PY1003: Introduction to Logic

Lecture 3

Sentential logic revisited II

(1) Truth-tables: a simple example

The sentence $\neg A$ is true if and only if the sentence *A* is false. And the sentence $\neg A$ is false if and only if the sentence *A* is true. We can express this point using a simple truth-table:

A	¬A
T	F
F	T

The column on the left represents all the possible ways that the world could be with regard to the truth or falsity of *A*. There are just two possibilities: *A* is true, or *A* is false. The column on the right tells us what the truth-value of $\neg A$ will be in each of these possible situations: if *A* is true, $\neg A$ is false; if *A* is false, $\neg A$ is true.

(2) Truth-functionality and some more truth-tables

The truth-value of $\neg A$ depends entirely upon the truth-value of *A*. So provided that we know what the truth-value of *A* is, we also know the truth-value of its negation. We say that sentential connectives like this are *truth-functional*. More precisely, a sentential connective is truth-functional iff the truth-value of the sentence that contains is a function of the truth-values of the sentences within the connective's scope.

All of the connectives that we looked at in the last lecture are truth-functional. Consider, for instance, the sentence $A \wedge B$. When is this true? Only when both *A* and B are true. When it is it false? When *A* is false, or B is false, or both *A* and B are false. So the truth-value of $A \wedge B$ depends entirely on the truth-value of *A* and of B, making the connective $\wedge$ truth-functional. Again, we can represent this information using a truth-table.

A	B	A ∧ B
T	T	T
T	F	F
F	T	F
F	F	F

(Notice that this time the table has four rows, rather than two. This because there are four possible ways that the world could be with regard to the truth-values of *A* and B.)

Here are the truth-tables for the other three sentential connectives that we looked at last week (these should be familiar from Reason and Argument):

A	**B**	**A v B**
T	T	T
T	F	T
F	T	T
F	F	F

A	**B**	**A → B**
T	T	T
T	F	F
F	T	T
F	F	T

A	**B**	**A ↔ B**
T	T	T
T	F	F
F	T	F
F	F	T

Notice that we use 'v' to represent the inclusive sense of 'or'. Thus '*A* or *B*' is true when both *A* *and* B are true. Notice too that '*A* → B' is true whenever *A* is false. Thus the sentence, 'If 2+2=5, the sky is pink' counts as true! This rather surprising result is sometimes known as the paradox of material implication.

(3) Truth-tables for more complicated formulae

Consider the sentence

(1) Either the president will accept the bribe or else he will refuse and in either case there will be a coup.

We shall adopt the following translation scheme:

A: The president will accept the bribe
B: There will be a coup

We can now translate (1) to the language of sentential logic:

(1') (*A* v ¬*A*) ∧ B

The truth-table for this formula is as follows:

A	**B**	**(A**	**v**	**¬**	**A)**	**∧**	**B**
T	T	T	T	F	T	T	T
T	F	T	T	F	T	F	F
F	T	F	T	T	F	T	T
F	F	F	T	T	F	F	F
[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]

Remember that the order in which we construct this truth-table is very important. We begin by filling in columns 1 and 2. It is then easy to fill in columns 3, 6, and 8. by referring back to 1 and 2. When we come to working out the truth value of the compound sentences, it is crucial to start with the connective which has the smallest scope. Thus we fill in the remaining columns in the following order: 5, 4, and 7. The final column that we fill in, 7, corresponds to the connective which has the widest scope. The truth-values that we write under this connective correspond to the truth-values of the whole formula. Comparing column 7 with columns 1 and 2, we see that the formula (1) is true if and only if B is true – that is, it is true that there will be a coup.

(4) Some more truth-functional connectives

So far, we have only considered the sentential connectives "it is not the case that…", "or", "and", "if… then…" and "if and only if", which we represent with the symbols $\neg$, v, $\wedge$, $\rightarrow$ and $\leftrightarrow$. But there are many other sentential connectives:

- The president will be assassinated *unless* he bribes the leader of the opposition.
- *It is neither the case that* the president will be assassinated *nor* that there will be a coup.
- The coup will succeed *but* the president won't be assassinated.
- The coup will succeed *only if* the president is assassinated.

We could introduce new symbols to represent these extra sentential connectives. But this would be unnecessary. For each of these connectives can be adequately expressed using the symbols that we already have. The translations of the sentential connectives above are as follows:

A unless B	$\neg B \rightarrow A$
It is neither the case that A nor the case that B	$\neg (A \vee B)$ or $\neg A \wedge \neg B$.
A but B	$A \wedge B$
A only if B	$A \rightarrow B$

In fact, we don't even need all the symbols that we have! For instance, instead of introducing a special symbol for the biconditional, we could represent the claim 'A if and only if B' as follows:

(3) $(A \rightarrow B) \wedge (B \rightarrow A)$.

In order to check that this is a good translation we can compare the truth table for (3) with the truth table for $A \leftrightarrow B$

A	B	(A	$\rightarrow$	B)	$\wedge$	(B	$\rightarrow$	A)
T	T	T	T	T	T	T	T	T
T	F	T	F	F	F	F	T	T
F	T	F	T	T	F	T	F	F
F	F	F	T	F	T	F	T	F
[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]

A	B	$A \leftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

The main connective for the table on the left is found in column (6). Looking here, we see that $(A \rightarrow B) \wedge (B \rightarrow A)$ is true in exactly the same circumstances that $A \leftrightarrow B$ is true. We therefore say that the two formulae are *logically equivalent.*

(4) Some sentential connectives that are not truth-functional

Consider some more sentential connectives:

- *The president believes that* there will be a coup
- *It is possible that* the president will be shot
- The president was shot *after* he accepted the bribe.

These sentential connectives are not truth-functional. As a result, they cannot be expressed using the language of sentential logic. For example, consider the compound sentence "The president was shot after he accepted the bribe." If we write "A" for "the president was shot" and B for "he accepted the bribe", we can express this sentence as follows: "A after B". When is this sentence true? Well, clearly it is true only when both A and B are true. If the president was not shot, or he did not accept the bribe, it would be false that the president was shot after he accepted the bribe. But this is not enough. Suppose that the president was shot *before* he accepted the bribe. Then it would be false that "A after B" even though both A and B are true.

In other words, the sentential connective *after* is not truth-functional. Its truth-value does not depend entirely on the truth-values of the sentences within its scope, but upon the times that the events described take place. In this module, we will not be dealing with such sentential connectives, although you will encounter them in the more advanced logic courses available in St Andrews.

(5) Summary and further reading

In this lecture, we've revisited truth-tables. We've reminded ourselves of the truth-tables for the logical constants, and we've seen how to construct truth-tables for more complicated formulae. We've also looked at the notions of *truth-functionality* and *logical equivalence.*

At this point, you should be feeling fairly confident about how to construct truth-tables. If you are not, I strongly suggest some extra reading. The following texts are helpful:

Colin Howson, *Logic With Trees,* ch.1, sections 3-8
Graeme Forbes, *Modern Logic*, ch. 3, sec 1

(If reading the Forbes, watch out for the different symbolism!)