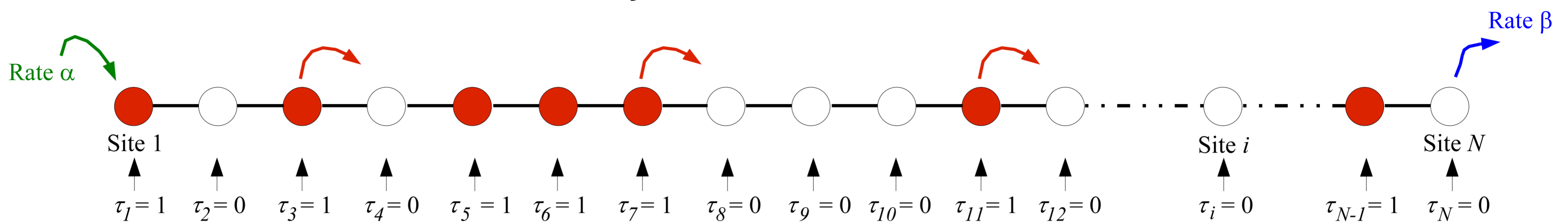


# Models of Traffic Flow: The Asymmetric Exclusion Process

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## Introduction

The **asymmetric exclusion process** is probably the most-studied model in non-equilibrium statistical mechanics. It is a one-dimensional lattice gas; particles hop randomly along a lattice of discrete sites, and only one particle can occupy a site at one time. The system is out of equilibrium because a current of particles is driven through the system even in the steady state.

This is a simplistic model for traffic flow on a single-lane road with no overtaking. Each particle represents a single car.

## The dynamics

For traffic we consider the **totally** asymmetric exclusion process, i.e. particles hop in one direction only (left to right). The dynamics are continuous and use **random sequential** updating.

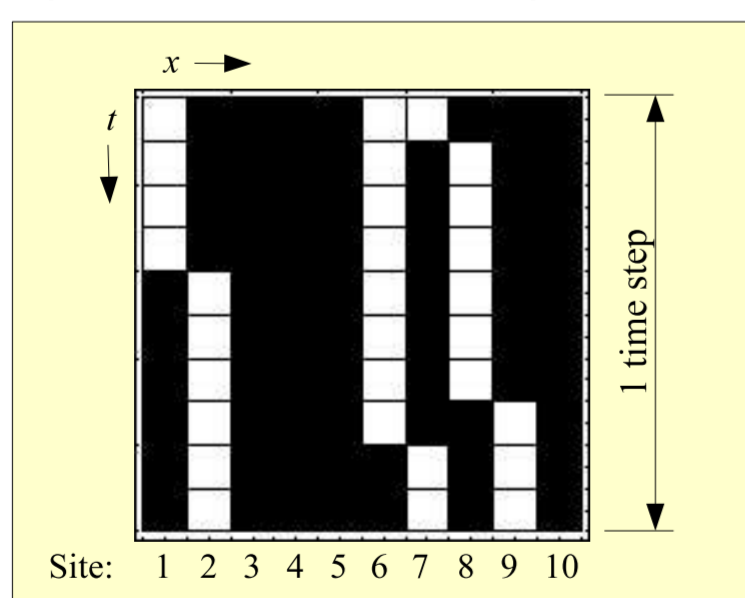
At each **update step** a site is chosen at random:

- If the site is empty nothing happens.
- If the site is occupied then the particle moves forward to the next site providing that site is empty.

We also need **boundary conditions**:

- If site 0 (left boundary) is chosen and site 1 is empty then a particle enters site 1 with probability  $\alpha$ .
- If site  $N$  (right boundary) is chosen, and it is occupied, then the particle leaves the lattice with probability  $\beta$ .

A single **time step** is made up of  $N$  update steps so that on average each particle moves one site per time step:



## Computer simulation

A major advantage of discrete models of traffic flow is that the process can easily be simulated, using for example *Mathematica*<sup>®</sup>. This is how the examples on this poster were produced.

## Macroscopic properties

These are the major quantities of analytical interest.

We define the **indicator variables** to record the state of site  $i$ :

$$\tau_i = \begin{cases} 1, & \text{if site } i \text{ is occupied;} \\ 0, & \text{if site } i \text{ is empty.} \end{cases}$$

## Density

The **density** at site  $i$  at time  $t$  is the average value of  $\tau_i(t)$  over all possible histories of the dynamics, written  $\langle \tau_i \rangle$ . The evolution of this is

$$\frac{d\langle \tau_i \rangle}{dt} = \underbrace{\langle \tau_{i-1} [1 - \tau_i] \rangle}_{\text{entry rate to site } i} - \underbrace{\langle \tau_i [1 - \tau_{i+1}] \rangle}_{\text{exit rate from site } i},$$

with appropriate modification for the boundary sites  $i = 1$  and  $i = N$ .

## Current

Whatever the initial conditions it can be shown that the system relaxes to a **steady state** where the probability of any configuration remains constant with time. In this state  $d\langle \tau_i \rangle / dt = 0$  for all  $i$  and there is a conserved **current**:

$$\alpha \langle (1 - \tau_1) \rangle = \langle \tau_1 (1 - \tau_2) \rangle = \dots = \langle \tau_{i-1} (1 - \tau_i) \rangle = \dots = \langle \tau_{N-1} (1 - \tau_N) \rangle = \beta \langle \tau_N \rangle = J$$

## Exact solution

The steady state solution can be found exactly using the **matrix method** of Derrida *et al.* [1] published in 1993. They showed that the steady state probabilities can be written in the **matrix product** form

$$P(\tau_1, \tau_2, \dots, \tau_N) = \frac{\langle W | \prod_{i=1}^N (\tau_i D + (1 - \tau_i) E) | V \rangle}{Z_N}$$

where  $D, E$  are infinite-dimensional matrices and  $\langle W |, |V \rangle$  are row and column vectors, so the result is a number. The product simply means "put  $D$  if the site is occupied and  $E$  if the site is empty". The matrices and vectors must satisfy the following rules:

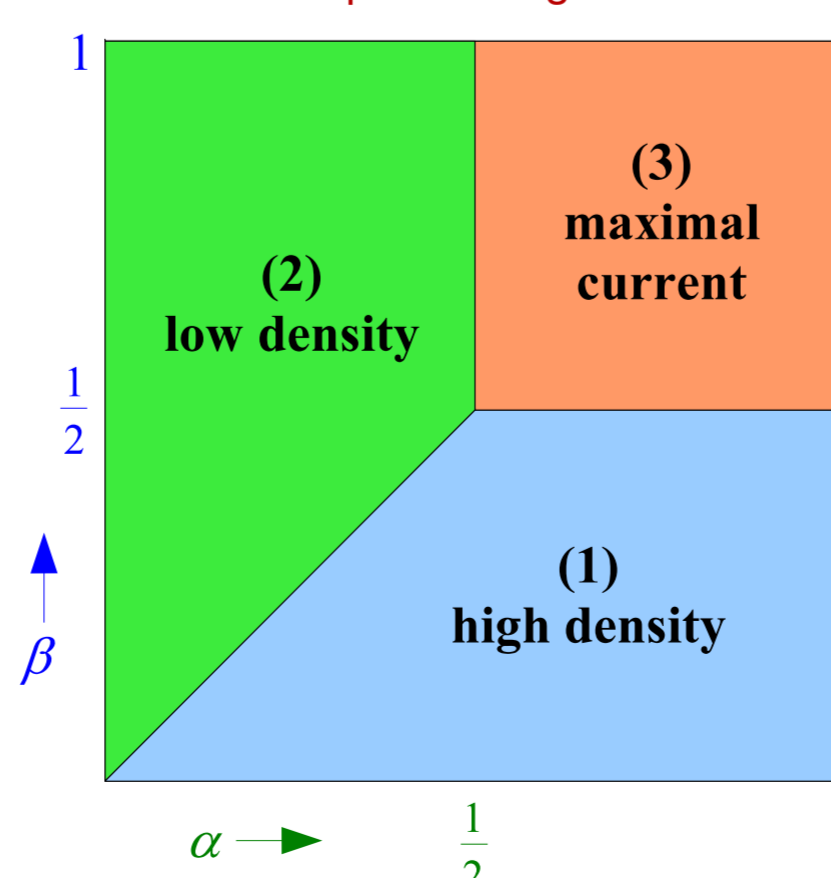
$$\begin{aligned} DE &= D + E, \\ \langle W | E &= \frac{1}{\alpha} \langle W |, \\ D | V &= \frac{1}{\beta} | V \rangle. \end{aligned}$$

Then calculations can be done using these algebraic rules directly, without using a specific representation of the matrices. In this way formulae for  $\langle W |, |V \rangle$  and  $J$  in the steady state can be found, in terms of the function

$$Z_N = \sum_{p=1}^N \frac{p(2N-1-p)!}{N!(N-p)!} \frac{(1/\beta)^{p+1} - (1/\alpha)^{p+1}}{1/\beta - 1/\alpha}.$$

## Phase behaviour

Taking the limit  $N \rightarrow \infty$  simplifies the analytical results and shows how the behaviour of the system is dependent on the parameters  $\alpha$  and  $\beta$ . There are three main **phases** where the behaviour of the system is qualitatively different, like water which can be ice, liquid or steam, depending on its temperature. This is shown on the **phase diagram**:

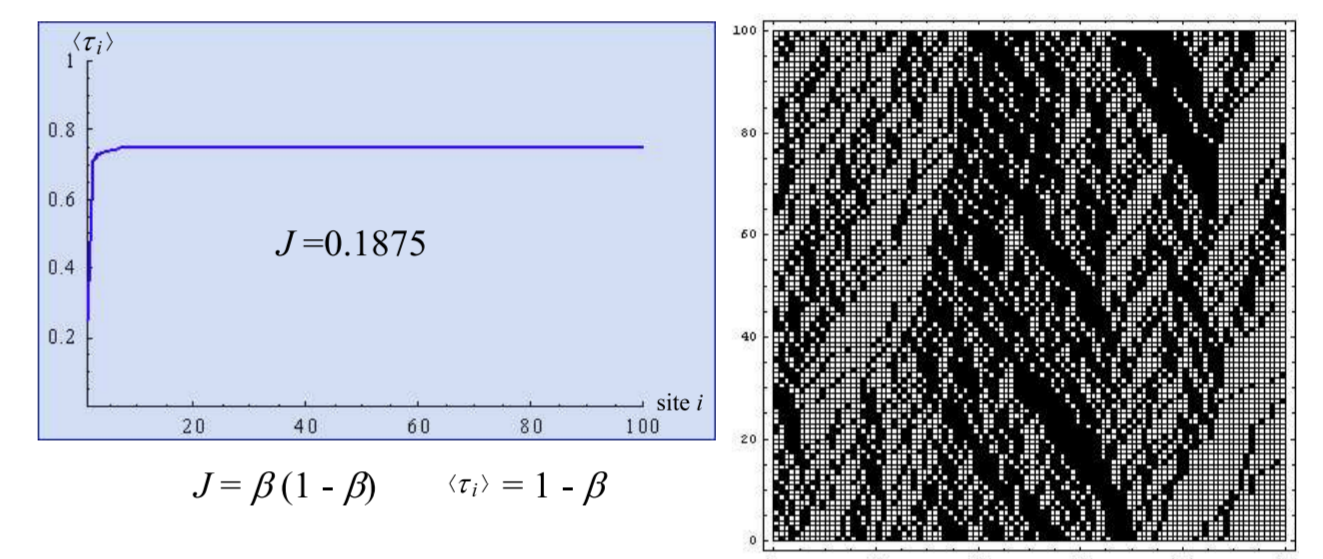


## Illustrations

The following plots illustrate the behaviour in each phase for a finite lattice of length  $N = 100$ . The exact solution for the density profile and an example simulation (with each line corresponding to one time step) are shown.

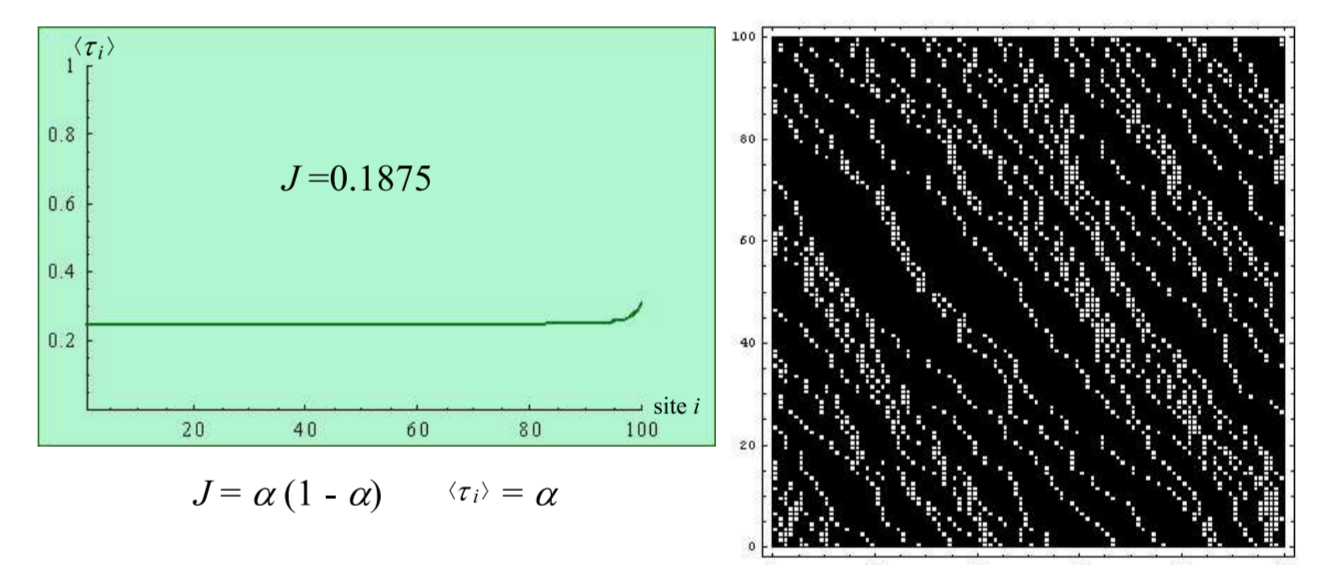
### (1) High density phase

The behaviour is controlled by the low exit rate  $\beta$ .



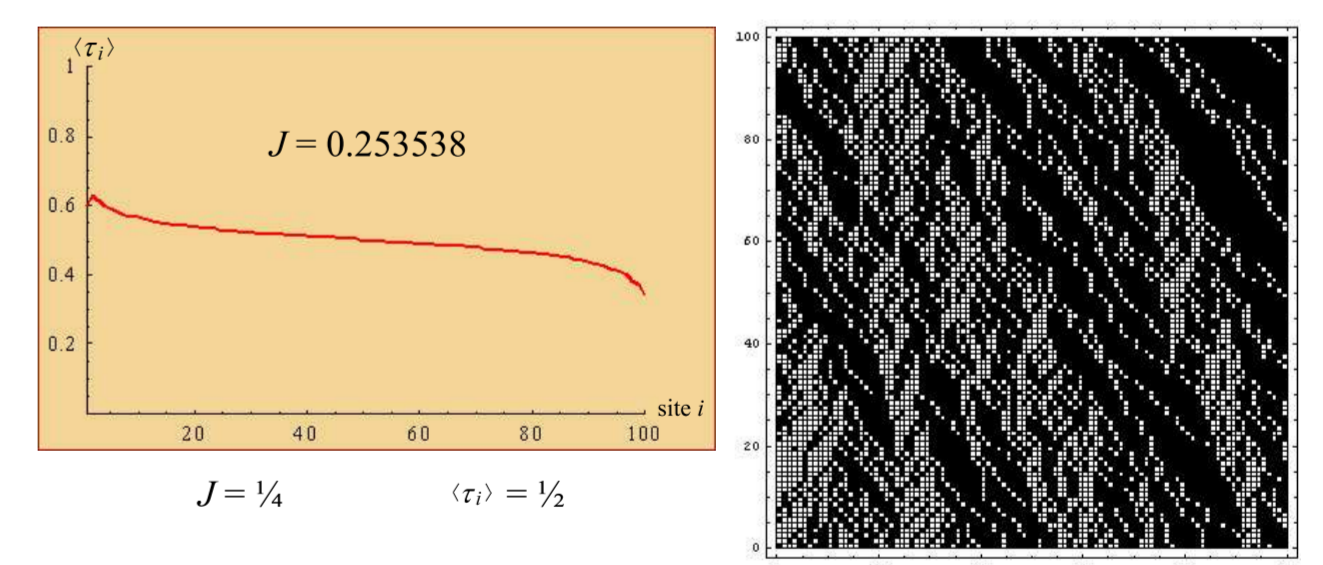
### (2) Low density phase

The behaviour is controlled by the low entry rate  $\alpha$ .



### (3) Maximal current phase

The current is saturated - increasing  $\alpha$  and  $\beta$  further doesn't increase the current.



## Extensions

Various extensions to the model have been studied, including:

- **Two species of particle** Cars and lorries are represented by two types of particle, which have different dynamical behaviour, e.g. cars may overtake lorries.
- **Nagel-Schreckenberg model** [2] A more realistic cellular automaton model of traffic where "cars" have a speed and can accelerate and decelerate. The updating is done in parallel.

## References

- [1] B. Derrida, M. R. Evans, V. Hakim and V. Pasquier, J. Phys. A, **26**, 1493 (1993).
- [2] K. Nagel and M. Schreckenberg, J. Phys. (France) I **2**, 2221 (1992).

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