Technology, Wage Dispersion and Inflation

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CDMA Working Paper Series No. 1403
12 Mar 2014
JEL Classification: E24, E52
Keywords: inflation, search frictions, wage dispersion
Technology, Wage Dispersion and Inflation

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March 23, 2014

Abstract

We study the effect of inflation on the wage dispersion due to firm heterogeneity and on-the-job search, in the context of a labour market à la Postel-Vinay and Robin (International Economic Review 43, 2002) and micro-founded money demand. The productivity distribution of firms is first assumed to be exogenously given and we show that a rise in inflation diminishes the wage dispersion. We then allow the firms to adjust their productivity level through investment and find that a higher inflation can reduce the dispersion of firms’ productivity. This is because the increase in the cost of holding money leads the fall of demand in the goods market, and hence make the least productive firms unprofitable and driving them out of business. The decrease in productivity dispersion furthermore also diminishes the wage dispersion.

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JEL codes: E24, E52

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*The author is indebted to Gerhard Sorger and Alejandro Cuñat of the University of Vienna for their sound supervision and patience. The author is also grateful to Emanuel Gasteiger, Christian Haefke, Frank Heinemann, Karl Schlag, Jean-Robert Tyran, Lutz Weinke as well as the participants of the seminars at Univeristy of Vienna, Technical University of Dortmund, University Institute of Lisbon and Workshop in Search and Matching Models (Aix-en-Provence) for valuable comments. All remaining errors are the responsibility of the author. Corresponding address: School of Economics and Finance, University of St Andrews, Castle Cliffe, The Scores, St Andrews, Fife, KY16 9AR, UK. Email: shoujian.zhang@st-andrews.ac.uk
1 Introduction

The existing empirical literature reports a negative correlation between the inflation rate and the dispersion of wages\(^1\). Within a fully micro-founded monetary framework, this paper studies the effect of inflation on the wage dispersions in the long run. Quite a few theoretical explanations have been advanced regarding why similar workers are paid differently. Mortensen (2003, Chapter 1.2) surveys these explanations within the search and matching framework of labour market. The first kind of explanation concerns on-the-job search, and was pioneered by Burdett and Mortensen (1989, 1998) and further exploited by many other wage-posting models as in Coles (2001) and Postel-Vinay and Robin (2002a)\(^2\). The second type of explanation takes account of firm heterogeneity. Idson and Oi (1999) point out that labour productivity varies across industries and also across firms within industries and this is indeed the potential reason for wage differentials. The third kind of explanation claims that the different bilateral bargaining powers over wages are a reason for the existence of wage differentials, which is an obvious implication of the standard search and matching literature, e.g., Pissarides (2000). The final type of explanation regards the compensation differentials as sources of wage dispersion (Hwang, Mortensen, and Reed (1998) and Lang and Mujumdar (2004)). Notably, the wage differentials caused by firm heterogeneity, especially by the productivity differences of firms, are empirically very important. Postel-Vinay and Robin (2002b) construct an equilibrium search model with on-the-job search augmented by worker and employer heterogeneity based on Postel-Vinay and Robin (2002a), and then calibrate it using French matched employer and employee panel data from the 1990s (DADS, Déclarations Annuelles des Données Sociales). They use this structural estimation to provide a decomposition of cross-employee wage variance and find that i) the share of the wage variance that is explained by person effects is significant only among high-skilled white collar workers, and quickly decreases to 0% as the observed skill level decreases; ii) market imperfections typically accounts for around 50% of wage dispersion; iii) the contribution of firm differentials to wage dispersion is typically 30% to 40% except in the high-skilled white collars group. Therefore, we believe that modelling firm heterogeneity in the context of a labour market with on-the-job search is quite important in wage dispersion studies.

This study analyses how in the long-run inflation rate can affect wage dispersions as a result of firm heterogeneity and on-the-job search. We introduce a search labour market à la Postel-Vinay and Robin (2002a) into a micro-founded model of money, namely, the search monetary framework\(^3\) of Lagos and Wright (2005). We confirm that a rise in

\(^1\)For instance, Hammermesh (1986) analyses the relationship between dispersion in the relative wage changes and the inflation rate using wage data from the period 1955–81 in the United States. He finds that higher inflation, especially unexpected inflation, reduced the relative wage dispersion. Erikson and Ichino (1995) study the effects of inflation on the wage differentials using the wage data from metal manufacture firms in Italy of the period 1976–90. They find that higher inflation rates significantly reduced the wage differentials.

\(^2\)Postel-Vinay and Robin (2002a) study the wage dispersions when both on-the-job search and firm heterogeneity are present.

\(^3\)Search monetary models were pioneered by Kiyotaki and Wright (1991, 1993) for the case of indivisible commodities and money. They impose the assumption of indivisible commodities and money to guarantee the tractability of the model, because otherwise the distribution of money holding becomes too complicated to obtain analytical results. The search monetary frameworks which are suitable for macro-economic and monetary policy analysis, i.e., the models that allow divisible commodities and money but circumvent the aforementioned difficulties concerning the distribution of money holdings, are developed along these lines: i) to introduce the assumption of large families and perfect risk sharing as in Shi
inflation diminishes wage dispersion due to firm heterogeneity and on-the-job search. Furthermore, we find an extra channel (productivity distribution) through which the inflation can affect the wage dispersion. For tractability reasons, we only check the effects of inflation on the upper and lower bounds of wages paid in the economy.

We first study the impact of inflation on the wage dispersions due to firm heterogeneity in the case of an exogenous productivity distribution. We find that the rising of inflation diminishes the wage dispersion in the sense that the lower wage bound increases while the upper wage bound decreases. The intuitions are as follows. First, the upper wage bound decreases since higher inflation causes a reduction in profits of the firms. This is because inflation is modelled as the cost of holding money in our micro-founded monetary exchange setting, and a rise in inflation increases households’ cost of holding money. Households then reduce their money holding in the decentralized goods market, which in turn reduces firms’ selling and profits in this market. As will be shown below, the upper wage bound is the zero-profit wage level (highest wage) paid by the most productive firms. Therefore, higher inflation reduces the upper wage bound as the most productive firms pay less to keep its employees due to the fall in profits. Second, the decrease in the highest wage that the firm could pay reduces the prospect of wage rising for workers. This will make the unemployed require a higher reservation wage (lowest wage) to compensate their home product.

We then allow the productivity level to be determined by firms’ investment decisions. Our model demonstrates that a rise in inflation first makes firms’ productivity less dispersed in the sense that the lower bound of productivity remains unchanged. Intuitively, the rise in inflation reduces firms’ profits, which drives the less productive firms out of the economy. This is again because an increase in inflation decreases the demand faced by the firms and hence in turn their profits. Therefore our model deduces a rise in the lower bound of productivity. The highest productivity in the economy is determined by the technology of productivity investment so a rise in inflation has no impact on the upper bound of productivity. Furthermore, the rise of inflation then diminishes wage dispersion in the sense that the lower bound of wages rises while the upper bound of wages drops. We thus identify three channels through which inflation changes the wage dispersion due to productivity differences. Two of the channels are the same as in the case of an exogenous productivity distribution. The last one is the mechanism of how inflation affects wage dispersion through its effect on productivity distribution.

This paper is closely related to the literature that incorporates goods and labour market search frictions in a single model. Berentsen, Menzio and Wright (2011) introduce a search and matching labour market into Lagos and Wright (2005). Their main findings are that the rise of inflation causes higher unemployment and the long-run Phillips curve slopes upwards. Shi (1998) introduces the standard search friction of labour markets into Shi (1997) and finds that an increase in the money growth rate increases steady-state employment and output when the money growth rate is low; but reduces steady-state employment and output when the money growth rate is already high. Furthermore, in general the Friedman rule is not optimal.

The remainder of the paper is organized as follows: we present the basic model in (1997); ii) to introduce an extra centralized good market and the assumption of linear negative utility of labour input in this centralized good market, as in Lagos and Wright (2005); iii) to introduce complete financial markets among groups e.g. Faig (2006). For further details about the development of search monetary models, please refer to the review by Shi (2006).
section 2. The stationary equilibrium of the economy with an exogenous productivity distribution, and the effects of inflation on wage dispersion in this case, are studied in section 3. Section 4 introduces endogenous productivity investment decisions and defines the new stationary equilibrium in this case. Section 5 proves the main propositions about the effects of inflation on wage dispersion in the case of endogenous productivity distribution. Section 6 concludes.

2 The Basic Model

The basic structure of the economy in the present model is the same as in Berentsen, Menzio and Wright (2011), the goods market of which is based on Lagos and Wright (2005), although our labour market setting is in the spirit of Postel-Vinay and Robin (2002a).

2.1 The Environment

There are two types of private agents: firms and households. There is a unit continuum of households; while the measurement of firms is $N$. Households work, consume and all households are equally skilled; while firms enroll workers, maximize profits and pay out dividends to households.

Time is discrete and continues forever. In each period, there are three markets that open sequentially. Market 1 is a labour market, Market 2 is a decentralized goods market and Market 3 is a centralized goods market. These three markets are indexed by $i = 1, 2, 3$, respectively.

Market 1 (labour market) is modeled in the spirit of Postel-Vinay and Robin (2002a), where households received job offers from the firms which operate constant returns to labour technologies to produce good $q$. The marginal productivity of labour $y$ differs across firms. We first assume that $y$ is exogenously given as being distributed over $[\bar{y}, \bar{y}]$ according to the continuous distribution $\Gamma$ (cdf). However, $\Gamma$ will be endogenized in section 4. A firm is willing to employ any worker so long as he/she could make a positive profit for the firm. Firms send job offers to the labour market during each period to enroll workers. Workers can either be employed or unemployed. We also allow workers to search for a better job while employed, so firms make offers to employed workers as well. The matching process in the labour market will be discussed later. Furthermore, we assume that lay-offs occur at a constant rate $\delta$. The wage-setting mechanism is completely adopted from Postel-Vinay and Robin (2002a), that is, it is assumed that firms can vary their wage offers according to the employment status of any particular worker instead of being bound to offer the same wage to all workers, and firms can counter the offers that their employees receive from competing firms instead of being completely passive in the face of such offers. In addition, wage contracts can be renegotiated only by mutual agreement. Finally, to integrate the setting of Postel-Vinay and Robin (2002a) into the

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4In this section and section 3, the productivity distribution is exogenously given, so we assume for the sake of simplicity that $N$ is also exogenous. However, we will assume $N$ to be endogenous when we have an endogenous productivity distribution in sections 4 and 5.

5We also follow Postel-Vinay and Robin (2002a) in assuming that there are no capacity constraints for firms, i.e., a firm could employ as many employees as it is willing to. As a result, no firm is ever induced to fire a given worker to replace him/her with a less costly unemployed worker in the context of this model.

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present model, we stipulate that working for a firm entails both the producing activity and participating in the matching process in the decentralized goods market.

Market 2 (decentralized goods market) is modeled in the spirit of the day-time market in Lagos and Wright (2005), where good $q$ is traded bilaterally between pairs consisting of a household (buyer as well as consumer) and an employee of a firm (seller as well as producer), due to anonymous matching. All private agents are anonymous, which generates an essential role for a medium of exchange. Household and firm then bargain over the terms of trade, $(q,d)$, where $d$ is the real balance paid by the household to the firm. The terms of trade are chosen so as to share the surplus from a buyer-seller match in fixed proportions. The household’s share is $\theta \in (0,1]$. Unsold good $q$ is transformed into another kind of good $x$ for Market 3, according to a linear transforming technology.

Market 3 (centralized goods market) is modeled in the spirit of the night-time market in Lagos and Wright (2005), where good $x$ is traded multilaterally. All the private agents take the market prices as given. Unsold good $x$ then vanishes between two periods. Also, without loss of generality we assume that agents discount at rate $\beta$ between Market 3 and the next Market 1, but not between the other markets.

We assume a central bank exists and controls the supply of fiat money. We denote the growth rate of the money supply by $\omega$, so that $M = (1 + \omega)M$, where $M$ denotes the per capita money stock in Market 3 and variable with a circumflex indicates the value of variables in the next period. Therefore, in steady states, $\omega$ is the inflation rate. The central bank implements its inflation goal by providing deterministic lump-sum injections of money, $\check{m}M$, to the household at the end of each period.

Throughout the discussion in the text, we assume policy and the productivity distribution to be constant, and focus only on steady states.

### 2.2 Households

We now consider optimal decisions of the household. Let $s = e, u$ index employment status: $e$ indicates that a household is employed; $u$ indicates that a household is unemployed. We adopt the convention of Berentsen, Menzio and Wright (2011) for measuring real balances $z$, i.e. the nominal balances of households in the current Market 3, the next Market 1 and Market 2, $m$, are deflated by $P$, the price level in the current centralized goods market (Market 3), which is the latest price known for that market. Thus, when an agent brings $\check{m}$ fiat money to the next period, we let $\check{z} = \check{m}/\check{P}$ denote his real balance. When he then takes this $\check{m}$ fiat money to the next Market 3, his real balance is then given by $\check{m}/\check{P} = \check{z}\check{p}$, where $\check{p} = P/\check{P}$ converts $\check{z}$ into the units of the numeraire in that market. Notice $\check{p} = 1/(1 + \omega)$, where $\omega$ is the inflation rate between this and the next centralized goods market.

#### 2.2.1 Centralized Goods Market

We start with Market 3. The household, which is employed in a firm of productivity $y$ at wage level $w$ and holds a real balance $z$, solves

\begin{equation}
W_{3,e}(z, y, w) = \max_{x, \check{z}} \{x + \beta W_{1,e}(\check{z}, y, w)\},
\end{equation}

\text{s.t. } x + \check{z} = w + \Delta + \frac{\pi M}{p} + z,
where $W$ denotes the value function of an employed household, $x$ is the consumption in Market 3, $\Delta$ is the dividend income.$^6$ Here we assume that utility is linear in $x$, as in Lagos and Wright (2005), to make all agents in the centralized markets choose the same real balance to enter into the next market. Substituting $x$ from the budget constraint into (1) gives

$$W_{3,e}(z, y, w) = w + \Delta + \frac{\pi M}{p} + z + \max_{\hat{z}} \{-\hat{z} + \beta W_{1,e}(\hat{z}, y, w)\}.$$  

(2)

Therefore, $W_{3,e}(z, y, w)$ is linear in $z$.

Analogously, the value function for an unemployment household reads

$$W_{3,u}(z) = b + \Delta + \frac{\pi M}{p} + z + \max_{\hat{z}} \{-\hat{z} + \beta W_{1,u}(\hat{z})\}.$$  

(3)

where $b$ is the home production of an unemployed household. $W_{3,u}(z)$ is also linear in $z$.

### 2.2.2 Decentralized Goods Market

We now move to Market 2. The value functions of an employed household and an unemployed household, respectively, with real balance $z$, read

$$W_{2,e}(z, y, w) = \alpha_h v(q) + \alpha_h W_{3,e}[\rho(z - d), y, w] + (1 - \alpha_h)W_{3,e}(\rho z, y, w),$$  

(4)

$$W_{2,u}(z) = \alpha_h v(q) + \alpha_h W_{3,u}[\rho(z - d)] + (1 - \alpha_h)W_{3,u}(\rho z),$$  

(5)

where $\alpha_h$ is the probability for a household to trade in the decentralized goods market. The matching process in the goods market will be discussed later. $v(\cdot)$ is the household’s utility function of consuming good $q$. $v(\cdot)$ is twice differentiable with $v(0) = 0$, $v' > 0$, $v'' < 0$, $\lim_{q \to 0} v'(q) = +\infty$, $\lim_{q \to -\infty} v'(q) < 1$. Define $q^*$ as the solution to $v'(q^*) = 1$. Because of the properties of $v(\cdot)$, $q^*$ is unique and $q^* > 0$.

Because of the linearity of $W_3$ ((2) and (3)), (4) and (5) become

$$W_{2,e}(z, y, w) = \alpha_h [v(q) - \rho d] + W_{3,e}(\rho z, y, w),$$  

(6)

$$W_{2,u}(z) = \alpha_h [v(q) - \rho d] + W_{3,u}(\rho z).$$  

(7)

$^6$We assume that the representative household holds the representative portfolio to omit the optimal investment decision of the households in the financial market. So the equilibrium dividend $\Delta$ equals the average profit across all the firms. We denote by $\pi(y)$ to be a firm’s periodic profit in steady state. Therefore, $\Delta = N \int^p \pi(y) d\Gamma(y)$
2.2.3 Labour Market

The value function of an employed household with real balance $z$ in Market 1, reads

$$W_{1,e}(z, y, w) = \delta W_{2,u}(z) + \lambda_h \int_y^0 W_{2,e}[z, p, \phi(y, p)]d\Gamma(p)$$

$$+ \lambda_h \int_y^y W_{2,e}[z, y, \phi(p, y)]d\Gamma(p)$$

$$+ \{1 - \delta - \lambda_h + \lambda_h \Gamma(Q(w, y))\}W_{2,e}(z, y, w),$$

(8)

where $\lambda_h$ is the probability of a worker being contacted by firms. We follow Mortensen (2003) in assuming that the arrival rate of offers to on-the-job searchers is the same as those of unemployed workers. $\phi(y, p)$ is the optimal wage offer of a firm with productivity $p$ when two firms with different productivity $(y, p$ and $y < p$) become competitors for a single worker. The threshold productivity level $Q(w, y)$ is defined as $\phi(Q(w, y), y) = w$. Therefore, the firm currently employing the worker has to raise the household’s wage to keep him if this household receives a offer from a firm with productivity higher than $Q(w, y)$. The first term of the right-hand side of (8) indicates the case of lay-off; the second term indicates the case where the current firm has to compete with a more productive firm; the third term indicates the case where the current firm competes with a less productive firm and the current firm has to raise the current wage; the last term indicates all other possibilities.

The value functions of unemployed household with real balance $z$ in Market 1, reads

$$W_{1,u}(z) = (1 - \lambda_h)W_{2,u}(z) + \lambda_h \int_y^0 W_{2,e}[z, p, \phi_0(p)]d\Gamma(p),$$

(9)

where $\phi_0(p)$ is optimal wage offer of a firm with productivity $p$ willing to hire an unemployed worker, i.e. the minimum wage that compensates this worker for the opportunity cost of employment. The second term of the right-hand side of (9) indicates the case of receiving an job offer.

It is convenient to summarize the three markets in a single equation. Substituting $W_{2,e}$ and $W_{2,u}$ from (6) and (7) into (8) and (9) while using the linearity of $W_{3,e}$ and $W_{3,u}$ yields expressions of $W_{1,e}(z, y, w)$ and $W_{1,u}(z)$ in terms of the values functions in Market 3. These expressions of $W_{1,e}(z, y, w)$ and $W_{1,u}(z)$ can then be inserted into (2), thus one gets an equation for the value function of households in Market 3 only,

$$W_{3,e}(z, y, w) = w + \Delta + \frac{\pi M}{p} + z + \max\{-\delta + \beta \hat{\alpha}_h[v(q) - \hat{\rho}d] + \beta \hat{\rho}z\}$$

$$+ \beta \delta W_{3,u}(0) + \beta \hat{\lambda}_h \int_y^p W_{3,e}[0, p, \phi(y, p)]d\Gamma(p)$$

$$+ \beta \hat{\lambda}_h \int_y^y W_{3,e}[0, y, \phi(p, y)]d\Gamma(p)$$

$$+ \beta \{1 - \delta - \hat{\lambda}_h + \hat{\lambda}_h \Gamma(Q(w, y))\}W_{3,e}(0, y, w),$$

(10)

Similar steps are applies to (3) and yield
\[
W_{3,u}(z) = b + \Delta + \frac{\pi M}{p} + z + \max \{-\hat{z} + \beta \hat{\alpha}_h[v(\hat{q}) - \hat{p}\hat{d}] + \hat{\beta}\hat{p}\hat{z}\}
+ \beta(1 - \hat{\lambda}_h)W_{3,u}(0) + \beta \hat{\lambda}_h \int_p^b W_{3,e}[0, p, \phi_q(p)]d\Gamma(p). \tag{11}
\]

This completes the description of the households’ problem.

### 2.3 Firms

We assume that the wage offer is fixed at the end of Market 1, although the wage is paid in Market 3. Therefore, it does not matter whether wages are paid in money or goods. Furthermore, we assume that the unsold goods \( q \) in Market 2 are transformed into \( y - q \) unit goods \( x \) when being taken into Market 3.

#### 2.3.1 Firms’ Profit

It is useful to start with computing \( S(y, w) \), the average periodic profit made by a firm with productivity \( y \) by employing a worker at wage \( w \), in term of goods in Market 3

\[
S(y, w) = \alpha_f(pd + y - q - w) + (1 - \alpha_f)(y - w) \tag{12}
= y + \alpha_f(pd - q) - w. \tag{13}
\]

The first term of the right-hand side of (12) corresponds to the case of being matched in Market 2; whereas the second term corresponds to the case of not being matched in Market 2. Let \( \phi_1(y) \) denote the highest wage that a firm with productivity level \( y \) could offer, i.e., the wage level that makes the firm obtain zero profit from hiring this employee. We have

\[
S[y, \phi_1(y)] = 0.
\]

Combining the above equation with (13) gives us,

\[
\phi_1(y) = y + \alpha_f(pd - q). \tag{14}
\]

If the firm has to offer a wage level higher than \( \phi_1(y) \) to win any competition over an employee, the firm would simply give up hiring this employee.

For any wage level where \( w \leq \phi_1(y) \), let \( L(w \mid y) \) denote the number of employees who are paid wages not higher than \( w \) in a type \( y \) firm. We also define \( L(y) \equiv L[\phi_1(y) \mid y] \) as the total employment in a firm of type \( y \). We also express a firm’s periodic profit, \( \pi(y) \), using the definitions from above,

\[
\pi(y) = \int_{\phi_0(y)}^{\phi_1(y)} S(y, w)dL(w \mid y). \tag{15}
\]

Firms’ profit-maximization decisions have been implicitly expressed as the firms’ wage-setting mechanism introduced later. That is because there is no capacity constraint for firms, i.e., a firm could employ as many employees as it is willing to.
2.3.2 Reservation Wage

We now describe how firms post their wage offers, starting with reservation wages. When a firm meets an unemployed worker, the firm would offer the reservation wage for the unemployed and the unemployed worker would take the job. The wage paid by a firm of type $y$ in order to hire an unemployed worker is the minimum wage $\phi_0(y)$ that compensates this worker for the opportunity cost of employment. $\phi_0(y)$ is then defined by

$$W_{2, e}[z, y, \phi_0(y)] = W_{2, u}(z).$$

Substituting $W_{2, e}$ and $W_{2, u}$ from (6) and (7) into (16) gives us

$$W_{3, e}[\rho z, y, \phi_0(y)] = W_{3, u}(\rho z).$$

Then substituting $W_{3, e}[z, y, \phi_0(y)]$ from the above equation into (11) and using the linearity of $W_{3, u}$ yields an expression for $W_{3, u}(0)$,

$$W_{3, u}(0) = \frac{b + \Omega}{1 - \beta},$$

where

$$\Omega = \Delta + \frac{\pi M}{p} + z + \max\{-\hat{z} + \beta \hat{\alpha} h[v(\hat{q}) - \hat{p} \hat{d}] + \beta \hat{\rho} \hat{z}\}.$$

2.3.3 Wage Offers and Counteroffers

When two firms with same productivity level enter competition for a single worker, the wage will increase until it reaches the highest wage that firms can offer, i.e., the wage that yields zero marginal profit for the firms. Therefore,

$$\phi(y, y) = \phi_1(y),$$

which could also be rewritten as, by definition,

$$Q[\phi_1(y), y] = y.$$

When two firms with different productivity are in competition for a single worker, then the more productive firm will keep/enroll the worker, because the more productive firm can offer a more attractive wage than that which yields zero marginal profit for the less productive firm. Therefore, assuming two firms with different productivity ($y, p$ and $y < p$) compete for a single worker, the worker would feel indifferent about working in the firm with higher productivity at the optimal wage offer in this competition, $\phi(y, p)$, and working in the firm with lower productivity at the highest wage offer $\phi_1(y)$, i.e.,

$$W_{2, e}[z, p, \phi(y, p)] = W_{2, e}[z, y, \phi_1(y)].$$

Substituting $W_{2, e}$ from (6) into (21) gives us

$$W_{3, e}[\rho z, p, \phi(y, p)] = W_{3, e}[\rho z, y, \phi_1(y)].$$
2.3.4 Derive Wage Offer Behavior Equations

Now we can derive 3 functions which can explicitly depict firms’ wage offer posting behavior, namely, $Q(w, y)$, $\phi(y, p)$ and $\phi_0(y)$, in addition to (14) regarding $\phi_1(y)$.

Evaluating (10) at $w = \phi_1(y)$ and using the result of (20) yields

$$W_{3,e}[z, y, \phi_1(y)] = \phi_1(y) + z + \Omega + \beta \delta W_{3,u}(0) + \beta \lambda_h \int_y^p W_{3,e}[0, p, \phi(y, p)]d\Gamma(p) + \beta[1 - \delta - \lambda_h + \lambda_h \Gamma(y)]W_{3,e}[z, y, \phi_1(y)]. \quad (23)$$

Plugging the evaluation of $W_{3,e}[z, p, \phi_1(y)]$ for $z = 0$ from (23) into (10) yields the general expression of $W_{3,e}(0, y, w)$

$$W_{3,e}(0, y, w) = w + \Omega + \beta \delta W_{3,u}(0) + \beta \lambda_h \int_{Q(w, y)}^y \frac{\phi_1(p) + \Omega + \beta \delta W_{3,u}(0)}{1 - \beta(1 - \delta)}d\Gamma(p) + \beta[1 - \delta - \lambda_h + \lambda_h \Gamma(Q(w, y))]W_{3,e}(0, y, w), \quad (24)$$

where $\tilde{\Gamma}(y)$ is defined as $\tilde{\Gamma}(y) = 1 - \Gamma(y)$. By virtue of the definition of $Q(w, y)$ and the evaluation of $W_{3,e}[z, p, \phi_1(y)]$ for $z = 0$ from (23), (6) and (7) give us

$$W_{3,u}(0, y, w) = W_{3,e}\{0, Q(w, y), \phi_1[Q(w, y)]\} = \frac{\phi_1[Q(w, y)] + \Omega + \beta \delta W_{3,u}(0)}{1 - \beta(1 - \delta)}. \quad (25)$$

Comparing (25) and (24) using the definition of $\phi_1(\cdot)$ in (14) yields

$$Q(w, y) = w - \alpha_f(\rho d - q) + \frac{\beta \lambda_h}{1 - \beta(1 - \delta)} \int_{Q(w, y)}^y \tilde{\Gamma}(p)d\Gamma(p). \quad (26)$$

Using the fact $Q[\phi(y, p), p] = y$ and (26), we have the expression of $\phi(y, p)$, when $y < p$,

$$\phi(y, p) = y + \alpha_f(\rho d - q) - \frac{\beta \lambda_h}{1 - \beta(1 - \delta)} \int_y^p \tilde{\Gamma}(x)dx. \quad (27)$$

We now turn to the unemployed workers’ reservation wages $\phi_0(y)$. Applying the evaluation of (25) at $w = \phi_0(y)$ and $W_{3,u}(0)$ from (18) into (17) gives us,

$$b = \phi_1\{Q[\phi_0(y), y]\}. \quad (28)$$

By the definition of $\phi_1$ in (14), (28) also reads as

$$b = Q[\phi_0(y), y] + \alpha_f(\rho d - q). \quad (29)$$
Evaluating (26) at \( w = \phi_0(y) \), gives us

\[
Q[\phi_0(y), y] = \phi_0(y) - \alpha_f (pd - q) + \frac{\beta \lambda_h}{1 - \beta (1 - \delta)} \int_q^y \bar{\Gamma}(p) dp.
\]

(30)

Plugging (29) into (30) and rearranging, we obtain the expression of \( \phi_0(y) \),

\[
\phi_0(y) = b - \frac{\beta \lambda_h}{1 - \beta (1 - \delta)} \int_{b - \alpha_f (pd - q)}^y \bar{\Gamma}(p) dp.
\]

(31)

To summarize, (14), (26), (27) and (31) explicitly describe firms' wage-posting behavior. This completes the description of the firms’ problem.

3 Equilibrium with Exogenous Productivity Distribution

3.1 Goods Market

We first describe the matching process in the decentralized goods market. We assume that the probability for a buyer (a household) to trade in the decentralized goods market \( \alpha_h \) is exogenous and constant\(^7\). Then, the probability for a seller (an employee of a firm) to trade in Market 2, \( \alpha_f \), is determined in equilibrium. The measurement of all employees of firms must equal the measurement of the employed households, \( 1 - u \), where \( u \) is the measurement of the unemployed households, i.e. the unemployment rate. The measurement of employees of firms matched in the decentralized goods market must equal the measurement of households matched in decentralized goods market, which is \( 1 \cdot \alpha_h = \alpha_f (1 - u) \). Therefore, we have

\[
\alpha_f = \frac{\alpha_h}{1 - u}.
\]

(32)

(32) means that the more employees of firms there are in the market, the harder it is for a firm to be matched in the decentralized goods market and to make profit.

We now use the generalized Nash solution\(^8\) to determine the terms of trade \((q, d)\) in Market 2. The buyer has bargaining power \( \theta \) and the threat points are given by continuation values. The surplus for an employed household is

\[
v(q) + W_{3,e}[^{\rho (z - d), y, w}] - W_{3,e}[^{\rho z, y, w}] = v(q) - \rho d,
\]

which is independent of \( y \) and \( w \) and is equal to the surplus for an unemployed household.

\(^{7}\)I make this assumption for simplicity, because it results in a unique equilibrium. Berensten, Menzio and Wright (2008) assume that \( \alpha_h \) depends endogenously on the unemployment rate, which opens the possibility of multiple equilibria. However, Berensten, Menzio and Wright (2008) analyse only the case of a unique equilibrium. Therefore, making \( \alpha_h \) exogenous does not change the analytical conclusions for properties of the equilibrium unemployment rate and the terms of trade.

\(^{8}\)I employ Nash bargaining as the price determination process because it is standard in the search literature (say, Shi (1995,1997), Trejos and Wright (1995), Lagos and Wright (2005)). However, using different price determination mechanisms, like price taking, and price posting with directed search, does not change the conclusions in goods market as Berensten, Menzio and Wright (2008) have shown and in turn does not change my conclusion on labour markets.
\[ v(q) + W_{3,u}[\rho(z - d)] - W_{3,u}(\rho z) = v(q) - \rho d. \]

The surplus for the firm in each match is

\[ \rho d - q, \]

which is also independent of \( y \).\(^9\) Hence, \((q, d)\) solves,

\[
\max_{q, d} [v(q) - \rho d]^{\theta}[\rho d - q]^{1-\theta}
\]

\( \text{s.t. } d \leq z, 0 \leq q. \)

It is obvious that Problem (TT) is independent of the firm’s productivity level and the household’s employment status. Therefore, the problems of the households’ optimal choice for \( \hat{z} \), the real balance taken to the next period, in (10) and (11) are identical and independent of \( z, y, w \) and the households’ employment status. The problem is rewritten as

\[
\max_{\hat{z}} \{-\hat{z} + \beta \hat{\alpha}_h[v(\hat{q}) - \hat{\rho} \hat{d}] + \beta \hat{\rho} \hat{z}\}
\]

We can conclude from problem (RB) that every household, irrespective of \( z \) (its real balance in Market 3) and \( s \) (employment status), will choose the same real balance \( \hat{z} \) to enter into the next period. This conclusion, in turn, implies in that problem (TT) for all the meetings in the decentralized market are identical and the terms of trade \((q, d)\) in the decentralized goods market are all the same.

**Assumption 1** We have \( 1 + \pi > \beta \), i.e. \( \rho < \frac{1}{\beta} \).

Assumption 1 means that the money growth rate is higher than what the Friedman rule would require. To see its application more clearly, we use the Fisher equation, which links the nominal interest rate and inflation in the long run. The Fisher equation reads,

\[ 1 + i = \frac{1 + \pi}{\beta} \]

where \( i \) is the nominal interest rate. Therefore, Assumption 1 simply means that \( i > 0 \), while the Friedman rule requires \( i = 0 \), i.e. \( \pi = \beta - 1 \). Assumption 1 makes sure that problem (RB) has a meaningful solution, as Lagos and Wright (2005) point out. Then, we only consider the equilibrium of the economy in the case \( 1 + \pi > \beta \).

**Proposition 1** Under Assumption 1, in stationary equilibrium \( d = z \) is always satisfied; \( q \) is the solution to

\[ \frac{v'(q)}{g'(q)} = \frac{i}{\alpha_h} + 1 \]

where \( g \) is defined as

\[ g(q) = \frac{\theta qv'(q) + (1 - \theta)v(q)}{\theta v'(q) + 1 - \theta}. \]

\(^9\)There is a feasibility constraint \( q \leq y \). Following Berensten, Menzio and Wright (2008), we assume for simplicity that this condition is always slack.
The household would set its next period real balance at
\[ \hat{z} = \beta (1 + i) g(q), \]  
(36)

Proof: see Appendix.
(34) fully characterizes the equilibrium in a decentralized goods market.

**Assumption 2** Let \( q^* \) solve \( v'(q^*) = 1 \). For all \( q \leq q^* \), we have that \( \frac{v'(q)}{g(q)} \) is strictly decreasing.

The above assumption is imposed simply to make sure that the solution to (34) is unique. Lagos and Wright (2005) also establish some sufficient conditions for Assumption 2, for instance that \( v'(\cdot) \) is log concave. We then have the following proposition for the goods market.

**Proposition 2** (34) is the equilibrium condition in the goods market. With Assumptions 1 and 2, the solution exists and is unique. Furthermore, \( q \) is decreasing in \( i \); the firm’s revenue in Market 2, \( R = \rho d - q = g(q) - q \), which is clearly increasing in \( q \), is also decreasing in \( i \).

Proof: see Appendix.
Because \( R = g(q) - q \) is a function of \( q \), we will later denote \( R \) as \( R(q) \).

### 3.2 Labour Market

We now describe the matching technology in the labour market. In the present model, where on-the-job search is allowed, the matching outcomes depend both on the measurement of the total labour force and the measurement of firms, the job offer suppliers. As we mentioned in section 2.1, the measurement of the total labour force is always 1 and the measurement of firms is exogenously given by \( N \). Therefore, we may assume the arrival rate of offers to all searchers, \( \lambda_h \), to be an exogenous constant in this section. However, we will endogenize \( \lambda_h \) and let it depend on \( N \) later. In steady state, we have that the unemployment rate is constant. Therefore, it is required that new job matches equal total lay-offs, i.e.

\[ u \lambda_h = (1 - u) \delta. \]  
(37)

We then have the following proposition for the labour market.

**Proposition 3** The equilibrium condition in the labour market with exogenous productivity distribution is characterized by equation (37). Furthermore, the equilibrium unemployment rate is constant.

### 3.3 Steady State Equilibrium

Proposition 1 and 2 show that goods market equilibrium is solely defined by \( q \) and the equilibrium condition (34), despite the situation in the labour market. Similarly, Proposition 3 shows that, labour market equilibrium is solely defined by \( u \) and the equilibrium condition (37), independently of the situation in the goods market. This leads us to a very simple definition of equilibrium.

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10The same exercise can also be found in Lagos and Wright (2005) and Rocheteau and Wright (2005).
**Definition 1** A steady state equilibrium of the economy with exogenous productivity distribution is a pair \( \{q, u\} \) that satisfies equilibrium conditions (34) and (37).

**Proposition 4** With Assumption 1 and 2, the steady state equilibrium of the economy with exogenous productivity distribution always exists and is unique. Furthermore, \( q \), the amount of goods traded in the decentralized goods market, is decreasing in interest rate \( i \), while the unemployment rate \( u \) is constant and independent of interest rate \( i \).

Although \( u \) is constant and independent of the interest rate \( i \), the wage distribution depends on the interest rate. In the following subsection, we will first express the wage distribution as a function of the distribution of productivity, and then we will check its dependence on the interest rate.

### 3.4 Wage Distribution

To derive the wage distribution in the economy, we begin with the value of \( L(w \mid y) \), the earning distribution within a firm, as a function of the distribution of productivity, \( \Gamma(\cdot) \).

Consider a set of workers who are paid less than \( w \) by firms with a productivity level \( y \), i.e., \( L(w \mid y)Nd\Gamma(y) \). Workers would leave this set either because they are laid off, which occurs at rate \( \delta \), or because they receive a more attractive offer, which grants them a wage increase. From previous sections we know that only those workers who receive an offer from a firm with productivity not less than \( Q(w, y) \) will either see their wage raised above \( w \), or leave their type \( y \) employer to go to a more productive firm. Such offers occur at rate \( \lambda_h\tilde{\Gamma}(Q(w, y)) \). Therefore, the total measurement of the outflow from this set is \( \{\delta + \lambda_h\tilde{\Gamma}(Q(w, y))\}L(w \mid y)Nd\Gamma(y) \). On the inflow side, workers enter the set either from a firm with productivity less than \( Q(w, y) \) or out of unemployment. Also note that, when \( w \) is too small (\( w \leq \phi(p, y) \)), it can only attract the unemployed. When \( w \) is high enough, it could also attract the workers who work for the firm with productivity lower than \( Q(w, y) \). Then, the number of workers hired by such a firm from firms with productivity less than \( Q(w, y) \) is \( \lambda_h \cdot Nd\Gamma(y) \cdot \int_p^{Q(w,y)} L(p)d\Gamma(p) \), while the inflow of unemployed workers into this category is \( \lambda_h \cdot u \cdot d\Gamma(y) \). Therefore, the stationarity of \( L(w \mid y) \) thus implies, for \( \phi_0(y) \leq w \leq \phi_1(y) \),

\[
\{\delta + \lambda_h\tilde{\Gamma}(Q(w, y))\}L(w \mid y)N = \begin{cases} 
\lambda_hu & \text{if } w \leq \phi(p, y), \\
\lambda_hu + \lambda_hNd\Gamma(y)\int_p^{Q(w,y)} L(p)d\Gamma(p) & \text{if } w > \phi(p, y).
\end{cases}
\]  

(38)

For small values of the wage \( w \), specifically, for \( w \leq \phi(p, y) \), the last integral term vanishes, because \( Q\{\phi(p, y), y\} = p \). Note that only workers just coming out of unemployment will accept offers in this case. Workers who have already experienced at least one period in employment have a reservation wage greater than \( \phi(p, y) \).

Note that to obtain an expression of \( L(w \mid y) \) in the case where \( w > \phi(p, y) \), we first need to determine \( L(y) \). This is done by considering the stock of workers employed at all firms with productivity levels less than \( y \), which equals \( N\int_y^\infty L(p)d\Gamma(p) \). This stock is also depleted at rate \( \delta + \lambda_h\tilde{\Gamma}(y) \), whereas it is fueled by hiring of the unemployed workers. The flow of unemployed workers hired into firms with productivity less than \( y \) is given by \( \lambda_hu\Gamma(y) \). Once again equating inflows and outflows for the stock of workers at hand...
always holds. Also noticing the evaluation of (40) at the proportion of workers earning less than \( y \) to

\[ \frac{\lambda_h u}{N} \frac{\delta + \lambda_h}{(\delta + \lambda_h \Gamma(y))^2} \]

and for \( y < p \),

\[ L(y) = 0. \]

Plugging the evaluation of (39) for \( y = Q(w, y) \) into (38) gives us, for \( \phi_0(y) \leq w \leq \phi_1(y) \),

\[ L(w \mid y) = \left\{ \begin{array}{ll}
\frac{\lambda_h u}{N} \frac{1}{\delta + \lambda_h} & \text{if } w \leq \phi(p, y), \\
\frac{\lambda_h u}{N} \frac{\delta + \lambda_h}{(\delta + \lambda_h \Gamma(Q(w, y)))^2} & \text{if } w > \phi(p, y).
\end{array} \right. \]

Also noticing the evaluation of (40) at \( l[Q(w, y)] \), we have, for \( \phi_0(y) \leq w \leq \phi_1(y) \),

\[ L(w \mid y) = l[Q(w, y)] \]

always holds.

Let \( G(w) \) denote the \( cdf \) of the aggregate earnings distribution. \( G(w) \) hence denotes the proportion of workers earning less than \( w \) in the economy. We have the following proposition for the wage distribution.

**Proposition 5** Given fixed exogenous productivity distribution \( \Gamma(y) \) over \([p, \bar{p}]\), we can state that the wages of all the workers comprise the segment \([\phi_0(p), \phi_1(p)]\). Furthermore,

\[ G(w) = \left\{ \begin{array}{ll}
0 & \text{if } w < \phi_0(\bar{p}) \\
\delta \int_{\phi_0^{-1}(w)}^{\phi_1^{-1}(w)} \frac{1}{(\delta + \lambda_h \Gamma(Q(w, y)))^2} d\Gamma(p) & \text{if } \phi_0(p) \leq w < \phi_0(p) \\
\delta \int_{\phi_0^{-1}(w)}^{\phi_1^{-1}(w)} \frac{1}{(\delta + \lambda_h \Gamma(Q(w, y)))^2} d\Gamma(p) & \text{if } \phi_0(p) \leq w < \phi_1(p) \\
\delta \int_{\phi_0^{-1}(w)}^{\phi_1^{-1}(w)} \frac{\delta + \lambda_h}{(\delta + \lambda_h \Gamma(Q(w, y)))^2} d\Gamma(p) + \delta \int_{\phi_1^{-1}(w)}^{\bar{p}} \frac{\delta + \lambda_h}{(\delta + \lambda_h \Gamma(Q(w, y)))^2} d\Gamma(p) & \text{if } \phi_1(p) \leq w < \phi_1(p) \\
1 & \text{if } \phi_1(p) \leq w
\end{array} \right. \]

where \( \phi_0(y) \) is defined in (31) and \( \phi_0^{-1} \) is its inverse function; \( \phi_1(y) \) is defined in (14) and \( \phi_1^{-1} \) is its inverse function; \( Q(w, y) \) is implicitly defined in (26).

Proof: see Appendix.

We have derived the wage accumulative distribution function \( G(w) \). The probability density function of the wage distribution can easily be found by differentiating \( G(w) \).

We now move to the effect of inflation on wage dispersion. In statistics, wage dispersion could be indicated by the variance of the wage, the 90th to 10th percentile ratio, or even the ratio of the highest wage to the lowest wage. Given the complexity of the wage distribution function \( G(w) \), it is very hard to derive analytically the the effect of inflation.
on the variance of wages. However, assuming that the shape of the wage distribution does not change much when inflation changes, it makes sense to check the effect of inflation on the bounds of wages paid in the economy.\textsuperscript{11} Denote by $w$ and $\bar{w}$ the lowest and the highest wages, respectively, paid in the economy. Proposition 5 shows that $w = \phi_0(\bar{p})$ and $\bar{w} = \phi_1(\bar{p})$.

**Proposition 6** Given fixed exogenous productivity distribution $\Gamma(y)$, it holds that $w$ increases and $\bar{w}$ decreases when the interest rate $i$ increases.

Proof: see Appendix.

In our model, both the highest wage and the lowest wage are paid by the most productive firms. The highest wage decreases when the interest rate $i$ increases. The reason is as follows. We use a micro-founded model of money to make money essential in the economy, therefore the goods market reaches the first optimal when the central bank sets the interest rate equal to zero, i.e., the Friedman rule. When the interest rate increases, there will be more distortion in the goods market, which implies that firms will suffer a loss in profit. We know that the highest wage in the economy is the wage level which makes the most productive firms obtain zero profit, and the profit loss reduces the zero-profit wage level. Therefore, a rise of the inflation rate reduces the highest wage level. The fact that the lowest wage increases when the interest rate $i$ increases, means that the reservation wage for the unemployed increases. This is because the decrease in the highest wage that the firm could pay reduces the prospects of wages rising and makes working opportunities less attractive for the unemployed. As a consequence, the unemployed require a higher reservation wage to compensate them for home production. In fact, the same logic also holds for the wage dispersion within any firm. We affirm that the wage dispersion within a firm becomes smaller when the interest rate becomes higher. Given the exogenous productivity distribution, we reinforce the conclusion, (which is drawn by Proposition 6 from an increasing lower bound and a decreasing upper bound) that higher inflation causes less wage dispersion in general.

Obviously, increasing $w$ and decreasing $\bar{w}$ also imply some other dispersion indications, such as that $\bar{w} - w$ or $\bar{w}/w$, also decrease unambiguously.

When we assume the productivity distribution to be exogenous, a constant unemployment rate is reached. In the next two sections, we will endogenize the productivity distribution to study how inflation affect productivity distribution and wage dispersion. Moreover, as we will see in the following sections, endogenizing the productivity distribution makes the unemployment rate positively dependent on the inflation rate, which is consistent with previous theoretical works about inflation and unemployment in the long run, such as Berentsen, Menzio and Wright (2011).

## 4 Equilibrium with Endogenous Productivity Distribution

We now adopt the idea of Acemoglu and Shimer (2000) of endogenous productivity dispersion, and apply it to our model. More specifically, we assume that a firm’s productivity follows from its investment choices such that the equilibrium productivity dispersion arises

\textsuperscript{11}I leave the analysis of other wage dispersion indicators, such as variance of wage, to future numerical exercises.
from the firms’ dispersed investment choices. Besides that, the optimization decisions of households as well as the wage-setting rules for firms after investment choices are exactly the same as we have depicted in section 2.2 and 2.3. Finally, the equilibrium conditions for the goods market also hold as in section 3.1.

4.1 Investment and Productivity

We assume from now on that the productivity \( y \) of a firm depends on its investment decision, which is made before entering into the labour market, and that the productivity \( y \) is fixed during its operation. More specifically, a firm with productivity \( y \) must pay a periodic cost \( cf(y) \) to maintain its productivity level, where \( c \) is a positive constant and \( f(y) \) is assumed to satisfy: \( f(0) = f'(0) = 0 \) and for all \( y > 0 \), \( f(y) > 0 \); \( f'(y) > 0 \); \( f''(y) > 0 \). The latter property of \( f(\cdot) \) means that the cost of productivity investment is increasing with respect to the productivity level and is also convex; while the former property is imposed to ensure the existence of equilibrium without loss of generality.

Some explanations must be given concerning the above form of the productivity investment cost. First, we assume that the costs are paid periodically instead of being paid once and for all when the firm is entering into the market, following Postel-Vinay and Robin (2002a). The reason is that employee dynamics within a firm are too complicated to trace in both models, hence the profit function of a firm and the free entry condition, which is important to pin down the measurement of firms later, could only be computed and imposed in periodic form in the steady state. Secondly, concerning the nature of such productivity investment cost, Postel-Vinay and Robin (2002a) ascribe it as the rental cost of capital used in the producing activity follows a Cobb-Douglas production function form. Therefore, their periodic productivity investment cost has the form of \( rg^{-1}(y) \), where \( r \) is a constant exogenous interest rate and \( g^{-1}(\cdot) \) is the inverse function of a Cobb-Douglas production function in per capita form. We would like to adopt their explanation of the source of the productivity investment cost, but cannot do so with an endogenous interest rate \( r = \frac{1}{3} \) directly. Instead, we have to exclude capital from our economy because otherwise both capital and fiat money can serve as the medium of exchange in the goods market and then capital holdings, which could influence the terms of trade in the goods market, making the analysis much more complicated\(^{12}\). Therefore, we simply assume that \( cf(y) \) is the cost of using such a production technology and that no private agent benefits from it. Thirdly, the productivity investment cost depends only on the productivity level but not on the scale of labour input of the firm, still following Postel-Vinay and Robin (2002a). Our explanation is that the technology modeled in the present study is merely a constant returns to labour technology. Once a firm gets the technology, it could be applied by any employee within the firm at no cost.

Before moving on, let us derive the expression of \( \pi(y) \). Note first, that the productivity investment decision is made before the firm enters the market. Therefore, our derivation of the wage distribution for a given productivity distribution is still valid for this section, i.e., (38) to (43) in section 3.4 also hold for the present case of an endogenous productivity distribution.

Lemma 1

\[
\pi(y) = \int_{y}^{\bar{y}} l(x)[1 + \frac{\beta\lambda_h \bar{\Gamma}(x)}{1 - \beta(1 - \delta)}]dx
\]

\(^{12}\)See Lagos and Rocheteau (2008) and others for detailed discussion.
Proof: see Appendix.

Given the specification of the productivity investment cost and \( \pi(y) \), we can now begin to derive the distribution of productivity \( \Gamma(y) \). Denote the periodic profit of a firm with productivity investment decision, \( y \), as \( \Pi(y) = \pi(y) - cf(y) \). In equilibrium, all firms must make the same (maximal) profit, say \( \Pi^* \). Thus, the following holds in equilibrium:

\[
\pi(y) - cf(y) = \Pi^*, \quad \text{if } y \in [\underline{p}, \overline{p}], \tag{45}
\]

\[
\pi(y) - cf(y) < \Pi^*, \quad \text{otherwise.} \tag{46}
\]

Since \( \pi(y) - cf(y) \) is a constant over the support \([\underline{p}, \overline{p}]\), and \( \pi(y) \) is differentiable as Lemma 1 shows, it is therefore true that, when \( y \in [\underline{p}, \overline{p}] \),

\[
\pi'(y) - cf'(y) = 0. \tag{47}
\]

Applying the expression of \( \pi(y) \) and \( l(y) \) from (44) and (40) in the above condition implies

\[
\frac{\lambda_h u}{N} \frac{\delta + \lambda_h}{\delta + \lambda_h \Gamma(y)} \left[ 1 + \frac{\beta \lambda_h \Gamma(y)}{1 - \beta(1 - \delta)} \right] = cf'(y). \tag{48}
\]

We need to know \( \lambda_h, u, \) and \( N \) to determine the shape of \( \Gamma(\cdot) \). These variables are discussed in the next subsection. We also need to impose two conditions concerning the definition of \( \Gamma(\cdot) \) to determine the lower and upper bound of \( y \). By definition,

\[
\Gamma(\underline{p}) = 1, \quad \Gamma(\overline{p}) = 0.
\]

When (48) is evaluated at \( y = \underline{p} \) and \( y = \overline{p} \), we get

\[
\frac{\lambda_h u}{N} \frac{1}{\delta + \lambda_h} \left[ 1 + \frac{\beta \lambda_h}{1 - \beta(1 - \delta)} \right] = cf'(\underline{p}), \tag{49}
\]

\[
\frac{\lambda_h u \delta + \lambda_h}{N \delta^2} = cf'(\overline{p}). \tag{50}
\]

### 4.2 Labour Market

We now describe the matching technology in the labour market. In this section, the measurement of firms, \( N \), is endogenously determined in equilibrium. In section 2.1, we assumed that \( \lambda_h \) depends on \( N \). Therefore, from now on, \( \lambda_h \) becomes endogenous. Following Mortensen (2003), we assume\(^{13}\) that,

\[
\lambda_h = \lambda N, \tag{51}
\]

\(^{13}\)In a similar continuous time setting, Mortensen (2003) (Chapter 2.2, page 38-39) claims that the arrival rate of offers for the unemployed follows a Poisson process with an arrival rate approximating to the firm-household ratio (it equals \( N \) in my case), when all the firms post their jobs offers, which are received by a particular worker purely randomly. In the current discrete time setting, I simply set the offer arriving probability being linear to the Poisson process arriving rate, just like other applications of a labour market search and matching model in a discrete time setting, e.g., Shi (1998), Berentsen, Menzio and Wright (2008) and so on.
where $\lambda$ is a positive constant.

In the steady state, the unemployment rate is constant. Therefore, it is required that new job matches equal total lay-offs and (39) still holds here.

The last condition required for labour market clearing is the free entry condition, which pins down the measurement of firms. In the long-run competitive economy, free entry and exit ensure that all the competing firms make zero profit, i.e.,

$$\Pi^* = 0.$$  \hspace{1cm} (52)

We then have the following proposition for the labour market.

**Proposition 7** The equilibrium condition in labour market with endogenous productivity distribution can be characterized by two equations for the unemployment rate $u$ and the lower bound of productivity $p$.

1. 
   $$b - \frac{\alpha_h}{1 - u} R = p - \frac{f(p)}{f'(p)},$$  \hspace{1cm} (53)

2. 
   $$\frac{(1 - \beta)\lambda u^2}{[1 - \beta(1 - \delta)]\delta} + \frac{\beta\lambda u}{1 - \beta(1 - \delta)} = cf'(p).$$  \hspace{1cm} (54)

The other labour market variables can be expressed by $u$ and $p$:

$$N = \frac{\delta}{\lambda} \left( \frac{1}{u} - 1 \right),$$  \hspace{1cm} (55)

$$\lambda_h = \frac{\lambda}{u} - \lambda.$$  \hspace{1cm} (56)

Furthermore, the upper bound of productivity $\bar{p}$ is the unique solution to,

$$\frac{\lambda}{\delta} = cf'(\bar{p}).$$  \hspace{1cm} (57)

Proof: see Appendix.

### 4.3 Steady State Equilibrium

Proposition 1 and 2 show that the goods market equilibrium is solely defined by $q$ and equilibrium condition (34), independently of the situation in the labour market. Firms’ revenue in Market 2, $R(q)$, is the link between the labour market and the goods market. This suggests the following definition.

**Definition 2** A steady state equilibrium of the economy with endogenous productivity distribution is a triple $\{q, u, p\}$ such that i) $\{q, u, p\}$ satisfies equilibrium conditions (34), (53) and (54); ii) (46) holds, i.e., no firm has an incentive to enter into the market with $y \notin [p, \bar{p}]$.

Before moving on to the existence and uniqueness of the equilibrium, we need to prove the following Lemma about the properties of equilibrium conditions in the labour market.

**Lemma 2** Given that $R(q)$ is constant, the locus of points of $(u, p)$ defined by (53) is downward sloping and (53) shifts up if $R(q)$ goes down. The locus defined by (54) slopes upward in $(u, p)$ space and does not shift when $R(q)$ changes.
Proof: see Appendix.

(53) states how the productivity low bound depends on the unemployment rate in the economy and we refer to it as productivity threshold condition (PT). Higher unemployment implies fewer firms and hence each firm is more likely to be matched in the goods market. Therefore, higher unemployment makes the less productive firm more likely to be profitable following each successful hire. Thus, the PT curve is downward sloping in the \((u, p)\) space. (54) describes how the unemployment rate determines the firms’ profits, represented by the investment input of the least productive firm (recall that every firm in the economy has the same net profit). We refer to (54) as the firms’ iso-profit condition (IP). The higher unemployment implies it is easier for each firm to hire and thus more profitable for each job posted. In turn, firms can choose higher productivity although the investment cost is higher. Thus, the IP curve is upward sloping in the \((u, p)\) space.

Since \(q\) is uniquely determined by (34), and is independent of the situation in the labour market, the above lemma implies that \(u\) and \(p\) are also uniquely determined by (53) and (54). Therefore, to establish the existence and uniqueness of the equilibrium, we only need to consider stability condition (46) to ensure that no firm would unilaterally deviate from the productivity investment strategy \(y \in [p,p]\). To this end, we establish the following lemma.

**Lemma 3** Given the labour market equilibrium conditions hold, (46) always holds.

Proof: see Appendix.

In the foregoing, we have established the existence and uniqueness of the steady state equilibrium. Now we study the properties of equilibrium when the interest rate changes. The effect of interest rate changes on \(q\) is clear as we proved in Proposition 2, \(q\) and \(R(q)\) becoming smaller as \(i\) rise. By virtue of Lemma 3, we can draw the curves of (53) and (54) in a \((u,p)\) space.

![Graph 1: joint determination of unemployment and productivity lower bound by productivity threshold condition (PT) and iso-profit condition (IP).](image)

It has been shown in Lemma 3 that when \(R(q)\) becomes smaller because the increase in \(i\), curve (54) does not move and curve (53) goes up. We then conclude that \(u\) becomes larger, i.e. \(\frac{\partial u}{\partial i} > 0\), and \(p\) becomes larger, i.e. \(\frac{\partial p}{\partial i} > 0\).
So with an endogenous productivity distribution, we confirm the conclusion in Berentsen, Menzio and Wright (2011), namely that a higher interest rate or a higher inflation rate leads to a higher unemployment rate in the long run; the Phillips Curve slopes upward.

To summarize, we have established the following results.

**Proposition 8** With Assumption 1 and 2, the steady state equilibrium always exist, and is unique. Furthermore, \( q \), the amount of goods traded in the decentralized goods market, is decreasing in the interest rate \( i \), while the lower bound of productivity \( \bar{p} \) and the unemployment rate \( u \) are increasing in \( i \), in the steady state equilibrium.

5 Technology, Wage Dispersion and Inflation

We now study the effect of inflation on technology dispersion and its further effect on wage dispersion. As in section 3.4, we only check the lower and upper bounds of productivity and wage distributions, while other indicators of dispersion are left to future numerical exercises.

**Proposition 9** In the steady state equilibrium with endogenous productivity distribution, a rise of inflation makes firms’ productivity less dispersed in the sense that the lower bound of productivity \( \bar{p} \) rises while the upper bound of productivity \( \bar{p} \) is unchanged.

The above proposition follows directly from Proposition 8 and the fact that no other endogenous variables appear in equation (57) which determines the upper bound of productivity.

The intuition for Proposition 9 is fairly clear. When the interest rate increases, it becomes more costly for the buyers to hold money. Therefore, it is hard for the firms to sell products, and a rise of inflation makes all the firms less profitable in the decentralized goods market. Thus, the firms with the lowest productivity are driven out of the economy as they make negative profits now, and only the firms with higher productivity are left. Furthermore, the highest productivity depends on the investment cost function and is thus independent of the inflation rate and the profit earned in the decentralized goods market.

**Proposition 10** In the steady state equilibrium with endogenous productivity distribution, a rise of inflation diminishes the wage dispersion in the sense that the lower bound of wages rises while the upper bound of wages drops.

Proof: see Appendix.

We find that there are three effects of higher inflation on the equilibrium wage dispersion, which are represented by three different terms which appear in the proof of Proposition 10. First of all, the rise in inflation reduces the revenue of firms from the goods market and therefore reduces the highest (and also higher) wages. This effect also exists in the case of an exogenous productivity distribution and thus higher inflation diminishes the wage dispersion for this very same reason. Second, the reduction of highest/higher wages means that there is less room for wage rises within the firm, i.e., it becomes less attractive to work for the firms than to be unemployed. So the reservation wages to compensate their home products, the unemployment benefit, go up. We could see reservation wages rise as a result (represented by the second term in (a.36)).
For the same reason, higher inflation diminishes the wage dispersion in the case of an exogenous productivity distribution. Third, the rise in inflation raises the lower bound of productivity and the gap between the highest productivity and lowest productivity is narrowed, as shown in Proposition 9. Therefore, it becomes harder for the firms with the highest productivity to enroll new workers compared to the situation when inflation was low. The reservation wages increase also due to this reason, (represented by the first term in (a.36)). However, this effect does not exist in the case of an exogenous productivity distribution.

6 Conclusion

We present a tractable model designed to study the effects of inflation on wage dispersion due to both productivity differences and on-the-job searches. The model contains explicit micro-foundations for money demand and for unemployment. We first study the case of an exogenously given productivity distribution and then the case where the productivity distribution is determined endogenously. For an exogenous productivity distribution, households reduce their money holding in the decentralized goods market when inflation increases, which in turn reduces firms’ selling and profits in this market. Higher inflation also brings down the upper wage bound as the most productive firms have lower profits and pay less to keep its employees. Moreover, the decrease in the highest wage that the firm could pay reduces the prospects of wages rising for workers, and then makes the unemployed require an increase in the lower bound of the wages. To summarize, wages become less dispersed when inflation rises. In the case of an endogenous productivity distribution, the rise of inflation reduces firms’ profits, which drives the less productive firms out of the economy. This is again because an increase in inflation decreases the demand faced by the firms and in turn their profits. Hence, our model predicts a rise of the lower bound of productivity. The highest productivity in the economy is determined by the technology of productivity investment which cannot remain unchanged by inflation. Furthermore, the rise in inflation also diminishes the wage dispersion.

We believe that it is desirable and important to search for empirical evidence of the link between inflation and productivity dispersion on the one hand and wage dispersion on the other. To the best of our knowledge, no such study has been undertaken. However, this task is beyond the focus of the current paper and we aim to pursue this idea in subsequent research.

Appendix: Proofs

Proof of Proposition 1

Proof. If the constraint $d \leq z$ does not bind, the first order condition for problem (TT) reads, with the definition of $q^*$,

\begin{align}
q &= q^* \quad \text{(a.1)} \\
\frac{d}{z} &= (1 - \theta) v(q^*) + \theta q^* \quad \text{(a.2)}
\end{align}
If the constrain \( d \leq z \) does bind, problem (TT) becomes

\[
\max_{q,d} [v(q) - \rho z]^\theta [\rho z - q]^{1-\theta}.
\]

The first order condition for above problem reads,

\[
\rho z = \frac{\theta q v'(q) + (1-\theta)v(q)}{\theta v'(q) + 1 - \theta} = g(q). \tag{a.3}
\]

(a.1) to (a.3) imply that

if \( z > z^* \), \( \frac{\partial q}{\partial z} = \frac{\partial d}{\partial z} = 0 \) \tag{a.4}

if \( z < z^* \), \( \frac{\partial q}{\partial z} = \frac{\rho}{g'(q)}, \frac{\partial d}{\partial z} = 1 \) \tag{a.5}

Problem (RB) of the last period reads,

\[
\max_z \{\beta \alpha_h v(q) - \rho d\} + (\beta \rho - 1)z.
\]

Denote the objective function as \( O(z) \equiv \beta \alpha_h [v(q(z)) - \rho d] + (\beta \rho - 1)z \).

(a.4) implies

if \( z > z^* \), \( O'(z) = \beta \rho - 1 < 0 \), \tag{a.6}

which means that the object function is decreasing in \( z \) for all \( z > z^* \). (a.5) implies

if \( z < z^* \), \( O'(z) = \beta \alpha_h \rho \frac{v'(q)}{g'(q)} - \beta \alpha_h \rho + \beta \rho - 1 \). \tag{a.7}

Furthermore,

\[
\frac{v'(q)}{g'(q)} - \frac{v'(q)[\theta v'(q) + 1 - \theta]^2}{v'(q)[\theta v'(q) + 1 - \theta] - \theta(1-\theta)[v(q) - q]v''(q)} \tag{a.8}
\]

\[
\lim_{z \to z^*} \frac{v'(q)}{g'(q)} = \frac{1}{1 - \theta(1-\theta)[v(q^*) - q^*]v''(q^*)} \leq 1. \tag{a.9}
\]

The inequality is strict except for \( \theta = 1 \); (a.9) uses the facts that \( v'' < 0 \) and \( v(q^*) > q^* \) (this is true because firstly \( v'' < 0 \) and \( v'(q^*) = 1 \) imply \( v'(q) > 1 \) whenever \( q < q^* \); secondly \( v(0) = 0 \) and above results imply \( v(q^*) \equiv \int_0^{q^*} v'(q) dq > \int_0^{q^*} 1 dq \equiv q^* \)). Therefore,

\[
\lim_{z \to z^*} O'(z) = \beta \alpha_h \rho \left( \lim_{z \to z^*} \frac{v'(q)}{g'(q)} - 1 \right) + \beta \rho - 1 < 0. \tag{a.10}
\]

(a.6) and (a.10) imply that the optimal \( z \) is reached when \( z < z^* \) and satisfies

\[
O'(z) = \beta \alpha_h \rho \frac{v'(q)}{g'(q)} - \beta \alpha_h \rho + \beta \rho - 1 = 0. \tag{a.11}
\]

The real balance choice satisfies (a.3), and \( d = z \) always holds. This is a natural conclusion because it is costly to carry cash when the Friedman rule does not hold.

Substituting \( \rho \) from \( \rho = \frac{1}{1+\theta} \) and the Fisher equation (33) into (a.11) and rearranging, we get (34). Similarly, substituting \( \rho \) into (a.3) and using the stationary condition (\( z = \dot{z} \)
for there is no economic growth in the present model, we get (36). ■

**Proof of Proposition 2**

**Proof.** Given that \( \frac{v'(q)}{g'(q)} \) is strictly decreasing for all \( q \leq q^* \) and \( \lim_{q \to 0} \frac{v'(q)}{g'(q)} = +\infty \), it is obvious that the solution of \( q \) in (34) exist and is unique. Furthermore, \( q \) is decreasing in \( i \), i.e., in equilibrium, \( \frac{\partial q}{\partial i} < 0 \). (34) implies that in equilibrium \( \frac{v'(q)}{g'(q)} \geq 1 \). (a.6) implies that \( \frac{v'(q)}{g'(q)} \) is strictly decreasing, in equilibrium, \( q \leq q^* \) must hold.

To see that \( R \) is also decreasing in \( i \), we have, by the virtue of Proposition 1,

\[
R = \rho d - q = g(q) - q = \frac{(1 - \theta)v(q) - q}{\theta v'(q) + 1 - \theta}.
\]

The computation then shows that

\[
\frac{\partial R}{\partial i} = \frac{\partial R}{\partial q} \frac{\partial q}{\partial i} = (1 - \theta) \frac{v'(q) - 1}{\theta v'(q) + 1 - \theta} - \theta v(q) - q v''(q) \frac{\partial q}{\partial i}.
\]

Note that in equilibrium, \( v'(q) \geq 1, v''(q) < 0 \) and \( v(q) > q \), thus \( \frac{\partial R}{\partial i} \leq 0 \), must always hold and the equality holds only when \( \theta = 1 \). ■

**Proof of Proposition 5**

**Proof.** We first look at the support of \( G(w) \). (31) implies that the slope of \( \phi_0(y) \) is

\[
\frac{d[\phi_0(y)]}{dy} = -\frac{\Gamma(y)\beta \lambda_h}{1 - \beta(1 - \delta)} \leq 0,
\]

which implies that the lowest wage which workers get to exit unemployment is \( \phi_0(p) \). Similarly, (14) implies that the slope of \( \phi_1(y) \) is

\[
\frac{d[\phi_1(y)]}{dy} = 1 > 0,
\]

which implies that the highest possible wage is \( \phi_1(p) \). The support of \( G(w) \) is the interval \([\phi_0(p), \phi_1(p)]\).

To determine \( G(w) \), we now count the measurement of workers earning less than \( w \) in any firm, and then integrate over the relevant set of firms. Therefore, we are going to use the expression of \( L(w \mid y) \) and \( L(y) \) when deriving \( G(w) \). Note that the functional forms of \( L(w \mid y) \) and \( L(y) \) we derived in section 3.4 are restricted in their meaningful segments. So we have to partition the support of \( \Gamma(y) \) to get an accurate expression of \( G(w) \).

**Case I:** \( \phi_0(p) \leq w < \phi_0(p) \). Here the wage is so low that the least productive firms are hardly attractive even to unemployed workers. Following the definition, firms with productivity greater than \( \phi_0^{-1}(w) \) can hire workers for less than \( w \). And with \( w \) in the range \( \phi_0(p) \leq w < \phi_0(p), \phi_0^{-1}(w) > p \), so not all firms will actually be able to have
employees paid less than \( w \). Accordingly, \( G(w) \) is given by,

\[
(1 - u)G(w) = N \int_{\phi_0^{-1}(w)}^{\bar{p}} L(w \mid p) d\Gamma(p). \tag{a.12}
\]

Then we substitute \( L(w \mid p) \) from (42) into (a.12) and use the expression of \( L(\cdot) \) in (40) to get

\[
G(w) = \frac{\lambda_h u}{1 - u} \int_{\phi_0^{-1}(w)}^{\bar{p}} \frac{\delta + \lambda_h}{\{\delta + \lambda_h \Gamma(Q(w, p))\}^2} d\Gamma(p). \tag{a.13}
\]

Then, substituting \( u \) from the labour market steady state condition (37) into (a.13) yields the equality in the second line of (43). Note that the continuity of \( G(w) \) at \( w = \phi_0(\bar{p}) \), is ensured by \( \phi_0^{-1}[\phi_0(\bar{p})] = \bar{p} \).

Case II: \( \phi_0(p) \leq w < \phi_1(p) \). In this case, all firms are productive enough to attract at least some workers by an offer of \( w \). \( G(w) \) is thus simply given by

\[
(1 - u)G(w) = N \int_{v}^{\bar{p}} L(w \mid p) d\Gamma(p). \tag{a.14}
\]

Then we substitute \( L(w \mid p) \) from (42) into (a.14) and use the expression of \( L(\cdot) \) in (40) to get

\[
G(w) = \frac{\lambda_h u}{1 - u} \int_{v}^{\bar{p}} \frac{\delta + \lambda_h}{\{\delta + \lambda_h \Gamma(Q(w, p))\}^2} d\Gamma(p). \tag{a.15}
\]

Similarly, substituting \( u \) from the labour market steady state condition (37) into (a.15) gets the equality in the third line of (43). Note that the continuity of \( G(w) \) at \( w = \phi_0(p) \), is ensured by \( \phi_0^{-1}[\phi_0(p)] = p \).

Case III: \( \phi_1(p) \leq w < \phi_1(\bar{p}) \). In this case, all firms have employees paid less than \( w \), but only those more productive than \( \phi_1(w) \) also have employees paid more than \( w \). We thus have to distinguish between those two categories of firms to define \( G(w) \):

\[
(1 - u)G(w) = N \int_{p}^{\phi_1^{-1}(w)} L(p) d\Gamma(p) + N \int_{\phi_1^{-1}(w)}^{\phi_0^{-1}(w)} L(w \mid p) d\Gamma(p). \tag{a.16}
\]

Then we substitute \( L(w \mid p) \) from (42) into (a.16) and use the expression of \( L(\cdot) \) in (40) to get

\[
G(w) = \frac{\lambda_h u}{1 - u} \int_{p}^{\phi_1^{-1}(w)} \frac{\delta + \lambda_h}{\{\delta + \lambda_h \Gamma(p)\}^2} d\Gamma(p) + \int_{\phi_1^{-1}(w)}^{\phi_0^{-1}(w)} \frac{\delta + \lambda_h}{\{\delta + \lambda_h \Gamma(Q(w, p))\}^2} d\Gamma(p). \tag{a.17}
\]

Similarly, substituting \( u \) from the labour market steady state condition (37) into (a.17) gets the equality in the fourth line of (43).

**Proof of Proposition 6**

**Proof.** It is obvious that (31) implies

\[
\frac{\partial w}{\partial i} = \frac{\partial \phi_0(\bar{p})}{\partial i} = \frac{\partial \phi_0(p)}{\partial R} \frac{\partial R}{\partial i} = \frac{\beta \lambda_h \bar{\Gamma}[b - \alpha_f(\rho d - q)]}{1 - \beta(1 - \delta)} \cdot (-\alpha_f) \frac{\partial R}{\partial i} > 0,
\]

25
where the last inequality uses $\tilde{\Gamma}(\cdot) > 0$ and the conclusion in Proposition 2 that $\frac{\partial R}{\partial i} < 0$.

Similarly, (14) implies that

$$\frac{\partial \bar{w}}{\partial i} = \frac{\partial \phi_1(\bar{p})}{\partial i} = \frac{\partial \phi_1(p)}{\partial R} \frac{\partial R}{\partial i} = \alpha_f \frac{\partial R}{\partial i} < 0.$$  

\[\Box\]

**Proof of Lemma 1**

**Proof.** Given $y$, define $\Lambda_y(\cdot)$ as $\Lambda_y[Q(w, y)] = w$. Then,

$$\frac{d\Lambda_y(x)}{dx}|_{x=Q(w, y)} = \frac{d\Lambda_y[Q(w, y)]}{dw} \frac{\partial w}{\partial Q(w, y)} = 1 : \left(\frac{\partial Q(w, y)}{\partial w}\right)^{-1}. \tag{a.18}$$

Differentiating both side of (26) with respect to $w$ gives us

$$\frac{\partial Q(w, y)}{\partial w} = 1 + \frac{\beta \lambda_h}{1 - \beta (1 - \delta)} (-1) \tilde{\Gamma}[Q(w, y)] \frac{\partial Q(w, y)}{\partial w},$$

which could be rearranged to get

$$\frac{\partial Q(w, y)}{\partial w} = \left[1 + \frac{\beta \lambda_h \tilde{\Gamma}[Q(w, y)]}{1 - \beta (1 - \delta)}\right]^{-1}. \tag{a.19}$$

(a.18) and (a.19) then implies

$$\frac{d\Lambda_y(x)}{dx}|_{x=Q(w, y)} = \frac{d\Lambda_y[Q(w, y)]}{dQ(w, y)} = 1 + \frac{\beta \lambda_h \tilde{\Gamma}[Q(w, y)]}{1 - \beta (1 - \delta)}. \tag{a.20}$$

Henceforth, (15), the definition of $\pi(y)$, and (13) imply,

$$\pi(y) = \int_{\phi_0(y)}^{\phi_1(y)} S(y, w) dL(w \mid y) = \int_{\phi_0(y)}^{\phi_1(y)} (y + \alpha_f R - w) dL(w \mid y)$$

$$= (y + \alpha_f R - w) L(w \mid y)|_{\phi_1(y)}^{\phi_1(y)} - \int_{\phi_0(y)}^{\phi_1(y)} L(w \mid y) d(y + \alpha_f R - w)$$

$$= \int_{\phi_0(y)}^{\phi_1(y)} L(w \mid y) dw \tag{a.21}$$

With the the definition of $\Lambda_y(\cdot)$, equation (42) shows that (a.21) is equivalent to

$$\pi(y) = \int_{\phi_0(y)}^{\phi_1(y)} l[Q(w, y)] dw = \int_{\phi_0(y)}^{\phi_1(y)} l[Q(w, y)] d\Lambda_y[Q(w, y)]$$

Now changing $Q(w, y)$ into $x$ in the integral in (g.4) yields (44).  

\[\Box\]
Proof of Proposition 7

**Proof.** (52) holds for all \( y \in [\bar{p}, \bar{p}] \). Therefore,

\[
\pi(p) - cf(p) = 0. \tag{a.22}
\]

Plugging \( \pi(\cdot) \) from (44) into (a.22) gives us,

\[
\int_{b-\alpha_j R}^{b} l(x)[1 + \frac{\beta \lambda_h \bar{\Gamma}(x)}{1 - \beta(1 - \delta)}]dx = cf(p). \tag{a.23}
\]

Then with the expression of \( l(\cdot) \) from (40), (a.23) becomes

\[
\int_{b-\alpha_j R}^{b} \frac{\lambda_h u}{N} \frac{\delta + \lambda_h}{\delta + \lambda_h \Gamma(x)} [1 + \frac{\beta \lambda_h \bar{\Gamma}(x)}{1 - \beta(1 - \delta)}]dx = cf(p). \tag{a.24}
\]

We know that \( \bar{\Gamma}(x) = 1 \), when \( x \leq \bar{p} \). Then simplifying the integrate in (a.24) yields

\[
\frac{\lambda_h u}{N} \frac{1}{\delta + \lambda_h} [1 + \frac{\beta \lambda_h}{1 - \beta(1 - \delta)}](p - b + \alpha_j R) = cf(p). \tag{a.25}
\]

Plugging \( \lambda_h \) from (51) into (37) gives us (55). Then plugging \( N \) from (55) into (51) gets (56). Now dividing (a.25) by (49) gives us

\[
p - b + \alpha_j R = \frac{f(p)}{f'(p)} \tag{a.26}
\]

Substituting \( \alpha_j \) from (32) into (a.26) and rearranging gives us (53). Substituting \( N \) and \( \lambda_h \) from (55) and (56) into (49) and rearranging gives us (54). Substituting \( N \) and \( \lambda_h \) from (55) and (56) into (50) and rearranging gives us (57). Furthermore, given the property of \( f(\cdot) \) that \( f'(0) = 0 \) and \( f'' > 0 \), there is a unique positive solution of \( \bar{p} \) for (57).

Proof of Lemma 2

**Proof.** It is obvious that the left-hand side of (53) is a decreasing function of \( u \) and independent of \( p \), for any fixed value of \( R \). Taking the derivative of right-hand side of (53) with respect of \( p \) yields that

\[
\frac{d[p - \frac{f(p)}{f'(p)}]}{dp} = 1 - \frac{f'(p)f''(p) - f(p)f''(p)}{[f'(p)]^2} = \frac{f(p)f''(p)}{[f'(p)]^2} > 0. \tag{a.27}
\]

Consequently, (53) slopes downward in \((u, p)\) space. Furthermore, the left-hand side of (53) is a decreasing function of \( R \) and independent of \( p \). This implies that (53) shifts up if \( R(q) \) goes down.

It is obvious that the left-hand side of (54) is an increasing function of \( u \) and independent of \( p \); while the right-hand side of (54) is an increasing function of \( p \) and independent of \( u \). Therefore, (54) slopes upward in \((u, p)\) space.
Proof of Lemma 3

Proof. We first consider the case where the productivity investment decision yields \( y > \bar{p} \). Suppose there is a firm with \( y > \bar{p} \) in the economy, then its labour force situation is as good as the firm with productivity \( \bar{p} \), in terms of employees’ measurement and their wage distribution. The firm’s profit can be expressed as

\[
\Pi(y) = \pi(y) - cf(y) = \pi(\bar{p}) + L(\bar{p})(y - \bar{p}) - cf(y).
\]

Computation then shows that

\[
\Pi'(y) = L(\bar{p}) - cf'(y),
\]
\[
\Pi''(y) = -cf''(y) < 0,
\]
\[
\Pi'(\bar{p}) = 0.
\]

So when \( y > \bar{p} \), it holds that \( \Pi'(y) < 0 \), which means that when \( y > \bar{p} \), \( \Pi(y) < \Pi(\bar{p}) = 0 \).

We now consider a productivity investment decision yielding \( y < \bar{p} \). Suppose there is a firm with \( y < \bar{p} \) in the economy. We claim that for \( y < \bar{p} \),

\[
\Pi(y) < \pi(y) - cf(y) \equiv \Xi(y),
\]

where we denote by \( \Xi(y) \) the profit of firms with productivity \( y \) in a economy with exogenous continuous productivity distribution over \( [y, \bar{p}] \). That is because the firm is the least attractive firm in the economy for workers and then its profit \( \pi(y) \) (without considering the productivity investment cost) is smaller than that of the firms of \( y \) in a economy with exogenous continuous productivity distribution over \( [y, \bar{p}] \). It is obvious that \( \Xi(\bar{p}) = \Pi(\bar{p}) \). Plugging the expression of \( \pi(y) \) gives us,

\[
\Xi(y) = \int_{b - \alpha f R}^{y} \frac{\lambda_h u}{N} \frac{\delta + \lambda_h}{[\beta + \lambda_h \Gamma(y)]^2} \left[ 1 + \frac{\beta \lambda_h \Gamma(x)}{1 - \beta (1 - \delta)} \right] dx - cf(y)
\]
\[
= \frac{\lambda_h u}{N} \left[ 1 + \frac{\beta \lambda_h}{1 - \beta (1 - \delta)} \right] \int_{b - \alpha f R}^{y} 1 dx - cf(y),
\]

(a.29)

Note that being less productive than \( b - \alpha f R \) means being unable to attract any worker and thus cannot be optimal. We thus focus on values of \( y \geq b - \alpha f R \). \( \Xi(y) \) is continuously differentiable, and such that

\[
\Xi'(y) = \frac{\lambda_h u}{N} \left[ 1 + \frac{\beta \lambda_h}{1 - \beta (1 - \delta)} \right] cf'(y),
\]
\[
\Xi'(\bar{p}) = \frac{\lambda_h u}{N} \left[ 1 + \frac{\beta \lambda_h}{1 - \beta (1 - \delta)} \right] cf'(\bar{p}) = 0,
\]
\[
\Xi''(y) = -cf''(y) < 0.
\]

So when \( y < \bar{p} \), \( \Xi'(y) > 0 \), which means that when \( y < \bar{p} \)

\[
\Xi(y) < \Xi(\bar{p}) = \Pi(\bar{p}) = 0.
\]

(a.30)

Combining (a.28) and (a.30), we get for \( y < \bar{p} \) that \( \Pi(y) < 0. \) ■
Proof of Proposition 10

Proof. Firstly note that \( w = \phi_0(\bar{p}) \) and \( \bar{w} = \phi_1(\bar{p}) \). Rewriting (31) and (14) down and substituting \( \alpha_f \) and \( \lambda_h \) from (32) and (51) yield,

\[
w = \phi_0(\bar{p}) = b - \frac{\beta \lambda N}{1 - \beta(1 - \delta)} \int_{\bar{p}}^{\bar{p}} \bar{\Gamma}(p) dp,
\]

\[
\bar{w} = \phi_1(\bar{p}) = \bar{p} + \frac{\alpha_h R}{1 - u},
\]

where \( u, p, N \) and \( \bar{\Gamma}(\cdot) \) are determined by (53), (54), (55), and (48).

We first determine \( \frac{\partial \bar{w}}{\partial i} \). The derivative of (a.32) with respect to \( i \) reads \( \frac{\partial \bar{w}}{\partial i} = \frac{\partial (\alpha_h R)}{\partial i} \). We know that \( \frac{\partial \bar{w}}{\partial i} > 0 \) from Proposition 8 and \( [p - \frac{f(p)}{f'(p)}]' > 0 \) from (a.27), then (53) implies that \( \frac{\partial (\alpha_h R)}{\partial i} < 0 \). Therefore, \( \frac{\partial \bar{w}}{\partial i} < 0 \) i.e., the upper bound of wages \( w \) drops when the inflation rises.

We then determine \( \frac{\partial w}{\partial i} \). Note that \( \frac{\partial \bar{w}}{\partial i} \) is always continuously differentiable within the integration interval of (a.31). Define \( \phi = \frac{\beta \lambda}{1 - \beta(1 - \delta)} \) and rewrite (a.31) as

\[
w = b - \phi \left[ \int_{\bar{p}}^{\bar{p}} N \bar{\Gamma}(p) dp + N(p - b + \frac{\alpha_h R}{1 - u}) \right]
\]

The derivative of (a.33) with respect to \( i \) reads

\[
\frac{\partial w}{\partial i} = -\phi \left[ \int_{\bar{p}}^{\bar{p}} \frac{\partial N \bar{\Gamma}(p)}{\partial i} dp + \frac{\partial N}{\partial i} (p - b + \frac{\alpha_h R}{1 - u}) + \frac{\partial N}{\partial i} \right].
\]

When \( y \in [p, \bar{p}], \bar{\Gamma}(y) \) is determined in (48). Solving \( u \) out of (37) and then substituting \( u \) into (48) while expressing \( \lambda_h \) with \( \lambda N \) yields

\[
\frac{\lambda \delta}{[\delta + \lambda N \bar{\Gamma}(y)]^2} \left[ 1 + \frac{\beta \lambda N \bar{\Gamma}(y)}{1 - \beta(1 - \delta)} \right] = cf'(y).
\]

Observation of (a.35) implies that \( N \bar{\Gamma}(y) \) is solely determined by (a.35) for any given \( y \in [p, \bar{p}], i.e., \frac{\partial N \bar{\Gamma}(y)}{\partial i} = 0 \). Thus, (a.34) implies

\[
\frac{\partial w}{\partial i} = -\phi \frac{\partial N}{\partial i} (p - b + \frac{\alpha_h R}{1 - u}) - \beta N \frac{\partial (\alpha_h R)}{\partial i}.
\]

The conclusion of \( \frac{\partial w}{\partial i} > 0 \) from Proposition 8 and (55) implies that

\[
\frac{\partial N}{\partial i} < 0.
\]

(a.26) implies that

\[
p - b + \frac{\alpha_h R}{1 - u} > 0
\]

always hold. From (a.36) to (a.38), it follows that \( \frac{\partial \bar{w}}{\partial i} > 0 \), i.e., the lower bound of wage
w rises when the inflation rises.

References


