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Preference Conditions for Invertible Demand Functions

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Preference Conditions for Invertible Demand Functions*

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Abstract

It is frequently assumed in several domains of economics that demand functions are invertible in prices. At the primitive level of preferences, however, the characterization of such demand functions remains elusive. We identify conditions on a utility-maximizing consumer's preferences that are necessary and sufficient for her demand function to be continuous and invertible in prices: strict convexity, strict monotonicity and differentiability in the sense of Rubinstein (2012). We show that preferences are Rubinstein-differentiable if and only if the corresponding indifference sets are smooth. As we demonstrate by example, these notions relax Debreu's (1972) preference smoothness.

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1 Introduction

Invertibility of demand is frequently assumed in several domains of economic inquiry that include consumer and revealed preference theory (Afriat, 2014; Matzkin and Ricther, 1991; Chiappori and Rochet, 1987; Cheng, 1985), non-separable and non-parametric demand systems (Berry et al., 2013), portfolio choice (Kübler and Polemarchakis, 2017), general equilibrium theory (Hildenbrand, 1994), industrial organization (Amir et al., 2017), and the estimation of discrete or continuous demand systems. In some of this work (e.g. Berry, Ghandi, and Haile, 2013; Cheng, 1985) the interest has naturally been on conditions that ensure invertibility of the relevant demand function/system. Focusing on a general consumer-theoretic domain, the present paper goes one step further and contributes to this large literature by providing the first complete *characterization* of classes of preference relations that generate consumer demand functions that are invertible in prices.

Certain smoothness conditions on either the demand system directly (e.g. Gale and Nikaido, 1965) or –closer to our analysis– on the utility function that generates it (e.g. Katzner, 1970) have been known for a long time to be sufficient for invertibility. However, the more foundational question of whether it is also possible to identify conditions on a consumer's preferences that are simultaneously *necessary and sufficient* for invertibility has remained unanswered. Perhaps surprisingly, our main result shows that a rational (i.e. one that is derived from utility maximization) demand function that is invertible in prices is characterized by three simple and behaviourally interpretable textbook conditions on the preferences that generate it: strict convexity, strict monotonicity and differentiability in the sense of Rubinstein (2012). The latter notion intuitively requires that for every bundle in the consumption set there exists a vector –which we refer to as the preference gradient– such that any (arbitrarily) small movement away from the bundle is in a direction that results to an improvement for the consumer if and only if the move is evaluated as positive by the vector.

Our characterization refers to invertible demand functions generated by continuous preference relations, and has two building blocks. The first establishes that strict convexity and strict monotonicity by themselves jointly characterize rational demand functions that have the *onto/surjectivity* property whereby for every bundle in the consumption set there are some (possibly non-unique) prices that rationalize the consumer's choice of that bundle. This result is of independent interest and although it involves relatively standard arguments, we have been unable to find a statement of it in the literature. The second building block establishes that for onto demand functions in this class the *one-to-one/injectivity* property is equivalent to Rubinstein differentiability of the preference relation that generates them. Proving this second part –and especially that preference differentiability is implied by injectivity– is not as straightforward and requires the use of novel arguments.

Importantly, as we demonstrate by example, within the realm of continuous, strictly convex and strictly monotonic preferences, the behaviourally interpretable notion of Rubinstein differentiability is weaker (hence, more general) than the requirement that the preferences admit a differentiable/smooth utility representation. In fact, Rubinstein differentiability turns out to be equivalent to the notion of weakly smooth (of order 1) preferences in the sense of Neilson (1991). This generalizes the original notion of smooth preferences due to Debreu (1972) –which is equivalent to the existence of a smooth utility representation– by restricting attention to what happens within any given indifference set, not along the entire indifference relation itself.

From the applications' point of view, our result provides transparency in –and guidance for– applied work that assumes demand functions that are invertible in prices. In particular, as is often the case in practice, the analyst may assume an invertible demand function directly. Our characterization clarifies that, at the more primitive level of the generating preferences, the analyst effectively assumes strict convexity, strict monotonicity and an intuitive notion of differentiability that is weaker than anything that could guarantee proper differentiability of the utility function. In this sense our analysis shows that micro-founding a model that features invertible demand can be done by imposing the relevant necessary and sufficient structure directly on preferences, thereby avoiding unnecessarily strong assumptions and maximizing the model's domain of application.

As a final note of motivation for our contribution we should also remark that invertibility of the aggregate demand function in a pure exchange economy is equivalent to uniqueness of Walrasian equilibrium relative prices. As pointed out in Jerison and Quah (2008) and Hildenbrand (1994), for example, when the Walrasian market demand function satisfies the "strict law of demand" (whereby the vectors of changes in prices and demanded quantities go in strictly opposite directions following a price change) then the above invertibility condition is satisfied and the equilibrium is unique and stable. Our analysis implies that if an individual (respectively, market) demand function satisfies the strict law of demand then the consumer's (respectively, the representative consumer's, if one exists) preferences are necessarily strictly convex, strictly monotonic and differentiable. This implication makes a non-obvious step in the direction of fully characterizing preference structures generating demand functions that satisfy the strict law of demand, which remains an open problem.

2 Main result

We consider a consumption set X which is a convex subset of $\mathbb{R}^n_+$ with $\operatorname{int}(X) \neq \emptyset$ (where $\operatorname{int}(A)$ is the interior of $A \subseteq \mathbb{R}^n_+$). For two consumption bundles $x, y \in X$ we write $x \geq y$ and $x \gg y$ whenever $x_i \geq y_i$ and $x_i > y_i$ for all $i \in \mathcal{N} := \{1, \ldots, n\}$, respectively. We also write $x > y$ whenever $x \geq y$ and $x \neq y$. The consumer's preferences are captured by a continuous weak order $\succsim$ on X (i.e., by a complete and transitive binary relation whose graph is closed in $X \times X$). Such preferences are *convex* if, for all $x, y \in X$ and any $\alpha \in [0,1]$, $x \succsim y$ implies $\alpha x + (1-\alpha)y \succsim y$. They are *monotonic* if, for all $x, y \in X$, $x \gg y$ implies $x \succ y$. We relax slightly the domain over which the standard notions of strict convexity and strict monotonicity apply so that it becomes operational in our more general environment. In particular, we will say that preferences are *strictly convex* if, for all $(x,y) \in \{X \times \operatorname{int}(X), \operatorname{int}(X) \times X\}$ and any $\alpha \in (0,1)$, $x \succsim y$ implies $\alpha x + (1-\alpha)y \succ y$. Similarly, we will say that preferences are *strictly monotonic* if, for all $(x,y) \in \{X \times \operatorname{int}(X), \operatorname{int}(X) \times X\}$, $x > y$ implies $x \succ y$.[1]

For a continuous weak order $\succsim$ on X and $X' \subseteq X$, we let $\max_{\succsim} X' := \{x \in X' : x \succsim y \ \ \forall y \in A\}$ denote

[1]Our slight weakening of the standard definitions of strict convexity (i.e. for all $x, y \in X$ and any $\alpha \in (0,1)$, $x \succsim y$ implies $\alpha x + (1-\alpha)y \succ y$) and strict monotonicity (i.e. for all $x, y \in X$, $x > y$ implies $x \succ y$) allows us to consider these notions on X without requiring that they hold also on $X \setminus \operatorname{int}(X)$. For instance, letting $X = \mathbb{R}^n_+$, our notions of strict convexity and strict monotonicity coincide with the standard ones for pairs of bundles when at least one bundle lies in $\mathbb{R}^n_{++}$, while they are not imposing any restrictions when either bundle lies on the boundary of $\mathbb{R}^n_+$. This allows our analysis to include preferences that are strictly convex and strictly monotonic on $\mathbb{R}^n_{++}$ but for which the boundary of $\mathbb{R}^n_+$ is an indifference set, a well-known example being the Cobb-Douglas preferences on $\mathbb{R}^n_+$ (see Section 2.1 below).

the set of all $\succsim$-greatest elements in X'. Given then some set $Y \subseteq \mathbb{R}^n_{++}$ of income-normalized strictly positive prices, the budget correspondence $B_{X'} : Y \twoheadrightarrow X'$ is defined by $B_{X'}(p) := \{x \in X' : px \leq 1\}$. We will say that $\succsim$ *generates* the demand correspondence $x : Y \twoheadrightarrow X'$ if the latter is defined by $x(p) := \max_{\succsim} B_{X'}(p)$. We will refer to such a demand correspondence as *rational*.

Clearly, even though the budget correspondence remains always nonempty- and convex-valued, it is not compact-valued whenever $X' \subseteq \mathbb{R}^n_{++}$. It is therefore no longer an immediate consequence of standard results such as the Maximum Theorem that the demand correspondence generated by $\succsim$ is well-defined. Nevertheless, as we explain below, the preference structure that we consider turns out to be sufficiently strong to overcome this technical difficulty because it ensures that only interior consumption bundles will ever be demanded.

A rational demand correspondence is *surjective* or *onto* if for all $x \in X'$, there exists $p \in Y$ such that $x \in x(p)$. If $x(\cdot)$ is single-valued (hence a demand *function*), it is said to be *injective* or *one-to-one* if for all $p, p' \in Y$, $p \neq p'$ implies $x(p) \neq x(p')$. A demand function $x : Y \to X'$ that is both injective and surjective is *bijective* or *invertible*. If $x(\cdot)$ has this property, then the inverse demand mapping given by $p(x) := \{p \in Y : \ x \in x(p)\}$ is itself a well-defined bijective function $p : X' \to Y$.

In addition to the standard properties of preferences that were introduced above, the problem under investigation naturally invites the introduction of some notion of preference differentiability or smoothness. The first such notion in the literature was proposed by Debreu (1972) who defined a preference relation on X as C^r (or *smooth of order r*) if its indifference graph is a C^r-manifold on $\mathrm{int}(X)$, and showed that a monotonic preference relation on X is C^r if and only if it is representable on $\mathrm{int}(X)$ by a C^r utility function. Later, Neilson (1991) defined a preference relation on X as *weakly-C^r* (or *weakly smooth of order r*) if its indifference sets -as opposed to the indifference relation itself- are C^r-manifolds on $\mathrm{int}(X)$. He established that this notion of smooth preferences suffices for the Hicksian demand function to be smooth. Following finally the more recent analysis in Rubinstein (2012), we define the preference relation $\succsim$ on X to be *differentiable* if, for every $x \in \mathrm{int}(X)$, there exists $p_x \in \mathbb{R}^n \setminus \{0\}$ that satisfies the set-equality

$$\{y \in \mathbb{R}^n : p_x \cdot y > 0\} = \left\{y \in \mathbb{R}^n : \text{there exists } \lambda^*_y > 0 \text{ such that } x + \lambda y \succ x \text{ for all } \lambda \in (0, \lambda^*_y)\right\}$$

If it exists, a vector p_x with the above property will be referred to as a *preference gradient* at x. An intuitive interpretation for the elements $p^1_x, \ldots, p^n_x$ of p_x is that they represent the consumer's "subjective values" of the different goods relative to the reference bundle x: *"Starting from x, any small move in a direction that is evaluated by this vector as positive is an improvement"* (see Rubinstein (2012) pp. 71). Rubinstein showed that, under strict convexity and strict monotonicity, the utility representation being partially differentiable suffices for the preference relation to be differentiable.

With the requisite concepts in place, our main result can now be formally stated.

Theorem 1 *The following are equivalent for a continuous weak order $\succsim$ on X:*
(i). $\succsim$ is strictly convex, strictly monotonic and differentiable.
(ii). $\succsim$ is strictly convex, strictly monotonic and weakly C^1.
(iii). There is a unique, open set $Y \subseteq \mathbb{R}^n_{++}$ and a unique, continuous, invertible demand function $x : Y \to \mathrm{int}(X)$ that is generated by $\succsim$.

The statement of the theorem is but a combination of Propositions 1-3, which are presented separately in the next section and proved in Section 4. To give the reader a better understanding of the interaction between the three preference axioms and the onto and one-to-one properties of the generated demand function, we decompose the theorem into its constituent parts and provide an outline of the relevant formal arguments. Before turning to this, however, an informal overview of our analysis would be instructive.

Naturally, the problem of finding prices that rationalize the demand of a given consumption bundle is a supporting-hyperplane existence question. Continuity and convexity of preferences in our framework do indeed ensure that such a hyperplane exists. Strict convexity, moreover, guarantees uniqueness of the demanded bundle at these prices, while strict monotonicity ensures that the prices cannot but be all strictly positive. Finally, Rubinstein differentiability implies that the (normalized) supporting prices are unique. This relatively straightforward argument establishes the part of Theorem 1 that claims the sufficiency of the postulated properties on preferences for the demand to be an invertible function - although not necessarily continuous, it takes a more involved argument to show that continuity of the demand function follows from preference differentiability (see Lemma 4.7). And it is considerably more challenging to establish the part of the theorem postulating that the invertibility of the demand function *necessitates* that the generating preference relation is differentiable. This amounts to showing that a given notion of smoothness for the functional representation of the graph of the indifference sets corresponds to the appropriate notion of smoothness for the preference relation, an abstract and highly non-trivial exercise.[2]

2.1 Characterization of onto demand functions

The first part of our decomposition characterizes onto demand functions by means of continuity, strict convexity and strict monotonicity alone.

Proposition 1 *The following are equivalent for a continuous weak order $\succsim$ on X:*
(i). $\succsim$ is strictly convex and strictly monotonic.
(ii). There exists a unique $Y \subseteq \mathbb{R}^n_{++}$ and a unique onto demand function $x : Y \to int\,(X)$ generated by $\succsim$.

Although much easier to establish compared to the characterization of one-to-one demand functions that we offer below, this result is of independent interest and, to our knowledge, novel. To outline briefly the intuition behind it, we introduce some additional notation. For each $x \in X$ we let $\mathcal{U}_x := \{z \in X : z \succsim x\}$ and $\mathcal{I}_x := \{z \in X : z \sim x\}$ denote the weak upper-contour and indifference sets of x, respectively. For any $A \subset \mathbb{R}^n$, moreover, we will say that $p \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ *supports A at x* if $px \leq pz$ for any $z \in A$, and that p supports A at x *properly* if $px < pz$ for any $z \in A \setminus \{x\}$. Take now any $x \in \operatorname{int}(X)$. Given continuity and strict convexity, it follows from the supporting hyperplane theorem there exists $p \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ that supports $\mathcal{U}_x$ at x; and hence that $x \in \max_{\succsim} \{z \in X : pz \leq px\}$. And given also strict monotonicity, any such p must be in fact a strictly positive price vector. Letting, therefore,

$$Y := \left\{ p \in \mathbb{R}^n_{++} : \text{ there exists } x \in \operatorname{int}(X) \text{ such that } x \in \max_{\succsim} B_X(p) \right\}$$

[2] A simple inspection of the expositions in Debreu (1972) or Section 2.3 in Mas-Colell (1985) attests to the technical nature of such an exercise. Of course, by restricting attention on what happens only along a given indifference set (not along the entire indifference relation), the present exposition is far more straightforward; it does remain though non-trivial.

the (unique) mapping $x : Y \twoheadrightarrow X$ given by $x(p) = \max_{\succsim} B_X(p)$ is an onto demand correspondence; in fact, an onto demand function since the strict convexity of $\succsim$ guarantees that $x(\cdot)$ is single-valued. Conversely, if $x : Y \to \text{int}(X)$ is an onto function, that $\succsim$ is strictly monotonic follows from the strict positivity of prices. And as $x(\cdot)$ is single-valued and generated by continuous and strictly monotonic preferences, that $\succsim$ is also strictly convex follows by the equivalence result in Bilancini and Boncinelli (2010).[3]

A special case of particular interest corresponds to the domain of the demand function being the entire orthant of strictly positive prices. When $X = \mathbb{R}^n_+$, we get that $Y = \mathbb{R}^n_{++}$ in Proposition 1 if and only if $\succsim$ is in addition *self-contained* in $\mathbb{R}^n_{++}$ in the sense that, for all $z \in \mathbb{R}^n_{++}$ and $x \in X$, $z \sim x$ implies $x \in \mathbb{R}^n_{++}$.[4] Under the maintained assumption of strict monotonicity, $\succsim$ being self-contained in $\mathbb{R}^n_{++}$ means that the boundary $\mathbb{R}^n_+ \setminus \mathbb{R}^n_{++}$ forms an indifference set. The latter property implies in turn that the preference relation is indeed self-contained in $\mathbb{R}^n_{++}$. For this is the relevant consumption set if one assumes –as is often the case in the literature– that the main interest of the analysis is in strictly positive consumption bundles: extending the preference domain from $\mathbb{R}^n_{++}$ to $\mathbb{R}^n_+$ adds an indifference set which lies at the very bottom of the preference ranking over indifference sets, leaving the non-trivial part of the ranking unaffected. Put differently, restricting the preference domain to $\mathbb{R}^n_{++}$ instead of $\mathbb{R}^n_+$ is without loss of generality because no additional information about demand can be obtained by examining how preferences "behave" on the boundary $\mathbb{R}^n_+ \setminus \mathbb{R}^n_{++}$.

Claim 1 *Let the onto demand function $x : \mathbb{R}^n_{++} \supseteq Y \to \mathbb{R}^n_{++}$ be generated by the continuous weak order $\succsim$ on $X = \mathbb{R}^n_+$. Then $Y = \mathbb{R}^n_{++}$ if and only if $\mathbb{R}^n_{++} \supseteq X$ while $\succsim$ is self-contained in $\mathbb{R}^n_{++}$.*

The requirement that $X = \mathbb{R}^n_+$ while the preference relation is self-contained in $\mathbb{R}^n_{++}$ has appeared in the literature as Assumption 3.1-4 in Katzner (1970).[5] It is satisfied, for example, by Cobb-Douglas and Leontieff preferences, but not by quasilinear preferences.

2.2 Characterization of invertible demand functions

To examine the second part of our decomposition, for the arbitrary $x \in \text{int}(X)$ consider the projection of the indifference set $\mathcal{I}_x$ along the typical dimension $i \in \mathcal{N}$

$$\mathcal{I}^i_x := \left\{ z_i \in \mathbb{R}_+ : \text{ there exists } z_{-i} \in \mathbb{R}^{n-1}_+ \text{ such that } z \in \mathcal{I}_x \right\}$$

and the decomposition $\mathcal{I}_x = \mathcal{I}^i_x \times \mathcal{I}^{-i}_x$. We can define then the correspondence $l_i(\cdot|x) : \mathcal{I}^{-i}_x \twoheadrightarrow \mathcal{I}^i_x$ by $z_i \in l_i(z_{-i}|x) \Leftrightarrow z \in \mathcal{I}_x$; whose graph is of course $\mathcal{I}_x$. As it turns out (see Lemma B.12), $l_i(\cdot|x)$ is actually a function that is locally convex (and thus also continuous) at x_{-i}. As a result, its *subdifferential* $\partial l_i(x_{-i}|x)$

[3] The result in Bilancini and Boncinelli (2010) states that, if a rational demand correspondence is generated by a strictly monotonic and continuous weak order, then the former is single-valued if and only if the latter is strictly convex.

[4] The full price domain necessitates that the consumption domain includes the entire orthant of strictly positive bundles (see Lemma B.11). We extend X to the entire $\mathbb{R}^n_+$ in order to obtain a sufficient condition for the full price domain. Claim 1 establishes that the extension can be done in only one way: the boundary $\mathbb{R}^n_+ \setminus \mathbb{R}^n_{++}$ must form an indifference set.

[5] Theorem 3.1-13 in Katzner (1970) states that a demand function $x : \mathbb{R}^n_{++} \to \mathbb{R}^n_{++}$ is bijective if it is generated by a preference relation $\succsim$ on $\mathbb{R}^n_+$ which is self-contained in $\mathbb{R}^n_{++}$ and representable by a strictly concave and strictly increasing utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ that is twice continuously differentiable on $\mathbb{R}^n_{++}$. Claim 1 underlines the limits of the scope of Katzner's theorem: the price domain of the demand function under consideration must be the entire $\mathbb{R}^n_{++}$.

–the collection of its subgradients at x_{-i}– is non-empty and linked fundamentally to its smoothness: $l_i(\cdot|x)$ is partially differentiable at x_{-i} if and only if $\partial l_i(x_{-i}|x)$ is singleton (in which case the unique subgradient is the gradient).[6]

Proposition 2 *Let the onto demand function $x : \mathbb{R}^n_{++} \supseteq Y \to int(X)$ be generated by the continuous weak order $\succsim$ on X. For any $x \in int(X)$ the following are equivalent:*
(i). For some $i \in \mathcal{N}$, $l_i(\cdot|x)$ is partially differentiable at x_{-i}.
(ii). $\succsim$ is differentiable at x.
(iii). $p(x)$ is a singleton.
(iv). For any $i \in \mathcal{N}$, $l_i(\cdot|x)$ is partially differentiable at x_{-i}.

That is, the (onto) demand function that is generated by a strictly convex and strictly monotonic continuous weak order $\succsim$ on X is also injective (and therefore invertible) if and only if $\succsim$ is differentiable on $\text{int}(X)$. More specifically, letting $\nabla l_i(x_{-i}|x)$, $i \in \mathcal{N}$ denote the gradient (the unique subgradient) at $x \in \text{int}(X)$, the inverse demand function is given via the following system of equations.

$$\begin{aligned} q_{-i}(x) &= \nabla l_i(x_{-i}|x) \in \mathbb{R}^n_{--} && (1)\\ q_i(x) &= 1/\left(x_i - q_{-i}(x) \cdot x_{-i}\right) \in \mathbb{R}_{++} && (2)\\ p(x) &= q_i(x)\left(1, -q_{-i}(x)\right) \in \mathbb{R}^n_{++} && (3) \end{aligned}$$

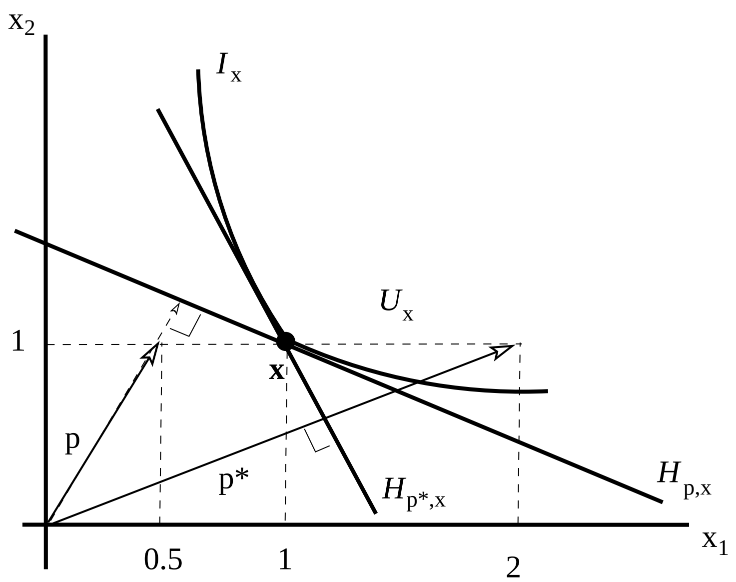


Figure 1: $\succsim$ is strictly convex, strictly monotonic but not differentiable at x. Either of p, p^* supports $\mathcal{U}_x$; yet neither is a preference-gradient at x.

Figures 1-2 provide some geometric intuition for the equivalence between $\partial l_i(x_{-i}|x)$, on the one hand, and $p(x)$, on the other, being singletons. The non-trivial part though (and in this sense the gist) of Proposition 2 is that $\partial l_i(x_{-i}|x)$ being singleton is also equivalent to $\succsim$ being differentiable at x. Regarding this equivalence, the concept of ordient –which was first introduced by Renou and Schlag (2014)–

[6]We refer the reader to Appendix A for some background on subgradients and subdifferentials.

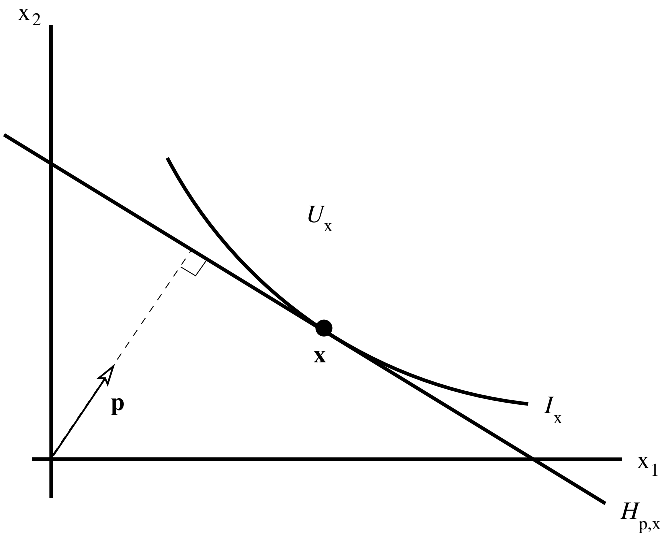


Figure 2: $\succsim$ is strictly convex, strictly monotonic and differentiable at x; p is a preference-gradient at x.

facilitates some intuition. For $z, x \in X$ with $z \neq x$ we will say that $z - x$ is an *improvement* [resp. *worsening*] *direction at* x if there exists $\lambda^* > 0$ such that $x + \lambda(z - x) \succ x$ [resp. $x \succ x + \lambda(z - x)$] for all $\lambda \in (0, \lambda^*)$ with $x + \lambda(z - x) \in X$. Considering also the plane $H_{p,x} := \{z \in X : pz = px\}$ as well as the (interior) half-planes $H^+_{p,x} := \{z \in X : pz > px\}$ and $H^-_{p,x} := \{z \in X : pz < px\}$, we will say also that $p \in \mathbb{R}^n \setminus \{0\}$ is an *increasing* [resp. *decreasing*] *ordient at* x if $z - x$ is an improvement [resp. worsening] direction at x for any $z \in H^+_{p,x}$ [resp. $z \in H^-_{p,x}$]. Being at the same time increasing and decreasing ordient will be referred to as being an *ordient at* x.[7] And it is easy to see that, if $p \in \mathbb{R}^n \setminus \{0\}$ is an increasing ordient at x such that $z - x$ is an improvement direction at x only if $z \in H^+_{p,x}$, then p is in fact a preference gradient at x.

For an arbitrary $x \in \operatorname{int}(X)$ take now $p \in \mathbb{R}^n_{++}$ such that $x = x(p)$. This means of course that p supports $\mathcal{U}_x$ at x (see Lemma B.8); hence, that p is a decreasing ordient at x (see Lemma B.1). To ensure though the uniqueness of $p \in \mathbb{R}^n_{++} : x = x(p)$, it must be an ordient at x (recall footnote 7). We require therefore that p be also an increasing ordient at x. Yet p may support $\mathcal{U}_x$ at x only if it does so properly (see again Lemma B.8). And to be an increasing order that supports $\mathcal{U}_x$ at x properly, p cannot but be a preference gradient at x (Lemma B.9); more precisely, the unique preference gradient at x.

Characterization of homeomorphic demand functions

It is also standard practice in the literature to work with invertible demand functions that are also continuous. We take next a closer look at the continuity of the demand function, from the perspective of the preferences themselves –as far as strictly convex and strictly monotonic, continuous weak orders on

[7] Renou and Schlag (2014) introduce the concept of ordients and undertake an in-depth study of its implications for optimization. Intuitively, if p is an ordient at x, then $\left\{H^-_{p,x}, H^+_{p,x}\right\}$ becomes a partition of $X \setminus H_{p,x}$ into improvement and worsening directions: $z - x$ is an improvement [resp. worsening] direction at x if and only if $z \in H^+_{p,x}$ [resp. $z \in H^-_{p,x}$]. It is straightforward to check that, for strictly convex preferences, this means in fact that $H_{p,x}$ (uniquely) separates locally at x the sets $\mathcal{U}_x = \{z \in X : z \succsim x\}$ and $X \setminus \mathcal{U}_x$. This separation being local, however, it does not guarantee that $H_{p,x}$ supports $\mathcal{U}_x$ at x. The latter property, which is sine qua non for x to be demanded at p, obtains if p is a preference gradient at x.

X are concerned. To this end, we borrow from Neilson (1991) for the appropriate notion of smoothness for the class of preferences under consideration. We will say that a weak order $\succsim$ on X is *weakly* C^1 at $x \in \text{int}(X)$ if $\mathcal{I}_x$ is a C^1 manifold (of dimension $n-1$) at x; i.e, if there exists $\epsilon > 0$ such that $\mathcal{B}_\epsilon(x) \cap \mathcal{I}_x$ is a C^1 manifold (of dimension $n-1$).[8] If this is moreover the case everywhere on $\text{int}(X)$, $\succsim$ will be called weakly C^1.

As it turns out, under strict convexity and strict monotonicity, the notions of differentiability and weak smoothness (weakly C^1) for a preference relation are equivalent, and fundamentally related to the continuity of both the ordinary and inverse demand functions.[9]

Proposition 3 *Let the onto demand function $x : \mathbb{R}^n_{++} \supseteq Y \to int(X)$ be generated by the continuous weak order $\succsim$ on X. The following are equivalent.*
(i). $\succsim$ is differentiable.
(ii). Y is open, and $x(\cdot)$ is injective and continuous (thus, an homeomorphism).
(iii). $\succsim$ is weakly C^1.

3 Preference differentiability is weaker than the standard notion of smooth preference: examples

Rather than considering the preferences themselves, the standard practice for applications in the literature is to work instead with the utility representations, taken to be sufficiently smooth to guarantee desirable properties for the generated demand function. It is straightforward to check that, for any $x \in \text{int}(X)$ and $i \in \mathcal{N}$, $l_i(\cdot|x)$ is guaranteed to be partially differentiable at x_{-i} if there exists some utility function $u : X \to \mathbb{R}$ for $\succsim$ that is partially differentiable at x.[10] Indeed, as long as the partial derivatives of $u(\cdot)$ are well defined at x, those of $l_i(\cdot|x)$ are given by (see Lemma B.15)[11]

$$\frac{\partial l_i(x_{-i}|x)}{\partial x_j} = -\left(\frac{\partial u(x)}{\partial x_i}\right)^{-1} \frac{\partial u(x)}{\partial x_j} \qquad x \in \text{int}(X), \quad i, j \in \mathcal{N} : i \neq j \tag{4}$$

Clearly, for a strictly convex and strictly monotonic weak order $\succsim$ on X to be differentiable at $x \in \text{int}(X)$, it suffices that it admits some utility representation that is partially differentiable at x.[12] Yet the latter property is by no means necessary for $\succsim$ to be differentiable at x. The notions of preference

[8]A C^r ($r \in \mathbb{N} \cup \{\infty\}$) function $f : \mathbb{R}^n \supseteq X \to \mathbb{R}^n$ (X open) is a C^r *diffeomorphism* if it is an homeomorphism with a C^r inverse. A set $M \subseteq \mathbb{R}^n$ is a C^r *k-dimensional* ($k \leq n$) *manifold* if for every $x \in M$ there is a C^r diffeomorphism $f : \mathbb{R}^n \supseteq X \to \mathbb{R}^n$ (X open) which carries the open set $X \cap \left(\mathbb{R}^k \times \left\{0^{n-k}\right\}\right)$ onto a neighborhood of x in M. For more details and some examples, see for instance Chapter 1.H in Mas-Colell (1985).

[9]Letting X, Y be two metric spaces, a function $f : X \to Y$ is an *homeomorphism* if it is injective, continuous, and its inverse function is continuous on $f(X)$.

[10]The utility representation may also take the more general form $u : X \to \mathbb{R} \cup \{-\infty, +\infty\}$. Nevertheless, under strict monotonicity, it cannot but be $u(x) \notin \{-\infty, +\infty\}$ for any $x \in \text{int}(X)$.

[11]Equation (4) depicts a well-known relation: the gradient of the indifference curve at an interior point is given by the marginal rates of substitution. Yet the standard derivation invokes the implicit function theorem, taking thus the utility function to be sufficiently smooth (at least C^1). By contrast, Lemma B.15 takes it to be only partially differentiable. Notice that the partial differentiability of a function does not necessarily imply its full differentiability, unless the partial derivatives are continuous –see for example Theorem 4.3.4 in de la Fuente (2000).

[12]The claim follows directly from (4) above and Proposition 2. It is also shown independently in Rubinstein (2012), pp. 60.

differentiability, on the one hand, and differentiability (even partial) of the utility representation, on the other, are by no means equivalent. As demonstrated by the examples below, the former property is weaker than the latter (see Example 1) and ordinal (while the latter is not - see Example 2). Equally importantly, it is independent from the fact that $\succsim$ generates an (onto) demand function: preference differentiability is implied neither by strict convexity nor by strict monotonicity (see Example 3).

Example 1.

Consider the weak order $\succsim$ on $X = (2e, +\infty) \times (0, 1)$ that is represented by the utility function

$$u(x) := \begin{cases} \ln x_1 + \ln x_2, & \text{if } x \in S := \{x \in X : x_1 x_2 \le e\} \\ \frac{\ln x_1}{1 - \ln x_2}, & \text{otherwise} \end{cases}$$

It is trivial to check that $\succsim$ is continuous and strictly increasing on X. For $\overline{u} \in \mathbb{R}$, moreover, the indifference set $\{x \in X : u(x) = \overline{u}\}$ is given by the graph of the function: $x_2 := e^{\overline{u}}/x_1$ on S and $x_2 := e x_1^{-(1/\overline{u})}$ on $X \setminus S$. For any $\overline{x} \in X$, therefore, we have $l_2(x_1|\overline{x}) = e^{u(\overline{x})}/x_1$ on S and $l_2(x_1|\overline{x}) = e x_1^{-(1/u(\overline{x}))}$ on $X \setminus S$. Clearly, $\succsim$ is also strictly convex on X (see Lemma B.13).

For any $x \in \mathcal{I}_{\overline{x}}$, we get $l_2'(x_1|\overline{x}) = -l_2(x_1|\overline{x})/x_1 = -x_2/x_1$ on S and $l_2'(x_1|\overline{x}) = -u(\overline{x})^{-1} l_2(x_1|\overline{x})/x_1 = -u(\overline{x})^{-1} x_2/x_1$ on $X \setminus S$. And as $u(\overline{x}) = 1$ on the set $S_0 := \{x \in S : x_1 x_2 = e\}$, $\succsim$ is also differentiable on X. Indeed, by equations (1)-(3), the preference gradient is given by

$$p(x) = \begin{cases} \frac{1}{2x_2}\left(\frac{x_2}{x_1}, 1\right), & \text{if } x \in S \\ \frac{\ln x_1}{x_2(1 + \ln(x_1/x_2))}\left(\frac{x_2(1 - \ln x_2)}{x_1 \ln x_1}, 1\right), & \text{otherwise} \end{cases}$$

However, $u(\cdot)$ is not even partially differentiable on S_0: we have $\nabla u(x) = \left(x_1^{-1}, x_2^{-1}\right)$ on $S \setminus S_0$ and $\nabla u(x) = \left((x_1(1 - \ln x_2))^{-1}, u(x)(x_2(1 - \ln x_2))^{-1}\right)$ on $X \setminus S$. In fact, no utility representation of $\succsim$ can be partially differentiable on S_0. To see this, let $f(u(\cdot))$ be such a representation for some strictly increasing transformation $f : \mathbb{R} \to \mathbb{R}$ that is continuously differentiable on $\mathbb{R} \setminus \{1\}$. Taking x approach a limit $\overline{x} \in S_0$ would have $f'(u(\cdot)) \nabla u(\cdot)$ approach $\lim_{z \nearrow 1} f'(z)\left(\overline{x}_1^{-1}, \overline{x}_2^{-1}\right)$ from within S but $\lim_{z \searrow 1} f'(z)(1 - \ln \overline{x}_2)^{-1}\left(\overline{x}_1, \overline{x}_2^{-1}\right)$ from outside of S. Alas $\lim_{z \nearrow 1} f'(z) = \lim_{z \searrow 1} f'(z)(1 - \ln \overline{x}_2)^{-1}$ cannot obtain but at most on a single point in S_0. $\Diamond$

Example 2.

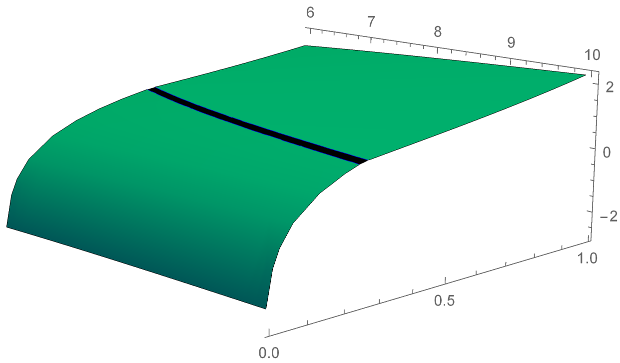
Consider the weak order $\succsim$ on $X = \mathbb{R}^2_{++}$ that is represented by the utility function

$$u(x) := \begin{cases} x_1^{\frac{1}{2}} x_2^{\frac{1}{4}}, & \text{if } x \in S := \left\{x \in X : x_1^2 x_2 \le 1\right\} \\ x_1^{\frac{2}{3}} x_2^{\frac{1}{3}}, & \text{otherwise} \end{cases}$$

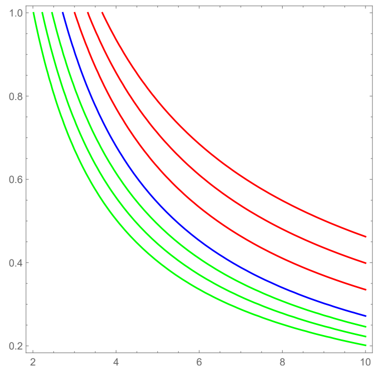
It is again trivial to check that $\succsim$ is continuous, strictly increasing, strictly convex and differentiable on X. The preference gradient is given here by $p(x) = \frac{1}{3x_2}\left(\frac{2x_2}{x_1}, 1\right)$ everywhere on X. The demand function $x : \mathbb{R}^2_{++} \to \mathbb{R}^2_{++}$ is given by $x(p) = \left(\frac{2}{3p_1}, \frac{1}{3p_2}\right)$.

And again as before, $u(\cdot)$ is not even partially differentiable on $S_0 := \left\{x \in S : x_1^2 x_2 = 1\right\}$. We have $\nabla u(x) = \frac{1}{4}\left(2u(x) x_1^{-1}, u(x) x_2^{-1}\right)$ on $S \setminus S_0$ and $\nabla u(x) = \frac{1}{3}\left(2u(x) x_1^{-1}, u(x) x_2^{-1}\right)$ on $X \setminus S$ while

Figure 3: Strictly convex, strictly monotonic and differentiable but non-smooth preferences (Example 1)



(a) The graph of the utility function is kinked when $x_1 \cdot x_2 = e$



(b) The indifference curves are smooth (blue curve for $\overline{u} = 1$)

$u(x) = 1$ on S_0.

In sharp contrast to the previous example, however, $\succsim$ does admit here another utility representation which is partially differentiable (in fact C^∞) everywhere on X. To see this, let $f(u(\cdot))$ be another representation for some strictly increasing transformation $f : \mathbb{R}_{++} \to \mathbb{R}_{++}$ that is continuously differentiable on $\mathbb{R}_{++} \setminus \{1\}$. Taking x approach a limit $\overline{x} \in S_0$ would have $f'(u(\cdot))\nabla u(\cdot)$ approach $\frac{1}{4}\lim_{z\nearrow 1} f'(z)\left(2\overline{x}_1^{-1}, \overline{x}_2^{-1}\right)$ from within S and $\frac{1}{3}\lim_{z\searrow 1} f'(z)\left(2\overline{x}_1^{-1}, \overline{x}_2^{-1}\right)$ from outside of S. It would suffice therefore that $\lim_{z\nearrow 1} f'(z) = \frac{4}{3}\lim_{z\searrow 1} f'(z)$. Indeed, defining $f(z) = z^4$, if $z \leq 1$, and $f(z) = z^3$, otherwise, gives the representation $f(u(x_1, x_2)) = x_1^2 x_2$, everywhere on X. $\diamondsuit$

Example 3.

Consider the weak order $\succsim$ on $X = \mathbb{R}^2_{++}$ that is represented by the utility function

$$u(x) := \begin{cases} x_1^2 x_2, & \text{if } (x_1, x_2) \in S := \{x \in X : x_1 \leq x_2\} \\ x_1 x_2^2, & \text{otherwise} \end{cases}$$

As before, $\succsim$ is continuous, strictly increasing and strictly convex on X. For any $\overline{x} \in X$, we have $l_2(x_1|\overline{x}) = u(\overline{x})/x_1^2$ on S and $l_2(x_1|\overline{x}) = \sqrt{u(\overline{x})/x_1}$ on $X \setminus S$. For any $x \in \mathcal{I}_{\overline{x}}$, therefore, we get $l_2'(x_1|\overline{x}) = -2x_2/x_1$ on $S \setminus S_0 := \{x \in S : x_1 = x_2\}$ and $l_2'(x_1|\overline{x}) = -x_2/(2x_1)$ on $X \setminus S$. As opposed to the previous examples, here $\succsim$ is not differentiable anywhere on S_0 (see Figure 1 for a graphical representation). Obviously, neither $u(\cdot)$ nor any other utility representation can be partially differentiable on

S_0. $\Diamond$

Clearly, the preference being weakly smooth (weakly C^1), on the one hand, and the utility representation being smooth, on the other, are by no means equivalent notions. For a strictly convex and strictly monotonic continuous weak order on X, being weakly C^1 is in general weaker than being representable by a C^1 utility function; thus, the former property is weaker than the standard notion of smoothness for preference relations.[13] Indeed, in either of Examples 1-2 above, the preference relation under consideration is weakly C^1 and generates a demand homeomorphism. It is C^1-smooth in the standard sense, however, only in Example 2.

The key difference between the two examples lies in that, even though in either case $u(\cdot)$ presents convex kinks everywhere on (the respective) S_0, the preference relation in Example 2 is homothetic.[14] And for strictly monotonic and strictly convex continuous weak orders on X, homotheticity implies that the preference is differentiable if and only if it admits a C^1 utility representation. The latter equivalence obtains also if the preference is quasilinear.

Claim 2 *Let $\succsim$ be a strictly convex, strictly monotonic, continuous weak order on X that is quasilinear or homothetic. At any $x \in int(X)$, the following are equivalent.*
(i) $\succsim$ is differentiable at x.
(ii) $\succsim$ is weakly C^1 at x.
(iii) $\succsim$ admits a utility function $u : X \to \mathbb{R}$ that is C^1 at x.
(iv) $\succsim$ admits a utility function $u : X \to \mathbb{R}$ that is partially differentiable at x.

The preferences being homothetic also in Example 3 above, we have yet another demostration of the claim. And the intuition in Example 3 applies for the more general formulation where $\succsim$ is represented by

$$u(x) := \begin{cases} f(x_1, x_2), & \text{if } (x_1, x_2) \in S := \{x \in X : x_1 \leq x_2\} \\ f(x_2, x_1), & \text{otherwise} \end{cases}$$

for some function $f : S \to \mathbb{R}$ that is strictly quasiconcave, strictly increasing and C^1 on $S \setminus S_0$ (where X, S and S_0 are as in Example 3). For as long as $\partial f(x)/\partial x_1 \neq \partial f(x)/\partial x_2$ on S_0, $\succsim$ will not be differentiable anywhere on S_0. Indeed, by (4) above and for any $\overline{x} \in X$ we have

$$l_2'(x_1|\overline{x}) = \begin{cases} -\frac{\partial f(x)/\partial x_1}{\partial f(x)/\partial x_2}, & \text{if } x \in \mathcal{I}_{\overline{x}} \cap S \setminus S_0 \\ -\frac{\partial f(x)/\partial x_2}{\partial f(x)/\partial x_1}, & \text{if } x \in \mathcal{I}_{\overline{x}} \cap (X \setminus S) \end{cases}$$

[13] Recall the standard definition of smoothness for a weak order $\succsim$ on $\mathbb{R}^n_{++}$: $\succsim$ is said to be C^r-smooth ($r \in \mathbb{N} \cup \{\infty\}$) if it is locally non-satiated while its indifference relation $\mathcal{I} := \{(x,z) \in \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} : x \sim z\}$ is a C^r manifold. If $\succsim$ is in addition monotonic, it is C^r-smooth if and only if it is representable by a C^r utility function - see Debreu (1972) and Section 2.3 in Mas-Colell (1985). Obviously, for a strictly convex and strictly monotonic continuous weak order on $\mathbb{R}^n_{++}$, being weakly C^1 is in general weaker than being C^1-smooth in the standard sense. The former notion restricts attention to movements within the same indifference set. The latter is concerned also with movements across different indifference sets.

[14] In either of examples 1-2, we have $\nabla u(x) < \nabla u(z)$ for any $(x,z) \in S \setminus S_0 \times X \setminus S$; hence, $u(\cdot)$ is locally convex everywhere on S_0.

Clearly, the quantity $\lim_{x_1 \to \overline{x}_1} l_2' (x_1 | \overline{x})$ takes different values as we approach $\overline{x} \in S_0$ from within S as opposed to from outside. The well-known example of homothetic preferences generating a non-invertible demand function in Hurwicz and Uzawa (1971) has $f(x) = x_2 \phi(x_1/x_2)$ with $\phi : [0,1] \to \mathbb{R}$ given by $\phi(t) = 3 - (1-t)\left(2 + \sqrt{1-t}\right)$.

4 Proofs

For $x \in \mathbb{R}^n$ and $\varepsilon > 0$, $\mathcal{B}_\varepsilon(x)$ denotes in what follows the open ball in $\mathbb{R}^n$ with center x and radius ε. For $i, j \in \mathcal{N}$ $(i < j)$, the vectors $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathbb{R}^{n-1}$ and $\left(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n\right) \in \mathbb{R}^{n-2}$ are depicted as x_{-i} and $x_{-(i,j)}$ respectively. We also define the index sets $\mathcal{N}_x^+ := \{i \in \mathcal{N} : x_i > 0\}$ and $\mathcal{N}_x^- := \{i \in \mathcal{N} : x_i < 0\}$. Moreover, for any finite $M \subset \mathbb{N}$, $|M|$ denotes its cardinality, while $||x||$ denotes the Euclidean norm of x. Finally, for $A \subset \mathbb{R}^n$, $\text{int}(A)$, $\text{bd}(A)$ and $\text{cl}(A)$ denote, respectively, the interior, boundary and closure of A. And for $i, j \in \mathcal{N}$ with $i \neq j$, we let A_i and A_j denote the projections of A on the real line along the respective dimension; we define also A_{-i} and $A_{-(i,j)}$ by $A = A_i \times A_{-i}$ and $A = A_i \times A_j \times A_{-(i,j)}$.

Proof of Proposition 1

(i) $\Rightarrow$ (ii). Let $x \in \text{int}(X)$. As $\succsim$ is continuous and convex, $\mathcal{U}_x$ is closed and convex. Moreover, since $\succsim$ is strictly monotonic, it cannot but be $x \in \text{bd}(\mathcal{U}_x)$ while $\text{int}(\mathcal{U}_x) \neq \varnothing$. To check the first claim, suppose instead that $x \in \text{int}(\mathcal{U}_x)$. There must be then $\varepsilon > 0$ such that $\mathcal{B}_\varepsilon(x) \subset \mathcal{U}_x$, and a contradiction obtains because $(x_i - \varepsilon/2, x_{-i}) \in \mathcal{B}_\varepsilon(x)$ while strict monotonicity necessitates that $x \succ (x_i - \varepsilon/2, x_{-i})$. For the second claim, notice that $x \in \text{int}(X)$ necessitates the existence of $\delta > 0$ such that $\mathcal{B}_\delta(x) \subset X$. Taking then e to denote the vector of ones in $\mathbb{R}^n$, strict monotonicity ensures that $x + \delta e/2 \succ x$ while $\mathcal{B}_{\delta/4}(x + \delta e/2) \subset \mathcal{U}_x$.

Given the observations above, it is well-known - see for instance Lemma 7.7 in Aliprantis and Border (2006) - that some $p \in \mathbb{R}^n \setminus \{0\}$ supports $\mathcal{U}_x$ at x. It must be in fact $x \in \max_\succsim \{z \in X : pz \leq px\}$ (see Lemma 4.1 if $X \setminus \text{int}(X) = \varnothing$; see Lemma 4.2 if $X \setminus \text{int}(X) \neq \varnothing$) while $p \in \mathbb{R}^n_{++}$ (see Lemma 4.3).

Lemma 4.1 *Let $\succsim$ be a continuous weak order on $X \subseteq \mathbb{R}^n_+$ with $X \setminus int(X) = \varnothing$.*[15] *For any $x \in X$, $p \in \mathbb{R}^n \setminus \{0\}$ supports $\mathcal{U}_x$ at x only if $x \in \max_\succsim \{z \in X : pz \leq px\}$.*

Proof. Let $p \in \mathbb{R}^n \setminus \{0\}$ support $\mathcal{U}_x$ at x. We need to show that $z \in X \setminus \{x\}$ and $pz \leq px$ implies $x \succsim z$. As this is obvious when $pz < px$, suppose that $pz = px$ and assume to the contrary that $z \succ x$; i.e., that $z \in \mathcal{U}_x \setminus \mathcal{I}_x$. Since $\succsim$ is continuous, the latter set is open; thus, $z' \succ x$ for all $z' \in \mathcal{B}_\varepsilon(z)$ for sufficiently small $\varepsilon > 0$. As $X \setminus \text{int}(X) = \varnothing$, however, it must be $z \in \text{int}(X)$ so that we have in fact $z' \succ x$ for all $z' \in \mathcal{B}_{\varepsilon_0}(z) \subset \text{int}(X)$ for sufficiently small ε_0. Let then z' be given by $z_i' = z_i + \frac{\varepsilon_0}{2}$ if $p_i \leq 0$ and $z_i' = z_i - \frac{\varepsilon_0}{2}$ if $p_i > 0$, for $i \in \mathcal{N}$. Since $p \neq 0$, it cannot but be $pz' < pz = px$. Which is however absurd given that $z' \in \mathcal{U}_x$ while p supports $\mathcal{U}_x$ at x. ■

Lemma 4.2 *Let $\succsim$ be a strictly convex, continuous weak order on $X \subseteq \mathbb{R}^n_+$ with $X \setminus int(X) \neq \varnothing$. For any $x \in int(X)$, $p \in \mathbb{R}^n \setminus \{0\}$ supports $\mathcal{U}_x$ at x only if $x \in \max_\succsim \{z \in X : pz \leq px\}$.*

[15] For any $A \subset \mathbb{R}^n$, it is necessarily the case that $A \setminus \text{int}(A) = \text{bd}(X)$ only if A is closed. In general, we have $A \setminus \text{int}(A) = A \cap \text{bd}(A)$.

Proof. Let $p \in \mathbb{R}^n \setminus \{0\}$ support $\mathcal{U}_x$ at x. It suffices to show that $z \in X \setminus \{x\}$ and $pz \leq px$ implies $x \succsim z$. As this is obvious when $pz < px$, suppose that $pz = px$ and assume to the contrary that $z \succ x$. Define $z^\lambda = \lambda z + (1-\lambda)x$ for $\lambda \in (0,1)$ and observe that, since $||z^\lambda - x|| = \lambda||z-x||$, for any given $\varepsilon > 0$ it cannot but be $z^\lambda \in \mathcal{B}_\varepsilon(x)$ for sufficiently small λ. And as $x \in \text{int}(X)$ we have in fact $z^\lambda \in \mathcal{B}_\varepsilon(x) \subset \text{int}(X)$ for sufficiently small ε. However, by the strict convexity of $\succsim$, it cannot but be $z^\lambda \succ x$. Which means in turn that $x \notin \max_{\succsim}\{z \in \text{int}(X) : pz \leq px\}$, a contradiction of Lemma 4.1. ■

Lemma 4.3 *Let $\succsim$ be a strictly monotonic and continuous [resp. strictly monotonic and strictly convex] weak order on $X \subseteq \mathbb{R}^n_+$ with $X \setminus int(X) = \varnothing$ [resp. $X \setminus int(X) \neq \varnothing$]. For any $x \in int(X)$, $p \in \mathbb{R}^n \setminus \{0\}$ supports $\mathcal{U}_x$ at x only if $p \in \mathbb{R}^n_{++}$.*

Proof. Take any $(p,x) \in \mathbb{R}^n \setminus \{0\} \times \text{int}(X)$ and let p support $\mathcal{U}_x$ at x. We will show first that this can be only if $p \in \mathbb{R}^n_+ \setminus \{0\}$. To argue ad absurdum, suppose that $p_i < 0$ for some $i \in \mathcal{N}$. Taking $z \in X$ such that $z_i > x_i$ and $z_j = x_j$ for $j \in \mathcal{N} \setminus \{i\}$, we obviously have $pz < px$. Since $\succsim$ is strictly monotonic, however, we also have $z \succ x$. Together these relations contradict that p supports $\mathcal{U}_x$ at x.
It remains to show that $p \notin \text{bd}(\mathbb{R}^n_+)$. To argue again ad absurdum, let $p_i = 0$ for some $i \in \mathcal{N}$ and take $z \in X$ such that $z_i > x_i$ and $z_j = x_j$ for $j \in \mathcal{N} \setminus \{i\}$. As this gives $z \succ x$ with $pz = px$ and $z \in \text{int}(X)$, the desired contradiction is due to Lemma 4.1 [resp. Lemma 4.2]. ■

There exists thus $\widetilde{p} \in \mathbb{R}^n_{++}$ with $x \in \max_{\succsim}\{z \in X : \widetilde{p}z \leq \widetilde{p}x\}$. Equivalently, since $x \in \text{int}(X) \subseteq \mathbb{R}^n_{++}$, there exists $p = \widetilde{p}/\widetilde{p}x \in \mathbb{R}^n_{++}$ such that $x \in \max_{\succsim}\{z \in X : pz \leq 1\}$. Defining therefore Y as in the main text, the mapping $x : Y \twoheadrightarrow \mathbb{R}^n_{++}$ defined by $x(p) = \max_{\succsim}\{z \in X : pz \leq 1\}$ is an onto demand correspondence. That it is also single-valued follows from the strict convexity of $\succsim$.
<u>(ii) ⇒ (i)</u>. That $\succsim$ must be strictly monotonic is due to the following result.

Lemma 4.4 *Let $\succsim$ be a weak order on $X \subseteq \mathbb{R}^n_+$. An onto demand function $x : \mathbb{R}^n_{++} \supseteq Y \to X$ is generated by $\succsim$ only if the latter is strictly monotonic on X.*

Proof. Let $x, z \in X$ be such that $x \geq z$ with $x \neq z$. Since $x(\cdot)$ is onto, single-valued, and generated by $\succsim$, there exists $p \in \mathbb{R}^n_{++}$ such that $x = x(p)$. Hence, $px \leq 1$ while $x \succ z'$ for all $z' \in X \setminus \{x\}$ with $pz' \leq 1$. Notice now that $x \neq z$, $z - x \leq 0$, and $p \gg 0$ together imply that $p(z-x) < 0$. We have therefore $pz < px \leq 1$ from which it follows that $x \succ z$. ■

Given now the strict monotonicity and continuity of $\succsim$ and the single-valuedness of $x(\cdot)$, that $\succsim$ must be also strictly convex follows from the main result in Bilancini and Boncinelli (2010). □

Proof of Claim 1

Observe first that $\succsim$ is necessarily strictly convex and strictly monotonic (Proposition 1).
<u>"if"</u>. Take an arbitrary $p \in \mathbb{R}^n_{++}$. Being a continuous weak order on $X = \mathbb{R}^n_+$, $\succsim$ can be represented by a continuous utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$. As a result, $B_X(p)$ being compact, the solution set of $\max_{z \in B_X(p)} u(z)$ is guaranteed to be non-empty; it is also a singleton by the strict convexity of $\succsim$. Let then $x_0 = \max_{z \in B(p)} u(z)$. We will make use of the following result.

Lemma 4.5 *Let $\succsim$ be a strictly monotonic, continuous weak order on $X = \mathbb{R}^n_+$. Then $\succsim$ is self contained in $\mathbb{R}^n_{++}$ if and only if $\mathcal{I}_0 = bd\left(\mathbb{R}^n_+\right)$.*

Proof. The "if" part being obvious, we need to show the "only if." To this end, let $x, x' \in \text{bd}\left(\mathbb{R}^n_+\right) \setminus \{0\}$ be such that $x \succ x'$, and let $x'' \in \mathbb{R}^n_{++}$ be given by $x''_i = x_i$ for $i \in \mathcal{N}^+_x$ and $x''_i = \varepsilon$ for $i \in \mathcal{N} \setminus \mathcal{N}^+_x$ and for some $\varepsilon > 0$. By the strict monotonicity of $\succsim$, we have that $x'' \succ x$. Define then the function $z : [0,1] \to \mathbb{R}^n_+$ by $z(\lambda) = \lambda x'' + (1-\lambda) x'$. Since $z(1) = x'' \succ x \succ x' = z(0)$, there exists $\lambda_0 \in (0,1)$ such that $z(\lambda_0) \in \mathcal{I}_x$ (see Lemma B.5(i)). As $z(\lambda_0) \in \mathbb{R}^n_{++}$, however, $\succsim$ cannot be self contained in $\mathbb{R}^n_{++}$. We just established that $\succsim$ is self contained in $\mathbb{R}^n_{++}$ only if $x \sim x'$ for any $x, x' \in \text{bd}\left(\mathbb{R}^n_+\right) \setminus \{0\}$. That it must be also $0 \sim x$ for any $x \in \text{bd}\left(\mathbb{R}^n_+\right) \setminus \{0\}$ follows from the continuity of $\succsim$. Thus, $\text{bd}\left(\mathbb{R}^n_+\right) \subseteq \mathcal{I}_0$. That also $\mathcal{I}_0 \subseteq \text{bd}\left(\mathbb{R}^n_+\right)$ follows by the weak monotonicity of $\succsim$. ■

By the preceding lemma, since $\succsim$ is self-contained in $\mathbb{R}^n_{++}$, we have that $\text{bd}\left(\mathbb{R}^n_+\right) = \mathcal{I}(0)$. As a result, it cannot but be $x_0 \in \mathbb{R}^n_{++}$. The latter being though the image set of the onto function $x(\cdot)$, that $p \in Y$ follows from the very definition of Y.
"only if". To establish the contrapositive statement, we will make use of the following result.

Lemma 4.6 *Let $\succsim$ be a strictly monotonic, continuous weak order on $X = \mathbb{R}^n_+$. Then $\succsim$ is self contained in $\mathbb{R}^n_{++}$ if it satisfies the following condition*[16]

A 1 *For all $z \in \mathbb{R}^n_{++}$ and $x \in bd(\mathbb{R}^n_+) \setminus \{0\}$, $z - x$ is an improvement direction at x.*

Proof. To establish the contrapositive statement, suppose that $\succsim$ is not self contained in $\mathbb{R}^n_{++}$. There must exist then $x, x' \in \text{bd}\left(\mathbb{R}^n_+\right) \setminus \{0\}$ with $x \succ x'$ (Lemma 4.5). Taking e to denote the vector of ones in $\mathbb{R}^n$ define the function $z : [0,1] \to \mathbb{R}^n_{++}$ by $z(\mu) = x' + \mu e$. By continuity, we have that $x \succ z(\mu_0)$ for small enough $\mu_0 > 0$ (see Lemma B.5(ii)). Which implies in turn (see Lemma B.5(iii)) that there exists $\lambda_0 \in (0,1)$ such that $x \succ \lambda x + (1-\lambda) z(\mu_0) = x + (1-\lambda)(z(\mu_0) - x)$ for all $\lambda \in (0, \lambda_0)$. Clearly, $z(\mu_0) - x$ is not an improvement direction at x. ■

Suppose that $\succsim$ is not self-contained in $\mathbb{R}^n_{++}$. By the preceding lemma, it does not satisfy condition A1 either. There exist therefore $(z, x) \in \mathbb{R}^n_{++} \times \text{bd}(\mathbb{R}^n_+) \setminus \{0\}$ such that $z - x$ is not an improvement direction at x. But then there exists also $p \in \mathbb{R}^n_{++}$ such that $x = \max_{\succsim} \{z \in \mathbb{R}^n_+ : pz \leq px\}$ (see Lemma B.10); equivalently, $x = \max_{\succsim} \{z \in \mathbb{R}^n_+ : \widetilde{p}z \leq 1\}$ where $\widetilde{p} = p/px$.
Let now $Y = \mathbb{R}^n_{++}$, and observe that $x \neq x(\widetilde{p}) \in \mathbb{R}^n_{++}$. We can define thus the function $\overline{x} : [0,1] \to \mathbb{R}^n_+$ by $\overline{x}(\lambda) = \lambda x + (1-\lambda) x(\widetilde{p})$. Clearly, $\widetilde{p}x(\widetilde{p}) = 1 = \widetilde{p}x$ means that also $\widetilde{p}\overline{x}(\lambda) = 1$ for all $\lambda \in (0,1)$. Moreover, as $\lim_{\lambda \to 0} \overline{x}(\lambda) = x(\widetilde{p})$, it cannot but be $\overline{x}(\lambda) \in \mathbb{R}^n_{++}$ for small enough λ. A contradiction now obtains because, due to the strict convexity of $\succsim$, $x \sim x(\widetilde{p})$ means that $\overline{x}(\lambda) \succ x(\widetilde{p})$. □

Proof of Proposition 2

Observe first that $\succsim$ is necessarily strictly convex and strictly monotonic (Proposition 1).
Clearly, (iv) ⇒ (i) obtains trivially while (iii) ⇒ (iv) follows immediately from the fact that $\partial l_i(x_{-i}|x) \to p(x) = \{p \in Y : x = x(p)\}$ is a bijective function (see Lemma B.14 in the Appendix).

[16] We thank Phil Reny for suggesting condition A1 to us.

(ii) $\Rightarrow$ (iii). We will establish first that $\succsim$ is differentiable at $x \in \text{int}(X)$ only if the collection of $p \in \mathbb{R}^n_{++}$ that support $\mathcal{U}_x$ at x is a singleton. To prove the contrapositive statement, observe first that the latter collection is non-empty: it coincides with the collection of $p \in \mathbb{R}^n \setminus 0$ that support $\mathcal{U}_x$ at x (Lemma 4.3), which is non-empty (recall the first part of the proof for Proposition 1). Suppose now that either of $p, \widetilde{p} \in \mathbb{R}^n_{++}$ with $p \neq \widetilde{p}$ supports $\mathcal{U}_x$ at x. Define $\overline{p} = \lambda p + (1-\lambda)\widetilde{p}$ for some $\lambda \in (0,1)$, and consider the hyperplane $H_{\overline{p},x}$. For any $z \in H_{\overline{p},x}$ we have $\overline{p}z = \overline{p}x$, which requires that at least one of $pz > px$ and $\widetilde{p}z > \widetilde{p}x$ fails to hold. As either of p and $\widetilde{p}$ supports $\mathcal{U}_x$ at x properly (Lemma B.7), it must be $x \succ z$ for any $z \in H_{\overline{p},x} \setminus \{x\}$. Take then $z \in H_{\overline{p},x} \setminus \{x\}$ with $p(z-x) > 0$. For $\mu \in (0,1)$ define also the point $z^{\mu} = x + \mu(z-x)$. As $z^{\mu} \in H_{\overline{p},x} \setminus \{x\}$, it cannot but be $x \succ z^{\mu}$. And this being the case for any $\mu \in (0,1)$, $z - x$ is a worsening direction at x. Yet, we do have $p(z-x) > 0$ and thus p cannot be a preference gradient at x. A trivially similar argument shows that $\widetilde{p}$ is not a preference gradient at x either. And as either price vector above was chosen arbitrarily, we have just established that no $p \in \mathbb{R}^n_{++}$ supporting $\mathcal{U}_x$ at x is a preference gradient at x. Recall though that the collection of $p \in \mathbb{R}^n_{++}$ that support $\mathcal{U}_x$ at x coincides with the collection of $p \in \mathbb{R}^n \setminus 0$ that support $\mathcal{U}_x$ at x. It follows therefore that no $p \in \mathbb{R}^n \setminus 0$ supporting $\mathcal{U}_x$ at x is a preference gradient at x. The claim now follows from the fact that the collection of preference gradients at x is a subset of the collection of $p \in \mathbb{R}^n \setminus 0$ that support $\mathcal{U}_x$ at x (Lemma B.3).
We have just shown that $\succsim$ being differentiable at x requires that the collection of $p \in \mathbb{R}^n_{++}$ that support $\mathcal{U}_x$ at x is singleton. As the latter collection coincides with the collection of $p \in \mathbb{R}^n_{++}$ that support $\mathcal{U}_x$ at x properly (recall again Lemma B.7), there is a unique $p \in \mathbb{R}^n_{++}$ such that $x = x(p)$ (Lemma B.8).
(i) $\Rightarrow$ (ii). To argue ad absurdum, let $l_i(\cdot|x)$ be partially differentiable at x_{-i} but suppose that $\succsim$ is not differentiable at x. Taking $q_{-i} \in \mathbb{R}^{n-1}_{--}$ to be the unique subgradient of $l_i(\cdot|x)$ at x_{-i} and $q_i = 1/(x_i - q_{-}x_{-})$, the vector $p = q_i(1, -q_{-i}) \in \mathbb{R}^n_{++}$ supports uniquely $\mathcal{U}_x$ at x. Our hypothesis being that p is not the preference gradient of $\succsim$ at x, there must exist $d \in \mathbb{R}^n \setminus \{0\}$ which is not an improvement direction for $\succsim$ at x even though $pd > 0$. Our argument will proceed hereafter in steps as follows.
Step 1. Take $\varepsilon > 0$ sufficiently small so that $\mathcal{B}_{\varepsilon}(x) \subset \text{int}(X)$ and let $r_d = \varepsilon/||d||$. As $x + rd \in \text{int}(X) \subseteq X$ for any $r \in (0, r_d)$, under the strict convexity of $\succsim$, the hypothesis that d is not an improvement direction for $\succsim$ at x necessitates that $x \succ x + rd$ for any $r \in (0, r_d)$. Moreover, $p \in \mathbb{R}^n_{++}$ and $pd > 0$ together imply that $\mathcal{N}^+_d \neq \varnothing$. And since $x \succ x + rd$, by the strict monotonicity of $\succsim$, there must be also $i \in \mathcal{N} \setminus \mathcal{N}^+_d$ with $d_i < 0$ (i.e., $i \in \mathcal{N}^-_d \neq \varnothing$). Let then $\widetilde{p} = (p_i/2, p_{-i}) \in \mathbb{R}^n_{++}$ which gives of course $(p - \widetilde{p})d < 0$. Define next the $[0,1] \to \mathbb{R}^n$ function $\widetilde{p}(\rho) = p - (1-\rho)\widetilde{p}$. Since $\widetilde{p}(0)d < 0 < \widetilde{p}(1)d$, there exists $\rho_0 \in (0,1)$ that gives $\widetilde{p}(\rho_0)d = 0$ where $\widetilde{p}(\rho_0) = ((1+\rho_0)p_i/2, \rho_0 p_{-i}) \in \mathbb{R}^n_{++}$. Define also the $(0, r_d) \times [0,1] \to \mathbb{R}^n_{++}$ function

$$x(r,\mu)_j = \begin{cases} x_j + rd_j & j \in \mathcal{N} \setminus \mathcal{N}^-_d \\ x_j + (1-\mu)rd_j & j \in \mathcal{N}^-_d \end{cases}$$

By the strict monotonicity of $\succsim$ and since $\mathcal{N} \setminus \mathcal{N}^-_d \supseteq \mathcal{N}^+_d \neq \varnothing$, this gives $x(r,1) \succ x \succ x + rd = x(r,0)$. Letting hence $u : X \to \mathbb{R}$ be a utility function for $\succsim$, we have $u(x(r,0)) < u(x) < u(x(r,1))$ and the intermediate-value theorem ensures the existence of $\mu_r \in (0,1)$ such that $u(x(r,\mu_r)) = u(x)$. By the strict monotonicity of $\succsim$ moreover the mapping $r \mapsto \mu(r) = \mu_r$ is a function. And as

$$\widetilde{p}(\rho_0)(x(r,\mu(r)) - x) = r\left(\widetilde{p}(\rho_0)d - \sum_{j \in \mathcal{N}^-_d} \widetilde{p}(\rho_0)_j \mu(r) d_j\right) = -r \sum_{j \in \mathcal{N}^-_d} \widetilde{p}(\rho_0)_j \mu(r) d_j > 0$$

we have just established the existence of a function $\mu : (0, r_d) \to (0,1)$ which gives $x(r, \mu(r)) \in \mathcal{I}_{\succsim}(x)$ with $\widetilde{p}(\rho_0)(x(r,\mu(r)) - x) > 0$ everywhere on its domain.

Step 2. The function $\mu(\cdot)$ is strictly increasing everywhere on its domain. To show this arguing ad absurdum, let $r < r'$ and suppose that $\mu(r) \geq \mu(r')$. Then $1-\mu(r) \leq 1-\mu(r')$ and, by the strict monotonicity of $\succsim$, it cannot but be $l_i\left(x(r,\mu(r'))_{-i}|x\right) \geq l_i\left(x(r,\mu(r))_{-i}|x\right)$ - the inequality being in fact strict unless $\mathcal{N}_d^- = \{i\}$. Moreover, the function $l_i(\cdot|x)$ being strictly convex, $r < r'$ implies also that[17]

$$\frac{l_i\left(x(r',\mu(r'))_{-i}|x\right) - l_i(x_{-i}|x)}{r'} > \frac{l_i\left(x(r,\mu(r'))_{-i}|x\right) - l_i(x_{-i}|x)}{r}$$

Putting these observations together, we must have

$$\begin{aligned}
(1-\mu(r'))\,d_i &= \frac{x(r',\mu(r'))_i - x_i}{r'} \\
&= \frac{l_i\left(x(r',\mu(r'))_{-i}|x\right) - l_i(x_{-i}|x)}{r'} \\
&> \frac{l_i\left(x(r,\mu(r'))_{-i}|x\right) - l_i(x_{-i}|x)}{r} \\
&\geq \frac{l_i\left(x(r,\mu(r))_{-i}|x\right) - l_i(x_{-i}|x)}{r} = (1-\mu(r))\,d_i
\end{aligned}$$

which is of course absurd given that $d_i < 0$.

Step 3. Observe next that, $l_i(\cdot|x)$ being partially differentiable at x_{-i} with gradient vector q_{-i}, for any direction $v_{-i} \in \mathbb{R}^{n-1}$ the quantity

$$[l_i(x_{-i} + \varepsilon v_{-i}|x) - x_i]/\varepsilon = [l_i(x_{-i} + \varepsilon v_{-i}|x) - l_i(x_{-i}|x)]/\varepsilon$$

approaches $q_{-i}v_{-i}$ from above as $\varepsilon \searrow 0$. Formally,

$$\forall \Delta > 0, \exists \epsilon_\Delta > 0 : \; l_i(x_{-i} + \varepsilon v_{-i}|x) - x_i < \varepsilon(\Delta + q_{-i}v_{-i}) \quad \forall \varepsilon \in (0, \varepsilon_\Delta)$$

Letting thus $v_{-i} = d_{-i}$ and $\Delta = -(1-\rho_0)(1-\mu(r_d/2))\,d_i/(2\rho_0)$, there exists $\varepsilon_0 \in (0, r_d/2)$ such that

$$\begin{aligned}
\rho_0 q_i\left(x(\varepsilon,\mu(\varepsilon))_i - x_i\right) &= \rho_0 q_i\left(l_i(x_{-i} + \varepsilon d_{-i}|x) - x_i\right) \\
&< \varepsilon\rho_0 q_i(\Delta + q_{-i}d_{-i}) \\
&= -\left(\frac{1-\rho_0}{2}\right) q_i\left(1-\mu\left(\frac{r_d}{2}\right)\right)\varepsilon d_i + \rho_0 q_i q_{-i}\varepsilon d_{-i} \\
&= -\left(\frac{1-\rho_0}{2}\right) q_i\left(1-\mu\left(\frac{r_d}{2}\right)\right)\varepsilon d_i - \widetilde{p}(\rho_0)_{-i}\,\varepsilon d_{-i} \\
&\leq -\left(\frac{1-\rho_0}{2}\right) q_i\left(1-\mu\left(\frac{r_d}{2}\right)\right)\varepsilon d_i - \widetilde{p}(\rho_0)_{-i}\,\varepsilon d_{-i} \\
&\quad - \sum_{j \in \mathcal{N}_d^- \setminus \{i\}} \widetilde{p}(\rho_0)_j (1-\mu(\varepsilon))\,d_j \\
&= -\left(\frac{1-\rho_0}{2}\right) q_i\left(1-\mu\left(\frac{r_d}{2}\right)\right)\varepsilon d_i + \widetilde{p}(\rho_0)_{-i}\left(x_{-i} - x(\varepsilon,\mu(\varepsilon))_{-i}\right) \qquad \forall \varepsilon \in (0,\varepsilon_0)
\end{aligned}$$

[17] Given $K \in \mathbb{N} \setminus \{0\}$ and a strictly convex function $f : S \to \mathbb{R}$ defined on an open and convex set $S \subseteq \mathbb{R}^K$, a vector $v \in \mathbb{R}^K$, and $\varepsilon \in \mathbb{R} \setminus \{0\}$, the ratio $[f(x+\varepsilon v) - f(x)]/\varepsilon$ is a strictly increasing function of ε (see Theorem 6.2.15 in de la Fuente (2000)). For the application of this result in the text, let $K = n-1$ and $v_j = d_j$ for $j \in \mathcal{N} \setminus \mathcal{N}_d^-$ while $v_j = (1-\mu(r'))\,d_j$ for $j \in \mathcal{N}_d^- \setminus \{i\}$.

where the weak inequality above is in fact strict unless $\mathcal{N}_d^- = \{i\}$. As though $\mu(\cdot)$ is strictly increasing on $(0, r_d/2)$ while $d_i < 0$, it cannot but be $-\left(1-\mu\left(\frac{r_d}{2}\right)\right)\varepsilon d_i < -\left(1-\mu(\varepsilon)\right)\varepsilon d_i = x_i - x(\varepsilon, \mu(\varepsilon))_i$. It follows therefore from above that

$$\rho_0 q_i \left(x(\varepsilon, \mu(\varepsilon))_i - x_i\right) < -\left(\frac{1-\rho_0}{2}\right) q_i \left(x(\varepsilon, \mu(\varepsilon))_i - x_i\right) + \widetilde{p}(\rho_0)_{-i}\left(x_{-i} - x(\varepsilon, \mu(\varepsilon))_{-i}\right)$$

Equivalently,

$$\begin{aligned} \widetilde{p}(\rho_0)_{-i}\left(x_{-i} - x(\varepsilon, \mu(\varepsilon))_{-i}\right) &> \left(\frac{1+\rho_0}{2}\right) q_i \left(x(\varepsilon, \mu(\varepsilon))_i - x_i\right) \\ &= \left(\frac{1+\rho_0}{2}\right) p_i \left(x(\varepsilon, \mu(\varepsilon))_i - x_i\right) \\ &= \widetilde{p}(\rho_0)_i \left(x(\varepsilon, \mu(\varepsilon))_i - x_i\right) \qquad \forall \varepsilon \in (0, \varepsilon_0) \end{aligned}$$

which contradicts though the very definition of the function $\mu(\cdot)$. □

Proof of Proposition 3

That (iii) ⇒ (i) holds of course trivially.
(i) ⇒ (ii). We will make use of the following result.

Lemma 4.7 *Let the onto demand function $x : \mathbb{R}^n_{++} \supseteq Y \to int(X)$ be generated by the continuous weak order $\succsim$ on X. For any $p \in Y$, $x(\cdot)$ is continuous at p if $\succsim$ is differentiable at $x(p)$.*

Proof. Observe first that $\succsim$ is necessarily strictly convex and strictly monotonic (Proposition 1). Take an arbitrary $p \in Y$ and let $x = x(p)$. Choose also $\epsilon > 0$ sufficiently small so that $\mathcal{B}_\epsilon(x) \subset \text{int}(X)$. Consider the problem $\max_{z \in \text{cl}(\mathcal{B}_{\epsilon/2}(x)): p'z \le 1} \succsim$ for $p' \in \mathbb{R}^n_{++}$. By Proposition 1, there exists a unique $\widetilde{Y} \subseteq \mathbb{R}^n_{++}$ and a unique onto solution function $\widetilde{x} : \widetilde{Y} \to \mathcal{B}_{\epsilon/2}(x)$. For any $p' \in \widetilde{Y}$, moreover, the constrained set is compact - being but the intersection of the compact sets $\text{cl}(\mathcal{B}_{\epsilon/2}(x))$ and $B_{\mathbb{R}^n_+}(p')$. Thus, by Berge's Maximum theorem (see for instance Theorems 7.2.1-7.2.2 in de la Fuente (2000)), $\widetilde{x}(\cdot)$ is continuous on $\widetilde{Y}$ - which is open since the function is onto an open set.
We will show next that $\widetilde{Y} \subseteq Y$ while $\widetilde{x}(\cdot)$ is the restriction of $x(\cdot)$ on $\widetilde{Y}$. To this end, observe that the arbitrary $p' \in \widetilde{Y}$ supports $\mathcal{U}_{\widetilde{x}(p')}$ at $\widetilde{x}(p')$. For if there exists $z \in \mathcal{U}_{\widetilde{x}(p')}$ with $p'z \le p'\widetilde{x}(p')$, then any $\lambda \in (0,1)$ would give $z^\lambda = \lambda z + (1-\lambda)\widetilde{x}(p') \succ \widetilde{x}(p')$ while $p'z^\lambda \le 1$. As though $\lim_{\lambda \to 0} z^\lambda = \widetilde{x}(p')$, for any $\varepsilon > 0$, we have that $z^\lambda \in \mathcal{B}_\varepsilon(\widetilde{x}(p'))$ for sufficiently small λ. Yet, a small enough ε ensures that $\mathcal{B}_\varepsilon(\widetilde{x}(p')) \subset \mathcal{B}_{\epsilon/2}(x)$, contradicting the optimality of $\widetilde{x}(p')$. Recall now Lemmas 4.2 -4.3. The fact that p' supports $\mathcal{U}_{\widetilde{x}(p')}$ at $\widetilde{x}(p')$ implies that $\widetilde{x}(p') \in \max_{x \in X: p'x \le 1} \succsim$. As the latter set though is the singleton $x(p')$ while $x(\cdot)$ is onto $\text{int}(X)$, it cannot but be $\widetilde{x}(p') = x(p')$ and $p' \in Y$.
Notice finally that, since $\widetilde{x}(\cdot)$ is onto $\mathcal{B}_{\epsilon/2}(x)$ and coincides on its domain with $x(\cdot)$, there exists $p'' \in \widetilde{Y}$ such that $\widetilde{x}(p'') = x = x(p'')$. However, $\succsim$ being differentiable at x, we cannot have $x(p'') = x = x(p)$ unless $p'' = p$ (recall Proposition 2). This establishes that $\widetilde{Y} = Y$ while $\widetilde{x}(\cdot)$ coincides with $x(\cdot)$. The claim now follows. ■

By the preceding lemma, for any $x \in \text{int}(X)$, if $\succsim$ is differentiable at x then $x(\cdot)$ is continuous at the unique (recall Proposition 2) $p \in Y$: $x = x(p)$. Given this and the Invariance of Domain Theorem, the claim follows.[18]

<u>(ii) $\Rightarrow$ (iii)</u>. We will make use of the following result.

Lemma 4.8 *Let $\succsim$ be a strictly convex, strictly monotonic, continuous weak order on X. Then $\succsim$ is weakly C^1 at $x \in int(X)$ if and only if, for any $i \in \mathcal{N}$, the level-set function $l_i(\cdot|x)$ is C^1 at x.*

Proof. For the "if" direction, take an arbitrary $i \in \mathcal{N}$ and let $l_i(\cdot|x)$ be C^1 at $x \in \text{int}(X)$. Observe that the graph of a C^r ($r \in \mathbb{N} \cup \{\infty\}$) function $f : \mathbb{R}^n \supseteq X \to \mathbb{R}^m$ (X open) is a C^r (n-dimensional) manifold - see Section 1.H.1 in Mas-Colell (1985). The graph of $l_i(\cdot|x)$ being the indifference set $\mathcal{I}_x$, it follows that the latter is a C^1 manifold at x.

For the "only if," letting $\mathcal{I}_x$ be a C^1 manifold at x, there exist $\epsilon > 0$ and a C^1 regular function $\xi : \mathcal{B}_\epsilon(x) \times \mathcal{B}_\epsilon(x) \to \mathbb{R}$ such that $\mathcal{B}_\epsilon(x) \cap \mathcal{I}_x = \xi^{-1}(0)$ (see for instance Section 1.H.2 in Mas-Colell (1985)). It follows then by the implicit-function theorem that, for any $i \in \mathcal{N}$, the mapping $l_i(\cdot|x)$ is a well-defined C^1 regular function on $\mathcal{B}_{\epsilon_0}(x) \cap \mathcal{I}_x$ for some $\epsilon_0 \in (0, \epsilon)$. ■

Observe now that, $x(\cdot)$ being injective, $p(\cdot)$ is a function. Hence, for any $i \in \mathcal{N}$ and any $x \in \text{int}(X)$, $l_i(\cdot|x)$ is partially differentiable at x_{-i} (Proposition 2). The unique subgradient (the gradient) of $l_i(\cdot|x)$ at x_{-i} is defined from the singleton $p(x)$ by the system of equations (1)-(3): $\nabla l_i(\cdot|x) = -p_{-i}(x) / p_i(x)$. And $x(\cdot)$ being in fact an homeomorphism, $p(\cdot)$ is continuous at x. Clearly, $l_i(\cdot|x)$ is C^1 at x_{-i}. The claim now follows by the preceding lemma. □

Proof of Claim 2

That ""(iii) $\Rightarrow$ (iv)" is trivial while we have already established that "(iv) $\Rightarrow$ (i)" (see Lemma B.15 and Proposition 2) as well as that "(i) $\Rightarrow$ (ii)" (see Proposition 3). Hence, only (ii) $\Rightarrow$ (iii) needs to be shown.

Homothetic preferences. As is well-known, a strictly convex, strictly monotonic, continuous, and homothetic weak order $\succsim$ on $X \subseteq \mathbb{R}^n_+$ admits a continuous, strictly monotonic, strictly quasiconcave, and homogenous of degree one utility representation $u : X \to \mathbb{R}_+$ with $u(x) > 0$ for any $x \in \text{int}(X)$.[19] Given these properties, $u(\cdot)$ is in fact concave on $\text{int}(X)$.[20] The claim now follows from the Lemma 4.9.

Lemma 4.9 *Let the strictly convex and strictly monotonic continuous weak order $\succsim$ on X be represented by the concave utility function $u : X \to \mathbb{R}$. For any $x \in int(X)$, $u(\cdot)$ is C^1 at x if and only if $\succsim$ is differentiable at x.*

Proof. The "only if" direction follows from Lemma B.15: if $u(\cdot)$ is C^1 at x so is the typical level function $l_i(\cdot|x)$. For the "if," observe first that $u(\cdot)$ being concave, $-u(\cdot)$ is convex. The subdifferential $\partial(-u(x))$ therefore will be non-empty at the arbitrary $x \in \text{int}(X)$. Yet any $p \in \partial(-u(x))$ gives $0 \geq u(x) - u(z) \geq p(x - z)$ for any $z \in \mathcal{U}_x$. That is, any $p \in \partial(-u(x))$ supports $\mathcal{U}_x$ at x. Which implies in turn that $\partial(-u(x)) \subset Y$ (see Lemmata 4.3 and B.8). By Proposition 2 then $\succsim$ is differentiable at x only if $\partial(-u(x))$

[18]Letting $X, Y \subseteq \mathbb{R}^n$ with X open, the Invariance of Domain Theorem - see Brouwer (1912) - states that a function $f : X \to Y$ being injective and continuous suffices for it to be an homeomorphism and for $f(X)$ to be open.

[19]A strictly monotonic and homogenous of degree one function $u : \mathbb{R}^n_+ \supseteq X \to \mathbb{R}$ must give $u(x) < u(\lambda x) = \lambda u(x)$ for any $\lambda > 1$ and any $x \in X \setminus \{0\}$. Clearly, it cannot but be $u(x) > 0$ for any $x \in \text{int}(X)$.

[20]See Theorem 1 in Prada (2011): if a function $f : X \subseteq \mathbb{R}^n_+ \to \mathbb{R}$ is quasiconcave, increasing and homogenous of degree γ with $0 < \gamma \leq 1$, then it is concave.

is a singleton. By Proposition 3, moreover, $\succsim$ is actually weakly C^1 at x so that the gradient $\partial\left(-u\left(x\right)\right)$ is also continuous. ■

Quasi-linear preferences. A quasilinear, strictly convex and strictly monotonic continuous weak order on X is represented by a utility function $u : X \to \mathbb{R}$ given by

$$u\left(x\right) = x_i + v\left(x_{-i}\right), \qquad i \in \mathcal{N} \tag{5}$$

for some continuous, strictly increasing and strictly quasiconcave function $v : X_{-i} \to \mathbb{R}$ (where X_{-i} denotes the projection of X on $\mathbb{R}^{n-1}$). For any $x \in X$, $l_i\left(\cdot|x\right)$ is given by $l_i\left(z_{-i}|x\right) = x_i + v\left(x_{-i}\right) - v\left(z_{-i}\right)$. The claim now follows immediately. □

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Appendices

A Subgradients

Recall that a set $S \subseteq \mathbb{R}^n$ is said to be locally convex if for every $x \in S$ there is $\epsilon > 0$ such that $\mathcal{B}_\epsilon(x)$ is convex. Given a locally convex $S \subseteq \mathbb{R}^n$, a function $f : S \to \mathbb{R}$ is said to be locally convex if for every $x \in S$ there exists $\epsilon > 0$ such that f is convex on $\mathcal{B}_\epsilon(x)$.

Consider now a non-empty, open, locally convex $S \subseteq \mathbb{R}^n$, and a locally convex function $f : S \to \mathbb{R}$. Taking $x_0 \in \operatorname{int}(S)$, the directional derivative of f at x_0 in any direction $h \in S$ is well-defined (see Theorem 3.3.4 in Jahn (2007)) and given by

$$f'(x_0)(h) := \lim_{\lambda \to 0_+} \frac{f(x_0 + \lambda h) - f(x_0)}{\lambda}$$

Letting moreover $\{e_1, \ldots, e_n\}$ denote the orthonormal basis of $\mathbb{R}^n$, the ith partial derivative of f at x_0 is given by

$$\partial f(x_0) / \partial x_i := \lim_{\lambda \to 0} \frac{f(x_0 + \lambda e_i) - f(x_0)}{\lambda}$$

if this limit exists. And f is said to be partially differentiable at x_0 if $\partial f(x_0)/\partial x_i$ exists for all $i \in \{1, \ldots, n\}$. Observe furthermore that

$$\begin{aligned}
\lim_{\lambda \to 0^+} \frac{f(x_0 + \lambda e_i) - f(x_0)}{\lambda} &= f'(x_0)(e_i) \\
\lim_{\lambda \to 0^-} \frac{f(x_0 + \lambda e_i) - f(x_0)}{\lambda} &= -\lim_{\lambda \to 0^+} \frac{f(x_0 - \lambda e_i) - f(x_0)}{\lambda} \\
&= -f'(x_0)(-e_i)
\end{aligned}$$

It follows thus that the ith partial derivative of f at x_0 exists if and only if either of the directional derivatives, $f'(x_0)(e_i)$ and $f'(x_0)(-e_i)$, exists and $f'(x_0)(e_i) = -f'(x_0)(-e_i)$.

We will let also $\partial f(x_0)$ denote the *subdifferential* of f at x_0. This is the set of vectors $q \in \mathbb{R}^n$ such that

$$f(x) \geq f((x_0) + q(x - x_0) \quad \forall x \in S$$

Each $q \in \partial f(x_0)$ is called a *subgradient* of f at x_0. And f being locally convex, it is also continuous at x_0 (see Theorem 6.2.14 in de la Fuente (2000)). Which ensures in turn that $\partial f(x_0) \neq \varnothing$ (see Theorem 3.26 in Jahn (2007)), while

$$f'(x_0)(h) = \max\{qh : q \in \partial f(x_0)\}$$

for any direction $h \in S$ (see Theorem 3.28 in Jahn (2007)). In particular, we have

$$f'(x_0)(e_i) = \max\{qe_i : q \in \partial f(x_0)\}$$

and, thus, also

$$f'(x_0)(-e_i) = -\min\{qe_i : q \in \partial f(x_0)\}$$

It is trivial then to check then f is partially differentiable at $x_0 \in \text{int}(S)$ if and only if $\partial f(x_0)$ is a singleton.[21]

Observe finally that, fixing any direction $h \in S$, we can write $x^\lambda = x_0 + \lambda h$ for $\lambda \in (0, 1)$. And as f is locally convex at x_0, there exists $\lambda_0 \in (0, 1)$ such that

$$\begin{aligned}
f\left(x_0 + \widetilde{\lambda} h\right) &= f\left(\widetilde{\lambda}(x_0 + h) + \left(1 - \widetilde{\lambda}\right) x_0\right) \\
&\leq \widetilde{\lambda} f(x_0 + h) + \left(1 - \widetilde{\lambda}\right) f(x_0) \qquad \forall \widetilde{\lambda} \in (0, \lambda_0)
\end{aligned}$$

or equivalently

$$f(x_0 + h) - f(x_0) \geq \frac{f\left(x_0 + \widetilde{\lambda} h\right) - f(x_0)}{\widetilde{\lambda}} \qquad \forall \widetilde{\lambda} \in (0, \lambda_0) \tag{6}$$

Letting now $\lambda \to 0$, this gives

$$f(x_0 + h) - f(x_0) \geq f'(x_0)(h) \tag{7}$$

[21] It should be pointed out that the four theorems cited above are stated for convex domains and functions in the respective sources. Yet, the derivative being itself a local concept, they obviously remain valid also for locally convex domains and functions.

More importantly for the purposes of the subsequent analysis, if f is (globally) strictly convex then (6) holds with strict inequality and for all $\lambda \in (0,1)$. It follows then that

$$\begin{aligned} f(x_0+h)-f(x_0) &> \frac{f(x_0+\lambda h)-f(x_0)}{\lambda} \\ &= \frac{f(x^\lambda)-f(x_0)}{\lambda} \\ &\geq \frac{f'(x_0)(h)(x^\lambda-x_0)}{\lambda} = f'(x_0)(h)\,h \end{aligned} \tag{8}$$

the weak inequality due to (7).

B Supporting results

Lemma B.1 *Let $\succsim$ be a weak order on X and $x \in int(X)$. The collection of $p \in \mathbb{R}^n \setminus \{0\}$ that support $\mathcal{U}_x$ at x is a subset of the collection of decreasing ordients at x.*

Proof. Since $x \in \operatorname{int}(X)$, for any $z \in X$, we have $x+\lambda(z-x) \in \operatorname{int}(X) \subseteq X$ for sufficiently small $\lambda > 0$. Let now $p \in \mathbb{R}^n \setminus \{0\}$ support $\mathcal{U}_x$ at x and take $z \in H^-_{p,x}$. As $p(z-x) < 0$ is equivalent to $p(x+\lambda(z-x)) < px$, it must be $x \succ x+\lambda(z-x)$ for any $\lambda > 0$. That is, $z-x$ must be a worsening direction at x and p a decreasing ordient at x. ■

Lemma B.2 *Let $\succsim$ be a convex weak order on X and $x \in int(X)$. The collection of $p \in \mathbb{R}^n \setminus \{0\}$ that support $\mathcal{U}_x$ at x coincides with the collection of decreasing ordients at x.*

Proof. Since $x \in \operatorname{int}(X)$, for any $z \in X$, we have $x+\lambda(z-x) \in \operatorname{int}(X) \subseteq X$ for sufficiently small $\lambda > 0$. By Lemma B.1, moreover, it suffices to establish the collection of decreasing ordients at x as subset of the collection of $p \in \mathbb{R}^n \setminus \{0\}$ that support $\mathcal{U}_x$ at x. To this end observe that, $\succsim$ being convex, any $\lambda \in [0,1]$ gives $x+\lambda(z-x) = \lambda z+(1-\lambda)x \in \mathcal{U}_x$ if $z \in \mathcal{U}_x$. Letting then $p \in \mathbb{R}^n \setminus \{0\}$ be a decreasing ordient at x, as $z-x$ is not a worsening direction at x, it cannot be $p(z-x) < 0$. That is, p cannot but support $\mathcal{U}_x$ at x. ■

Lemma B.3 *Let $\succsim$ be a strictly convex weak order on X and $x \in int(X)$. The collection of preference gradients at x is a subset of the collection of $p \in \mathbb{R}^n \setminus \{0\}$ that support $\mathcal{U}_x$ at x properly. As a result, the collection of preference gradients at x is a subset of the collection of ordients at x.*

Proof. Since $x \in \operatorname{int}(X)$, for any $z \in X$, we have $x+\lambda(z-x) \in \operatorname{int}(X) \subseteq X$ for sufficiently small $\lambda > 0$. For the first part of the claim, observe that, $\succsim$ being strictly convex, for any $\lambda \in (0,1)$ we get that $x+\lambda(z-x) = \lambda z+(1-\lambda)x \in \mathcal{U}_x \setminus \mathcal{I}_x$ if $z \in \mathcal{U}_x$. Given, therefore, $z \in \mathcal{U}_x$, $z-x$ is an improvement direction at x and thus $p(z-x) > 0$ if p is a preference gradient at x.
Recall now the definitions for preference gradients and ordients. The second part of the claim follows from the first part and Lemma B.2. ■

Lemma B.4 *Let $\succsim$ be a continuous weak order on $X \subseteq \mathbb{R}^n_+$ and suppose that $p \in \mathbb{R}^n \setminus \{0\}$ supports $\mathcal{U}_x$ at $x \in int(X)$. Then $v \in \mathbb{R}^n \setminus \{0\}$ is an improvement direction at x only if $pv > 0$.*

Proof. Obviously, since $x \in \operatorname{int}(X)$ we have $\mathcal{B}_x(\epsilon_0) \subset \operatorname{int}(X)$ for sufficiently small $\epsilon_0 > 0$. Suppose now that $p \in \mathbb{R}^n \setminus \{0\}$ supports $\mathcal{U}_x$ at x, and let $v \in \mathbb{R}^n \setminus \{0\}$ be an improvement direction at x. There exists then $\lambda^* > 0$ such that $x + \lambda v \succ x$ for all $\lambda \in (0, \lambda^*)$. Taking thus $\lambda_0 \in (0, \min\{\lambda^*, \epsilon_0/||v||\})$, we have $x + \lambda_0 v \succ x$ while $x + \lambda_0 v \in \operatorname{int}(X)$. Yet $x + \lambda_0 v \succ x$ can be only if $pv = (1/\lambda_0)\lambda_0 pv = (1/\lambda_0)p(\lambda_0 v) = (1/\lambda_0)p(x + \lambda_0 v - x) \geq 0$.
It suffices therefore to rule out the case $pv = 0$. To establish this ad absurdum recall first that, $\succsim$ being complete and continuous, $\mathcal{U}_x \setminus \mathcal{I}_x$ is open for any $x \in \operatorname{int}(X)$. As a result, $x + \lambda_0 v \succ x$ necessitates that $z \succ x$ for all $z \in \mathcal{B}_{x+\lambda_0 v}(\epsilon_1)$ for sufficiently small $\epsilon_1 > 0$. More specifically, letting $\epsilon_2 \in (0, \min\{\epsilon_0, \epsilon_1\})$ we have $u(z) > u(x)$ for all $z \in \mathcal{B}_{x+\lambda_0 v}(\epsilon_2) \subset \operatorname{int}(X)$. Taking then $z = x + \lambda v - \varepsilon p$ for some $\varepsilon \in (0, \epsilon_2/||p||)$ ensures that $x + \lambda v - \varepsilon p \in \operatorname{int}(X)$ while $x + \lambda v - \varepsilon p \succ x$. And the latter relation implies in turn that $0 \leq p(x + \lambda v - \varepsilon p - x) = p(\lambda v - \varepsilon p) = \lambda pv - \varepsilon pp = -\varepsilon p^\intercal p$, a contradiction. ■

Lemma B.5 *Let $\succsim$ be a continuous weak order $\succsim$ on X. Take also $x^0, x^1, x^2 \in X$ with $x^2 \succ x^1 \succ x^0$. The following hold*
(i). There exists $\lambda_0 \in (0,1)$ such that $\lambda_0 x^0 + (1-\lambda_0)x^2 \sim x^1$.
(ii). If $x^0 + z \in X$, there exists $\lambda_1 \in (0,1)$ such that $x^1 \succ x^0 + \lambda z$ for any $\lambda\,(0, \lambda_1)$.
(iii) There exists $\lambda_2 \in (0,1)$ such that $x^1 \succ \lambda x^1 + (1-\lambda)x^0$ for any $\lambda\,(0, \lambda_2)$.

Proof. Being a continuous weak order on $X \subseteq \mathbb{R}^n_+$, $\succsim$ can be represented by a continuous utility function $u : X \to \mathbb{R}$. Moreover, X being convex, $\lambda x + (1-\lambda)z \in X$ for any $\lambda \in (0,1)$ and any $x, z \in X$.
(i). Define the function $z : [0,1] \to X$ by $z(\lambda) = \lambda x^0 + (1-\lambda)x^2$. Given that $u(z(0)) < u(x^1) < u(z(1))$, and the continuity of $u(\cdot)$, the claim follows from the intermediate-value theorem.
(ii). Observe that $x^0 + \lambda z = \lambda(x^0 + z) + (1-\lambda)x^0 \in X$ for any $\lambda\,(0,1)$. And as $\lim_{\lambda\to 0} u(x^0 + \lambda z) = u(x^0) < u(x^1)$, the claim follows from the continuity of $u(\cdot)$.
(iii). As $\lim_{\lambda\to 0} u(\lambda x^1 + (1-\lambda)x^0) = u(x^0) < u(x^1)$, the claim follows again from the continuity of $u(\cdot)$. ■

Lemma B.6 *Let $\succsim$ be a strictly monotonic, continuous weak order on $X \subseteq \mathbb{R}^n_+$. For any $x \in X$, $p \in \mathbb{R}^n_{++}$ supports $\mathcal{I}_x$ at x only if it supports $\mathcal{U}_x \setminus \mathcal{I}_x$ at x properly.*

Proof. Let $p \in \mathbb{R}^n_{++}$ be such that $px \leq pz$ for any $z \in \mathcal{I}_x$. We need to show that $px < pz'$ for any $z' \in X$ such that $z' \succ x$. To this end, take any such z' and let $\epsilon = z' - x$. By strict monotonicity, $z' \succ x$ necessitates that $\mathcal{N}^+_{z'-x} \neq \varnothing$. Notice also that, the claim being obvious if $\mathcal{N}^+_{z'-x} = \mathcal{N}$, we may take $\mathcal{N} \setminus \mathcal{N}^+_{z'-x} = \{i \in \mathcal{N} : \epsilon_i \leq 0\}$ to be non-empty. And as the claim is again obvious if $\epsilon_i = 0$ for all $i \in \mathcal{N} \setminus \mathcal{N}^+_{z'-x}$, we may take in fact that $\mathcal{N}^-_{z'-x} \neq \varnothing$.
Let now $\mu \in [0,1]$ and define $x^\mu \in X$ by $x^\mu_i = x_i + \mu\epsilon_i$ for $i \in \mathcal{N} \setminus \mathcal{N}^-_{z'-x}$ and $x^\mu_i = x_i + \epsilon_i$ for $i \in \mathcal{N}^-_{z'-x}$. As $\mathcal{N}^-_{z'-x} \neq \varnothing$ while $x^1 = z'$, we have $x^1 \succ x \succ x^0$ due to strict monotonicity. There exists thus $\mu_0 \in (0,1)$ such that $\mu_0 x^1 + (1-\mu^0)x^0 \in \mathcal{I}_x$ (Lemma B.5(i)). The latter point is given though by $x_i + \mu^0\epsilon_i$ for $i \in \mathcal{N} \setminus \mathcal{N}^-_{z'-x}$ and $x^\mu_i = x_i + \epsilon_i$ for $i \in \mathcal{N}^-_{z'-x}$. That is, $\mu_0 x^1 + (1-\mu^0)x^0 = x^{\mu_0}$ and the claim now follows

since

$$\begin{aligned} px^{\mu_0} &= \sum_{i \in \mathcal{N} \setminus \mathcal{N}^-_{z'-x}} p_i \left(x_i + \mu^0 \epsilon_i\right) + \sum_{i \in \mathcal{N}^-_{z'-x}} p_i \left(x_i + \epsilon_i\right) \\ &< \sum_{i \in \mathcal{N} \setminus \mathcal{N}^-_{z'-x}} p_i \left(x_i + \epsilon_i\right) + \sum_{i \in \mathcal{N} \setminus \mathcal{N}^-_{z'-x}} p_i \left(x_i + \epsilon_i\right) \\ &= \sum_{i \in \mathcal{N}} p_i \left(x_i + \epsilon_i\right) = pz' \end{aligned}$$

where the inequality is because $\varnothing \neq \mathcal{N}^+_{z'-x} \subseteq \mathcal{N} \setminus \mathcal{N}^-_{z'-x}$. ■

Lemma B.7 *Let $\succsim$ be a strictly convex and strictly monotonic, continuous weak order on $X \subseteq \mathbb{R}^n_+$. For any $x \in X$, $p \in \mathbb{R}^n_{++}$ supports $\mathcal{I}_x$ at x only if it supports $\mathcal{U}_x$ at x properly.*

Proof. From Lemma B.6, it follows immediately that there cannot be $z \in \mathcal{I}_x \setminus \{x\}$ with $px = pz$ if $\succsim$ is strictly convex. For otherwise, taking $\lambda \in (0,1)$ and letting $x^\lambda = \lambda x + (1-\lambda) z$, we would have $px^\lambda = px$ while $x^\lambda \in \mathcal{U}_x \setminus \mathcal{I}_x$. ■

Lemma B.8 *Let the (onto) demand function $x : \mathbb{R}^n_{++} \supseteq Y \to int\,(X)$ be generated by the strictly convex and strictly monotonic continuous weak order $\succsim$ on $X \subseteq \mathbb{R}^n_+$. For any $(p, x) \in \mathbb{R}^n_{++} \times int\,(X)$, the following are equivalent.*
(i). p supports $\mathcal{U}_x$ at x properly.
(ii). p supports $\mathcal{U}_x$ at x.
(iii). $x = x\,(p)$.

Proof. That "(i) ⇒ (ii)" is of course trivially true while we have established already that "(ii) ⇒ (i)" (see Lemma B.7). That "(ii) ⇒ (iii)" follows from Lemmata 4.1-4.2 and 4.3. To establish the contrapositive of "(iii) ⇒ (ii)," suppose that p does not support $\mathcal{U}_x$ at x. Obviously, it cannot support $\mathcal{I}_x$ at x either. There exist that is $z \in \mathcal{I}_x \setminus \{x\}$ such that $pz < px$; clearly, it cannot be $x = \max_{\succsim} \{z \in X : pz \leq px\} = x\,(p)$. ■

Lemma B.9 *Let $\succsim$ be a strictly convex and strictly monotonic, continuous weak order on $X \subseteq \mathbb{R}^n_+$ and $x \in int\,(X)$. The collection of preference gradients at x coincides with the collection of increasing ordients at x that support $\mathcal{U}_x$ at x properly. And either collection coincides also with the collection of ordients at x.*

Proof. By definition $p \in \mathbb{R}^n \setminus \{0\}$ is a preference gradient at x if it is an increasing ordient at x that satisfies "$v \in \mathbb{R}^n \setminus \{0\}$ is an improvement direction at x only if $pv > 0$." For the first part of the claim, that the collection of preference gradients at x is a superset of the collection of increasing ordients at x that support $\mathcal{U}_x$ at x follows from Lemma B.4. The opposite set inclusion follows from Lemma B.3.
For the second part of the claim, we only need to show that the collection of ordients at x is a subset of the collection of preference gradients at x (see again Lemma B.3). And by the first part of the claim, we only need to show actually that the collection of decreasing ordients at x is a subset of the collection of $p \in \mathbb{R}^n \setminus \{0\}$ that support $\mathcal{U}_x$ at x properly. The former collection coincides with the collection of $p \in \mathbb{R}^n_{++}$ that support $\mathcal{U}_x$ at x (recall Lemmata B.2 and 4.3), which coincides in turn with the collection of $p \in \mathbb{R}^n_{++}$ that support $\mathcal{U}_x$ at x properly (Lemma B.7). ■

Lemma B.10 *Let $\succsim$ be a strictly monotonic and convex [resp. strictly convex] continuous weak order on $X = \mathbb{R}^n_+$. Let also $x \in bd\left(\mathbb{R}^n_+\right) \setminus \{0\}$. There exists $z \in \mathbb{R}^n_{++}$ such that $z - x$ is not an improvement direction at x only if there exists $p \in \mathbb{R}^n_{++}$ such that $x \in \max_{\succsim} \{z \in X : pz \leq px\}$ [resp. $x = \max_{\succsim} \{z \in X : pz \leq px\}$].*

Proof. Observe first that, $\succsim$ being convex and continuous, $\mathcal{U}_x$ is convex and closed. Suppose now that, for some $z \in \mathbb{R}^n_{++}$, $z - x$ is not an improvement direction at x. We must have then $x \succsim (1-\lambda)\, x + \lambda z \in X$ for arbitrarily small $\lambda > 0$. Given this and the strict monotonicity of $\succsim$, it is trivial to check that the convex and compact set

$$\mathcal{L}_{x,z} := \left\{ z' \in \mathbb{R}^n_{++} : z'_i \in \left\{ \begin{array}{ll} [x_i/2, x_i] & i \in \mathcal{N}^+_x \\ [z_i/2, z_i] & i \in \mathcal{N} \setminus \mathcal{N}^+_x \end{array} \right. \right\}$$

gives $x \succ (1-\lambda)\, x + \lambda z'$ for any $z' \in \mathcal{L}_{x,z}$. That is, $(1-\lambda)\, x + \lambda \mathcal{L}_{x,z} \cap \mathcal{U}_x = \varnothing$ and by the separating hyperplane theorem (see for instance Theorem 1.F.2.2 in Mas-Colell (1985)) there exists $\widetilde{p} \in \mathbb{R}^n \setminus \{0\}$ such that $\widetilde{p}z' < \widetilde{p}x'$ for any $(z', x') \in (1-\lambda)\, x + \lambda \mathcal{L}_{x,z} \times \mathcal{U}_x$.
It must be in fact $\widetilde{p} \in \mathbb{R}^n_+ \setminus \{0\}$. To see this arguing ad absurdum, let $\widetilde{p}_i < 0$ for some $i \in \mathcal{N}$. Let also $z'' \in \mathcal{L}_{x,z}$ be given by $z''_i = x_i/2$, $z''_j = x_j$ for $j \in \mathcal{N}^+_x \setminus \{i\}$, and $z''_j = z_j/2$ for $j \in \mathcal{N} \setminus \mathcal{N}^+_x$ while $x'' \in \mathcal{U}_x$ is given by $x''_i = x_i$ for $i \in \mathcal{N}^+_x$ and $x''_j = z_j/2$ for $j \in \mathcal{N} \setminus \mathcal{N}^+_x$. It is trivial to verify that this leads to the absurdity $\widetilde{p}\left((1-\lambda)\, x + \lambda z'\right) > \widetilde{p}x'$.
Take now any $i \in \mathcal{N}^+_x$ and z'' as in the preceding paragraph. Since $\widetilde{p}\left((1-\lambda)\, x + \lambda z'\right) < \widetilde{p}x$ reads $\widetilde{p}_i x_i > \sum_{j \in \mathcal{N} \setminus \mathcal{N}^+_x} \widetilde{p}_j z_j \geq 0$, it cannot but be $\widetilde{p}_i > 0$. That is, we have established the existence of $\widetilde{p} \in \mathbb{R}^n_+ \setminus \{0\}$ which supports $\mathcal{U}_x$ at x and has $\widetilde{p}_i > 0$ for any $i \in \mathcal{N}^+_x$.
Let now $p \in \mathbb{R}^n_{++}$ be given by $p_i = \widetilde{p}_i$ for $i \in \mathcal{N}^+_x$ and $p_i = \widetilde{p}_i + \varepsilon$ for $i \in \mathcal{N} \setminus \mathcal{N}^+_x$ and some $\varepsilon > 0$. As $p \gg \widetilde{p}$, we have $(p - \widetilde{p})\, z \geq 0$ for any $z \in X$. Moreover, $x_i = 0$ for any $i \in \mathcal{N} \setminus \mathcal{N}^+_x$ means that $px = \widetilde{p}x$. Since therefore $\widetilde{p}$ supports $\mathcal{U}_x$ at x, it cannot but be $pz \geq \widetilde{p}z \geq \widetilde{p}x = px$ for any $z \in \mathcal{U}_x$. Clearly, p also supports $\mathcal{U}_x$ at x. That $x \in \max_{\succsim} \{z \in X : pz \leq px\}$ is now due to Lemma B.6 [resp. Lemma B.7]. ■

Lemma B.11 *Let the demand correspondence $x : \mathbb{R}^n_{++} \supseteq Y \twoheadrightarrow X$ be generated by the strictly monotonic, continuous weak order $\succsim$ on $X \subseteq \mathbb{R}^n_+$. Then $Y = \mathbb{R}^n_{++}$ only if $\mathbb{R}^n_{++} \subseteq X$.*

Proof. We will show first that X_i is unbounded above. To establish this arguing ad absurdum, suppose that $x_i \leq b$ for all $x \in X$ and some $b > 0$. Take an arbitrary $p \in Y$ and let $x \in x(p)$. As the strict monotonicity of $\succsim$ guarantees Walras' law, we have $1 = px \leq bp_i + p_{-i}x_{-i}$; equivalently, $p_i \geq (1 - p_{-i}x_{-i})/b$. Recall though that, by hypothesis, $Y = \mathbb{R}^n_{++}$. That is, $Y_{-i} = \mathbb{R}^{n-1}_{++}$ and we can take $|p_{-i}|$ to be arbitrarily small. For the last inequality above, therefore, to hold for all $p_{-i} \in Y_{-i}$, it cannot but be $p_i \geq 1/b$ for all $p \in Y$. But this is absurd given that $Y_i = \mathbb{R}_{++}$.
We will show next that $\inf_{x \in X} x_i = 0$. To establish this arguing again ad absurdum, suppose that $x_i \geq a$ for all $x \in X$ and some $a > 0$. Taking again an arbitrary $p \in Y$ and letting $x = x(p)$, we now have $1 = px \geq ap_i + p_{-i}x_{-i}$; equivalently, $p_i \leq (1 - p_{-i}x_{-i})/a < 1/a$. Which is again absurd given that $Y_i = \mathbb{R}_{++}$.
Observe finally that, X being convex, so is X_i. It follows thus that X_i is path-connected; hence, connected. Clearly, X_i is an interval on $\mathbb{R}_{++}$ - see for instance Theorem 6.76 in Bowder (1996). And given the observations in the preceding two paragraphs, the claim follows. ■

Lemma B.12 *Let $\succsim$ be a strictly convex and strictly monotonic continuous weak order on $X \subseteq \mathbb{R}^n_+$. Take also any $(i,x) \in \mathcal{N} \times int\,(X)$. The mapping $l_i\,(\cdot|x) : int\,(X) \cap \mathcal{I}_x^{-i} \to \mathcal{I}_x^i$ is a function that is locally convex at x_{-i} and such that $\partial l_i\,(x_{-i}|x) \subseteq \mathbb{R}^{n-1}_{--}$.*

Proof. $\succsim$ being a strictly convex and strictly monotonic continuous weak order, it can be represented by a continuous, strictly monotonic, and strictly quasi-concave utility function $u : X \to \mathbb{R}$. Consider then an arbitrary $(i,x) \in \mathcal{N} \times \text{int}\,(X)$. To see that $l_i\,(\cdot|x)$ is a function, take $z \in \mathcal{I}_x \cap \text{int}\,(X)$ and observe that it cannot be $z_i, z_i' \in l_i\,(z_{-i}|x)$ with $z_i > z_i'$. For this would imply that $z \sim x \sim (z_i', z_{-i})$ which is absurd under strict monotonicity.
We will establish next that $l_i\,(\cdot|x)$ is locally strictly convex at x_{-i}. To this end, take any $\lambda \in (0,1)$ and, for $\varepsilon > 0$ sufficiently small, let $z, y \in \mathcal{I}_x \cap \mathcal{B}_\varepsilon\,(x) \subset \text{int}\,(X)$. Since $u\,(z) = u\,(x) = u\,(y)$, by the strict quasi-concavity of $u\,(\cdot)$, it cannot but be

$$\begin{aligned} u\,(x) &< u\,(\lambda z + (1-\lambda)\,y) \\ &= u\,(\lambda z_i + (1-\lambda)\,y_i, \lambda z_{-i} + (1-\lambda)\,y_{-i}) \end{aligned} \tag{9}$$

Choose now $k \in (1, +\infty)$ such that $(x_i/k, x_{-i}) \in \mathcal{B}_\varepsilon\,(x)$. Letting $\Delta = u\,(x) - u\,(x_i/k, x_{-i})$, that $\Delta > 0$ follows from $u\,(\cdot)$ being strictly monotonic. Moreover,

$$\begin{aligned} ||\lambda z_{-i} + (1-\lambda)\,y_{-i} - x_{-i}|| &= ||\lambda\,(z_{-i} - x_{-i}) + (1-\lambda)\,(y_{-i} - x_{-i})\,|| \\ &\leq \lambda||z_{-i} - x_{-i}|| + (1-\lambda)\,||y_{-i} - x_{-i}|| < \varepsilon \end{aligned}$$

By the continuity of $u\,(\cdot)$, therefore, we get that

$$\left| u\left(\frac{x_i}{k}, \lambda z_{-i} + (1-\lambda)\,y_{-i}\right) - u\left(\frac{x_i}{k}, x_{-i}\right)\right| < \Delta$$

and, thus,

$$u\left(\frac{x_i}{k}, \lambda z_{-i} + (1-\lambda)\,y_{-i}\right) < u\,(x) \tag{10}$$

for ε sufficiently small. Given (9)-(10), however, the intermediate-value theorem ensures that

$$\exists \widetilde{x}_i \in \mathbb{R}_{++} : \ (\widetilde{x}_i, \lambda z_{-i} + (1-\lambda)\,y_{-i}) \in \mathcal{B}_\varepsilon\,(x)\,,\ u\,(\widetilde{x}_i, \lambda z_{-i} + (1-\lambda)\,y_{-i}) = u\,(x) \tag{11}$$

This establishes that the set $\mathcal{I}_x^{-i}$ is locally convex at x_{-i}.
Observe next that together (9) and (10) give

$$u\,(\widetilde{x}_i, \lambda z_{-i} + (1-\lambda)\,y_{-i}) < u\,(\lambda z_i + (1-\lambda)\,y_i, \lambda z_{-i} + (1-\lambda)\,y_{-i})$$

which, by the strict monotonicity of $u\,(\cdot)$, holds only if $\widetilde{x}_i < \lambda z_i + (1-\lambda)\,y_i$; that is, only if

$$l_i\,(\lambda z_{-i} + (1-\lambda)\,y_{-i}|x) < \lambda l_i\,(z_{-i}|x) + (1-\lambda)\,l_i\,(y_{-i}|x) \tag{12}$$

as required for $l_i\,(\cdot|x)$ to be locally strictly convex at x_{-i}.
To see finally that the subgradients of $l_i\,(\cdot|x)$ at x_{-i} cannot but all lie in $\mathbb{R}^{n-1}_{--}$, let $q \in \mathbb{R}^{n-1}$ be a subgradient: i.e., let

$$l_i\,(z_{-i}|x) - l_i\,(x_{-i}|x) \geq \sum_{j \in \mathcal{N}\setminus\{i\}} q_j\,(z_j - x_j) \qquad \forall z_{-i} \in \mathcal{I}_x^{-i} \tag{13}$$

To argue ad absurdum, suppose that $q_k \geq 0$ for some $k \in \mathcal{N} \setminus \{i\}$. For $\lambda \in [0,1]$ we can define the vector $x^\lambda \in \operatorname{int}(X)$ by $x_i^\lambda = x_i - (1-\lambda)\varepsilon$, $x_k^\lambda = x_k + \lambda\varepsilon$, and $x_j^\lambda = x_j$ for $j \in \mathcal{N} \setminus \{i,k\}$. As $u\left(x^0\right) < u(x) < u\left(x^1\right)$ due to strict monotonicity, the intermediate value theorem ensures the existence of $\lambda_0 \in (0,1)$ such that $u\left(x^{\lambda_0}\right) = u(x)$. Yet $x^{\lambda_0} \in \mathcal{I}_x$ means that (13) gives

$$\begin{aligned}(1-\lambda_0)\,\varepsilon = x_{0i} - x_i^\lambda &= l_i\left(x_{-i}|x\right) - l_i\left(x_{-i}^{\lambda_0}|x\right) \\ &\leq \sum_{j \in \mathcal{N}\setminus\{i\}} q_j\left(x_j - x_j^{\lambda_0}\right) = -\lambda q_k \varepsilon \leq 0\end{aligned}$$

a contradiction. ■

Lemma B.13 *Let $\succsim$ be a strictly monotonic continuous weak order on X. Suppose also that there exist $i \in \mathcal{N}$ such that $\mathcal{I}_x^{-i} = X^{-i}$ for all $x \in X$. Then $l_i(\cdot|x)$ is strictly convex on X^{-i} if and only if $\succsim$ is strictly convex on X.*

Proof. Let $u : X \to \mathbb{R}$ be a utility function for $\succsim$.
For the "only if" it suffices to show that $u(\cdot)$ is strictly quasi-concave on X. To establish this by contradiction, take $x, y \in X$: $y \succsim x$ and suppose that $u(\lambda x + (1-\lambda)y) \leq u(x)$ for some $\lambda \in (0,1)$. Recall also that, $\succsim$ being strictly monotonic, $l_i(\cdot|x)$ is a well-defined function (Lemma B.12). And as its domain is the entire X^{-i}, there exists $z_i \in X^i$ such that $z_i = l_i(y_{-i}|x)$. By the strict monotonicity of $\succsim$, however, $u(z_i, y_{-i}) = u(x) \leq u(y)$ necessitates that $z_i \leq y_i$. And by strict monotonicity again, it cannot but be

$$\begin{aligned}u\left(\lambda x_i + (1-\lambda) z_i, \lambda x_{-i} + (1-\lambda) y_{-i}\right) &\leq u\left(\lambda x_i + (1-\lambda) y_i, \lambda x_{-i} + (1-\lambda) y_{-i}\right) \\ &= u(\lambda x + (1-\lambda) y) \leq u(x)\end{aligned} \tag{14}$$

Yet, X^{-i} being convex, there exists also $z_i' \in X^i$ such that

$$z_i' = l_i\left(\lambda x_{-i} + (1-\lambda) y_{-i}|x\right) < \lambda l_i\left(x_{-i}|x\right) + (1-\lambda)\, l_i\left(y_i|x\right) = \lambda x_i + (1-\lambda) z_i$$

the inequality due to the strict-convexity of $l_i(\cdot|x)$. Which implies in turn that

$$u\left(\lambda x_i + (1-\lambda) z_i, \lambda x_{-i} + (1-\lambda) y_{-i}\right) > u\left(z_i', \lambda x_{-i} + (1-\lambda) y_{-i}\right) = u(x) \tag{15}$$

by the strict monotonicity of $\succsim$. Needless to say, (14) and (15) give the desired contradiction.
To show the "if" direction, take any $x \in X$ and recall that, $\succsim$ being strictly convex, $u(\cdot)$ is strictly quasi-concave. We will establish that $l_i(\cdot|x)$ is strictly convex arguing ad absurdum. To this end, take $\lambda \in (0,1)$ and $z, y \in \mathcal{I}_x$. The domain of $l_i(\cdot|x)$ being the entire X^{-i}, there exists $\widetilde{z}_i \in X^i$ such that $\widetilde{z}_i = l_i\left(\lambda z_{-i} + (1-\lambda) y_{-i}|x\right)$. Suppose now that

$$\widetilde{z}_i = l_i\left(\lambda z_{-i} + (1-\lambda) y_{-i}|x\right) \geq \lambda l_i\left(z_{-i}|x\right) + (1-\lambda)\, l_i\left(y_{-i}|x\right) = \lambda z_i + (1-\lambda) y_i$$

Which implies though the absurdity that

$$\begin{aligned}u(x) = u\left(\widetilde{z}_i, \lambda z_{-i} + (1-\lambda) y_{-i}\right) &\geq u\left(\lambda z_i + (1-\lambda) y_i, \lambda z_{-i} + (1-\lambda) y_{-i}\right) \\ &= u(\lambda z + (1-\lambda) y) > u(z) = u(x)\end{aligned}$$

the two inequalities due to the strict monotonicity and strict quasi-concavity of $u(\cdot)$. ■

Lemma B.14 *Let $\succsim$ be a strictly convex and strictly monotonic continuous weak order on $X \subseteq \mathbb{R}^n_+$. For any $(i,x) \in \mathcal{N} \times int(X)$, the mapping $\partial l_i(x_{-i}|x) \mapsto p(x) := \{p \in Y : x = x(p)\}$ is a bijective function.*

Proof. We will establish first that the mapping in question is a function. To this end, take any $x \in \text{int}(X)$ and recall that, for any $i \in \mathcal{N}$, $l_i(\cdot|x)$ is a locally-convex function (see Lemma B.12); thus, the set $\partial l_i(x_{-i}|x)$, the subdifferential of $l_i(\cdot|x)$ at x_{-i}, is non-empty. Letting then $q_{-i} \in \mathbb{R}^n_{--}$ be a subgradient of $l_i(\cdot|x)$ at x_{-i}, we can define (uniquely) the quantities $q_i = 1/(x_i - q_{-i}x_{-i}) \in \mathbb{R}_{++}$ and $p = q_i(1, -q_{-i}) \in \mathbb{R}^n_{++}$. This gives

$$\begin{aligned} p_i(z_i - x_i) = q_i(l_i(z_{-i}|x) - l_i(x_{-i}|x)) &= q_i(l_i(z_{-i}|x) - l_i(x_{-i}|x)) \\ &\geq q_i \sum_{j \in \mathcal{N}\setminus\{i\}} q_j(z_j - x_j) \\ &= \sum_{j \in \mathcal{N}\setminus\{i\}} p_j(x_j - z_j) \qquad \forall z \in \mathcal{I}_x \end{aligned}$$

where the inequality follows from the very definition of the subgradient. Clearly, p supports $\mathcal{I}_x$ at x. And as it also supports $\mathcal{U}_x \setminus \{x\}$ at x properly (see Lemma B.7), it cannot but be $x = x(p)$ (see Lemma B.8) as required.

To show now that the function in question is onto, take any $p \in Y$. We have of course $x = x(p)$ for some $x \in \text{int}(X)$. And as p supports $\mathcal{U}_x \setminus \{x\}$ at x properly (see again Lemma B.8), we have in fact $px < pz$ for any $z \in \mathcal{I}_x \setminus \{x\}$. That is,

$$l_i(z_{-i}|x) = z_i > x_i - \sum_{j \in \mathcal{N}\setminus\{i\}} \frac{p_j}{p_i}(z_j - x_j) = l_i(x_{-i}|x) - \sum_{j \in \mathcal{N}\setminus\{i\}} \frac{p_j}{p_i}(z_j - x_j), \quad i \in \mathcal{N}$$

and thus $-p_{-i}/p_i \in \mathbb{R}^n_{--}$ cannot but be a subgradient of $l_i(\cdot|x(p))$ at $x(p)_{-i}$.

To show finally that the function is also injective, arguing ad absurdum, take $p \in Y$ with $x = x(p)$ for some $x \in \text{int}(X)$ and suppose that it is the image of two different subgradients: $q_{-i}, \widetilde{q}_{-i} \in \partial l_i(x_{-i}|x)$ with $q_{-i} \neq \widetilde{q}_{-i}$. By the definition of the function $l_i(\cdot|x)$, we have

$$\widetilde{q}_i x_i = 1 + \widetilde{q}_i \sum_{j \in \mathcal{N}\setminus\{i\}} \widetilde{q}_j x_j = 1 - \sum_{j \in \mathcal{N}\setminus\{i\}} p_j x_j = 1 + q_i \sum_{j \in \mathcal{N}\setminus\{i\}} q_j x_j = q_i x_i$$

That is, $q_i = \widetilde{q}_i$ which implies in turn that $q_{-i} = -p_{-i}/q_i = -p_{-i}/\widetilde{q}_i = \widetilde{q}_{-i}$. ■

Lemma B.15 *Let the strictly monotonic, strictly convex, and continuous weak order $\succsim$ on $X \subseteq \mathbb{R}^n_+$ be represented by the utility function $u : X \to \mathbb{R}$. For every $x \in int(X)$ there exists $\varepsilon > 0$ such that $u(\cdot)$ is partially differentiable at $z \in \mathcal{I}_x \cap \mathcal{B}_\varepsilon(x) \subset int(X)$ only if $l_i(\cdot|x)$ is partially-differentiable at z_{-i} for any $i \in \mathcal{N}$; more specifically,*

$$\frac{\partial l_i(z_{-i}|x)}{\partial z_j} = -\left(\frac{\partial u(z)}{\partial z_i}\right)^{-1} \frac{\partial u(z)}{\partial z_j} \qquad i,j \in \mathcal{N} \quad i \neq j$$

Proof. Take any $x \in \text{int}(X)$ and any $(i,j) \in \mathcal{N} \times \mathcal{N} \setminus \{i\}$. By Lemma B.12, there exists some $\varepsilon_0 > 0$ such that the mapping $l_i(\cdot|x)$ is a well-defined, strictly convex (and, thus, also continuous) function on $\mathcal{I}_x \cap \mathcal{B}_{\varepsilon_0}(x) \subset \text{int}(X)$.

Consider then an arbitrary $z \in \mathcal{I}_x \cap \mathcal{B}_{\varepsilon_0/2}(x)$. For $\varepsilon \in (-\varepsilon_0/2, 0) \cup (0, \varepsilon_0/2)$ and $\mu \in [0,1]$ let $z(\varepsilon,\mu) \in$

$\mathcal{B}_{\varepsilon_0/2}(x)$ be given by $z(\varepsilon,\mu)_i = z_i + \mu\varepsilon$, $z(\varepsilon,\mu)_j = z_j - (1-\mu)\varepsilon$, and $(\varepsilon,\mu)_k = z_k$ for $k \in \mathcal{N} \setminus \{i,j\}$. Since $u(z(\varepsilon,0)) < u(z) < u(z(\varepsilon,1))$ if $\varepsilon > 0$ [resp. $u(z(\varepsilon,0)) > u(z) > u(z(\varepsilon,1))$ if $\varepsilon < 0$], given that $u(\cdot)$ must be continuous, for any $\varepsilon \in (-\varepsilon_0,0) \cup (0,\varepsilon_0)$ the intermediate-value theorem establishes the existence of $\mu_\varepsilon \in (0,1)$ such that $z(\varepsilon,\mu_\varepsilon) \in \mathcal{I}_x \cap \mathcal{B}_{\varepsilon_0/2}(x)$. By the strict monotonicity of $\succsim$ (and thus also of $u(\cdot)$) moreover, the mapping $\varepsilon \mapsto \mu(\varepsilon) = \mu_\varepsilon$ is a function.
Let now $\mu_i(\varepsilon) = \mu(\varepsilon)\varepsilon = \varepsilon - \mu_j(\varepsilon)$ and define the function $g : (-\varepsilon_0,0) \cup (0,\varepsilon_0) \times [0,1] \to \mathbb{R}$ by

$$g(\varepsilon,\lambda) = u(z_i + \lambda\mu_i(\varepsilon), z_{-i}) - u\left(z_j + (1-\lambda)\mu_j(\varepsilon), z_{-j}\right)$$

As $g(\varepsilon,0) < 0 < g(\varepsilon,1)$ if $\varepsilon > 0$ [resp. $g(\varepsilon,0) > 0 > g(\varepsilon,1)$ if $\varepsilon < 0$], for any $\varepsilon \in (-\varepsilon_0,0) \cup (0,\varepsilon_0)$ the intermediate-value theorem ensures again the existence of $\lambda_\varepsilon \in (0,1)$ such that $g(\varepsilon,\lambda_\varepsilon) = 0$. That is, for any $\varepsilon \in (-\varepsilon_0,0) \cup (0,\varepsilon_0)$ there exists $\lambda_\varepsilon \in (0,1)$ such that

$$\begin{array}{ll} u(z_i + \lambda_\varepsilon\mu_i(\varepsilon), z_{-i}) = u\left(z_j + (1-\lambda_\varepsilon)\mu_j(\varepsilon), z_{-j}\right) > u(z) & \text{if } \varepsilon > 0 \\ u(z_i + \lambda_\varepsilon\mu_i(\varepsilon), z_{-i}) = u\left(z_j + (1-\lambda_\varepsilon)\mu_j(\varepsilon), z_{-j}\right) < u(z) & \text{if } \varepsilon < 0 \end{array}$$

the last inequality due to the strict monotonicity of $u(\cdot)$. This implies of course that

$$\forall\varepsilon \in (-\varepsilon_0,0) \cup (0,\varepsilon_0),\ \exists\lambda_\varepsilon \in (0,1) : \ \frac{|u(z_i + \lambda_\varepsilon\mu_i(\varepsilon), z_{-i}) - u(z)|}{|u\left(z_j + (1-\lambda_\varepsilon)\mu_j(\varepsilon), z_{-j}\right) - u(z)|} = 1$$

Putting these observations together, we have

$$\begin{aligned} & \frac{l_i\left(z_j - \mu_j(\varepsilon), z_{-(i,j)}|x\right) - l_i(z_{-i}|x)}{-\mu_j(\varepsilon)} \\ & = \frac{z_i + \mu_i(\varepsilon) - z_i}{-\mu_j(\varepsilon)} \\ & = -\frac{\mu_i(\varepsilon)}{\mu_j(\varepsilon)} \\ & = -\frac{|u\left(z_j + (1-\lambda_\varepsilon)\mu_j(\varepsilon), z_{-j}\right) - u(z)|/\mu_j(\varepsilon)}{|u(z_i + \lambda_\varepsilon\mu_i(\varepsilon), z_{-i}) - u(z)|/\mu_i(\varepsilon)} \\ & = -\frac{|u\left(z_j + (1-\lambda_\varepsilon)\mu_j(\varepsilon), z_{-j}\right) - u(z)/\mu_j(\varepsilon)|}{|u(z_i + \lambda_\varepsilon\mu_i(\varepsilon), z_{-i}) - u(z)/\mu_i(\varepsilon)|} \end{aligned} \tag{16}$$

By the strict monotonicity of $u(\cdot)$ though we also have

$$\begin{array}{ll} 0 < u(z_k + \lambda\varepsilon_k, z_{-k}) - u(z) < u(z_k + \varepsilon_k, z_{-k}) - u(z) & \text{if } \varepsilon_k > 0 \\ 0 > u(z_k + \lambda\varepsilon_k, z_{-k}) - u(z) > u(z_k + \varepsilon_k, z_{-k}) - u(z) & \text{if } \varepsilon_k < 0 \end{array}$$

and thus

$$\begin{aligned} \left|\frac{u(z_k + \lambda\varepsilon_k, z_{-k}) - u(z)}{\varepsilon_k}\right| &= \frac{u(z_k + \lambda\varepsilon_k, z_{-k}) - u(z)}{\varepsilon_k} \\ &< \frac{u(z_k + \varepsilon_k, z_{-k}) - u(z)}{\varepsilon_k} \quad \forall\lambda \in (0,1)\ \forall k \in \mathcal{N} \end{aligned}$$

For any $\lambda \in (0,1)$ and any $k \in \mathcal{N}$, as $\varepsilon_k \to 0$ the quantity $|u(z_k + \lambda\varepsilon_k, z_{-k}) - u(z)/\varepsilon_k|$ approaches the well-defined (since $u(\cdot)$ is partially-differentiable) limit $\partial u(z)/\partial z_k > 0$ (the inequality since $u(\cdot)$ is strictly monotonic). Notice finally that, since $|\mu_k(\varepsilon)| < |\varepsilon|$, $\varepsilon \to 0$ necessitates that also $\mu_k(\varepsilon) \to 0$ for $k \in \{i,j\}$. Clearly, as $\varepsilon \to 0$ the last (and hence also the first) quantity in (16) approaches the well-defined limit $-\left[\partial u(z)/\partial z_j\right] / \left[\partial u(z)/\partial z_i\right]$. ■