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What order?
Perturbation methods for stochastic volatility asset pricing and business cycle models*

PRELIMINARY & INCOMPLETE †

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September 7, 2016

Abstract

When a DSGE model features stochastic volatility, is a third-order perturbation approximation sufficient? The answer is often no. A key parameter—the standard deviation of stochastic volatility innovations—does not appear in the coefficients of the decision rules of endogenous variables until a fourth- or sixth-order perturbation approximation (depending on the functional form of the stochastic volatility process). This paper shows analytically this general result and demonstrates, using three models, that important model moments can be imprecisely measured when the order of approximation is too low. i) In the Bansal–Yaron long-run risk model, the equity risk premium rises from 4.5% to 10% by going to sixth-order. ii) In a workhorse real business cycle model, the welfare cost of business cycles also rise when a fourth-order approximation properly accounts for the presence of stochastic volatility. iii) In a canonical New-Keynesian model, the risk-aversion parameter can be lowered while matching the term premium when a fourth-order approximation is used.

Keywords: Numerical solution methods, Time-varying uncertainty, Equity premium, DSGE models, Welfare.

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1 Introduction

Stochastic volatility has become a popular device in both finance and macroeconomics to account for stylized asset pricing facts and heteroscedasticity in business cycles, respectively. Among macroeconomists, third-order perturbation techniques have become the standard method for solving DSGE models featuring stochastic volatility. This is because perturbation methods can be very efficient compared to alternative, global, solution techniques and are a natural extension of the (log)-linear approximation methods that have historically been popular in macroeconomics since the introduction of real business cycle models in the 1980s. It is also well known that third-order perturbation is the lowest order perturbation that captures movements in endogenous variables resulting from changes in uncertainty.

Is third-order sufficient though? This paper documents important cases in which one would—if choosing to use perturbation methods—want to solve for decisions rules up to fourth- or even sixth-order. The main contributions of this paper are twofold. First, it shows analytically that the coefficients of third-order decision rules are not functions of a key parameter—the standard deviation of stochastic volatility innovations (henceforth denoted ω). It shows that the necessary order of approximation is fourth- or sixth-order and depends on the functional form of the stochastic volatility process. Second, it demonstrates, using three models, that three important model moments—equity risk premium, bond term premium, and welfare cost of business cycles—can be mis-measured if the order of approximation is too low.

Consider the following exogenous stochastic process

$$z_t = \rho_z z_{t-1} + m(x_t)\,\varepsilon_{z,t} \quad \varepsilon_{z,t} \sim \mathbb{N}iid\,(0,1) \tag{1}$$

$$x_t = (1-\rho_x)\,x + \rho_x x_{t-1} + \omega\varepsilon_{x,t} \quad \varepsilon_{x,t} \sim \mathbb{N}iid\,(0,1)\,. \tag{2}$$

Think of equation (1) as a typical process for the level of technology, z_t, in a DSGE model, or as log dividend growth in an endowment asset pricing model. z_t follows an AR(1) with innovations $\varepsilon_{z,t}$. The innovations are scaled by $m(x_t)$, where $m(\cdot)$ is an (as yet unspecified) function of x_t. The term x_t also follows an exogenous stochastic AR(1) process (henceforth, the stochastic volatility process).

This paper is concerned with the parameter ω, the standard deviation of stochastic volatility innovations. It first appears in the constant term of either a fourth- or sixth-order approximation using standard perturbation methods. Whether fourth- or sixth-order depends on the functional form of $m(\cdot)$ (which is discussed below).

Why do we need at least fourth-order? For intuition, suppose we are interested in approximating the decision rule for some endogenous variable y_t using perturbation methods.

A zeroth-order decision rule for y_t is simply its deterministic steady state, $y_t = y$ (the steady state of y_t if agents faced no uncertainty today or in the future), which will be a function of steady state z. A first-order decision rule, in addition, captures how y_t deviates linearly from steady state in response to movements in the state variables (i.e. movements of z_t from z). A second-order decision rule adds terms quadratic in the state vector but also introduces a risk-correction to the constant term for the uncertainty agents face, and is a function of $m(x)$. A third-order decision rule captures deviations of y_t from y in response to movements in x_t, the level of uncertainty agents face. So far so good, all results that are well understood in the literature. But, what if we also want the decision rule to depend on the volatility of uncertainty? The coefficients of the decision rule up to third-order are independent of ω. A reasonable conjecture would be that a fourth-order approximation is necessary, with the volatility of uncertainty appearing in the constant term of decision rules.[1] A schematic of the information added to the approximate decision rule as the order of approximation increases in given in Table 1.

Table 1: Information added in higher-order perturbations

Perturbation order:	0^{th}	$\rightarrow$	1^{st}	$\rightarrow$	2^{nd}	$\rightarrow$	3^{rd}	$\rightarrow$	4^{th} (or 6^{th})
Decision rule depends on:	z	$\rightarrow$	z, z_t	$\rightarrow$	z, z_t, x	$\rightarrow$	z, z_t, x, x_t	$\rightarrow$	z, z_t, x, x_t, ω

A twist to this intuition is that sometimes even a fourth-order approximation is insufficient. Macroeconomists like to use the functional form $m(\cdot) \equiv \exp(\cdot)$ for stochastic volatility. In contrast, finance papers like to use $m(\cdot) \equiv \sqrt{\cdot}$. I show analytically that for $\exp(\cdot)$ a fourth-order approximation is sufficient, while for $\sqrt{\cdot}$ a sixth-order approximation is necessary because fourth-order terms involving ω cancel out.

Does this risk-correction for the volatility of uncertainty matter quantitatively? This paper demonstrates the quantitative importance of not ignoring ω using three "workhorse" models from the literature: An endowment asset pricing model, a real business cycle model, and a New-Keynesian model.

First, Bansal and Yaron (2004) incorporated long-run risk and recursive preferences in an endowment asset pricing model and showed that stochastic volatility not only generated time-variation in risk premiums but also significantly increased the mean equity risk premium

[1] The value of ω only affects the constant term but not linear terms. If n^* is the order of approximation necessary for ω to affect the constant term, then one needs an approximation of at least order $n^* + 1$ for ω to affect linear terms.

from around 4.5% to 6%. The key parameter is ω. I show using standard perturbation methods that a fourth-order approximation delivers an equity risk premium close to 4.5% and is unaffected by the volatility of uncertainty while a sixth-order approximations delivers a greatly increased equity risk premium highly sensitive to the calibration of ω.

Second, within the macroeconomics literature, a longstanding open question regards the welfare cost of business cycles. With more and more models now adding stochastic volatility to explain heteroscedasticity in business cycle fluctuations, it is important to accurately account for the welfare costs of stochastic volatility. Using the Caldara et al. (2012) real business cycle model, one gets much closer to welfare measures from global solutions techniques by going from a third- to a fourth-order approximation.

Third, a growing literature is attempting to bring the insights from endowment based asset pricing models into macro models to jointly account for asset pricing and business cycle facts. Rudebusch and Swanson (2012) added recursive preferences to a New-Keynesian model to try and match the bond term premium. They matched the term premium but at the expense of an unrealistically high risk-aversion coefficient. Andreasen (2012) developed the New-Keynsian model further by adding stochastic volatility but the term premium did not change. Both papers used third-order perturbation methods and hence the risk-correction due to stochastic volatility was not captured. Solving the model up to fourth-order, it is possible to lower the risk-aversion parameter while still matching the term premium.

The rest of the paper is organized as follows. Section 2 presents a simple asset pricing model with closed-form solution that reveals the relationship between ω and the auxiliary perturbation parameter. Section 3 shows analytically under what conditions a fourth-order approximation is sufficient. Section 4 demonstrates the quantitative importance of ω in an endowment asset pricing model, New-Keynesian model, and real business cycle model, respectively. Section 5 discusses the results, Section 6 reviews the related literature and Section 7 concludes.

2 A closed-form solution

This section closely follows de Groot (2015) in that it specifies a simple endowment asset pricing model featuring stochastic volatility for which the price-dividend ratio decision rule has a closed-form representation. The closed-form reveals the way in which the auxiliary perturbation parameter appears in the exact decision rule and hence what order of approximation ω, the parameter measuring the standard deviation of stochastic volatility innovations, first appears.

2.1 The model

A representative agent has preferences over consumption, c_t, given by

$$V_t = \max_{c_t} \left(\frac{c_t^{1-\gamma}}{1-\gamma} + \beta \mathbb{E}_t V_{t+1} \right), \tag{3}$$

where V_t is the value function, β is the discount factor, and γ is the coefficient of relative risk aversion. The agent's budget constraint is

$$c_t + s_{t+1} p_t = (d_t + p_t) s_t, \tag{4}$$

where s_t denotes units of an asset with price p_t and dividends d_t. The dividend growth rate, $z_t \equiv \log (d_t / d_{t-1})$, follows the process

$$z_t = (1 - \rho_z) z + \rho_z z_{t-1} + \sqrt{x_t} \sigma \varepsilon_{z,t} \quad \varepsilon_{z,t} \sim \mathbb{N}iid(0,1), \tag{5}$$

where z is the steady state of z_t and ρ_z is the persistence parameter. The innovations to z_t are scaled by $\sqrt{x_t}$, where x_t is the time-varying conditional variance of dividend growth and follows the process

$$x_t = (1 - \rho_x) x + \rho_x x_{t-1} + \omega \sigma \varepsilon_{x,t} \quad \varepsilon_{x,t} \sim \mathbb{N}iid(0,1), \tag{6}$$

where x is the steady state of x_t, ρ_x is the persistence parameter, and ω the standard deviation of stochastic volatility innovations. In both equation (5) and (6), σ is the auxiliary perturbation parameter and takes the value 0 or 1. The perturbation parameter thus links the deterministic version of the model ($\sigma = 0$) with its fully stochastic counterpart ($\sigma = 1$).

After imposing market clearing, $s_t = 1$, which implies $c_t = d_t$, and after further manipulation, the first-order equilibrium condition becomes

$$y_t = \sum\nolimits_{i=1}^{\infty} \beta^i \mathbb{E}_t \exp \left((1-\gamma) \sum\nolimits_{j=1}^{i} x_{t+j} \right), \tag{7}$$

where $y_t \equiv p_t / d_t$ is the price-dividend ratio.

2.2 Perturbation methods: A primer

With any macroeconomic model, the aim is to find the decision rule $\mathbf{y}_t = \mathbf{g}(\mathbf{x}_t)$ (or a good approximation of it) that satisfies the model's equilibrium conditions (where $\mathbf{y}_t$ and $\mathbf{x}_t$ are vectors of endogenous variables and state variables, respectively).

Perturbation methods find an approximation to $\mathbf{g}(\cdot)$ by starting from the exact solution of a related simpler problem—the deterministic steady state (or zeroth-order solution)—and taking a Taylor expansion with respect to the state variables and an auxiliary scale (or perturbation) parameter, σ, that switches uncertainty on and off. The expansion is taken in the neighbourhood of the deterministic steady state, where $\mathbf{x_t} = \mathbf{x}$ and $\sigma = 0$. Hence, when using perturbation methods, the decision rule is rewritten $\mathbf{y}_t = \mathbf{g}(\mathbf{x}_t, \sigma)$. The N^{th}-order approximation for the k^{th} variable in $\mathbf{y}_t$ is given by

$$y_t^k = \sum_{n=0}^{N} \left(\frac{1}{n!} \left[g^k_{(\mathbf{x}\sigma)^n} \right]_{\alpha_1 \ldots \alpha_n} \prod_j^i \begin{bmatrix} \mathbf{x_t} - \mathbf{x} \\ 1 \end{bmatrix}^{\alpha_j} \right), \tag{8}$$

where the above expression is written in tensor notation and where each coefficient $g_{(\mathbf{x}\sigma)^i}$ is evaluated at $(\mathbf{x}_t, \sigma) = (\mathbf{x}, 0)$. The question of this paper is what is the lowest n for which ω appears in a $g_{(\mathbf{x}\sigma)^n}$ coefficient.

2.3 Solution

Since, in general, the decision rule of macroeconomic models are unknown, perturbation methods construct an approximate decision rule by acting indirectly on the model's equilibrium conditions. In the special case where the exact decision rule exists we can use it to derive the perturbation approximation directly. In the model described by equations (5)–(7), there is one endogenous variable, $\mathbf{y}_t = y_t$, the state vector is $\mathbf{x}_t = [z_t, x_t]'$, and the exact decision rule, denoted $y_t = g(z_t, x_t, \sigma)$, is

$$y_t = \sum\nolimits_{i=1}^{\infty} \beta^i \exp\left(A_i z + B_i (z_t - z) + C_i \sigma^2 x + D_i \sigma^2 (x_t - x) + F_i \sigma^6 \omega^2 \right), \tag{9}$$

where A_i, B_i, C_i, D_i, and F_i are coefficients that are complicated functions of the structural parameters $\{\gamma, \rho_z, \rho_x\}$.[2] Notice that the parameter of interest, ω, is multiplied by σ^6. Since the coefficients of the perturbation approximation are evaluated at the deterministic steady state (i.e. when $\sigma = 0$), it is clear from equation (9) that ω will drop out of all coefficients of the Taylor expansion that are of fifth-order or below. This result is summarized as follows:

Result 1 *The decision rule for the price-dividend ratio in the model described by equations (5)–(7) is only a function of the conditional standard deviation of the stochastic volatility process, ω, if the perturbed approximation around the deterministic steady state is sixth-order or above.*

[2] See de Groot (2015)

The full perturbation approximation, even in this simple model, is unwieldy. However, in a sixth-order approximation, the parameter ω is absent in all terms except the constant term

$$g_{\sigma^6} = \sum\nolimits_{i=1}^{\infty} \beta^i \exp\left(A_i z\right) \left(x^3 C_i^3 + 720 F_i \omega^2\right). \tag{10}$$

The computational difficulties of sixth-order perturbation methods, for anything other than the simplest models, are discussed in Section 5. At this stage though, it is worth identifying the source of this result. A sixth-order approximation is required because standard perturbation methods construct an approximation in the neighbourhood of the deterministic steady state. Suppose we took a Taylor approximation of equation (9) around $(\mathbf{x}_t, \sigma) = (\mathbf{x}, 1)$ instead. This is not possible in general, but it is when the exact decision rule exists. A first-order approximation of equation (9) is then given by

$$y_t = \tilde{g} + \tilde{g}_z \left(z_t - z\right) + \tilde{g}_x \left(x_t - x\right), \tag{11}$$

where

$$\tilde{g} \equiv \sum_{i=1}^{\infty} G_i, \quad \tilde{g}_z \equiv \sum_{i=1}^{\infty} B_i G_i, \quad \tilde{g}_x \equiv \sum_{i=1}^{\infty} D_i G_i, \quad \text{and} \quad G_i \equiv \beta^i \exp\left(A_i z + C_i x + F_i \omega^2\right). \tag{12}$$

and where the tilde denotes that the coefficients of the Taylor expansion have been evaluated at $(\mathbf{x}_t, \sigma) = (\mathbf{x}, 1)$ rather than at the deterministic steady state.

Notice that in this first-order approximation, both the constant term $\tilde{g}$ and the coefficients of the terms linear in the state vector ($\tilde{g}_z$ and $\tilde{g}_x$) depend on ω. In contrast, with a standard perturbation approach in the neighbourhood of the deterministic steady state, a sixth-order approximation is required to risk-correct the constant term for ω. An even higher-order approximation is required for coefficients of linear terms to depend on ω.

This simple model valuably demonstrates the issue of incorporating ω. However, the decision rule in this model with CRRA utility is not quantitatively sensitive to the value of ω. Section 4.1 demonstrates its quantitative importance for risk premiums in an extension of this model with long-run risk and recursive preferences.

But first, one may still be surprised by the result that a sixth-order approximation is necessary. As described in the Introduction, it might be intuitive that ω not appear until fourth-order. But why sixth-order? The next section shows analytically, again with a simple model, that the necessary order of approximation depends on the functional form of the stochastic volatility process.

3 Fourth- or sixth-order?

To demonstrate why and when a fourth-order approximation is insufficient and a sixth-order approximation is necessary, consider a model described by the following single equilibrium condition

$$y_t = \mathbb{E}_t f\left(z_{t+1}\right), \tag{13}$$

where z_t follows the process given by equations (1) and (2). The functions $f(\cdot)$ and $m(\cdot)$ are continuous and 6-times differentiable. Think of y_t as the price-dividend ratio and z_t as log dividend growth as in Section 2 but agents, for some "behavioural" reason, care only about expected dividend growth at $t+1$ and not its entire future path. Unlike the model in Section 2 there is no convenient closed-form solution. The perturbation method therefore needs to act indirectly on the equilibrium condition, equation (13).

Substituting the exogenous processes into the equilibrium condition, we get

$$y_t = \mathbb{E}_t f\left(\rho_z z_t + m\left(\left(1-\rho_x\right)x + \rho_x x_t + \omega\sigma\varepsilon_{x,t+1}\right)\sigma\varepsilon_{z,t+1}\right). \tag{14}$$

While the expression is now in the form $y_t = g\left(\mathbf{x}_t, \sigma\right)$, the decision rule is still unknown because of the expectations operator.[3] Since $g(\cdot,\cdot)$ is unknown, we use perturbation methods to find an approximation, with the coefficients of the Taylor expansion evaluated at $(z_t, x_t, \sigma) = (0, x, 0)$. The full Taylor expansion is straightforward but tedious so it is not show here. Instead, I only show three coefficients that are plausible candidates for ω to appear in. These are the even-ordered terms g_{σ^2}, g_{σ^4}, and g_{σ^6}.[4]

The second-derivative of y_t with respect to σ is:[5]

$$\frac{\partial^2 y_t}{\partial\sigma^2} = \mathbb{E}_t f''(\cdot)\left(m'(\cdot)\omega\varepsilon_x\sigma\varepsilon_z + m(\cdot)\varepsilon_z\right)^2 + f'(\cdot)\left(m''(\cdot)\omega^2\varepsilon_x^2\sigma\varepsilon_z + 2m'(\cdot)\omega\varepsilon_x\varepsilon_z\right). \tag{15}$$

Setting $(z_t, x_t, \sigma) = (0, x, 0)$ and taking expectations gives

$$\left.\frac{\partial^2 y_t}{\partial\sigma^2}\right|_{(0,x,0)} = g_{\sigma^2} = f''(0)\, m(x)^2, \tag{16}$$

which is a risk-correction of the constant term of the decision rule and is as a function of the steady state conditional volatility of dividend growth, $m(x)$. The term is the same as if stochastic volatility was absent—the stochastic volatility parameter ω never enters second-

[3] $\varepsilon_{z,t+1}$ and $\varepsilon_{x,t+1}$ are not decision rule variables but rather arguments of integration in the expectations operator.

[4] Since the innovations are drawn from symmetric distributions, g_{σ^n} where n is odd are all generically zero.

[5] The time subscripts on ε_z and ε_x terms have been dropped for brevity.

order terms. The fourth-order term is

$$\left.\frac{\partial^4 y_t}{\partial \sigma^4}\right|_{(0,x,0)} = g_{\sigma^4} = 3\left(4f''(0)\left(m'(x)^2 + m(x)\,m''(x)\right)\omega^2 + f^{(4)}(0)\,m(x)^4\right), \tag{17}$$

where $f^{(4)}$, for example, denotes the fourth-derivative. This shows that, in general, ω appears in a fourth-order approximation. However, when $m(\cdot) \equiv \sqrt{\cdot}$ the term $m'(x)^2 + m(x)\,m''(x)$ becomes zero and the fourth-order term reduces to

$$g_{\sigma^4} = 3f^{(4)}(0)\,m(x)^4, \tag{18}$$

where ω has dropped out. In contrast, when $m(\cdot) \equiv \exp(\cdot)$, the fourth-order term becomes

$$g_{\sigma^4} = 3\left(8f''(0)\,e^{2x}\omega^2 + f^{(4)}(0)\,e^{4x}\right). \tag{19}$$

This result is summarized in the following:

Result 2 *When an exogenous innovation term in a DSGE model has time-varying standard deviation $m(x_t)$, where $m(\cdot)$ is at least twice differentiable, a fourth-order perturbation approximation risk-corrects the constant term of decision rules for stochastic volatility iff $\mathbb{S} \equiv m'(x)^2 + m(x)\,m''(x) \neq 0$. If $\mathbb{S} = 0$, at least a sixth-order approximation is necessary.*

This result is important because the finance and macro literature have largely specified stochastic volatility processes differently. The finance literature prefers to use $m(\cdot) \equiv \sqrt{\cdot}$. The benefit of this specification is that the stochastic process is still conditionally normal and can be exploited to generate a conditionally log-normal linear approximation that accounts for risk as in Campbell and Shiller (1988). The drawback of this functional form is that it is possible to get a negative standard deviation. Macroeconomists largely use $m(\cdot) \equiv \exp(\cdot)$. This functional form ensures the standard deviation remains strictly positive but, as pointed out by Andreasen (2010), has the drawback that the level of the process does not have any moments.

Finally, the sixth-order term with $m(\cdot) \equiv \sqrt{\cdot}$ is given by

$$\left.\frac{\partial^6 y_t}{\partial \sigma^6}\right|_{(0,x,0)} = g_{\sigma^6} = 15\left(6f^{(4)}(0)\,\omega^2 + f^{(6)}(0)\,x^3\right). \tag{20}$$

Clearly, the parameter of interest, ω, appears. With $m(\cdot) \equiv \exp(\cdot)$ there would also be a ω^4 term in g_{σ^6} but this also drops out under the $m(\cdot) \equiv \sqrt{\cdot}$ specification.

This section and the last have shown that the parameter that controls the standard deviation of stochastic volatility innovations is absent in a third-order perturbation approximation—

the standard solution method within the macro literature. In the next section, I will demonstrate the quantitative importance of approximations above third-order using three models featuring stochastic volatility: An endowment asset pricing model, a real business cycle model, and a New-Keynesian model, respectively.

4 Three quantitative examples

This section demonstrates the quantitative significance of ω for three important model moments—the equity risk premium, the welfare cost of business cycles, and the bond term premium—in three workhorse economic models—the long-run risk model of Bansal and Yaron (2004), the real business cycle model studied by Caldara et al. (2012), and the New-Keynesian model studied by Andreasen (2012)—and shows how these moments can be mis-measured when a perturbation approximations is not of sufficiently high order.

4.1 Bansal and Yaron (2004) asset pricing model

The seminal Bansal and Yaron (2004) paper provides a compelling resolution of asset pricing puzzles. In particular, the paper shows that time-varying uncertainty in consumption and dividend growth is important for quantitatively matching equity risk premiums.

The model is an extension of the simple endowment asset pricing model from Section 2 with three additional features. First, consumption and dividend growth rates are modelled to contain a small long-run predictable component. Second, consumption and dividend growth rates are modelled to exhibit stochastic volatility. Third, these dynamics are combined with an agent with recursive Epstein and Zin (1989) and Weil (1989) preferences.

This type of finance model is usually solved using global methods or the Campbell and Shiller (1988) conditionally log-normal linear approximation. Here the model will will be solved using perturbation methods. The model is calibrated using the estimated version of the model from Bansal et al. (2012). The model's equations and table of calibrated parameter values are relegated to the Online Appendix. The functional form for the stochastic volatility process is $m(\cdot) \equiv \sqrt{\cdot}$, which from Section 2 and 3 suggests a sixth-order approximation is necessary.

Table 2 shows the equity risk premium using perturbation methods of second-, third-, fourth- and sixth-order. The premium is calculated as the conditional expected equity risk premium, evaluated with the exogenous state variables at their respective steady states.

Using the notation from equation (8), the calculation is

$$y = \sum_{n=0}^{N} g_{\sigma^n}, \tag{21}$$

where y_t is the expected equity risk premium, $\mathbb{E}_t r_{t+1}^m - r_t^f$, where r_t^m is the market return on equity and r_t^f is the risk-free rate. The values in brackets in Table 2 are the unconditional mean equity risk premium, $\mathbb{E}(y_t)$. The final column, BKY, gives the equity risk premium using the solution method in Bansal et al. (2012).

Table 2: Equity risk premium (Annualized %)

	2^{nd}	3^{rd}	4^{th}	6^{th}	BKY
$\omega = 0$	5.12	5.12	4.54	4.65	4.65
	(5.12)	(5.12)	(4.54)	(4.65)	—
ω = 1.31e-6	5.12	5.12	4.54	6.04	5.05
	(5.12)	(4.98)	(4.44)	(5.94)	—
ω = 2.62e-6	5.12	5.12	4.54	10.19	5.97
	(5.12)	(4.77)	(4.35)	(10.00)	—

Note: The model is Bansal and Yaron (2004). The calibration is Bansal et al. (2012) Table II except $\beta = 0.9985$ rather than .9989. The values in column BKY are based on the solution method in Bansal et al. (2012) Table III. The first row for each ω reports the conditional equity risk premium. The values in brackets reports the unconditional mean equity risk premium.

Consider the first row results without stochastic volatility when $\omega = 0$. To a first-order perturbation approximation, the model is certainty equivalent and the premium is always zero. The second-order approximation delivers an approximate annual equity risk premium of 5%, a little high compared to BKY. Since the innovations are drawn from a symmetric (Normal) distribution, odd-order g_{σ^i} terms are zero. Hence, the equity risk premium for the third- and fifth-order approximation are the same as those for second- and fourth-order. At fourth-order, the equity risk premium drops a little to 4.5%. Notice that, even in the absence of stochastic volatility, a sixth-order approximation is required to get close to the BKY value of 4.65%.

The results from Section 2 and 3 suggest that a fourth-order approximation is insufficient to capture the role of stochastic volatility for the equity risk premium. This is borne out in Table 2. As ω increases the second-, third-, and fourth-order approximate values are unchanged relative to their $\omega = 0$ values. This is in contrast to the sixth-order approximation. The equity risk premium is increasing in ω when a sixth-order approximation used, qualitatively similar to BKY. Quantitatively, the value rises too much. When ω =2.62e-6, the equity risk premium is 10% in the sixth-order approximation as opposed to 6% under BKY. Possibly an even higher-order approximation is necessary to deliver an accurate equity risk premium in the presence of stochastic volatility in this model.[6]

Table 2 also reports unconditional means. Notice that these unconditional means are sensitive to changes in ω even at lower order-approximation. At third-order for example, the unconditional mean equity risk premium (actually) falls from 5.12% to 4.98% to 4.77% as ω increases from zero to 1.31e-6 to 2.62e-6, respectively. This is consistent with Proposition 4 (p300) in Andreasen (2012) which states that "In a third-order approximation around the deterministic steady state, stochastic volatility may affect the level of risk premia". This comes about because a decision rule at third-order has nonlinear terms in the endogenous state variables. However, as Table 2 demonstrates, this curvature does not capture the quantitative importance of stochastic volatility for risk premiums. In fact, the sign of the change is incorrect in this calibration with the equity risk premium falling as stochastic volatility increases.

In summary, Table 2 results suggest that ω needs to enter the decision rule explicitly, which requires for the Bansal and Yaron (2004) model at least a sixth-order approximation, if not higher, when perturbation methods are used.

4.2 Caldara et al. (2012) real business cycle model

What is the welfare cost of business cycle fluctuations? is an age old question in macroeconomics. This section looks at measuring the stochastic volatility component of the welfare cost of business cycles.

To do this the Caldara et al. (2012) real business cycle model featuring recursive preferences and stochastic volatility is studied. Since the real business cycle model is the core of almost all modern macroeconomic models, the key insights from this exercise will likely follow to richer models. The model equations and calibration of parameter values are again related to the Online Appendix. The model is a standard real business cycle model except households have recursive Epstein and Zin (1989) and Weil (1989) preferences and technology shocks exhibit stochastic volatility.

Table 3 reports the conditional welfare cost of business cycles in terms of percentage consumption equivalents.[7] The columns titled Value function and Chebyshev give the welfare

[6] Computer memory issues prevented checking whether the equity risk premium would fall from 10% closer to 6% with an eight-order approximation.

[7] The calculation is the same as equation (21) where y_t is instead the value of the representative household's

cost of business cycles using global solution methods while the first and second columns give welfare cost measures using perturbation methods of various orders. Since Caldara et al. (2012) use the functional form $m(\cdot) \equiv \exp(\cdot)$ for the stochastic volatility process, a fourth-order should be sufficient to capture the effect of stochastic volatility on welfare.

Table 3: Welfare cost of business cycles (% consumption equivalent)

	2nd & 3nd-order pert.	4th-order pert.	Value function	Chebyshev
$\omega = 0$	1.1278	1.1278	—	—
$\omega = 0.1$	1.1278	1.2389	1.2838	1.2855
$\omega = 0.2$	1.1278	1.5800	—	—

Note: The model and calibrated parameter values are from Caldara et al. (2012). The calibration is the "extreme" calibration from Caldara et al. (2012) Table 1 and Section 4. The Value function iteration and Chebyshev polynomial numbers are from Caldara et al. (2012) Table 5.

The Value function number in Table 3, for example, says that households would forgo 1.28% of steady state consumption to eliminate business cycle fluctuations in this model. In contrast, the welfare cost value is 1.13% using both second- and third-order perturbation methods. Since innovations are drawn from a symmetric (Normal) distribution, all $g_{\sigma^n} = 0$ terms for n odd are zero, and hence second- and third-order give the same welfare result.

The second- (and third-)order perturbation measures of welfare suffer in two ways. First, they under-measure the welfare cost of business cycles relative to the global solution by over 0.15 percentage points. Second, the welfare cost measure is unchanged when the level of stochastic volatility is raised or lowered. As the conditional standard deviation of the stochastic volatility process, ω, is increased from 0 to 0.2, the second- (and third-)order welfare cost measures remain at 1.13%.

In contrast, the fourth-order measure of welfare costs rise in ω, reflecting households' dislike of uncertainty. This occurs because the fourth-order approximation is the lowest order of approximation at which the parameter ω appears in the decision rule. Second- (and third-)order welfare measures underreport the welfare cost of business cycles because the effect of stochastic volatility on welfare is absent. For $\omega = 0.1$, the welfare cost of business cycles for the fourth-order perturbation approximation is 1.23% as opposed to 1.13%, much closer to the values given by the global solutions.

welfare function.

4.3 Andreasen (2012) New-Keynesian model

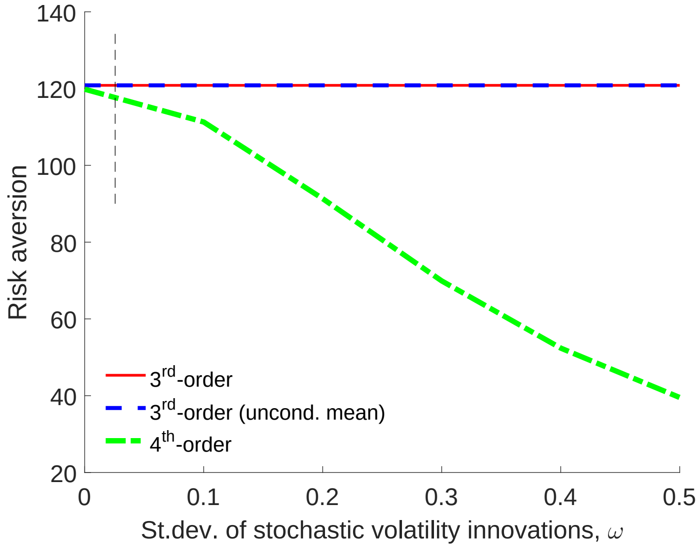
Rudebusch and Swanson (2012) developed a workhorse New-Keynesian model with recursive Epstein and Zin (1989) and Weil (1989) preferences and showed that the model could match both stylized business cycle and asset pricing facts. The drawback, however, was that the calibrated risk-aversion parameter was an order of magnitude larger than values used in the endowment economy finance literature. Andreasen (2012) further extended the model of Rudebusch and Swanson (2012) by adding stochastic volatility and showed that the model, solved used third-order perturbation methods, could generate an increase in the variability of the bond term premia but not its level. Hence, the calibrated risk-aversion parameter remained elevated.

By solving the Andreasen (2012) model to fourth-order, it is possible to reduce the risk-aversion parameter while still matching the mean term premium measured in the data. The model equations and calibrated parameter values are presented in the Online Appendix. The model size has been reduced by switching off the government spending and monetary policy shocks and removing the smoothing term in the monetary policy reaction function.

For the exercise in Figure 1, the aim was to match the spread between the nominal 10 year government bond yield and the 3 month Treasury bill yield. The value to match was a spread of 1.4 annualized percent. Figure 1 plots ω on the horizontal axis and shows the corresponding value for the risk aversion parameter necessary to generate a term premium of 1.4%. Without stochastic volatility ($\omega = 0$) the risk aversion parameter needs to be approximately 120, an order of magnitude larger than the values used in the finance literature (for example, risk aversion is 7.4 in estimated long-run risk model in Bansal et al. (2012) model). The solid red line shows how much risk aversion can be reduced when stochastic volatility is introduced using a third-order approximation—the solution method used by Rudebusch and Swanson (2012) and Andreasen (2012). Since the third-order approximation does not capture the effects of stochastic volatility, however, the risk-aversion parameter remains unchanged. However, when a fourth-order approximation is applied (the dot-dash green line), the term premium rises with ω and hence the the risk-aversion parameter can be reduced in order to maintain a constant term premium.

The range of ω in Figure 1 is greater than economically plausible values. The upper value used by Andreasen (2012) is indicated with the dashed black line. However, the qualitative result is striking. Even if the term premium in a New-Keynesian model was sensitive to stochastic volatility, standard solution methods would not have detected that relationship. In trying to jointly match stylized business cycle and asset pricing facts in production economies using perturbation methods, an order of approximation above third-order is clearly necessary.

Figure 1: Matching term premiums in New-Keynesian models



Note: The model and calibration is Andreasen (2012). Changes to the calibration include $\sigma_g = \sigma_r = \rho_r = 0$. The figure shows combinations of ω and risk aversion $\gamma + \theta (1 - \gamma)$ (by varying θ) for the difference between the nominal 10 year government bond yield and the 3 month Treasury bill yield is 1.40 annualized percent. The black dashed line shows the upper value for ω used in Andreasen (2012).

5 Discussion

In light of the results from Section 4, this section does two things. First, it calculates the computational costs of using higher-order perturbation methods. Second, it shows an alternative way of using perturbation methods to risk-correct a first-order order approximation in order to incorporate the role of ω.

5.1 Complexity of higher-order perturbation solutions

Low-order perturbation solutions can be computed efficiently. And, for many applications, the low-order approximate solutions can be very accurate. Third-order perturbation methods efficiently and accurately capture the time variation in endogenous variables resulting from time varying uncertainty. However, the inability of third-order approximations to risk-correct for the conditional variance of stochastic volatility is problematic. While fourth- and sixth-order solutions are possible in small-scale models, as the order of approximation rises, the complexity of the perturbation solution expands exponentially.

Table 4 shows how quickly complexity increases. The term s_i in the second column shows the number of coefficient that a perturbation solution has at each order of approximation i, where n_y are the number of endogenous variables and n_x are the number of state variables. For example, the long-run risk model from Section 4.1 has nine endogenous variables $\left\{g_t, g_{d,t}, x_t, r_t^f, r_t^m, \mathbb{E}_t r_{t+1}^m, \chi_t, y_{a,t}, y_{m,t}\right\}$ and seven exogenous variables $\{y_{m,t-1}, x_{t-1}, \chi_{t-1}, \varepsilon_{g,t}, \varepsilon_{d,t}, \varepsilon_{\chi,t}, \varepsilon_{x,t}\}$. The number of coefficients increases by two orders of magnitude in going from a third- to a sixth-order solution, and the time taken to compute the solution increases by one order of magnitude. Even for this relatively small-scale model, a seventh-order approximation was not possible on a standard laptop.

Table 4: Complexity of higher-order perturbation solutions

Order (i)	Number of elements (s_i)	Example with $n_y = 9, n_x = 7$	Time (sec)
$i = 1$	$s_1 = n_y n_x$	63	0.99
$i = 2$	$s_2 = s_1 + n_y\left(1 + n_x^2\right)$	513	2.87
$i = 3$	$s_3 = s_2 + n_y n_x\left(1 + n_x^2\right)$	3,663	11.29
$i = 4$	$s_4 = s_3 + n_y\left(1 + n_x^2 + n_x^4\right)$	25,722	39.77
$i = 5$	$s_5 = s_4 + n_y n_x\left(1 + n_x^2 + n_x^4\right)$	180,135	118.19
$i = 6$	$s_6 = s_5 + n_y\left(1 + n_x^2 + n_x^4 + n_x^6\right)$	1,261,035	344.91
$i = 7$	$s_7 = s_6 + n_y n_x\left(1 + n_x^2 + n_x^4 + n_x^6\right)$	8,827,335	—†
$i = 8$	$s_8 = s_7 + n_y\left(1 + n_x^2 + n_x^4 + n_x^6 + n_x^8\right)$	61,791,444	—†

Note: Times based on solving the Bansal and Yaron (2004) using `Dynare++` on a 2.1GHz Intel laptop with Windows 7. † Lack of memory prevented a solution being found.

5.2 Risk-adjusted linear perturbation

One potential alternative to using higher-order standard perturbation methods is to construct a risk-adjusted—or non-certainty equivalent—linear perturbation. The idea is to "fold" second-order terms into a linear structure. Despite only needing to compute second-order derivatives and no higher, ω as well as the first-order effects of time-varying uncertainty survives in this risk-adjusted linear solution. This builds on work by Coeurdacier et al. (2011), Juillard (2011), and de Groot (2013) on solving DSGE models around the risky steady state. Here I sketch the idea using the simple asset pricing model from Section 2. For an extensive presentation of the methodology, see de Groot (2016).

Consider the equilibrium conditions

$$\mathbb{E}_t \mathbf{f}\left(\mathbf{y}_{t+1}, \mathbf{y}_t, \mathbf{x}_{t+1}, \mathbf{x}_t\right) = \mathbf{0} \quad \text{and} \quad \mathbf{x}_{t+1} = \mathbf{h}\left(\mathbf{x}_t, \sigma\varepsilon_{t+1}\right), \tag{22}$$

where, for the sake of simplicity, (but without loss of generality) we assume all state variables, $\mathbf{x_t}$, are exogenous. The unknown decision rules for the endogenous variables are denoted $\mathbf{y}_t = \mathbf{g}\left(\mathbf{x}_t\right)$. To find a non-certainty equivalent linear solution, we need to simultaneously solve for $\mathbf{y}$ (n_y unknown steady state values) and $\mathbf{g_x}$ ($n_y \times n_x$ unknown coefficients). Higher order terms like $\mathbf{g_{xx}}$ are ignored but known terms like $\mathbf{h_{xx}}$ are exploited in the method. In particular, for

$$z_t = z + \rho_z\left(z_{t-1} - z\right) + \sqrt{x_t}\varepsilon_{z,t} \quad \text{we have} \quad h_{x\varepsilon_z} \equiv \frac{1}{2\sqrt{x}}. \tag{23}$$

Substituting the decision rules into the equilibrium conditions gives

$$\begin{gathered}\mathbf{F}\left(\mathbf{x}_t, \mathbf{y}, \mathbf{g_x}, \sigma\right) \equiv \\ \mathbb{E}_t\left[\mathbf{f}\left(\mathbf{y} + \mathbf{g_x}\left(\mathbf{h}\left(\mathbf{x}_t, \sigma\varepsilon_{t+1}\right) - \mathbf{x}\right), \mathbf{y} + \mathbf{g_x}\left(\mathbf{x}_t - \mathbf{x}\right), \mathbf{h}\left(\mathbf{x}_t, \sigma\varepsilon_{t+1}\right), \mathbf{x}_t\right)\right] = \mathbf{0}.\end{gathered} \tag{24}$$

The steady state conditions (when $\mathbf{x_t} = \mathbf{x}$) are

$$\mathbf{F}\left(\mathbf{x}, \mathbf{y}, \mathbf{g_x}, \sigma\right) \equiv \mathbb{E}\left[\mathbf{f}\left(\mathbf{y} + \mathbf{g_x}\left(\mathbf{h}\left(\mathbf{x}, \sigma\varepsilon\right) - \mathbf{x}\right), \mathbf{y}, \mathbf{h}\left(\mathbf{x}, \sigma\varepsilon\right), \mathbf{x}\right)\right] = \mathbf{0}. \tag{25}$$

Unlike standard perturbation methods, these conditions are not going to be evaluated at the deterministic steady state. But, since we can't evaluate $\mathbb{E}\left(.\right)$ exactly, we take a second-order approximation around $\sigma = 0$ which gives

$$\underbrace{\left[\mathbf{F}\left(\mathbf{x}, \mathbf{y}, ., 0\right)\right]^i}_{A} + \underbrace{\frac{1}{2}\left[\mathbf{F}_{\sigma\sigma}\left(\mathbf{x}, \mathbf{y}, \mathbf{g_x}, 0\right)\right]^i}_{B} = 0, \tag{26}$$

where i denotes the i^{th} endogenous variable. Term A alone would be the deterministic steady state (with no $\mathbf{g_x}$ terms). Term B is an additive scalar risk-adjustment to each steady state equation. The risk-adjustment is a function of $\mathbf{g_x}$, that is a function of the conditional variance-covariance matrix of the endogenous variables.

To apply this to the model from Section 2, (and for further simplicity set $\rho_z = 0$), we have a problem in two unknowns, y and g_x. We therefore need two equations. One is the steady state equation, which for the model is

$$0 = \underbrace{\beta e^{\theta z}\left(y+1\right) - y}_{A} + \underbrace{\frac{1}{2}\beta e^{\theta z}\left(\underbrace{\left(y+1\right)x\theta^2}_{\text{V(z) risk-adj.}} + \underbrace{g_x^2\omega^2}_{\text{SV risk-adj.}}\right)}_{B}, \tag{27}$$

where $\theta \equiv 1-\gamma$. Notice there are two risk-adjustments. First, for the conditional dividend growth rate variance. Second, for the presence of stochastic volatility. If g_x is non-zero (i.e. stochastic volatility creates time variation in y_t), then the risk-adjusted steady state equation includes ω, the parameter that only appears in a sixth-order approximation using standard perturbation methods.

The second equation to solve for the two unknowns comes from the first-derivatives of equation (24), namely

$$\left[\mathbf{F_x}\left(\mathbf{x_t}, \mathbf{y}, \mathbf{g_x}, \sigma\right)\right]^i_j = 0, \tag{28}$$

where j denotes the j^{th} state variable. This matrix of equations also has an expectations operator which requires another second-order approximation around $\sigma = 0$ (and evaluated at $\mathbf{x_t} = \mathbf{x}$). This gives

$$\underbrace{\left[\mathbf{F_x}\left(\mathbf{x}, \mathbf{y}, \mathbf{g_x}, 0\right)\right]^i_j}_{C} + \underbrace{\frac{1}{2}\left[\mathbf{F}_{\mathbf{x}\sigma\sigma}\left(\mathbf{x}, \mathbf{y}, \mathbf{g_x}, 0\right)\right]^i_j}_{D} = 0. \tag{29}$$

Term C alone generates a linear-homogenous system for the j corresponding to the stochastic volatility variable x_t resulting in $g_x = 0$. The D term is another additive scalar risk-adjustment that breaks the linear-homogenous property and ensures g_x is non-zero.[8].

For the simple model, the risk-adjusted first-derivative equation is

$$0 = \underbrace{\beta e^{\theta z} g_x \rho_x - g_x}_{C} + \underbrace{\frac{1}{2}\beta e^{\theta z}\rho_x\left(\frac{2}{\sqrt{x}}h_{x\varepsilon_z}\theta^2\left(y+1\right) + g_x y\left(x\theta^2 + g_x^2\omega^2\right)\right)}_{D}. \tag{30}$$

In a standard perturbation solution, we have term C only resulting in $g_x = 0$. With the risk-adjustment, we get term D. Since g_x in this equation is, in general, non-zero, by equation (27), both the (risk-adjusted) steady state, y, and the linear coefficient, g_x, are functions of ω.

In summary, given computational complexity it is unclear whether higher-order perturbation methods are the way forward to accurately incorporate the effect of ω when solving

[8]Since these are simply scalar adjustment this is still a standard matrix quadratic system

macro models. This section sketches an alternative based on the idea of a risk-adjusted perturbation method that incorporates ω with a linear decision rule.

6 Related literature

Perturbation methods were first extensively applied to dynamic stochastic models by Judd and co-authors (see Judd (1998)). (Log)-linear approximation techniques (equivalent to a first-order perturbation solution) have been common place in macroeconomics since Kydland and Prescott (1982). Schmitt-Grohé and Uribe (2004) showed analytically that in a second-order expansion the coefficients linear and quadratic in the state vector are independent of the volatility of the exogenous shocks and therefore that the presence of uncertainty affects only the constant term of the decision rules.

The use of time-varying uncertainty has for a long time been popular in finance but only recently has the role of time-varying uncertainty been applied to macroeconomic variables. For example, Bloom (2009) investigates how uncertainty affects investment and labor demand decisions while Fernández-Villaverde et al. (2011) investigates how interest rate volatility can explain output and investment in emerging economies.

Third-order perturbation methods are the standard solution method used for stochastic volatility macro models, as surveyed in Fernández-Villaverde and Rubio-Ramírez (2013). Benigno et al. (2013) develop perturbation methods in a nonstandard way to construct a second-order approximation with time-varying risk. The toolbox `Dynare` can handle third-order approximation while `Dyanre++` can handle higher-order approximations. However, the properties of these higher-order approximations have not been documented in the literature.

de Groot (2015) presents a closed-form model featuring stochastic volatility that develops the first insights into the relationship between ω and standard perturbation methods. de Groot (2016) develops a risk-adjusted linear solution that incorporates ω into a linear approximation of the decision rules.

7 Conclusion

In this paper, I have shown that for models with stochastic volatility, perturbation methods only capture the risk-correction due to the standard deviation of stochastic volatility innovations when the approximation is taken to fourth- or sixth-order, depending on the functional form of the stochastic volatility process. I have shown the important role that stochastic volatility plays in quantitatively driving risk premiums and the welfare costs of business cycles.

However, asset pricing insights from partial equilibrium endowment models do not easily translate to general equilibrium production economy models. While getting the solution techniques for DSGE models right is important, the missing ω in third-order perturbation solutions does not, unfortunately, appear to resolve the difficulty of simultaneously matching asset pricing and business cycle stylized facts as shown with the New-Keynesian model of Section 4.3.

Low-order perturbation methods often offer a good trade-off between efficiency and accuracy in solving DSGE models. However, the need for higher-order perturbation methods for certain classes of model increases the computational challenges, especially when working with large-scale models. New methods such as Fernández-Villaverde and Levintal (2016), den Haan et al. (2015), and de Groot (2016) provide promising alternatives.

References

Andreasen, Martin M. (2010) 'Stochastic volatility and DSGE models.' *Economics Letters* 108(1), 7–9

— (2012) 'On the effects of rare disasters and uncertainty shocks for risk premia in non-linear DSGE models.' *Review of Economic Dynamics* 15(3), 295–316

Bansal, Ravi, and Amir Yaron (2004) 'Risks for the long run: A potential resolution of asset pricing puzzles.' *Journal of Finance* 59(4), 1481–1509

Bansal, Ravi, Dana Kiku, and Amir Yaron (2012) 'Risks for the long run: Estimation with time aggregation.' *Unpublished manuscript*

Benigno, Gianluca, Pierpaolo Benigno, and Salvatore Nistic (2013) 'Second-order approximation of dynamic models with time-varying risk.' *Journal of Economic Dynamics and Control* 37(7), 1231–1247

Bloom, Nicholas (2009) 'The impact of uncertainty shocks.' *Econometrica* 77(3), 623–685

Caldara, Dario, Jesús Fernández-Villaverde, Juan F. Rubio-Ramírez, and Wen Yao (2012) 'Computing DSGE models with recursive preferences and stochastic volatility.' *Review of Economic Dynamics* 15(2), 188–206

Campbell, John Y., and Robert J. Shiller (1988) 'The dividend-price ratio and expectations of future dividends and discount factors.' *Review of Financial Studies* 1(3), 195–228

Coeurdacier, N., H. Rey, and P. Winant (2011) 'The risky steady state.' *American Economic Review* 101(3), 398–401

de Groot, Oliver (2013) 'Computing the risky steady state of DSGE models.' *Economics Letters* 120(3), 566–569

— (2015) 'Solving asset pricing models with stochastic volatility.' *Journal of Economic Dynamics and Control* 52, 308–321

— (2016) 'Stochastic volatility for asset pricing and business cycles.' *Unpublished manuscript*

den Haan, Wouter, Michal L. Kobielarz, and Pontus Rendahl (2015) 'Exact present solution with consistent future approximation: A gridless algorithm to solve stochastic dynamic models.' *Unpublished manuscript*

Epstein, Larry G., and Stanley E. Zin (1989) 'Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework.' *Econometrica* 57(4), 937–969

Fernández-Villaverde, Jesús, and Juan F. Rubio-Ramírez (2013) 'Macroeconomics and volatility: Data, models, and estimation.' In 'Advances in Economics and Econometrics: Tenth World Congress,' vol. 3 Cambridge University Press

Fernández-Villaverde, Jesús, and Oren Levintal (2016) 'Solution methods for models with rare disasters.' *Unpublished manuscript*

Fernández-Villaverde, Jesús, Pablo Guerrón-Quintana, Juan F. Rubio-Ramírez, and Martin Uribe (2011) 'Risk matters: The real effects of volatility shocks.' *American Economic Review* 101(6), 2530–2561

Judd, Kenneth L. (1998) *Numerical Methods in Economics* (MIT press)

Juillard, M. (2011) 'Local approximation of DSGE models around the risky steady state.' *Unpublished manuscript*

Kydland, Finn E., and Edward C. Prescott (1982) 'Time to build and aggregate fluctuations.' *Econometrica* 50(6), 1345–1370

Rudebusch, Glenn D., and Eric T. Swanson (2012) 'The bond premium in a DSGE model with long-run real and nominal risks.' *American Economic Journal: Macroeconomics* 4(1), 105–43

Schmitt-Grohé, Stephanie, and Martin Uribe (2004) 'Solving dynamic general equilibrium models using a second-order approximation to the policy function.' *Journal of Economic Dynamics and Control* 28(4), 755–775

Weil, Philippe (1989) 'The equity premium puzzle and the risk-free rate puzzle.' *Journal of Monetary Economics* 24(3), 401–421

A ONLINE APPENDIX (NOT FOR PUBLICATION)

This appendix presents the models used in Section 4. Notation has been changed from the original so as to, where possible, provide consistent notation across the three models. For full details, please consult the original papers. Replication code is available on request.

A.1 Bansal and Yaron (2004)

The asset pricing restriction for asset i with log gross return $r_{i,t} \equiv \log(R_{i,t})$ for a representative agent with recursive preferences is given by

$$\beta^{\theta}\mathbb{E}_t \exp\left(-\frac{\theta}{\psi}g_{t+1} + (\theta-1)r_{a,t+1} + r_{i,t+1}\right) = 1, \tag{31}$$

where $g_t \equiv (G_t)$ is the log aggregate gross growth rate of consumption and $r_{a,t} \equiv \log(R_{a,t})$ is the log gross return on an asset that delivers aggregate consumption as its dividend. The parameter β is the discount factor and $\theta \equiv (1-\gamma)/(1-1/\psi)$ is a function of γ, the risk-aversion parameter and ψ, the intertemporal elasticity of substitution. The dynamics of consumption and dividends are as specified as follows

$$\chi_t = \rho_\chi \chi_{t-1} + \varphi_e \sqrt{x_{t-1}}\varepsilon_{\chi,t}, \quad \varepsilon_{\chi,t} \sim \mathbb{N}iid(0,1) \tag{32}$$

$$g_t = \mu + \chi_{t-1} + \sqrt{x_{t-1}}\varepsilon_{g,t}, \quad \varepsilon_{g,t} \sim \mathbb{N}iid(0,1) \tag{33}$$

$$g_{d,t} = \mu_d + \phi\chi_{t-1} + \varphi_d\sqrt{x_{t-1}}\varepsilon_{d,t}, \quad \varepsilon_{d,t} \sim \mathbb{N}iid(0,1) \tag{34}$$

$$x_t = x + \rho_x(x_{t-1} - x) + \omega\varepsilon_{x,t}, \quad \varepsilon_{x,t} \sim \mathbb{N}iid(0,1), \tag{35}$$

where both consumption and dividend growth rates, g_{t+1} and $g_{d,t+1}$, respectively, contain a small persistent predictable component, χ_t, which determines the conditional expectation of consumption growth. The parameters ϕ and φ_d control the relative volatility of dividends and correlation with consumption, respectively. The parameter ρ_χ determines the persistence of the expected growth rate process. Finally, x_t represents time-varying uncertainty incorporated in consumption growth, with x its unconditional mean, ρ_x its persistence, and ω its conditional standard deviation.

The calibrated parameter values are given in Table 5. The decision frequency $h = 11$ implies 11 periods per year.

Table 5: Long-run risk model: Calibrated parameters

Parameter	β	μ	μ_d	$\sqrt{x}$	ϕ	φ_e	φ_d
Value	.9985	.0012	.002	.0073	4.45	.0306	5
Parameter	ρ_χ	ρ_x	ω	γ	ψ	h	
Value	.9812	.9983	2.61e-6	7.42	2.05	11	

Note: Parameter values are from Bansal et al. (2012) Table III except β which has been lowered from .9989 to .9985 to aid convergence of the perturbation solution.

A.2 Caldara et al. (2012)

A representative household has recursive preferences over streams of consumption, c_t, and leisure, l_t, given by

$$V_t = \max_{c_t, l_t} \left((1-\beta) \left(c_t^{\nu} (1-l_t)^{1-\nu} \right)^{\frac{1-\gamma}{\theta}} + \beta \left(\mathbb{E}_t V_{t+1}^{1-\gamma} \right)^{\frac{1}{\theta}} \right)^{\frac{\theta}{1-\gamma}}, \tag{36}$$

where β is the discount factor, ν controls labor supply, $\theta \equiv (1-\gamma)/(1-1/\psi)$, where γ controls risk-aversion and ψ is the elasticity of intertemporal substitution. When $\theta = 1$ preferences are CRRA and the elasticity of intertemporal substitution and risk aversion coincide. The household's budget constraint is

$$c_t + i_t + \frac{b_{t+1}}{R_t^f} = w_t l_t + r_t k_t + b_t, \tag{37}$$

where i_t is investment, R_t^f is the risk-free gross interest rate, b_t is the holding of a bond that pays one unit of consumption in period $t+1$, w_t is the wage, l_t is labor, r_t is the rental rate of capital, and k_t is capital. Households accumulate capital according to the law of motion $k_{t+1} = (1-\delta) k_t + i_t$, where δ is the depreciation rate. Competitive firms produce final goods, y_t, with Cobb-Douglas technology $y_t = \exp(z_t) k_t^{\alpha} l_t^{1-\alpha}$. The level of productivity, z_t, evolves exogenously and the processes is given by

$$z_t = \rho_z z_{t-1} + \exp(x_t) \varepsilon_{z,t} \quad \varepsilon_{z,t} \sim \mathbb{N}iid(0,1) \tag{38}$$

$$x_t = (1-\rho_x) x + \rho_x x_{t-1} + \omega \varepsilon_{x,t} \quad \varepsilon_{x,t} \sim \mathbb{N}iid(0,1). \tag{39}$$

The innovations to the level of productivity, $\varepsilon_{z,t}$, are scaled by the stochastic volatility term, $\exp(x_t)$. The stochastic volatility process has persistence ρ_x and the stochastic volatility innovations have standard deviation of ω. Finally, the economy satisfies the aggregate resource constraint $y_t = c_t + i_t$.

In order to highlight the role of stochastic volatility, the "extreme" calibration from Caldara et al. (2012) is used. The calibrated parameter values are given in Table 6.

Table 6: RBC model: Calibrated parameters

Parameter	β	ν	α	δ	ρ_z	$\exp(x)$	ρ_x	ω
Value	.991	.362	.3	.0196	.95	.021	.9	.1

Note: Calibrated parameter values are from Caldara et al. (2012) Table 5. The value of ν is chosen so that steady state labor supply, $l = 1/3$

A.3 Andreasen (2012)

This is a standard New-Keynesian DSGE model extended with Epstein and Zin (1989) and Weil (1989) preferences and stochastic volatility. The value function V_t for the representative household is given by

$$V_t \equiv \max_{c_t, l_t} \left(\frac{\left(c_t^{\nu} (1 - l_t)^{1-\nu}\right)^{1-\gamma}}{1-\gamma} - \begin{cases} -\beta \left(\mathbb{E}_t \left(V_{t+1}^{1-\theta}\right)\right)^{\frac{1}{1-\theta}} & \text{for} \quad \gamma < 1 \\ \beta \left(\mathbb{E}_t \left(-V_{t+1}^{1-\theta}\right)\right)^{\frac{1}{1-\theta}} & \text{for} \quad \gamma \geq 1 \end{cases} \right), \tag{40}$$

where β is the discount factor, and c_t and l_t denote consumption and labor supply, respectively. The intertemporal elasticity of substitution is given by $1/(1 - \nu(1-\gamma))$ and $\gamma + \theta(1-\gamma)$ is a measure of risk-aversion that accounts for the leisure decision. The households real budget constraint is given by

$$\mathbb{E}_t M_{t+1} b_{t+1} + c_t = \frac{b_t}{\pi_t} + w_t l_t + T_t, \tag{41}$$

where M_{t+1} is the stochastic discount factor, b_t is a nominal state-contingent claim, π_t is inflation, w_t is the real wage, and T_t is a real lump-sum transfer.

Final output is produced by perfectly competitive firms using a continuum of intermediate goods, $y_t(i)$ and technology

$$y_t = \left(\int_0^1 y_t(i)^{\frac{\eta-1}{\eta}} di \right)^{\frac{\eta}{\eta-1}}, \quad \text{which implies} \quad y_t(i) = \left(\frac{p_t(i)}{p_t} \right)^{-\eta} y_t, \tag{42}$$

Table 7: New-Keynesian model: Calibrated parameters

Parameter	β	γ	ν	θ	α	η	ξ	δ	ϕ_π	ϕ_y
Value	.9995	2.5	.35	-110	.36	6	260	.025	1.5	.3
Parameter	ρ_r	l	g/y	π	μ_a	ρ_z	ρ_g	$\exp(x)$	σ_g	σ_r
Value	0	.38	.17	1.008	1.005	.98	.9	.0075	0	0

Note: Calibrated parameters values are from Andreasen (2012) Table 1. To reduce the size of the model, $\{\rho_r, \sigma_g, \sigma_r\}$ have been changed from their original values $\{.85, .004, .003\}$.

where the aggregate price level is given by $p_t = \left(\int_0^1 p_t(i)^{1-\eta} di\right)^{\frac{1}{1-\eta}}$. Intermediate firms produce slightly differentiated goods using technology

$$y_t(i) = a_t \exp(z_t) k^\alpha l_t(i)^{1-\alpha}, \tag{43}$$

where k denotes fixed physical capital and $l_t(i)$ is labor used by the firm i, z_t is an exogenous stationary technology process, and a_t is a deterministic technology trend with $\mu_a \equiv a_t/a_{t-1}$. Intermediate firms choose $l_t(i)$ and $p_t(i)$ to solve

$$\max_{l_t(i), p_t(i)} \mathbb{E}_t \sum_{j=0}^{\infty} M_{t,t+j} \left(\frac{p_{t+j}(i)}{p_{t+j}} y_{t+j}(i) - w_{t+j} l_{t+j}(i) - \frac{\xi}{2} \left(\frac{p_{t+j}(i)}{p_{t+j-1}\pi} - 1 \right)^2 y_{t+j} \right), \tag{44}$$

subject to equation (43). The central bank sets the nominal interest rate, r_t, using a standard Taylor rule

$$r_t = r(1-\rho_r) + \rho_r r_{t-1} + \phi_\pi \ln\left(\frac{\pi_t}{\pi}\right) + \phi_y \ln\left(\frac{y_t}{z_t y}\right) + \sigma_r \varepsilon_{r,t}. \tag{45}$$

Aggregation implies $y_t = a_t z_t k^\alpha l_t{}^{1-\alpha}$. Each period $g_t a_t$ units of output are used for public consumption, where g_t is exogenously given by

$$\log(g_t/g) = \rho_g \log(g_{t-1}/g) + \sigma_g \varepsilon_{g,t}. \tag{46}$$

The aggregate resource constraint is $y_t = c_t + g_t a_t + \delta k a_t$, where $\delta k z_t$ units of output are used every period to maintain the fixed capital stock. Finally, the exogenous process for technology, z_t, is given by

$$z_t = \rho_z z_{t-1} + \exp(x_t)\,\varepsilon_{z,t} \tag{47}$$

$$x_t = (1-\rho_x)\,x + \rho_x x_{t-1} + \omega\varepsilon_{x,t}. \tag{48}$$

The calibrated parameter values are given in Table 7.