PY1003: Introduction to Logic
Lecture 16
Identity

1. A problem

Consider the following sentence:

(1) Victoria Beckham is identical to Posh Spice

How are we to translate this sentence into the language of predicate logic? At first glance, identity seems to be a relation that holds between two or more objects. Using 'a' to translate the name Victoria Beckham, 'b' to translate the name Posh Spice, and 'Ixy' to express the claim that x is identical to y, we might therefore offer the following translation of the sentence above:

(1*) Iab

The problem is that this translation will render invalid many arguments which seem to be valid. For instance, consider the following intuitively valid argument:

Posh Spice is rich
<u>Victoria Beckham is identical to Posh Spice</u>
Therefore, Victoria Beckham is rich

Using the translation scheme suggested above, this argument becomes

Ra, Iab |- Rb

However, this sequent is *invalid.* You can quickly check this is the case by attempting to construct a truth tree for the argument.

2. Identity

In order to resolve this problem, we need to find a new way of expressing claims about identity. We will use the symbol '=' to mean 'is identical to'. Then we can express the sentence (1) above as follows

(1') a = b

We will also use the identity sign to translate the following sentences: 'Jekyll and Hyde are the same person'; 'the Morning Star and the Evening Star are the same object'; 'Sting and Gordon Sumner are one and the same'.

WARNING: in English, the phrase 'a is identical to b' can mean two distinct things. First, it may mean that a and b are one and the same object, or person. This sort of identity is called *quantitative* or *numerical* identity. Second, it may mean that a and b share almost all of their properties – that is, that they are very similar. (It is in this sense that identical twins are identical.) This second sort of identity is called *qualitative* identity. The identity symbol should only ever be used to express claims about numerical identity.

3. The tree rules for identity

We now have a better way of translating our argument about Posh Spice. Using the same translation key, we can express the argument as follows:

Ra, a=b |- Rb

But how are we to show that this argument is valid? In order to do so we need to learn two new rules for manipulating truth trees. These rules are as follows:

Identity Rule 1: For any constants a and b, if a sentence of the form a=b appears on a branch of a tree, we may uniformly substitute a for b, and b for a in any sentence that occurs on that branch.

Identity Rule 2: For any constant a, we may write 'a=a' on any point on any open branch of a tree.

The justification behind these two rules is quite simple. If a and b are identical, then anything that can be truly said of a can also be said of b. For instance, if it is true that a is rich, it will also be true that b is rich. Thus if the sentence Ra appears on that branch, we can also safely write Rb on that branch. This justifies the first rule. The second rule is justified by the observation that everything is identical with itself. Thus a sentence of the form 'a = a' will always be true. So it can be safely written on any open branch of any tree.

We can now establish that our simple argument involving Victoria Beckham is valid:

Ra, a=b |- Rb

Ra
a=b
¬Rb
 |
 Rb (Identity Rule 1)

NOTE: Note that a sentence does not get 'used up' after we apply the First Identity Rule to it. In this respect, the First Identity Rule is like the rule for Universal Quantification Instantiation. Thus we never tick off sentences after applying the First Identity Rule.

4. Some more examples

Consider the following argument, which intuitively appears to be valid:

The Morning Star is identical to the Evening Star
<u>The Evening Star is identical to Venus</u>
Therefore, the Morning Star is identical to Venus

We will adopt the following translation key: (a: the morning star, b: the evening star, c: Venus.) We can now formalise the argument as follows:

$a = b, b = c \mid\!- a = c$

A very simple truth tree shows us that this argument is valid:

```
a = b
b = c
¬(a = c)
  |
 a = c            (Identity Rule 1)
```

Or here is a slightly more complicated example. Suppose that we want to assess the validity of the following sequent:

$\forall x\ (Rx \rightarrow Qx), Ra, \neg Qb \mid\!- \neg a=b$

We can show that the sequent is valid as follows:

```
      ∀x (Rx → Qx) √a
      Ra
      ¬Qb
      ¬¬a=b √
        |
       a=b
        |
     Ra → Qa  √
     /       \
   ¬ Ra      Qa
              |
             Qb      First Identity Rule
```

5. The properties of identity

We can, if we like, continue to think of identity as a relation. However, it is a relation which has some very special properties. The identity relation is *reflexive, symmetric* and *transitive.* These terms are understood as follows:

Reflexivity: a binary relation R is reflexive iff $\forall x\ (Rxx)$
Symmetry: a binary relation R is symmetric iff $\forall x\ \forall y\ (Rxy \rightarrow Ryx)$
Transitive: a binary relation R is transitive iff $\forall x\ \forall y\ \forall z\ ((Rxy \wedge Ryz) \rightarrow Rxz)$

A relation which is reflexive, transitive and symmetric is called an *equivalence relation.* Identity, then, is an equivalence relation.

We can prove that the identity is reflexive, transitive and symmetric using truth trees. In order to establish that identity is symmetric, for example, we begin by assuming the *negation* of symmetry. We then show that this assumption is a contradiction:

$$\neg \forall x \forall y (x=y \rightarrow y = x)$$
|
$$\exists x \neg \forall y (x=y \rightarrow y = x)$$
|
$$\neg \forall y (a = y \rightarrow y = a)$$
|
$$\exists y \neg (a = y \rightarrow y = a)$$
|
$$\neg (a = b \rightarrow b = a)$$
|
$$a = b$$
$$\neg b = a$$
|
$$\neg a=a$$ Identity Rule 1
|
$$\underline{a=a}$$ Identity Rule 2

6. Opaque contexts

In justification of the First Identity Rule, we noted that if two objects a and b are identical, then anything that can truly be said of a can truly be said of b. That is, identicals can be substituted *salve veritatae.* However, there are some unusual contexts in which this claim is false. Suppose, for instance, that Fred believes that Posh Spice is rich, and that Posh Spice is identical to Victoria Beckham. Does it follow that Fred believes that Victoria Beckham is rich? Not at all – if Fred does not *know* that Victoria and Posh are identical he may not believe that Victoria is rich.

What is going in here? The quick explanation is that phrases such as 'Fred knows that…' create what is called an *opaque context.* Other opaque contexts include 'Fred believes that…', 'Everyone thinks that…' and 'It is generally accepted that…'. Within opaque contexts, identicals cannot be substituted *salve veritate.* As a result, it would be inappropriate to apply the First Identity Rule to a constant that appears within an opaque context. Opaque contexts are a topic of much philosophical interest. For the purposes of this module, however, you merely need to be aware of their existence.

7. Conclusion and further reading.

In this lecture, we have encountered the symbol for identity, and learnt two new tree rules for identity. We have noted that the identity relation is an equivalence relation (that is, it is reflexive, symmetric and transitive) and that we must be wary of opaque contexts when dealing with identity. Further reading: Howson, C, *Logic with Trees,* 115-121.