal and what is the maximal velocity $v_{\max}$? When is the velocity zero? **[4]**

B.2. Integrate v to find the position x of the electron. What is the maximal range $x_{\max}$ of the electron in the limit $\alpha \to \infty$? **[4]**

B.3. Suppose you could ignore the relativistic mass factor $(1-v^2/c^2)^{-1/2}$ in the equation of motion, such that $\mathrm{d}v/\mathrm{d}t = a\cos\omega t$. Calculate $x_{\max}$ and compare it with the correct relativistic result. **[2]**

Homework problem, due 3rd November

C. Moving Dipoles. Atoms are electrically neutral, but most atoms are polarizable in electric fields, developing an induced electric dipole moment. The potential of such an induced dipole is $V = -(\alpha/2)E'^2$, where α denotes the polarizability and $\vec{E}'$ is the electric field as seen in the frame of the atom moving with velocity $\vec{v}$. We will show in the course that $\vec{E}' \approx \vec{E} + \vec{v} \times \vec{B}$ for $v/c \ll 1$. Suppose that the magnetic field B is such that $c^2B^2 \ll E^2$ (Even fairly strong magnetic fields are normally much weaker that easily produced electric fields, because magnetism is relativistic in nature.)

C.1. • Write down the Lagrangian $L = (m/2)v^2 - V$, neglecting terms quadratic in the magnetic field. • Show that the atom appears to move as if it were a charged particle in effective electric and vector potentials $\mathcal{U}$ and $\vec{\mathcal{A}}$, such that

$$q\mathcal{U} = -\frac{\alpha}{2}E^2\,, \quad q\vec{\mathcal{A}} = -\alpha\vec{E} \times \vec{B}.$$

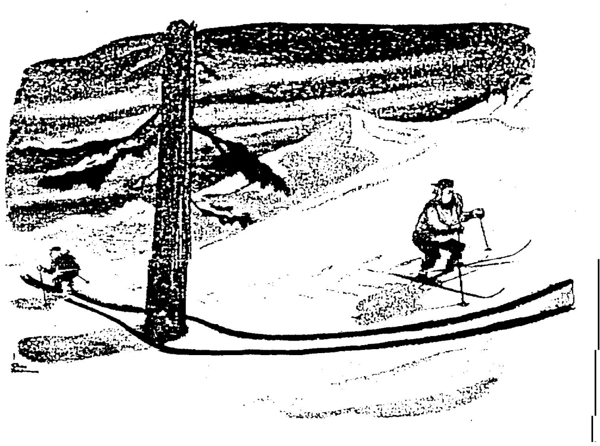
[4]

C.2. Suppose you use the following setup: a charged straight wire and a homogeneous magnetic field $\vec{B}$ with field lines parallel to the wire, say in the z-direction. The wire produces the electric field

$$\vec{E} = \frac{Q}{2\pi\varepsilon_0 r^2}(x, y, 0)\,, \quad r^2 = x^2 + y^2.$$

• Calculate $q\vec{\mathcal{A}}$ and show that the effective "magnetic field" $\nabla \times \vec{\mathcal{A}}$ vanishes. **[2]**

C.3. You see from the result of C.2. that there is no classical force due to the magnetic field on the atom (for the particular setup we are considering). However, atoms are quantum particles and a single atom can go around the wire similar to the skier in the picture. In the semiclassical approximation, you can represent the wavefunction ψ of the atom as $\psi = |\psi| \exp(iS/\hbar)$, where S is the classical action for which, as you know, $\nabla S = \vec{p} = m\vec{v} + q\vec{\mathcal{A}}$ in the non-relativistic limit. Therefore, the



phase difference $\Delta\phi$ between the left and right paths of the atom around the wire is

$$\Delta\phi = \frac{\Delta S}{\hbar} = \frac{1}{\hbar}\oint \vec{p}\cdot d\vec{r} = \frac{q}{\hbar}\oint \vec{\mathcal{A}}\cdot d\vec{r}.$$

• Why does $\Delta\phi$ not depend on the particular integration path (*i.e.* the trajectory)?
• Calculate $\Delta\phi$ for a convenient path. You found a remarkable result; although there is no classical force, there is a quantum phase shift. **[4]**

C.4. The classical motion of the atom is interesting as well. If the atom gets too close to the wire, it will become fatally attracted, like a particle falling into a black hole. You can describe the trajectory in the (x, y) plane using complex numbers $z = x + iy$. The effective potential $\mathcal{U}$ leads to the equation of motion

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = -\frac{a^2 z}{|z|^4}, \quad a^2 = \frac{\alpha Q^2}{4\pi^2\varepsilon_0 m}.$$

• Show that

$$z = w_+^{(1+\mu)/2}\, w_-^{(1-\mu)/2}, \quad w_\pm = -vt \pm i\frac{b}{\mu}, \quad \frac{1}{\mu^2} = 1 - \frac{a^2}{v^2 b^2}$$

is a solution for real b and v with $v > 0$. • Show that initially, for $t \sim -\infty$, the trajectory $z(t)$ approaches $-vt + ib$, which implies that the particle is incident from the far right moving to the left with velocity $-v$ in x-direction and with the offset b in y-direction (the impact parameter b). For $a^2 < v^2 b^2$ the parameter μ is real, while μ is imaginary for $a^2 > v^2 b^2$. In the latter case, the particle falls irresistibly towards the wire at $z = 0$ until it gets stuck and engaged in chemical reactions such that the equation of motion is not applicable anymore. • Why does imaginary μ correspond to a fatal crash course and why does the particle escape when μ is real? [Hint: Discuss $|z|^4 = |w_+ w_-|^2 = (v^2 t^2 + b^2 \mu^{-2})^2$.] • Show that in the limiting case of $a^2 = v^2 b^2$, you get

$$z = -vt \exp\left(-i\frac{b}{vt}\right).$$

• Draw the trajectory for this case. **[10]**

Homework problem, due 24th November

D. Spinor Representation of Maxwell's Equations. Maxwell's equations may appear in various forms. Maxwell's original form reads, in contemporary SI units,

$$\begin{aligned}
\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}&=0\,, & \frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}&=\frac{\rho}{\epsilon_0},\\
\frac{\partial B_x}{\partial t}+\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}&=0\,, & \frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}-\frac{1}{c^2}\frac{\partial E_x}{\partial t}&=\frac{j_x}{\epsilon_0 c^2},\\
\frac{\partial B_y}{\partial t}+\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}&=0\,, & \frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}-\frac{1}{c^2}\frac{\partial E_y}{\partial t}&=\frac{j_y}{\epsilon_0 c^2},\\
\frac{\partial B_z}{\partial t}+\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}&=0\,, & \frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}-\frac{1}{c^2}\frac{\partial E_z}{\partial t}&=\frac{j_z}{\epsilon_0 c^2}.
\end{aligned}$$

The traditional 3D form is

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}\,,\quad \nabla\cdot\vec{B}=0\,,\quad \nabla\times\vec{B}=\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}+\frac{1}{\epsilon_0 c^2}\vec{j}\,,\quad \nabla\cdot\vec{E}=\frac{\rho}{\epsilon_0}.$$

The 4D tensor form is

$$\partial_\alpha F^{*\alpha\beta}=0\,,\quad \epsilon_0\partial_\alpha F^{\alpha\beta}=j^\beta,$$

and there are integral representations as well. In this exercise you will derive one of the most concise forms of Maxwell's equations, the spinor representation

$$\sigma^\alpha(\epsilon_0\partial_\alpha\phi-j_\alpha)=0\,.$$

Here

$$\phi=(\vec{E}+ic\vec{B})\cdot\vec{\sigma}\,,\quad \sigma^\alpha=(\mathbb{1},\vec{\sigma})\,,\quad \vec{\sigma}=(\sigma_x,\sigma_y,\sigma_z)\,,$$

$\mathbb{1}$ is the identity matrix and the σ's are the Pauli matrices

$$\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

It is understood that $\vec{a}\cdot\vec{\sigma}$ abbreviates $a_x\sigma_x+a_y\sigma_y+a_z\sigma_z$.

D.1. Write ϕ in matrix form, to get an impression how it looks like. **[2]**

D.2. Prove the following identity for any vectors $\vec{a}$ and $\vec{b}$ and the vector of the Pauli matrices $\vec{\sigma}$:

$$(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma})=(\vec{a}\cdot\vec{b})\mathbb{1}+i(\vec{a}\times\vec{b})\cdot\vec{\sigma}.$$

Here $(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma})$ means the matrix product of $\vec{a}\cdot\vec{\sigma}$ and $\vec{b}\cdot\vec{\sigma}$. **[8]**

D.3. Use this identity and the traditional form of Maxwell's equations to prove the validity of the spinor representation. **[5]**

Homework problem, due 8th December

E. Electromagnetic media. Transparent materials such as glass or water are linear dielectric media. In such media, the electromagnetic field induces atomic dipoles, generating a macroscopic electric polarization and/or magnetization that, in non-dispersive media, is proportional to the instantaneous electric and/or the magnetic field, respectively. In isotropic media the polarization and magnetization vectors point in the directions of the local electric and magnetic fields. Non-dispersive isotropic linear media are described in the constitutive equations in SI units

$$\vec{D}' = \varepsilon_0 \varepsilon \vec{E}' \,, \quad \vec{B}' = \mu_0 \mu \vec{H}' \,, \quad \varepsilon_0 \mu_0 = c^{-2}$$

that supplement Maxwell's equations

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \,, \quad \nabla \cdot \vec{B} = 0 \,, \quad \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \,, \quad \nabla \cdot \vec{D} = 0 \,.$$

For simplicity, we consider the case where no external charges and currents are present. The primes indicate that the constitutive equations are valid in coordinate frames that are locally co-moving with the medium. In such frames the medium is locally at rest. Note that the velocity of the medium may vary in space and time. For example, a rotating glass disk moves with non-uniform velocities that are proportional to the distance from the centre of rotation. Therefore a locally co-moving frame is only valid in one point of the medium, in general. In this exercise you should express the electromagnetism in media in a form that is valid in all inertial frames.

E.1. As a first step, construct the tensor

$$H^{\alpha\beta} = \begin{pmatrix} 0 & -D_x & -D_y & -D_z \\ D_x & 0 & -H_z/c & H_y/c \\ D_y & H_z/c & 0 & -H_x/c \\ D_z & -H_y/c & H_x/c & 0 \end{pmatrix} .$$

Explain briefly why the Maxwell equations $\nabla \times \vec{H} = \partial \vec{D}/\partial t$ and $\nabla \cdot \vec{D} = 0$ are equivalent to $\partial_\alpha H^{\alpha\beta} = 0$. **[1]**

E.2. Consider the tensor

$$G^{\alpha\beta} = g^{\alpha\beta} + (\varepsilon\mu - 1) u^\alpha u^\beta \,,$$

where u^α denotes the local four-velocity of the medium. Show that in a locally co-moving frame where the medium is locally at rest,

$$H^{\alpha\beta} = \frac{\varepsilon_0}{\mu} G^{\alpha\alpha'} G^{\beta\beta'} F_{\alpha'\beta'} \,.$$

Since $G^{\alpha\beta}$ is given in an expression that is valid in all frames, this form of the constitutive equations must be universally valid. **[4]**

E.3. In part E.2., you derived a remarkable result. The medium appears to modify the procedure to raise indices from the fundamentally covariant field tensor $F_{\alpha\beta}$ to the tensor $H^{\alpha\beta}$ that appears in Maxwell's equations. One can show in General Relativity that a gravitational field acts in precisely the same way, where $G^{\alpha\beta}$ would be the contravariant metric tensor that, in General Relativity, is a function of the coordinates and not the constant $g^{\alpha\beta}$ of Special Relativity. Media act like gravity and often gravity acts like a medium, for example in gravitational lensing. • Prove that the inverse of $G^{\alpha\beta}$ is the metric tensor

$$G_{\alpha\beta} = g_{\alpha\beta} + \left(\frac{1}{\varepsilon\mu} - 1\right) u_\alpha u_\beta \,.$$

This formula shows how the medium modifies the measure of space and time for the electromagnetic field. The correction to the ordinary flat $g_{\alpha\beta}$ is proportional to the product of the four-velocities of the medium. The prefactor $(\varepsilon\mu)^{-1} - 1$ is Fresnel's dragging coefficient that we discussed in the course in connection with the Fizeau experiment. **[2]**

E.4. • Show how to derive Maxwell's equations for $H^{\alpha\beta}$ from the Lagrangian density

$$\mathscr{L} = -\frac{1}{4} H^{\alpha\beta} F_{\alpha\beta} \,.$$

[2]

E.5. • Show that

$$\mathscr{L} = \frac{\vec{E}\cdot\vec{D}}{2} - \frac{\vec{B}\cdot\vec{H}}{2} \,.$$

[1]

E.6. Consider the tensor that is known as the Minkowski energy-momentum tensor of the electromagnetic field in a medium,

$$T^\alpha_\beta = H^{\alpha\nu} F_{\nu\beta} - \mathscr{L}\delta^\alpha_\beta \,.$$

• Use the 4D form of Maxwell's equations for $H^{\alpha\beta}$ and the Bianchi identity for $F_{\alpha\beta}$ to show that T^α_β is conserved in a uniform medium where ε, μ and u^α are constant,

$$\partial_\alpha T^\alpha_\beta = 0.$$

[5]