Two-stage bargaining solutions

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Abstract

We introduce some procedural considerations in axiomatic bargaining theory. To this effect we characterize a new class of bargaining solutions: those which can be obtained by sequentially applying two binary relations, which include the Pareto relation, to eliminate alternatives. As a by-product we obtain as a particular case a partial characterization result by Zhou [17] of an extension of the Nash axioms and solution to domains including non-convex problems, as well as a complete characterization of solutions that satisfy Pareto optimality, Covariance with positive affine transformations, and Independence of irrelevant alternatives.

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1 Introduction

Imagine an arbitrator who can rank alternatives on the basis of a fairness criterion. He chooses an alternative from the feasible set by means of the following procedure. First, he discards all alternatives which are Pareto dominated. Then, among the remaining ones, he picks the fairest alternative. In this paper we introduce a new class of bargaining solutions that generalizes (to arbitrary criteria) this intuitive two-stage procedure. A two-stage bargaining solution is a solution which can be constructed by sequentially applying two asymmetric binary relations P_1 and P_2. More precisely, the solution point from each feasible set determines the (single-valued) set within the P_1-maximizers that P_2-dominates all the other P_1-maximizers. We provide a complete characterization of two-stage bargaining solutions for which P_1 and P_2 include the Pareto relation..

There are several features of interest in our concept and characterizations. Firstly, the class of sequential procedures by which we model the arbitrator's decisions is natural. Indeed, Tadenuma ([13], [14]) has pioneered the sequential application of exactly the two above criteria, efficiency and fairness, in social choice.[1] Our contribution generalizes and abstracts this idea within an axiomatic bargaining framework á la Nash [10].[2]

Secondly, we consider solutions on domains that include non-convex problems. In this respect our paper is related to a number of papers on the ex-

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[1] Tadenuma studies in particular the effect of the order of application of the two criteria.

[2] See Thomson [16] for an overview of axiomatic bargaining theory. See Houy and Tadenuma [15] for a different generalisation of the idea of sequentially applying two (or more) criteria.

tension of bargaining solutions satisfying the original Nash axioms (discussed below) on larger domains. Our framework delivers a major by-product by yielding a generalization of a theorem of Zhou [17], which states that, on the domain of comprehensive problems, any solution that satisfies Independence of Irrelevant Alternatives (IIA), Pareto optimality (PAR) and Covariance with positive affine transformations (COV) is a selection from some asymmetric Nash multivalued solution.[3] We show that any solution that satisfies a certain weakening of IIA (discussed below), in addition to PAR and COV, sequentially maximizes two relations that are invariant under affine transformations (that is xP_iy if and only if $\tau(x)P_i\tau(y)$ for any positive affine transformation τ). If the solution satisfies IIA in full, then the two relations collapse into a single transitive relation: this provides a complete characterization of PAR, COV and IIA bargaining solutions. Zhou's partial characterization result then follows easily.

Thirdly, our approach illustrates concretely the fact that reasonable procedures based on well-behaved (though incomplete) criteria can lead to violations of the main consistency criterion in bargaining theory, namely IIA. On a standard domain, a bargaining solution maximizes a single binary relation if and only if it satisfies IIA (Peters and Wakker [12]), and on many domains including non-convex problems this single relation must be transitive when the solution also satisfies Pareto optimality (Denicoló and Mariotti [2]). With our framework, the sequential application of the two relations can generate solutions which violate IIA, even when the relations themselves are

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[3] Zhou's theorem is also obtained with different techniques by Denicoló and Mariotti [2] and in a recent paper by Peters and Vermeulen [11].

well-behaved in the sense of being transitive. For example, suppose that, under some fairness concept, c is fairer than a which is fairer than b, and that the only possible Pareto comparison is that b Pareto dominates c. Then the arbitrator chooses a from $\{a, b, c\}$ (first discarding c by Pareto dominance and then b by fairness), but he chooses c from $\{a, c\}$ (by applying fairness).

Our weakening of IIA is achieved through two consistency axioms that, together with PAR, characterize two-stage bargaining solutions. The first is Expansion (EXP): if x is the solution point of each problem in a class of bargaining problems then it is the solution of their union. The second axiom is Weak IIA (WIIA): if x is the solution point of two nested bargaining problems R and T which both contain y, then y is not the solution point of any 'intermediate' problem S. WIIA allows some 'menu effects' excluded by IIA. EXP and WIIA are both implied by IIA while the converse is not true (as the Pareto/fairness example in the opening paragraph illustrates).

In the next section we introduce the notation. Section 3 contains the main characterization results. Section 4 focusses on solutions that satisfy COV, in addition to PAR and the independence axioms. Section 5 contains some brief comments on interpretation.

2 Preliminaries

A *(bargaining) problem* is a pair (S, d), where $S \subset \mathcal{R}^n$ and $d \in S$. The set S is interpreted as the set of feasible alternatives (welfare or utility vectors for n agents) from which an arbitrator must choose, and d is a distinguished point relevant for the arbitrator's decision. Following common usage, we call

d the *disagreement point.*

$\mathcal{R}^n$ is viewed as a vector space, with the origin and the unit vector denoted 0 and e, respectively. The vector inequalities are: $s > t$ iff $s_i > t_i$ for all i; $s \geq t$ iff $s_i \geq t_i$ for all i. A positive affine transformation is a function $\tau : \mathcal{R}^n \rightarrow \mathcal{R}^n$ such that, for some real numbers $\alpha_i > 0$ and β_i, $i = 1, ..., n$, $\tau_i(s) = \alpha_i s_i + \beta_i$ for all i. Given $S \subset \mathcal{R}^n$ and and a transformation τ, denote $\tau(S) = \{t \in \mathcal{R}^n | t = \tau(s) \text{ for some } \in S\}$. For a bargaining problem (S, d), denote $\tau(S, d) = (\tau(S), \tau(d))$.

Given a domain of bargaining problems Σ, a *solution on* Σ is a function $\gamma : \Sigma \rightarrow \mathcal{R}^n$ such that $\gamma(S, d) \in S$ for all $(S, d) \in \Sigma$. Sometimes we will refer to *multi-solutions*, for which γ is allowed to be a correspondence.

We consider a very general class of domains of bargaining problems.[4] Say that a domain Σ of bargaining problems is *admissible* if the following assumptions hold:

D1: For all $(S, d) \in \Sigma$: S is compact, there exists $s \in S$ such that $s > d$, and $s \geq d$ for all $s \in S$.

D2: For all $d \in \mathcal{R}^n$, for all $s, t \in \{u \in \mathcal{R}^n | u > d\}$, there exists a unique $(M(s,t), d) \in \Sigma$ such that: (a) $s, t \in M(s,t)$ and for all $u \in M(s,t)$ with $u \neq s, t$, $s \geq u$, or $t \geq u$, or both. (b) for any $(S, d) \in \Sigma$ with $s, t \in S$, $M(s,t) \subseteq S$.

D3: For any class $\{S^k, d\} \in \Sigma$ of problems, $\left(\bigcup_k S^k, d\right) \in \Sigma$.

All domains of non-convex problems considered in the literature are particular cases of admissible domains. For example the set of comprehensive problems (Zhou [17], Peters and Vermeulen [11]), the set of finite problems

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[4] This class was essentially introduced in Denicolo and Mariotti [2].

(Mariotti [7], [8], Peters and Vermeulen [11]), the set of all problems satisfying D1 (Kaneko [3]), the set of d-star shaped problems.[5] While D1 is standard and D3 straightforward, D2 deserves a special note. Its role in the analysis is to guarantee the existence of a 'minimal' problem containing any two given alternatives, and such that the solution point for the minimal problem is (under PAR) one of those two alternatives.

From now on, unless specified otherwise, fix an admissible domain Σ. We consider the following axioms on solutions, intended for all $(S,d),(R,d) \in \Sigma$:

COV: For any positive affine transformation τ, $\gamma(\tau(S,d)) = \tau(\gamma(S,d))$.

PAR: For all $s \in \mathcal{R}^n$ with $s \geq \gamma(S,d)$ and $s \neq \gamma(S,d)$: $s \notin S$.

IIA: If $\gamma(S,d) \in R \subset S$, then $\gamma(R,d) = \gamma(S,d)$.

EXP: Given a class of problems $\{S^k, d\}$, if $s = \gamma(S^k, d)$ for all k, then $s = \gamma\left(\bigcup_k S^k, d\right)$.

WIIA: If $\gamma(R,d) = \gamma(T,d) \neq t$ and $t \in R \subset S \subset T$, then $\gamma(S,d) \neq t$.

The first three axioms are standard in bargaining theory. EXP is standard in choice theory. Only the last axiom is relatively new. It can be interpreted as follows. Start with IIA: one way of reading it is that if new alternatives are added to the problem, then either the solution point is unchanged, or the new solution point is one of the new alternatives. In other words, there are no 'menu-effects': the effect of a new alternative cannot be to change the solution point to one of the 'old' alternatives. By contrast, WIIA allows for some such menu effects. However, *suppose* that adding a large set of new alternatives does not produce any effect. Then adding a *smaller* set of new alternatives does not make an old alternative a solution point. A stronger

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[5] That is, those problems (S,d) for which the convex hull of $\{d,s\}$ is in S for all $s \in S$.

version of this axiom states that if $R \subset S \subset T$ and $\gamma(R,d) = \gamma(T,d) = s$ then $\gamma(S,d) = s$. This stronger version was introduced (with the name of 'Independence of Revealed Irrelevant Alternatives') in Mariotti [6]. The present weakening appears in Lombardi [4].

3 Two-stage bargaining solutions

As standard, we consider only solutions that satisfy translation-covariance (as most known solutions do).[6] This permits to simplify notation by normalizing the disagreement point of all problems to the origin. A bargaining problem will then be defined simply as a subset of $\mathcal{R}^n_+$ containing the origin, and a bargaining solution can be denoted accordingly. The main new definition of this paper is the following (where $\max(S,P)$ denotes the set of maximal elements of the relation P in the set S).

Definition 1 *A solution γ is a two-stage solution if there exist two asymmetric relations P_1 and P_2 on $\mathcal{R}^n_{++}$ such that, for all $S \in \Sigma$,*

$$\{\gamma(S)\} = \left\{ s \in \max\left(S \cap \mathcal{R}^n_{++}, P_1\right) | sP_2t \textit{ for all } t \in \max\left(S \cap \mathcal{R}^n_{++}, P_1\right), t \neq s \right\}$$

In this case we say that P_1 and P_2 rationalize γ.

If P_1 and P_2 can be chosen so that $P_1 = P_2 = P$ we say that the solution is a degenerate two-stage solution, rationalized by P.

We have discussed the idea behind this definition in the introduction, so we shall not repeat it here.

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[6] In obvious notation, translation covariance means $\gamma(S+t, d+t) = \gamma(S,d) + t$ for all $t \in \mathcal{R}^n$.

Example 1 (Nash solutions): The symmetric Nash bargaining multi-solution ν is defined by the correspondence which associates with each problem the maximizers of the symmetric Nash product, namely

$$\nu\left(S\right) = \underset{S\cap\mathcal{R}^n_+}{\arg\max} \prod_i s_i \text{ for all } S \in \Sigma$$

A symmetric Nash selection is a solution that coincides with a selection from ν.[7] Some symmetric Nash selections (e.g. those satisfying IIA) can be naturally thought of as a two-stage solution for which sP_1t iff $\prod_i s_i > \prod_i t_i$ and the relation P_2 is used to break the ties between Nash product maximizers within each set. For a specific case, consider $n = 2$ and sP_2t iff $s_1 > t_1$. However this is a degenerate two-stage solution, as by taking the union of P_1 and P_2 one can rationalize the solution in one stage.

Example 2 (first efficiency, then equality): let sP_1t iff $s \geq t$ and $s \neq t$. Let sP_2t iff

either

$$\sum \left(\frac{s_i}{\sum s_j}\right)^2 < \sum \left(\frac{t_i}{\sum t_{ji}}\right)^2;$$

or

$$\sum \left(\frac{s_i}{\sum s_j}\right)^2 = \sum \left(\frac{t_i}{\sum t_j}\right)^2,$$

and a suitable tie-breaking rule, left undefined here, is met. The resulting two-stage solution picks the alternative that maximizes a measure of equality over the set of strongly Pareto optimal alternatives. As we have seen in the introduction, this procedure can generate cycles. Together with this

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[7] Note that the adjective ‘symmetric’ refers to the objective function to be maximized, obviously not to the selection itself, which on the usual domains will not be symmetric.

observation, our theorem 3 below shows that this two-stage solution is not degenerate.

Example 3 (first goodness, then priority): fix a set $G \subset \mathcal{R}^n_{++}$ of 'good' alternatives. Let sP_1t (with $s \neq t$) iff either

$$s \in G,\ t \in \mathcal{R}^n \backslash G \text{ and not } t \geq s$$

or

$$s, t \in G \text{ and } s \geq t$$

Let sP_2t iff $s_1 > t_1$ or $s_1 = t_1$ and $s_2 > t_2$ etcetera. The resulting two-stage solution picks a Pareto optimal and 'good' alternative, provided there exists a feasible alternative with these characteristics. Otherwise, it lexicographically maximizes the welfare of the agents (and this priority method is also used to break ties between Pareto optimal good alternatives when they exist). As a specific example, let

$$G = \left\{ s \in \mathcal{R}^n_{++} | s = \lambda e \text{ for some scalar } \lambda > 0 \right\}$$

In this case goodness is equality: if the Pareto frontier intersects the 45^0 line, the solution is egalitarian. If equality is not achievable in a Pareto optimal way, a priority list is followed. This solution does not satisfy IIA. Take the domain of finite problems (similar examples can be constructed in other admissible domains), with $n = 2$. Let $S = \{(4,2), (2,2), (5,1), (0,0)\}$. The solution selects $(4,2)$ from S (by definition $(2,2)\, P_1\, (5,1)$ and $(2,2)\, P_1\, (0,0)$. Then among the surviving alternatives $(4,2)$ is best according to P_2). However, it selects $(5,1)$ from $\{(4,2), (5,1), (0,0)\} \subset S$ (now P_1 restricted to the set is empty, and $(5,1)$ is best according to P_2). Therefore this, too, is a non-degenerate two-stage solution.

The main result of this section is a complete characterization of two-stage solutions.

Say that a relation P on $\mathcal{R}^n$ is *Pareto consistent* if it contains the strong Pareto relation. Then:

Theorem 2 *A solution is a two-stage solution, which can be rationalized by Pareto consistent* P_1 *and* P_2*, if and only if it satisfies PAR, EXP and WIIA.*

Proof: Sufficiency. Let γ be a solution that satisfies the axioms. Note first that by D2 and PAR, given $s, t \in \mathcal{R}^n_{++}$ there exist a minimal (in the order of set inclusion) problem $M(s,t) \in \Sigma$ with the property that either $\gamma(M(s,t)) = s$ or $\gamma(M(s,t)) = t$.

Now we explicitly construct the relations P_1 and P_2. Define sP_1t iff there is no $S \in \Sigma$ such that $t = \gamma(S)$ and $s \in S$. Define sP_2t iff $s = \gamma(M(s,t))$. The relation P_1 is asymmetric since sP_1t and tP_1s could be both true only if both $s \neq \gamma(S)$ for all $S \in \Sigma$ with $t \in S$ and $t \neq \gamma(S)$ for all $S \in \Sigma$ with $s \in S$: But, as observed, $\gamma(M(s,t)) \in \{s,t\}$. The asymmetry of P_2 is guaranteed by the single valuedness of γ. That P_1 and P_2 are Pareto consistent follows immediately from PAR.

For any $S \in \Sigma$, obviously there exists no $s \in S$ for which $sP_1\gamma(S)$. Take any $s \in S$ for which $sP_2\gamma(S)$: we show that then s is eliminated in the first round. Suppose to the contrary that there is no $t \in S$ with tP_1s. Therefore, by the definition of P_1, for all $t \in S \backslash s$ there exists $T_t \in \Sigma$ such that $t \in T_t$ and $s = \gamma(T_t)$. By D3, $\bigcup_t T_t \in \Sigma$. By EXP, $s = \gamma\left(\bigcup_t T_t\right)$. Since $sP_2\gamma(S)$, $s = \gamma(M(s, \gamma(S)))$. Since we have $M(s, \gamma(S)) \subseteq S \subset \bigcup_t T_t \in \Sigma$, WIIA is contradicted. We can conclude that there exists $t \in S$ such that tP_1s.

Observe finally that $\gamma(S)\, P_2 s$ for any $s \in \max\left(S \cap \mathcal{R}^n_{++}, P_1\right)$: in fact, by D2 and PAR, P_2 is a complete relation,[8] and by the previous argument it cannot be $s = \gamma(M(s, \gamma(S)))$ for any $s \in \max\left(S \cap \mathcal{R}^n_{++}, P_1\right)$.

Necessity. Let γ be a two-stage solution rationalized by P_1 and P_2 which are Pareto consistent. PAR holds obviously by the Pareto consistency of P_1 and P_2. Let $\left\{S^k\right\}$ be a class of problems. Suppose that $s = \gamma\left(S^k\right)$ for all k. Then tP_1s for no $t \in \bigcup_k S^k$. Moreover sP_2t for all $t \in \max\left(S^k \cap \mathcal{R}^n_{++}, P_1\right)$, with $t \neq s$, for all k. Therefore sP_2t for all $t \in \max\left(\left(\bigcup_k S^k\right) \cap \mathcal{R}^n_{++}, P_1\right)$ (since $\max\left(\left(\bigcup_k S^k\right) \cap \mathcal{R}^n_{++}, P_1\right) \subseteq \bigcup_k \max\left(S^k \cap \mathcal{R}^n_{++}, P_1\right)$), and EXP is satisfied. Next, suppose that $s = \gamma(R) = \gamma(T)$ with $t \in R \subset T$. Suppose by contradiction that $t = \gamma(S) \neq s$ for some $S \in \Sigma$ with $R \subset S \subset T$. Since $t = \gamma(S)$ there cannot exist $u \in R \subset S$ for which uP_1t. Then sP_2t (since $s = \gamma(R)$), and there exists $u \in S$ such that uP_1s. But this contradicts $s = \gamma(T)$. ■

The result that follows makes precise the difference between the combination of WIIA and EXP on the one hand, and IIA on the other.

Theorem 3 *A solution is a degenerate two-stage solution, which can be rationalized by a complete, transitive and Pareto consistent relation P, if and only if it satisfies PAR and IIA.*

Proof: Let γ be a solution that satisfies the axioms. Since IIA is easily seen to imply WIIA and EXP, asymmetric relations P_1 and P_2 can be constructed as in the proof of the previous theorem. Note that for any $s, t > 0$, if sP_1t then (by D2 and PAR) $s = \gamma(M(s,t))$ so that sP_2t. And using IIA

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[8] By complete we mean here that either sP_2t or tP_2s for any *distinct* s and t.

and the definition of $M(s,t)$, if sP_2t then sP_1t: otherwise, $t = \gamma(T)$ for some $T \in \Sigma$ with $s \in T$ would imply $t = \gamma(M(s,t))$, that is tP_2s. Therefore $P_1 = P_2 = P$.

To see that P is transitive, suppose that $sPtPu$. This means (viewing P as P_2) that $s = \gamma(M(s,t))$ and $t = \gamma(M(t,u))$. Suppose by contradiction that it is not the case that sPu, so that $u = \gamma(M(s,u))$. Let $T = M(s,t) \cup M(t,u) \cup M(s,u)$. By D3, $T \in \Sigma$. By PAR, $\gamma(T) \in \{s,t,u\}$, so that IIA applied to T and one of the sets $M(s,t)$, $M(t,u)$ or $M(s,u)$ is contradicted.

P is clearly complete and Pareto consistent by D2 and PAR. To conclude, it is easy to show that a solution that can be rationalized as in the statement satisfies the axioms. ■

We can see, then, that weakening IIA to the combination of WIIA and EXP has *two* distinct effects. First, it permits solutions that are rationalized by two relations rather than a single one. Second, it permits to relax the transitivity of the rationalizing relations.

The axioms used in theorem 2 are independent. The disagreement point solution satisfies WIIA and EXP but not PAR. The following solution (defined for $n = 2$ for simplicity) satisfies PAR and WIIA but not EXP. For a given $A \subset \mathcal{R}_+^n$,

$$\gamma_A(S) = \begin{cases} \arg\max\limits_{s \in \nu(S)} s_1 \text{ if } A \cap S \neq \varnothing \\ \arg\max\limits_{s \in \nu(S)} s_2 \text{ if } A \cap S = \varnothing \end{cases}$$

In words, if the feasible set intersects a fixed set A, the solution picks the Nash maximizer of maximum utility for player 1, and otherwise it picks the Nash maximizer of maximum utility for player 2. To see that EXP is violated consider the domain of finite problems and let $A = \{(0,1)\}$. We have

$$\gamma_A \{(0,1), (1,2), (0,0)\} = (1,2) = \gamma_A \{(2,1), (1,2), (0,0)\}$$

and yet

$$\gamma_A \{(0,1), (2,1), (1,2), (0,0)\} = (2,1)$$

WIIA is satisfied: suppose that $s = \gamma_A(R) = \gamma_A(T)$ with $t \in R \subset S \subset T$. Then if $A \cap R \neq \varnothing$ we have

$$s = \underset{r \in \nu(R)}{\arg\max} s_1 = \underset{r \in \nu(T)}{\arg\max} s_1$$

and also

$$\gamma_A(S) = \underset{r \in \nu(S)}{\arg\max}\ s_1$$

Therefore $\gamma_A(S) = s$, which implies the desired conclusion. Similarly if $A \cap T = \varnothing$. The remaining possibility is that $A \cap R = \varnothing$ but $A \cap T \neq \varnothing$. In this case it must be

$$s = \underset{r \in \nu(R)}{\arg\max} s_2 = \underset{r \in \nu(T)}{\arg\max} s_1$$

This implies that $|\nu(R) \cap \nu(T)| = 1$ (for otherwise the last displayed inequality could not hold). Since $\nu(R) \subseteq \nu(T)$ this means that $|\nu(R)| = 1$. Because $\gamma_A(S) \in \nu(S)$ and $R \subset S \subset T$, either $\gamma_A(S) = s$ or $\gamma_A(S) \notin R$, and in both cases WIIA is not violated.

Finally, the following solution (defined for $n = 2$ for simplicity) satisfies PAR and EXP but not WIIA. Fix x and y in $\mathcal{R}^n_{++}$ with $x \neq y$. Define the Pareto frontier of a set $S \in \Sigma$ as

$$PF(S) = \{s \in S | \nexists s' \in S \text{ such that } s' \neq s \text{ and } s' \geq s\}$$

Then:

$$\gamma(S)=\begin{cases} x & \text{if } x\in PF(S) \text{ and } y\notin PF(S)\\ \underset{s\in\nu(S)}{\arg\max} s_1 & \text{otherwise}\end{cases}$$

This bargaining solution is obviously Pareto optimal. To see that it satisfies EXP, suppose $\gamma(S)=\gamma(T)=x$. Then $x\in PF(S\cup T)$. If $y\notin PF(S)$ or $y\notin PF(T)$, then $y\notin PF(S\cup T)$ and $x=\gamma(S\cup T)$ as desired. If instead $y\in PF(S)\cap PF(T)$, then

$$x=\underset{s\in\nu(S)}{\arg\max} s_1=\underset{s\in\nu(T)}{\arg\max}\ s_1,$$

and therefore

$$x=\underset{s\in\nu(S\cup T)}{\arg\max} s_1.$$

If instead $\gamma(S)=\gamma(T)=z\neq x$, then the last two displayed equalities apply substituting z in place of x. However this solution fails WIIA. Fix $x=(1,10)$ and $y=(4,9)$, and let $z=(7,8)$, as in figure 1. Consider the bargaining sets $R=M(x,z)$, $S=0axyzc$ and $T=0axzc$, so that $R\subset S\subset T$. Here $\gamma(R)=x=\gamma(T)$, while $\gamma(S)=z\in R\subset S\subset T$, violating WIIA.

By using arguments essentially identical to those in the proof of theorem 2, one can obtain an extension of that theorem to even more general domains. To do so we consider a property that merges WIIA and EXP into a single axiom:

WIIA*: Let $\mathcal{S}$ be a class of problems . If $s=\gamma(R,d)=\gamma(S,d)$ for all $(S,d)\in\mathcal{S}$, and $t\in R\subset T\subset\bigcup_{(S,d)\in\mathcal{S}}S$, then $\gamma(T,d)\neq t$.

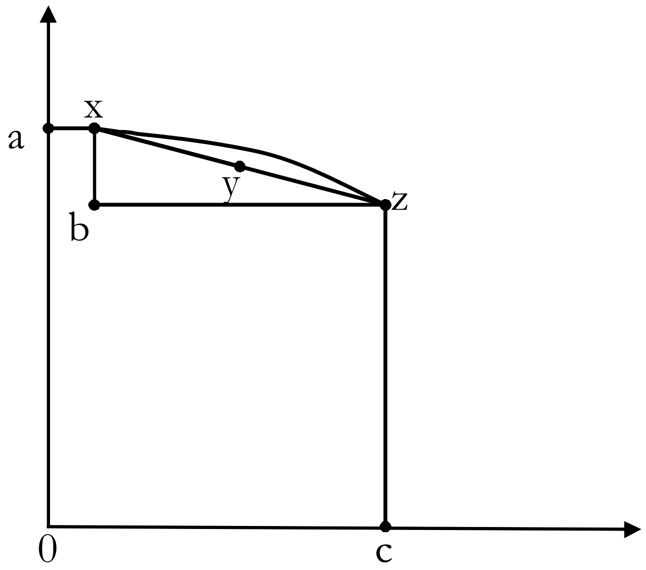

Figure 1: A solution satisfying PAR and EXP but not WIIA

Theorem 4 *Consider a domain* Σ *that satisfies D1 and D2. A solution on* Σ *is a two-stage solution, which can be rationalized by Pareto consistent* P_1 *and* P_2*, if and only if it satisfies PAR and WIIA*.*

4 Covariant solutions

In this section we consider solutions that satisfy COV. To this end, we need to make two further domain assumptions (formulated for normalized problems):
D4: For all $S \in \Sigma$, for all positive affine transformation τ: $\tau(S) \in \Sigma$.
D5: For all $s, t \in \mathcal{R}^n_{++}$, for all positive affine transformation τ: $\tau(M(s,t)) = M(\tau(s), \tau(t))$

As before, fix an admissible domain Σ, which satisfies in addition D4 and D5. A relation P on $\mathcal{R}^n_{++}$ is invariant with positive affine transformations (or *pat-invariant* in short) iff, for all positive affine transformations τ, $\tau(s) P \tau(t)$ whenever sPt.

We can now provide a characterization of two-stage and degenerate two-stage COV solutions:

Theorem 5 *(i) A solution is a two-stage solution, which can be rationalized by* P_1 *and* P_2 *that are Pareto consistent and pat-invariant, if and only if it satisfies PAR, EXP, WIIA and COV.*

(ii) A solution is a degenerate two-stage solution, which can be rationalized by a complete, transitive, Pareto consistent and pat-invariant P*, if and only if it satisfies PAR, IIA and COV.*

Proof: Let γ be a two-stage solution that satisfies COV, and define P_1 and P_2 as in the proof of theorem 2. Let τ be a positive affine transformation.

Let sP_1t. Suppose by contradiction that it is not the case that $\tau(s)\,P_1\tau(t)$. Then there exists $S \in \Sigma$ such that $\tau(t) = \gamma(S)$ and $\tau(s) \in S$. By COV, $t = \gamma(\tau^{-1}(S))$ (where $\gamma(\tau^{-1}(S))$ is well-defined by D4), so that (since $s \in \tau^{-1}(S) \in \Sigma$) sP_1t is contradicted. Next, let sP_2t. By the definition of P_2, D5 and COV it is immediate that $\tau(s)\,P_2\tau(t)$.

The statement now follows from theorems 2 and 3. ■

The interest and novelty of part (ii) of the theorem is that it is a *complete* characterization of PAR, COV and IIA solutions. In the literature only partial characterizations are stated for such solutions.

The theorem yields as an easy corollary a generalization (to different domains) of a partial characterization theorem by Zhou [17] given by Denicoló and Mariotti [2] and more recently by Peters and Vermeulen [11]. Define the asymmetric, *α-weighted Nash multi-solution*, by

$$\nu^{\alpha}(S) = \arg\max_{S\cap\mathcal{R}^n_+} \prod_i s_i^{\alpha_i}$$

for all $S \in \Sigma$, for some vector of non-negative weights $\alpha = (\alpha_1, ..., \alpha_n) \in \mathcal{R}^n_+$.

Corollary 6 *A solution that satisfies IIA, PAR and COV is a selection from some asymmetric Nash multisolution.*

Proof: It follows from theorems 3.3.3. in d'Aspremont [1], reformulated transforming the variables in logs as in Moulin [9], that for any relation P on $\mathcal{R}^n_{++}$ that is complete, transitive, Pareto consistent and pat-invariant the following holds: If $\sum_i \alpha_i \log s_i > \sum_i \alpha_i \log t_i$, then sPt. ■

5 Concluding remarks

The class of bargaining solutions we have introduced in this paper allows a wide range of *procedural* considerations to be taken into account in co-operative decision making. We argue that this is especially plausible if the compromises entailed by the bargaining solution express the decisions of an arbitrator or an external adjudicating institution. Two-stage sequential procedures of the type we have considered (in which the 'best' alternative is chosen from a 'shortlist') are common both in individual and committee decision making.

We elaborate this point at length in Manzini and Mariotti [5]. In that paper we introduce a related two-stage procedure for individual decision making from finite sets. At the technical level there are some notable differences, of which we highlight four. Firstly, in the current paper much hinges on the Pareto optimality assumption: given our very general domain, this is what allows the construction of binary relations from minimal problems (see the proof of theorem 2). Secondly, the definition of the procedure itself is different: whereas here we ask that in the second stage the selected alternative is the 'best' (it dominates all other surviving ones), which seems appropriate in a collective decision context, in the choice theory paper the chosen alternative is just the maximal one (it is undominated by all other surviving ones). Thirdly, the characterizing set of axioms is different (relying in the choice theory paper on a different weakening of IIA, which, contrary to WIIA, is well-defined only for finite sets). Fourthly, the study of covariant solutions is obviously peculiar to the bargaining framework. In respect of this last point, our technique really pays off with a simple characterization of COV, PAR and IIA solutions.[9]

Another angle from which our work can be considered is that of 'revealed group preference', as pioneered in Peters and Wakker [12]. In this interpretation, much as individual decisions may 'reveal' a preference which is being maximized, so can a negotiating group 'reveal' its collective preference.[10] The axioms characterizing a solution provide a test by means of which the existence of such a group preference can, at least in principle, be detected by observable data. From this perspective, one may note that it is hard to expect a group to act on the basis of standard preferences. It seems at least as plausible for the group to reveal an aggregating procedure, rather than an aggregating preference. Our axioms, Expansion and Weak Independence of Irrelevant Alternatives, are as easy a test of the group following a two-stage procedure as Independence of Irrelevant Alternatives is a test of the group maximizing an aggregate preference.

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[9] The technique recently introduced by [11] allows one to indentify more precisely the selections from Nash multisolutions, to address in addition issues of constructibility.

[10] This is also the interpretation given in the work by Zhou [17] which we have discussed in this paper.

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