PY1003: Introduction to Logic
Lecture 10
Predicate Logic II

1 Some more limits to predicate logic

Consider the following argument:

<u>Everyone is mortal</u>
Therefore, David is mortal

This argument is clearly valid. Yet sentential logic leaves us unable to explain why this is the case. To see this, suppose that we adopt the following translation scheme:

A: Everyone is mortal
B: David is mortal

Then we can symbolise the argument into sentential logic as follows:

A |- B

Yet this argument is clearly *in*valid, since the interpretation {A=T, B=F} provides a counterexample. Again, sentential logic has let us down. The aim of this lecture is to learn how to express the argument above in the language of predicate logic.

2. The universal quantifier

Consider again the sentence:

(1) Everyone is mortal

This sentence could also be expressed as:

(1′) Every person is such that he/she is mortal

If we introduce the individual variable, 'x' we can rewrite this sentence as follow:

(1″) For all x, x is mortal

And if we use the predicate letter 'M' in place of the predicate 'is mortal', we can write:

(1‴) For all x, Mx

The next stage is to introduce a new symbol, '∀'. This symbol is called the *universal quantifier,* and it means 'For all…'. Using the universal quantifier, we can finally complete our translation:

(1⁗) $\forall xMx$

Sentence which begin with a universal quantifier are called *universally quantified sentences.* Here are some universally quantified sentences in English, along with their translations into predicate logic:

Each object is green: $\forall xGx$
Everything is purple: $\forall xPx$
Everyone is happy: $\forall xHx$

4. Universal quantifiers and compound sentences

Consider the sentence:

(2) Everyone is happy and everyone is rich

We can express this using the universal quantifier together with the symbol for conjunction:

(2′) $\forall xHx \wedge \forall xRx$

Contrast this with the distinct sentence:

(3) Everyone is happy and rich

We translate this as follows:

(3″) $\forall x(Hx \wedge Rx)$

Notice, though, that (2) and (3) are logically equivalent. That is, they are true in exactly the same possible situations: if everyone is happy and everyone is rich then everyone is happy and rich, and *vice versa.* Nonetheless, we translate (2) as (2′) and (3) as (3′) in order to respect the different syntactic form of the two sentences.

Now consider the sentence:

(4) Everybody who is rich is happy

How are we to tranlate this into the language of sentential logic? The key is to note that a different way of expressing (4) is as follows:

(4′) Every person is such that if they are rich, then they are happy

Translating (4′) into predicate logic we arrive at the following:

(4″) $\forall x (Rx \rightarrow Hx)$

Here are some more compound sentences involving the universal quantifier:

It is not the case that all rich people are happy: $\neg\forall x (Rx \rightarrow Hx)$
Everything is black or white: $\forall x (Bx \vee Wx)$
All happy people do yoga: $\forall x (Hx \rightarrow Yx)$

Notice that we can also use the logical constants to combine universally quantifier sentences with existentially quantified sentences. For instance:

If someone is rich, then everyone is happy: $\exists xRx \rightarrow \forall xHx$
Either everyone is lying or else someone is a thief: $\forall xLx \vee \exists xTx$
Someone is a thief and everyone was shocked: $\exists xTx \wedge \forall xSx$

4. Negated universally quantified sentences and quantifier interdefinability

Suppose that we wish to assert the following sentence:

(5) It is not the case that everyone is rich

We can do this by placing the negation symbol in front of the universal quantifiers:

(5′) $\neg\forall xRx$

Another way of expressing the thought behind (5) is as follows:

(6) There is someone who is not rich

In predicate logic, we express this sentence as follows:

(6′) $\exists x\neg Rx$

Notice that (5′) and (6′) are logically equivalent – that is, they are true in exactly the same possible situations. This point illustrates a more general truth about the existential quantifier and the universal quantifier. Take P to be any predicate. Then:

$\forall xPx$ iff $\neg\exists x\neg Px$
$\exists xPx$ iff $\neg\forall x\neg Px$

We express this point by saying that the two quantifiers are *interdefinable.* This means that we could, in theory, adopt a version of predicate logic that contained only one quantifier. In practice, however, it is more convenient to use them both.

5. A complication: domains

Suppose that we are at a party. Looking around at the guests, we utter the following sentence: "Everyone is happy." Clearly, we do not mean to assert that everyone in the whole world is happy. Rather, our sentence is naturally intended to be understood as meaning "Everyone at the party is happy". In other words, the quantifier "everyone" is *restricted* or *relativised* to the people at the party. And my sentence counts as true if and only if every one of them is happy.

Here is some terminology to express this point. The collection of people or objects to whom the quantifiers in a sentence are relativised is called the *domain of discourse.* Thus in the example above, the domain of discourse contains the people at the party, and nothing else. If a quantifier is intended to be relativised to a certain domain of discourse, we say that it *ranges over* that domain. Thus in the example above, the quantifier "everyone" ranges over the people at the party.

In real life, the range of a quantifier is usually made clear by the context. When learning predicate logic, however, we will often be considering sentences in isolation from their contexts. It is therefore useful to adopt the following convention: the only domains of discourse that we will recognise are the domains of, respectively, people, things, places and times. Thus the sentence

(7) Everyone is happy

which is translated as

(7') $\forall x Hx$

the universal quantifier is understood as ranging over all people, everywhere. Suppose that we wish to express the more plausible claim:

(8) Everyone at the party is happy

We should do so by introducing the further predicate 'P' which means 'is at the party'. This enables us to translate (8) as follows:

(8') $\forall x\ (Px \rightarrow Hx)$

Here are a few more examples that illustrate the difference:

Everything in my bag is wet: $\forall x(Bx \rightarrow Wx)$
Everything that exists is material: $\forall x Mx$
Somebody at the party is smoking: $\exists x\ (Px \wedge Sx)$

6. Arguments using universal quantifiers

Remember the argument with which we began this lecture:

Everyone is mortal
Therefore, David is mortal

We are now able to translate this argument into the language of predicate logic. The translation is as follows:

$\forall x Mx \vdash Ma$

Since the validity of an argument depend upon its logical form, we know that any argument with the same logical form will be valid. We do not, yet, have a way of demonstrating that this argument is valid. We will be developing a method for assessing the validity of arguments in predicate logic in the next few lectures.

7. Summary and further reading

In this lecture we have learnt a new symbol to add to the language of predicate logic. This is the universal quantifier, $\forall$, which means, 'For all…'. We have also been introduced to the notion of a domain of discourse. The further reading for this lecture is the same as that suggested for Lecture 9.