PY1003: Introduction to Logic
Lecture 4

Sentential logic revisited III

(1) Truth-trees and validity

Recall the simple argument with which we began the module:

1. Ross in his office or Ross is in the library
2. It is not the case that Ross is in his office
3. Therefore, Ross is in the library

We can now determine the validity of this argument using a truth table. We adopt the following translation scheme:

A: Ross is in his office
B: Ross is in the library

This enables us to write out the argument in sequent form:

A v B, ¬A |- B

Recall that an argument is valid iff it is impossible for the premisses to be true, and the conclusion false. So to assess the validity of this argument, we need to work out whether there are any possible situations in which 'A v B' is true, '¬A' is true, but B is false. We do this using a truth-table.

A	**B**	**A**	**v**	**B**	**¬**	**A**	**B**	
T	T	T	T	T	F	T	T	[1]
T	F	T	T	F	F	T	F	[2]
F	T	F	T	T	T	F	T	[3]
F	F	F	F	F	T	F	F	[4]
[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	

Columns 1 and 2 represent all the ways that the world could be with regard to the truth-values of A and B. Columns 4, 6, and 8 are the important ones, because they tell us the truth-values of the premisses and the conclusion in each of these situations. We can see from the truth-table that there is only one possible situation in which both premisses are true. This occurs when A is false and B is true and is represented by row 3 of the table. Looking at across to column 8, we see that in these circumstances the conclusion, too, is true. So we can conclude that the argument is valid. That is, in all possible circumstances in which the premisses are true, the conclusion is also true.

Now consider another argument:

1. Ross is either in his office or in the library
2. Ross is in his office
3. Therefore, Ross is not in the library

Using the same translation scheme, we can write this out as follows:

A v B, A |- ¬B

And here is the truth table for this argument:

A	B	A	v	B	A	¬	B	
T	T	T	T	T	T	F	T	[1]
T	F	T	T	F	T	T	F	[2]
F	T	F	T	T	F	F	T	[3]
F	F	F	F	F	F	T	F	[4]
[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	

Again, remember that columns 4, 6, and 7 are the important ones, since they show the truth-values of the premisses and the conclusion. Looking at row [1], we see that there are circumstances in which the two premisses are true, and yet the conclusion is false. This happens when both A and B is true (perhaps Ross' office is in the library!). So we know that the argument is invalid.

If there is a possible situation in which the premisses of an argument are true and the conclusion is false, we say that this situation provides a *counterexample* to the argument. We can express the counterexample to the argument above as follows: {A=T, B=T}

(2) Interpretations

Here it is helpful to introduce some more terminology. An *interpretation* of a formula of sentential logic is an assignment of truth-values to the sentential letters that occur in that formula. So columns 1 and 2 above list all the possible interpretations of the formula A v B, A |- ¬B – that is, they list all of the possible assignments of truth-values to the sentences A and B.

Using the notion of an interpretation, we can now give neater definitions of some of the notions that we have been employing so far:

- An argument is *valid* iff there is no interpretation in which the premisses are true and the conclusion is false.
- A *counterexample* to an argument is an interpretation which makes the premisses are true and the conclusion false.

- Two formulae of sentential logic are *logically equivalent* if they have the same truth value in all interpretations.

(3) More complicated arguments

We didn't really need a truth-table to test the validity of the argument about Ross. A bit of common sense would have told us that it was valid. However, when it comes to more complicated arguments, truth-tables can be very useful. Consider:

1. If the president refuses the bribe, then either he'll be shot or there will be a coup.
2. There won't be a coup.
3. Therefore, the president will accept the bribe or he will be shot.

It is not so easy to tell whether this argument is valid. But a truth-table helps us out. First we need to translate the argument into sentential logic. Our translation scheme will be as follows:

A: The president accepts the bribe
B: The president will be shot
C: There will be a coup

We can now write out the argument in sequent form:

$\neg A \rightarrow (B \vee C), \neg C$ |- $A \vee B$

And the truth-table for the argument looks like this:

A	**B**	**C**	**¬**	**A**	**→**	**(B**	**v**	**C)**	**¬**	**C**	**A**	**v**	**B**
T	T	T	F	T	T	T	T	T	F	T	T	T	T
T	T	F	F	T	T	T	T	F	T	F	T	T	T
T	F	T	F	T	T	F	T	T	F	T	T	T	F
T	F	F	F	T	T	F	F	F	T	F	T	T	F
F	T	T	T	F	T	T	T	T	F	T	F	T	T
F	T	F	T	F	T	T	T	F	T	F	F	T	T
F	F	T	T	F	T	F	T	T	F	T	F	F	F
F	F	F	T	F	F	F	F	F	T	F	F	F	F
[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]	[12]	[13]	[14]

From a quick look at column thirteen, we see that there are only two interpretations in which the conclusion of the argument is false. These correspond to the bottom two rows of the table. Looking leftwards along these rows, we see that in each of these interpretations, one of the premisses is false. We can conclude that there are no interpretations in which the premisses are true and the conclusion false. So the argument is valid.

(4) Tautology, contradiction and consistency

We classify sentences as follows: a sentence is a *tautology* iff it is true in all interpretations. A sentence is a *contradiction* iff it is true in no interpretations. A sentence is *consistent* iff it is true in at least one interpretation. (Note that, according to these definitions, all tautologies are consistent.)

We can use truth-tables to test for tautology, contradiction and consistency. If a sentence is a tautology then the column under the main connective will contain nothing but 'T's. For instance, the sentence '$A \vee \neg A$' is a tautology

A	**A**	**v**	**¬**	**A**
T	T	T	F	T
F	F	T	T	F

If a sentence is contradictory, then the column under the main connective will contain nothing but 'F's. For instance, the sentence '$A \wedge \neg A$' is a contradiction:

A	**A**	**∧**	**¬**	**A**
T	T	F	F	T
F	F	F	T	F

And if a sentence is consistent then there will be at least some 'T's under the main connective. For instance, the sentence '$A \rightarrow (A \wedge B)$' is consistent:

A	**B**	**A**	**→**	**(A**	**∧**	**B)**
T	T	T	T	T	T	T
T	F	T	F	T	F	F
F	T	F	T	F	F	T
F	F	F	T	F	F	F

(4) Summary and reading

After this lecture you should be confident in using truth-tables to test the validity of an argument, and to determine whether a sentence is tautologous, contradictory or consistent. You should understand the notion of an interpretation of a formula of sentential logic.

If you want to do some further reading, I'd recommend: Graeme Forbes, *Modern Logic*, ch.3, sec 2 and 3.