Manipulation of Choice Behavior

Paola Manzini* Marco Mariotti† Christopher J. Tyson‡

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Abstract

We introduce and study the problem of manipulation of choice behavior. In a class of two-stage models of decision making, with the agent's choices determined by three "psychological variables," we imagine that a subset of these variables can be selected by a "manipulator." To what extent does this confer control of the agent's behavior? Within the specified framework, which overlaps with two existing models of choice under cognitive constraints, we provide a complete answer to this question.

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1 Introduction

Recent work in the theory of individual decision making has relaxed the classical preference maximization hypothesis to allow non-preference factors to impact behavior.[1] In the same spirit, we consider an abstract model in which the agent's choices result from the interaction of three functions, termed "psychological variables." These functions could be preference-related, measuring utility or aspiration levels; they could be cognitive, measuring salience levels or attention thresholds; or they could have some other interpretation. The question we address — which in the context of axiomatic choice theory appears to be novel — is the following: If one or more of the psychological variables is subject to outside influence, to what degree can the agent's choices be manipulated?

*School of Economics and Finance; University of St. Andrews; Castlecliffe, The Scores, St. Andrews KY16 9AL, U.K and IZA, Bonn, Germany. Email: [paola.manzini@st-andrews.ac.uk]

†School of Economics and Finance; University of St. Andrews; Castlecliffe, The Scores, St. Andrews KY16 9AL, U.K. Email: [marco.mariotti@st-andrews.ac.uk]. Manzini and Mariotti gratefully acknowledge financial support from ESRC grant RES–000–22–3474.

‡School of Economics and Finance; Queen Mary, University of London; Mile End Road, London E1 4NS, U.K. Email: [c.j.tyson@qmul.ac.uk].

[1] In addition to the contributions cited below, this literature includes, e.g., Ambrus and Rozen [2], Baigent and Gaertner [3], Bossert and Sprumont [4], Caplin and Dean [5], Cherepanov et al. [6], Eliaz et al. [7]. Kalai et al. [9], Mandler et al. [11], Manzini and Mariotti [13], Masatlioglu and Nakajima [14], Masatlioglu and Ok [16], Rubinstein and Salant [18], Salant and Rubinstein [19], and Spears [21].

For instance, imagine that the decision maker is a utility maximizer over the alternatives that he or she notices, but is aware only of those options with a sufficiently high level of salience (with regard to the visual or another sensory system). In this case the psychological variables are the utility, salience, and salience-threshold functions that interact to determine behavior. Now suppose that a second agent, the "manipulator," can control the salience of the alternatives but not their utilities or the salience thresholds. In this case we would like to know what varieties of choice behavior can be induced by varying the salience function while holding the other two psychological variables fixed.

The preceding example might relate to a decision maker scanning items in a shop display or on a web page, with their arrangement determined by the manipulator. Other instances of manipulation include a newspaper editor choosing the prominence of news reports, a moral or intellectual authority (such as a parent, teacher, or religious leader) shaping aspects of an agent's world-view, and a financial advisor suggesting to a client what would be an acceptable level of risk or return.

For the purposes of this paper we deemphasize the manipulator's motives and do not describe the precise mechanism used to influence behavior. One manipulator might wish to induce the purchase of high-margin or slow-selling items from a product line; another might hope to ensure that her son marries a particular type of person; and another might aim to elicit cyclical choices so as to transform the decision maker into a money pump. Rather than these objectives, our focus is on determining which choice patterns can be generated given different assumptions about the manipulator's capabilities.

To give some structure to our enterprise we need a general model of how choices are determined by the interaction of different factors. We employ an abstract two-stage framework involving three psychological variables: Given a menu A of alternatives, the decision maker's potential choices are contained in the set

$$C(A) = \operatorname*{argmax}_{x \in A} g(x) \text{ subject to } f(x) \geq \theta(A), \tag{1}$$

where f and g are defined on the space of alternatives and θ on the space of menus (with all three functions being real-valued).

This framework has no fixed interpretation, and overlaps with two existing theories of choice based on very different hypotheses about the decision maker's mental processes. On the one hand, f could be interpreted as an attention function, θ as an attention-threshold map, and g as a utility function, as in Masatlioglu et al. [15] (and also Lleras et al. [10]). On the other hand, f could be interpreted as the utility function, θ as a utility-threshold map, and g as a measure of salience, as in Tyson [24, 25]. The first of these interpretations captures the concept of a "consideration set" familiar from the marketing literature, while the second operationalizes Simon's [20] notion of "satisficing."[2] And the model of course also allows for a standard agent with utility function g and an inactive constraint $\langle f, \theta \rangle$ that lies dormant (e.g., with both $f(x) \equiv 0$ and $\theta(A) \equiv 0$) until activated by a manipulator.

The two-stage model described by Equation 1 can accommodate a wide range of possible behaviors. Indeed, if each choice set $C(A)$ is single-valued, then we will see that *any* choice

[2] Consideration-set models are also studied by Eliaz and Spiegler [8]. Moreover, Manzini and Mariotti's "rational shortlist method" [12] can support this interpretation.

data can result from the postulated structure. More generally, the only restriction imposed by the framework itself has to do with the alternatives contained in certain "behavioral indifference classes." (See Section 2.2 for further discussion of these points.) And it follows that a manipulator with access to all three psychological variables has nearly complete control over the decision maker's choices.

It is clear that for a given interpretation of the model some psychological variables will be more naturally assumed to be manipulable than others. Not wishing to commit to any particular viewpoint, we provide a complete and hence interpretation-free analysis of the extent of manipulability in the present framework: For any subset of the variables f, θ, and g, we give necessary and sufficient conditions for choice behavior to be attainable with the specified subset of variables fixed and the remaining variables free (from the perspective of the manipulator).

Broadly speaking, our method of characterization is to determine the information about the free variables that would be revealed by the choices the manipulator wishes to induce, and to state conditions that rule out the possibility of a contradiction. For example, suppose that both f and θ are fixed and known, while g can be manipulated.[3] If alternatives x and y are both on menu A and if furthermore $f(x), f(y) \geq \theta(A)$, then inducing $x \in C(A)$ and $y \notin C(A)$ requires the manipulator to set $g(x) > g(y)$ in view of Equation 1. If in addition x and y are on a menu B such that $\theta(B) \leq \theta(A)$, then plainly the manipulator *cannot* induce $y \in C(B)$. This is the sense in which the psychological variables that are assumed to be fixed, together with the structure of the model, constrain the choice data that can be generated by manipulation. And conditions that capture all such constraints characterize the manipulator's capabilities in each of the cases we consider.

2 Model

2.1 Preliminaries

We take as given a nonempty, finite set X, together with a set $\mathcal{D} \subseteq 2^X \setminus \{\emptyset\} =: \mathcal{A}$ such that each $\{x\} \in \mathcal{D}$. Each $x \in X$ is an *alternative*, and each $A \in \mathcal{D}$ is a *menu*. A *choice function* is any $\xi : \mathcal{D} \to \mathcal{A}$ such that $\forall A \in \mathcal{D}$ we have $\xi(A) \subseteq A$. Here $\xi(A)$ is the *choice set* assigned to A, with the interpretation that those and only those alternatives in $\xi(A)$ could be chosen from this menu.

In the context of our characterization results we fix a "target" choice function, denoted by C, with the interpretation that the manipulator would like to induce the decision maker to behave according to this rule. Hence the alternatives that the manipulator wishes to be choosable from menu A are contained in the set $C(A)$.

Functions $f, g : X \to \Re$ are *criteria*, while $\theta : \mathcal{D} \to \Re$ is a *threshold map*. Collectively these objects are referred to as *psychological variables*. Any triplet $\langle f, \theta, g \rangle$ is a *psychological profile*, with f and g referred to, respectively, as the *primary* and *secondary* criterion.

Definition 1. ***A.*** *Given $\langle f, \theta \rangle$, the* threshold set *of each $A \in \mathcal{D}$ is defined by*

$$\Gamma(A|f, \theta) := \{x \in A : f(x) \geq \theta(A)\} \subseteq A. \tag{2}$$

[3] Note that this case is covered by Theorem 5 below.

B. *Given θ, the* critical threshold *of each $x \in X$ is defined by*

$$M(x|\theta) := \max\{\theta(A) : A \in \mathcal{D} \wedge x \in C(A)\} \in \Re. \tag{3}$$

The threshold set $\Gamma(A|f,\theta)$ contains the alternatives on menu A that pass the relevant threshold according to the primary criterion, while the critical threshold $M(x|\theta)$ is the highest of the thresholds of those menus to whose choice sets alternative x belongs. Note that since for each $x \in X$ we have both $\{x\} \in \mathcal{D}$ and $x \in C(\{x\})$, the critical threshold is always well defined. And observe also that for each $A \in \mathcal{D}$ we have $x \in C(A)$ only if $M(x|\theta) \geq \theta(A)$.

Given any (binary) relation R on X, we write R' for its converse, $\overline{R}$ for its complement, and R^* for its transitive closure.[4] Such a relation is said to be a strict partial order if it is irreflexive ($x\overline{R}x$) and transitive ($xRyRz$ only if xRz), a weak order if it is also negatively transitive ($x\overline{R}y\overline{R}z$ only if $x\overline{R}z$), and a linear order if in addition it is weakly complete ($x\overline{R}y\overline{R}x$ only if $x = y$). An equivalence is a relation that is transitive, reflexive (xRx), and symmetric (xRy only if yRx).

2.2 Two-stage threshold representations

As discussed in Section 1, we study the following model of behavior.

Definition 2. *A* two-stage threshold *(or* TST*)* representation *of C is a psychological profile $\langle f,\theta,g\rangle$ such that $\forall A \in \mathcal{D}$ we have*

$$C(A) = \underset{x\in\Gamma(A|f,\theta)}{\operatorname{argmax}}\, g(x). \tag{4}$$

When this relationship holds we say that C is *induced* by the profile $\langle f,\theta,g\rangle$.

Our first aim is to identify which choice functions are induced by psychological profiles, independently of whether the psychological variables are controlled by the decision maker or the manipulator. We shall then proceed to study manipulation per se, determining which of these inducible functions remain achievable when one or more of the variables is fixed.

To answer the question of which choice functions are consistent with our basic model, we introduce several binary relations that are "behavioral" in the sense of being derived from the target function C.

Definition 3. *The* separation relation *S is defined by xSy if and only if $\exists A \in \mathcal{D}$ such that both $x \in C(A)$ and $y \in A \setminus C(A)$.*

Thus xSy if there exists some menu from which x is choosable and y is not. Under classical assumptions this relation would be a weak order and would reveal strict preference on the part of the decision maker, but in general we can be sure only that it is irreflexive.

Definition 4. ***A.*** *The* togetherness relation *T is defined by xTy if and only if $\exists A \in \mathcal{D}$ such that $x, y \in C(A)$.* ***B.*** *The* extended togetherness relation *E is defined by $E = T^*$.*

[4] That is to say, $\forall x, y \in X$ we have $xR'y$ iff yRx; $x\overline{R}y$ iff $\neg[xRy]$; and xR^*y iff $\exists n \geq 2$ and $x_1, \ldots, x_n \in X$ such that $x = x_1Rx_2R\cdots Rx_n = y$.

Thus xTy whenever x and y are simultaneously choosable, and xEy whenever there exists a chain of alternatives linking x to y and related sequentially by T. Clearly T is both reflexive (since $x \in C(\{x\})$ for all x) and symmetric, and it follows that E is an equivalence. In the classical case E-equivalence classes of course amount to revealed indifference curves.

In general the separation and extended togetherness relations are not mutually exclusive. A choice function with $x[S \cap E]y$ (i.e., both xSy and xEy) is given in the following example, which shows that not all target functions can be induced by psychological profiles.

Example 1. *Let $C(xyz) = xyz$, $C(xy) = x$, $C(yz) = y$, and $C(xz) = z$.*[5] *If C were induced by $\langle f, \theta, g \rangle$, then $C(xyz) = xyz$ would imply $g(x) = g(y) = g(z)$. But then the remaining choice data would imply $f(x) > f(y) > f(z) > f(x)$, a contradiction.*

On the other hand, $S \cap E \neq \emptyset$ does not guarantee inconsistency with the model, as a small modification to the above example confirms.

Example 2. *Let $C(xyz) = xy$, $C(xy) = x$, $C(yz) = y$, and $C(xz) = z$. Then C is induced by $\langle f, \theta, g \rangle$ with $f(x) = 2$, $f(y) = 1$, $f(z) = 0$, $\theta(xyz) = \theta(yz) = 1$, $\theta(xy) = 2$, $\theta(xz) = 0$, $g(x) = g(y) = 0$, and $g(z) = 1$.*

Both examples above exhibit the separation cycle $xSySzSx$. Example 2 shows that this, as well as $S \cap E \neq \emptyset$, can be reconciled with the model, so what is the feature of Example 1 that generates the inconsistency? The key difference between these examples is that the first combines the two non-classical phenomena just noted, while the second keeps them apart. More precisely, Example 1 contains a cycle in the relation $S \cap E$; or, equivalently, an S-cycle within an E-equivalence class. And the absence of such cycles is just what is needed for our baseline characterization.

Theorem 1. *The target function C is induced by a psychological profile if and only if $S \cap E$ is acyclic.*[6]

To reiterate, a necessary and sufficient condition for the target function to be inducible is the absence of S-cycles within E-equivalence classes. Any other type of S-cycle is permitted, and no monotonicity (e.g., contraction or expansion consistency) conditions are imposed on C. In particular, no constraint links pairwise choices to those from larger menus; so that, e.g., the data $C(xyz) = x$, $C(xy) = y$, and $C(xz) = z$ are allowed despite "pairwise dominated" x being choosable from xyz. (Indeed, pairwise choice data may not even be available.) Note also that whenever the target function is single-valued (i.e., $|C(A)| = 1$ for each $A \in \mathcal{D}$) we have $\emptyset = T = E = S \cap E$ and so $S \cap E$ is trivially acyclic. Hence:

Corollary 1. *Any single-valued target function is induced by a psychological profile.*

Theorem 1 and Corollary 1 tell us that the general model restricts behavior very little, and a decision maker or manipulator who controls all three psychological variables can generate a large class of choice functions. Previous work relating to TST representations reduces this freedom by imposing additional restrictions thought to be appropriate under a particular

[5] Note the multiplicative notation for enumerated sets, used here and in other examples for conciseness.

[6] Recall that acyclicity of R is the property that xR^*y only if $x \neq y$.

interpretation of the model: Masatlioglu et al. [15] require that the first stage — which we express via the operator $\Gamma(\cdot|f,\theta)$ — be an "attention filter," while Tyson [24, 25] requires that the structure $\langle f,\theta\rangle$ be "expansive."[7] Both of these assumptions have significant behavioral consequences beyond acyclicity of $S \cap E$, even in the single-valued case.

In the present paper we take a different approach to narrowing the class of choice functions covered by Theorem 1. Instead of constraining the allowable profiles with assumptions that seem natural under a preferred interpretation of the TST framework, we simply take one or more psychological variables as given. The remaining variables are those that can be freely chosen by the manipulator, and we seek to characterize the resulting behavior.[8]

Theorem 1 is proved in Appendix A, as are the results to follow. To appreciate how the argument works, consider the following direct proof of Corollary 1. Given any single-valued target C, for each $A \in \mathcal{D}$ write $C(A) = \{x_A\}$. Let f be any one-to-one function, define θ by $\theta(A) = f(x_A)$, and let $g(x) \equiv -f(x)$. Now by construction each $x \in A$ with $f(x) > f(x_A)$ has $g(x) < g(x_A)$, and it follows that C is induced by $\langle f,\theta,g\rangle$.

To extend this construction to choice functions that are not single-valued, we must allow the primary criterion to distinguish between alternatives that are both separated and either chosen together or linked by a chain of alternatives that are chosen together sequentially. We thus require that f order X in accordance with $S \cap E$, with the acyclicity condition ensuring that no contradiction arises at this stage. The threshold map θ can then be defined by $\theta(A) \equiv \min f[C(A)]$. And for a secondary criterion g that is constant on E-equivalence classes and otherwise defined in opposition to f, the profile $\langle f,\theta,g\rangle$ will induce the target C.

A notable feature of this argument is that in both cases the psychological profile we construct is highly non-unique: In proving Corollary 1 we can take f to be *any* one-to-one function, and in the general case substantial arbitrariness remains. This freedom is important for the results to follow.

3 Manipulation results

3.1 Stage-one manipulation

We consider first a situation in which the secondary criterion g is fixed and known, while the primary criterion f and the threshold map θ are under the manipulator's control. As noted above, $\Gamma(\cdot|f,\theta)$ can in this case be interpreted as a manipulable "consideration set" operator, with g playing the role of the decision maker's utility function.

With g fixed, we can define the weak order that this real-valued function represents.

Definition 5. *Given g, the relation G is defined by xGy if and only if $g(x) > g(y)$.*

[7] A function $\xi : \mathcal{D} \to \mathcal{A}$ is an attention filter if $\forall A, B \in \mathcal{D}$ such that $\xi(B) \subseteq A \subseteq B$ we have $\xi(A) = \xi(B)$. A criterion-threshold pair $\langle f,\theta\rangle$ is expansive if $\forall A, B \in \mathcal{D}$ such that $A \subseteq B$ and $\max f[A] \geq \theta(B)$ we have $\theta(A) \geq \theta(B)$.

[8] These two approaches to specializing Theorem 1 could also be combined. For example, we could study manipulation of TST choice behavior (designating fixed and free psychological variables) under the attention filter assumption. Although much of our technique is likely to be transferrable to such settings, for the time being we investigate manipulation in the general case to avoid committing to a particular interpretation of psychological profiles.

Under any TST representation two alternatives can be choosable together only if the secondary criterion does not distinguish between them. More generally, consistency of the target C with the observed criterion g requires that xEy only if $g(x) = g(y)$, which is to say that $E \subseteq \overline{G}$. In addition, since xSy implies either $f(x) > f(y)$ or $g(x) > g(y)$, the joint observation xSy and $g(x) = g(y)$ implies $f(x) > f(y)$. From the latter implication we can conclude that the relation $S \cap \overline{G}$ must be acyclic. And the pair of conditions we have just identified turns out to characterize the class of choice functions that can be induced given a pre-specified secondary criterion g.

Theorem 2. *Given g, there exist $\langle f, \theta \rangle$ such that the target function C is induced by $\langle f, \theta, g \rangle$ if and only if both $E \subseteq \overline{G}$ and $S \cap \overline{G}$ is acyclic.*

To understand the content of this result, assume first that the secondary criterion g is one-to-one, which can be interpreted as the decision maker's preferences linearly ordering the set of alternatives. Whatever psychological variables $\langle f, \theta \rangle$ are selected by the manipulator, the resulting choice function will of course be single-valued. Indeed, it is apparent from our discussion of Corollary 1 that *any* single-valued target is achievable in this case. For such a C we have $E = \emptyset \subseteq \overline{G}$, and so need only confirm that $S \cap \overline{G}$ is acyclic. But a cycle in this relation would amount to an S-cycle within a g-indifference class — an impossibility since g is one-to-one and S is irreflexive.

When g is not one-to-one, the constraint faced by the manipulator is essentially that the target function C cannot have behavioral indifference (i.e., E-equivalence) classes coarser than those of the unmanipulated choice function that simply maximizes g. The manipulator can use the first-stage variables to separate alternatives that are g-indifferent, provided the new separations create no cycles in the relation $S \cap \overline{G}$. But he or she can do nothing to manufacture a new behavioral indifference xEy in the event that $g(x) \neq g(y)$.

In summary, a manipulator who controls both stage-one variables can exert tremendous influence over the decision maker's behavior, to the extent that the choices resulting from linear order preferences can be distorted to yield any single-valued target function. Several examples of such manipulation follow: In the first, the agent's preferences are behaviorally reversed; in the second, the manipulator induces a binary choice cycle; and in the third, a globally-indifferent agent ends up exhibiting a somewhat incoherent pattern of choices.

Example 3. *Let $g(x) = 2$, $g(y) = 1$, and $g(z) = 0$. For $f(x) = 0$, $f(y) = 1$, and $f(z) = 2$; and for $\theta(xyz) = \theta(xz) = \theta(yz) = 2$ and $\theta(xy) = 1$; the profile $\langle f, \theta, g \rangle$ induces the target function C given by $C(xyz) = C(xz) = C(yz) = z$ and $C(xy) = y$.*

Example 4. *Let $g(x) = g(y) = 1$ and $g(z) = 0$. For $f(x) = 1$, $f(y) = 0$, and $f(z) = 2$; and for $\theta(xyz) = \theta(yz) = 0$, $\theta(xy) = 1$, and $\theta(xz) = 2$; the profile $\langle f, \theta, g \rangle$ induces the target function C given by $C(xyz) = xy$, $C(xy) = x$, $C(xz) = z$, and $C(yz) = y$.*

Example 5. *Let $g(x) = g(y) = g(z) = 0$. For $f(x) = 1$, $f(y) = 0$, and $f(z) = 2$; and for $\theta(xyz) = 1$, $\theta(xy) = \theta(yz) = 0$, and $\theta(xz) = 2$; the profile $\langle f, \theta, g \rangle$ induces the target function C given by $C(xyz) = xz$, $C(xy) = xy$, $C(xz) = z$, and $C(yz) = yz$.*

Note that in Example 4 the target function would no longer be achievable if $C(yz) = yz$, since we would then have $y[E \cap G]z$. Similarly, in Example 5 the target function would cease to be achievable if $C(xyz) = x$, since we would then have $x[S \cap \overline{G}]z[S \cap \overline{G}]x$.

We consider next the case of both θ and g fixed, with the manipulator controlling only f. This could apply, for instance, if the thresholds returned by θ were cognitive characteristics of the decision maker, and if the primary criterion f measured a property of the alternatives to which some sensory system responds. How does the manipulator's loss of control over the threshold map affect his or her ability to influence the agent's behavior?

The following example shows that manipulability is more limited in this case.

Example 6. *Let $\theta(xy) \geq \theta(xyz)$, $g(y) \geq g(x)$, $C(xyz) = x$, and $y \in C(xy)$. If for some f the profile $\langle f, \theta, g\rangle$ were to induce C, then $y \in C(xy)$ would imply that $f(y) \geq \theta(xy) \geq \theta(xyz)$. But then $g(y) \geq g(x)$ and $x \in C(xyz)$ would imply that $y \in C(xyz)$, a contradiction.*

To capture the constraint illustrated in Example 6, we define a new relation in terms of both the target function C and the observable variable θ.

Definition 6. *Given θ, the relation H_θ is defined by $xH_\theta y$ if and only if $\exists A \in \mathcal{D}$ such that $x \in C(A)$, $y \in A \setminus C(A)$, and $M(y|\theta) \geq \theta(A)$.*

Since here we have $f(y) \geq M(y|\theta) \geq \theta(A)$ and hence $g(x) > g(y)$ if C is induced by $\langle f, \theta, g\rangle$, the relation H_θ reveals strict dominance according to the secondary criterion and must be consistent with the observed G. Indeed, this consistency can replace the acyclicity condition in Theorem 2 to yield the desired characterization result.

Theorem 3. *Given $\langle \theta, g\rangle$, there exists an f such that the target function C is induced by $\langle f, \theta, g\rangle$ if and only if both $E \subseteq \overline{G}$ and $H_\theta \subseteq G$.*

Observe once again that if C is single-valued then $E \subseteq \overline{G}$ holds vacuously. A significant difference from Theorem 2, however, is that in this case not every single-valued target can be achieved even with g one-to-one. The condition $H_\theta \subseteq G$ prohibits contradictions between revealed and observed secondary-criterion dominance, and Example 6 demonstrates that this requirement is germane whether or not C is single-valued. The scope for manipulation using f is thus substantially narrower than that using the entire first stage $\langle f, \theta\rangle$. In particular, when θ is constant the manipulator can only partially reverse the agent's preferences:

Example 7. *Let $\theta(xyz) = \theta(xy) = \theta(xz) = \theta(yz) = 1$; and let $g(x) = 2$, $g(y) = 1$, and $g(z) = 0$. For $f(x) = 0$ and $f(y) = f(z) = 1$, the profile $\langle f, \theta, g\rangle$ induces the target function C given by $C(xyz) = C(xy) = C(yz) = y$ and $C(xz) = z$.*

And similarly some variation in θ is needed to induce a binary cycle:

Example 8. *Let $\theta(xyz) = \theta(xy) = \theta(yz) = 0$ and $\theta(xz) = 1$; and let $g(x) = 2$, $g(y) = 1$, and $g(z) = 0$. For $f(x) = f(y) = 0$ and $f(z) = 1$, the profile $\langle f, \theta, g\rangle$ induces the target function C given by $C(xyz) = C(xy) = x$, $C(yz) = y$, and $C(xz) = z$.*

To complete our analysis of stage-one manipulation, we now consider the possibility that θ rather than f is externally controlled. This would correspond to scenarios where a shortlist is formed based on the primary criterion and where the stringency of the membership rule can be controlled, though the f-values themselves must be taken as given. For instance, we could have a search engine returning results based on some objective measure of relevance, but with the manipulator free to choose the number of results displayed.

In the same way that G is represented by g, we can define the weak order represented by f when the latter is observable.

Definition 7. *Given f, the relation F is defined by xFy if and only if $f(x) > f(y)$.*

And just as C and θ together have implications for the secondary criterion (recorded in H_θ) in the context of Theorem 3, C and f together have such implications in the present case.

Definition 8. *Given f, the relation H_f is defined by $H_f = S \cap \overline{F}$.*

The condition $H_f \subseteq G$ replaces $H_\theta \subseteq G$ in the earlier result, counterbalancing the change in logical quantification of the psychological variables.[9]

Theorem 4. *Given $\langle f, g\rangle$, there exists a θ such that the target function C is induced by $\langle f, \theta, g\rangle$ if and only if both $E \subseteq \overline{G}$ and $H_f \subseteq G$.*

Manipulation of θ alone is illustrated in the following example.

Example 9. *Let $f(x) = f(z) = 1$ and $f(y) = 0$; and let $g(x) = g(y) = 1$ and $g(z) = 0$. For $\theta(xyz) = 0$ and $\theta(xy) = \theta(xz) = \theta(yz) = 1$, the profile $\langle f, \theta, g\rangle$ induces the target function C given by $C(xyz) = xy$, $C(xy) = C(xz) = x$, and $C(yz) = z$.*

Here $yExH_fz$, and as required $y\overline{G}xGz$. Note that the target function would no longer be achievable if $C(xyz) = y$, since we would then have both yH_fx and $y\overline{G}x$.

3.2 Stage-two and cross-stage manipulation

Imagine now that the secondary criterion can be manipulated, while both first-stage variables are fixed. As noted above, this corresponds to an interpretation of TST representations as modeling satisficing behavior with salience effects (see Tyson [25]), with salience controlled by the manipulator.

When the full first-stage structure $\langle f, \theta\rangle$ is observable, one obvious necessary condition for C to be inducible is that any alternative choosable from a menu pass the relevant threshold; that is, for all $x \in A \in \mathcal{D}$ we can have $x \in C(A)$ only if $f(x) \geq \theta(A)$. Indeed, any violation of this condition would directly contradict the definition of a TST representation. As in the two previous cases we can also search for information about the secondary criterion revealed by C in conjunction with the fixed variables.

Example 10. *Let $C(xy) = x$, $C(xz) = z$, $C(yz) = yz$, $f(y) \geq \theta(xy)$, and $f(x) \geq \theta(xz)$. If for some g the target C were induced by $\langle f, \theta, g\rangle$, then $f(y) \geq \theta(xy)$ and $C(xy) = x$ would imply $g(x) > g(y)$, and similarly $f(x) \geq \theta(xz)$ and $C(xz) = z$ would imply $g(z) > g(x)$. But also $C(yz) = yz$ would imply $g(y) = g(z)$, a contradiction.*

Capturing the constraint on C seen in Example 10 leads us to define a third and final relation of revealed secondary-criterion dominance.

Definition 9. *Given $\langle f, \theta\rangle$, the relation $H_{f\theta}$ is defined by $xH_{f\theta}y$ if and only if $\exists A \in \mathcal{D}$ such that $x \in C(A)$, $y \in A \setminus C(A)$, and $f(y) \geq \theta(A)$.*

[9]Note that $H_f \subseteq G$ is equivalent to $S \subseteq F \cup G$, which clearly holds when C is induced by $\langle f, \theta, g\rangle$ and can be checked directly if both f and g are observable.

In the present case g is a free variable, so we cannot impose $H_{f\theta} \subseteq G$ as a behavioral axiom. Instead we employ a "congruence" condition of the sort introduced by Richter [17], which serves to rule out the type of contradiction in Example 10. Since in the context of a TST representation $xH_{f\theta}y$ implies $g(x) > g(y)$ and xEy implies $g(x) = g(y)$, we have that $x[H_{f\theta} \cup E]y$ implies $g(x) \geq g(y)$. It follows that $x[H_{f\theta} \cup E]^*y$ has the same implication. But $g(x) \geq g(y)$ is the same as $\neg[g(y) > g(x)]$, which implies $y\overline{H_{f\theta}}x$. Together with stage-one consistency, this form of congruence characterizes manipulation of the secondary criterion.

Theorem 5. *Given $\langle f, \theta \rangle$, there exists a g such that the target function C is induced by $\langle f, \theta, g \rangle$ if and only if both $C(\cdot) \subseteq \Gamma(\cdot|f, \theta)$ and $[H_{f\theta} \cup E]^* \subseteq [\overline{H_{f\theta}}]'$.*

The following example illustrates manipulation of the secondary criterion.

Example 11. *Let $f(x) = 2$, $f(y) = 1$, and $f(z) = 0$; and let $\theta(xyz) = 1$, $\theta(xy) = 2$, and $\theta(xz) = \theta(yz) = 0$. For $g(x) = 0$, $g(y) = 2$, and $g(z) = 1$, the profile $\langle f, \theta, g \rangle$ induces the target function C given by $C(xyz) = C(yz) = y$, $C(xy) = x$, and $C(xz) = z$.*

Here $yH_{f\theta}zH_{f\theta}x$, and as required $\neg[xH_{f\theta}y]$. Setting $C(xyz) = xy$ would render the target function unachievable, since we would then have both $zH_{f\theta}xEy$ and $yH_{f\theta}z$.

We complete our study of manipulation in the TST environment by considering the cases in which the free variables are the secondary criterion and either the primary criterion or the threshold map (but not both). For example, manipulation of $\langle f, g \rangle$ might be plausible in the consideration-set framework if the manipulator were able to display alternatives more or less prominently (thereby determining f) and at the same time make them more or less intrinsically desirable (thereby determining g). Needless to say, this case is also compatible with two independent manipulators, such as a manufacturer influencing g and a retailer influencing f.

Each of the two cases in question is characterized by a congruence axiom analogous to that in Theorem 5, with the appropriate revealed secondary-dominance relation substituted for $H_{f\theta}$.

Theorem 6. ***A.*** *Given θ, there exist $\langle f, g \rangle$ such that the target function C is induced by $\langle f, \theta, g \rangle$ if and only if $[H_\theta \cup E]^* \subseteq [\overline{H_\theta}]'$.* ***B.*** *Given f, there exist $\langle \theta, g \rangle$ such that the target function C is induced by $\langle f, \theta, g \rangle$ if and only if $[H_f \cup E]^* \subseteq [\overline{H_f}]'$.*

Our final example illustrates manipulation of f and g together.

Example 12. *Let $\theta(xyz) = 0$ and $\theta(xy) = \theta(xz) = \theta(yz) = 1$. For $f(y) = f(z) = 1$ and $f(x) = 0$; and for $g(x) = g(y) = 1$ and $g(z) = 0$; the profile $\langle f, \theta, g \rangle$ induces the target function C given by $C(xyz) = xy$, $C(xy) = C(yz) = y$, and $C(xz) = z$.*

Here $xEyH_\theta z$, and as required $\neg[zH_\theta x]$. Note that if $C(yz) = yz$ then we would have both $zEyEx$ and $xH_\theta z$, so the target function would no longer be achievable.

Our various characterizations of manipulability are summarized in Table 1.

	variables		
Theorem	fixed	free	conditions on C and fixed variables
1	—	f, θ, g	$S \cap E$ acyclic
2	g	f, θ	$E \subseteq \overline{G}$ and $S \cap \overline{G}$ acyclic
3	θ, g	f	$E \subseteq \overline{G}$ and $H_\theta \subseteq G$
4	f, g	θ	$E \subseteq \overline{G}$ and $H_f \subseteq G$
5	f, θ	g	$C(\cdot) \subseteq \Gamma(\cdot\|f, \theta)$ and $[H_{f\theta} \cup E]^* \subseteq [\overline{H_{f\theta}}]'$
6A	θ	f, g	$[H_\theta \cup E]^* \subseteq [\overline{H_\theta}]'$
6B	f	θ, g	$[H_f \cup E]^* \subseteq [\overline{H_f}]'$

Table 1: A summary of our characterization results.

4 Discussion

To the best of our knowledge two-stage threshold representations have not previously been studied in the general form of Definition 2. As has already been mentioned, Masatlioglu et al. [15] and Tyson [25] formulate models that overlap with this definition, employing specific interpretations and associated restrictions on the psychological variables. The literature also deals with one-stage threshold representations of the following sort.

Definition 10. *A* threshold representation *of C is a pair $\langle f, \theta \rangle$ such that $C(\cdot) = \Gamma(\cdot|f, \theta)$.*

These are trivially a special case of TST representations, and correspondingly Theorem 1 generalizes the following result attributed to Aleskerov and Monjardet [1].

Proposition 1. *The target function C admits a threshold representation if and only if S is acyclic.*

Indeed, when $g(x) \equiv 0$ and thus the second stage is inconsequential, our Theorem 2 amounts to a restatement of Proposition 1 (since in this case $G = \emptyset$ and $S \cap \overline{G} = S$).

As is apparent from Table 1, our results provide a complete answer to the manipulability question as it is posed in this paper. The analysis is, however, limited to TST representations, and naturally similar questions can be asked in the context of the many models of choice that do not fit into the present framework (or that impose additional assumptions; see Footnote 8). To the extent that these models have features in common — such as a multi-stage structure — with the class we consider, our techniques may be transferrable. But even so we consider our present findings only a first pass at the manipulability problem, and view the framing of this problem for axiomatic choice theory as potentially an equally-important contribution of the paper.

In conclusion, a word about the nature of the manipulator. While both the name we have given this agent and illustrations relating to marketing may suggest interference by entities with dubious motives, manipulation could also be benevolent. Under this interpretation our analysis could be seen as a step in the direction of formalizing Thaler and Sunstein's [23] notion of "nudging": A well-informed and paternalistic manipulator could steer an imperfect choice procedure towards improved outcomes. Results such as ours would then show how the possibilities for improvement depend on the instruments available to the nudger.

A Proofs

Theorems are proved in order of their appearance, although some earlier proofs exploit later results. Write $K(x)$ for the E-equivalence class of x, and let $\mathcal{K} = \{K(x) : x \in X\} \subseteq \mathcal{A}$.

Proof of Theorem 1. If C is induced by $\langle f, \theta, g\rangle$, then $G \subseteq \overline{E}$. Moreover, by Theorem 4 we have $S \cap \overline{F} = H_f \subseteq G \subseteq \overline{E}$, so $S \cap E \subseteq F$. But then since F is acyclic, $S \cap E$ must also be acyclic.

Conversely, suppose that $S \cap E$ is acyclic. Let $\gg$ be any linear order on $\mathcal{K}$. Define a relation R by xRy iff either $K(x) \gg K(y)$ or $x[S \cap E]y$, and define a second relation G by xGy iff $K(y) \gg K(x)$. Since $\gg$ is a linear order (on $\mathcal{K}$), G is a weak order (on X) and has a representation g. Note that $E \subseteq \overline{G}$ by construction. Furthermore, since $\gg$ is a linear order and $S \cap E \subseteq E$, we have that xR^*y only if $K(y)\overline{\gg}K(x)$.

Suppose that $\exists x_1, \ldots, x_n \in X$ such that $x_1 R x_2 R \cdots R x_n = x_1$. Since $S \cap E$ is acyclic, there must then exist some $k < n$ such that $K(x_k) \gg K(x_{k+1})$. But since also $x_{k+1} R^* x_k$, we have $K(x_k)\overline{\gg}K(x_{k+1})$, a contradiction. It follows that R is acyclic.

For $x, y \in X$ we have $x[S \cap \overline{R}]y$ only if both $K(x)\overline{\gg}K(y)$ and $x\overline{E}y$, which implies that $K(y) \gg K(x)$ and xGy. This shows that $S \cap \overline{R} \subseteq G$, or $S \cap \overline{G} \subseteq R$. But then since R is acyclic, $S \cap \overline{G}$ must also be acyclic, and the existence of $\langle f, \theta\rangle$ such that $\langle f, \theta, g\rangle$ induces C follows by Theorem 2. □

Proof of Theorem 2. If C is induced by $\langle f, \theta, g\rangle$, then $E \subseteq \overline{G}$. Moreover, by Theorem 4 we have $S \cap \overline{F} = H_f \subseteq G$, so $S \cap \overline{G} \subseteq F$. But then since F is acyclic, $S \cap \overline{G}$ must also be acyclic.

Conversely, suppose that both $E \subseteq \overline{G}$ and $S \cap \overline{G}$ is acyclic. Define a relation R by xRy iff either yGx or $x[S \cap \overline{G}]y$, and observe that then $R^* \subseteq [G' \cup \overline{G}]^* = [\overline{G}]^* \subseteq \overline{G}$ since G is a weak order.

Suppose that $\exists x_1, \ldots, x_n \in X$ such that $x_1 R x_2 R \cdots R x_n = x_1$. Since $S \cap \overline{G}$ is acyclic, there must then exist some $k < n$ such that $x_{k+1} G x_k$. But since also $x_{k+1} R^* x_k$, we have $x_{k+1} \overline{G} x_k$, a contradiction. It follows that R is acyclic and R^* is a strict partial order, and thus by Szpilrajn's [22] Embedding Theorem there exists a weak order $F \supseteq R^* \supseteq R$ with representation f.

We now have $E \subseteq \overline{G}$ by assumption and $H_f = S \cap \overline{F} \subseteq S \cap \overline{R} \subseteq G$ by the definition of R, so the existence of a θ such that $\langle f, \theta, g\rangle$ induces C follows by Theorem 4. □

Proof of Theorem 3. If C is induced by $\langle f, \theta, g\rangle$, then $E \subseteq \overline{G}$. Moreover, $xH_\theta y$ means that $\exists A \in \mathcal{D}$ such that $x \in C(A)$, $y \in A \setminus C(A)$, and $M(y|\theta) \geq \theta(A)$, and since $f(y) \geq M(y|\theta)$ we then have $y \in \Gamma(A|f, \theta)$ and $g(x) > g(y)$. Hence $H_\theta \subseteq G$.

Conversely, suppose that $E \subseteq \overline{G}$ and $H_\theta \subseteq G$. For each $x \in X$, let $f(x) = M(x|\theta)$. Given $A \in \mathcal{D}$ and $x \in C(A)$, we then have $x \in \Gamma(A|f, \theta)$ by construction. For $y \in C(A)$ we have $xEyEx$ and hence $x\overline{G}y\overline{G}x$, so that $g(y) = g(x)$. Moreover, for $z \in \Gamma(A|f, \theta) \setminus C(A)$ we have $M(z|\theta) = f(z) \geq \theta(A)$, so that $xH_\theta z$, xGz, and $g(z) < g(x)$. Therefore C is induced by $\langle f, \theta, g\rangle$. □

Proof of Theorem 4. If C is induced by $\langle f, \theta, g\rangle$, then $E \subseteq \overline{G}$. Moreover, $xH_f y$ means that $f(y) \geq f(x)$ and $\exists A \in \mathcal{D}$ such that $x \in C(A)$ and $y \in A \setminus C(A)$, and since $f(x) \geq \theta(A)$ we then have $y \in \Gamma(A|f, \theta)$ and $g(x) > g(y)$. Hence $H_f \subseteq G$.

Conversely, suppose that $E \subseteq \overline{G}$ and $H_f \subseteq G$. For each $A \in \mathcal{D}$, let $\theta(A) = \min f[C(A)]$. Given $A \in \mathcal{D}$ and $x \in C(A)$, we then have $x \in \Gamma(A|f, \theta)$ by construction. For $y \in C(A)$ we have $xEyEx$ and hence $x\overline{G}y\overline{G}x$, so that $g(y) = g(x)$. Now select any $w \in C(A)$ such that $f(w) = \theta(A)$. For $z \in \Gamma(A|f, \theta) \setminus C(A)$ we have $f(z) \geq \theta(A) = f(w)$, so that wH_fz, wGz, and $g(z) < g(w)$. Therefore C is induced by $\langle f, \theta, g\rangle$. □

Proof of Theorem 5. If C is induced by $\langle f, \theta, g\rangle$, then $C(\cdot) \subseteq \Gamma(\cdot|f, \theta)$. Moreover, we have both $H_{f\theta} \subseteq G \subseteq [\overline{G}]'$ and $E \subseteq [\overline{G}]'$, and so $[H_{f\theta} \cup E]^* \subseteq [\overline{G}]'^* \subseteq [\overline{G}]' \subseteq [\overline{H_{f\theta}}]'$ since G is a weak order.

Conversely, suppose that $C(\cdot) \subseteq \Gamma(\cdot|f, \theta)$ and $[H_{f\theta} \cup E]^* \subseteq [\overline{H_{f\theta}}]'$. Define a relation $\gg$ on $\mathcal{K}$ by $K_1 \gg K_2$ iff there exist $x_1 \in K_1$ and $x_2 \in K_2$ such that $x_1H_{f\theta}x_2$.

Suppose that $\exists K_1, \ldots, K_n \in \mathcal{K}$ such that $K_1 \gg K_2 \gg \cdots \gg K_n \gg K_1$. There must then exist $x_k, y_k \in K_k$ for each k such that $x_1H_{f\theta}y_2Ex_2H_{f\theta}y_3E \cdots H_{f\theta}y_nEx_nH_{f\theta}y_1Ex_1$. But then both $y_2[H_{f\theta} \cup E]^*x_1$ and $x_1H_{f\theta}y_2$, contradicting $[H_{f\theta} \cup E]^* \subseteq [\overline{H_{f\theta}}]'$. It follows that $\gg$ is acyclic, that $\gg^*$ is a strict partial order, and by Szpilrajn's Theorem that there exists a linear order $\ggg \supseteq \gg^* \supseteq \gg$ on $\mathcal{K}$.

Define a relation G by xGy iff $K(x) \ggg K(y)$, so that $E \subseteq \overline{G}$. Since $\ggg$ is a linear order (on $\mathcal{K}$), G is a weak order (on X) and has a representation g. Moreover, we have $xH_{f\theta}y$ only if $K(x) \gg K(y)$, $K(x) \ggg K(y)$, and xGy. Hence $H_{f\theta} \subseteq G$.

Given $A \in \mathcal{D}$ and $x \in C(A)$, we have $x \in \Gamma(A|f, \theta)$ by assumption. For $y \in C(A)$ we have $xEyEx$ and hence $x\overline{G}y\overline{G}x$, so that $g(y) = g(x)$. Finally, for $z \in \Gamma(A|f, \theta) \setminus C(A)$ we have $f(z) \geq \theta(A)$, so that $xH_{f\theta}z$, xGz, and $g(z) < g(x)$. Therefore C is induced by $\langle f, \theta, g\rangle$. □

Proof of Theorem 6. **A.** If C is induced by $\langle f, \theta, g\rangle$, then $C(\cdot) \subseteq \Gamma(\cdot|f, \theta)$ and so $\forall x \in X$ we have $f(x) \geq M(x|\theta)$. It follows that $H_\theta \subseteq H_{f\theta}$, and moreover we have $[H_{f\theta} \cup E]^* \subseteq [\overline{H_{f\theta}}]'$ by Theorem 5. Hence we can conclude that $[H_\theta \cup E]^* \subseteq [H_{f\theta} \cup E]^* \subseteq [\overline{H_{f\theta}}]' \subseteq [\overline{H_\theta}]'$.

Conversely, suppose that $[H_\theta \cup E]^* \subseteq [\overline{H_\theta}]'$. For each $x \in X$, let $f(x) = M(x|\theta)$. Given $A \in \mathcal{D}$ and $x \in C(A)$, we then have $x \in \Gamma(A|f, \theta)$ by construction. In this case we have also $H_\theta = H_{f\theta}$ and thus $[H_{f\theta} \cup E]^* \subseteq [\overline{H_{f\theta}}]'$, so the existence of a g such that $\langle f, \theta, g\rangle$ induces C follows by Theorem 5.

B. If C is induced by $\langle f, \theta, g\rangle$, then $E \subseteq [\overline{G}]'$. We have also $H_f \subseteq G \subseteq [\overline{G}]'$ by Theorem 4, and so $[H_f \cup E]^* \subseteq [[\overline{G}]'^* \subseteq [\overline{G}]' \subseteq [\overline{H_f}]'$ since G is a weak order.

Conversely, suppose that $[H_f \cup E]^* \subseteq [\overline{H_f}]'$. Define a relation $\gg$ on $\mathcal{K}$ by $K_1 \gg K_2$ iff there exist $x_1 \in K_1$ and $x_2 \in K_2$ such that $x_1H_fx_2$.

Suppose that $\exists K_1, \ldots, K_n \in \mathcal{K}$ such that $K_1 \gg K_2 \gg \cdots \gg K_n \gg K_1$. There must then exist $x_k, y_k \in K_k$ for each k such that $x_1H_fy_2Ex_2H_fy_3E \cdots H_fy_nEx_nH_fy_1Ex_1$. But then both $y_2[H_f \cup E]^*x_1$ and $x_1H_fy_2$, contradicting $[H_f \cup E]^* \subseteq [\overline{H_f}]'$. It follows that $\gg$ is acyclic, that $\gg^*$ is a strict partial order, and by Szpilrajn's Theorem that there exists a linear order $\ggg \supseteq \gg^* \supseteq \gg$ on $\mathcal{K}$.

Define a relation G by xGy iff $K(x) \ggg K(y)$, so that $E \subseteq \overline{G}$. Since $\ggg$ is a linear order (on $\mathcal{K}$), G is a weak order (on X) and has a representation g. Moreover, we have xH_fy only if $K(x) \gg K(y)$, $K(x) \ggg K(y)$, and xGy. Hence $H_f \subseteq G$, and so the existence of a θ such that $\langle f, \theta, g\rangle$ induces C follows by Theorem 4. □

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