Relevant Logic

A Philosophical Examination of Inference

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To my wife, Gill,
without whose encouragement and support
it could not have been completed

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Introduction

The logician's concern is with validity, with the relation of consequence between premises and conclusion. In order to justify an assertion, we may adduce other statements, from which we claim the assertion follows. But what is the criterion by which to decide if the conclusion really does follow? The question has two aspects: concretely, to decide in particular cases whether the conclusion follows from the premises—in technical language, whether a consequence relation holds; and abstractly, to understand in general what the relation between premises and conclusion in a valid argument is.

The purpose of this book is to explore and defend a particular answer to this abstract question, an answer which will be characterised in chapter 6 as the Relevant Account of Validity. Relevant logic evolved as a general framework of logical investigation in the early 1960s, out of work by Alan Anderson and Nuel Belnap, themselves extending ideas of Wilhelm Ackermann, Alonzo Church and others (see chapter 6 below). In a valid argument, the premises must be relevant to the conclusion. My aim in this book is to show why this must be so, and what the content of this claim is.

Interest in relevant logic has grown enormously in the last decade, following publication of the first volume of Anderson and Belnap's encyclopedic work *Entailment: the logic of relevance and necessity.*[1] Nonetheless, interested readers will experience some difficulty in finding out what relevant logic involves. *Entailment* itself is a large work containing a wealth of detail. Yet, for example, it includes nothing on the possible worlds semantics (see chapter 6 below) for relevant logic developed in the early 1970s, that topic being reserved for the second volume, whose publication has been badly, and sadly, delayed. Details of the semantics can certainly be unearthed from *Relevant Logics and their rivals* by Richard Routley and others,[2] but

[1] A.Anderson and N.Belnap, *Entailment*, vol.I (Princeton, 1975).

[2] F.R. Routley et al., *Relevant Logics and their rivals* (Atascadero, Calif., 1982).

many readers, particularly those not already well-versed in the subject, will find that work an uncompromising and difficult one. By far the best introduction so far to the subject is Michael Dunn's survey article, 'Relevant Logic and Entailment'.[3]

The present book does not attempt to compete with Dunn's survey. I have set myself two aims here. The first is to lay forth what I believe are the correct philosophical reasons for rejecting classical logic and adopting a relevant logic as a correct description of the basis of inference. These are not, in general, the reasons which led historically to the development of the subject, and are not those emphasised in the writings of Anderson, Belnap and Dunn. In many ways relevance as a core notion is displaced in the present framework by direct consideration of inference itself. Secondly, I have tried to keep the work intelligible to anyone who has some training in logic and in philosophical analysis, but no more than the average second or third year undergraduate with an introductory course on formal and philosophical logic behind them. Anyone who has read and understood Lemmon's *Beginning Logic* or Guttenplan's *The Languages of Logic* should have no difficulty with the technical material in chapters 4 and 5. The most difficult material, on the completeness proof, I have relegated to an appendix to chapter 5. Later chapters do not presuppose that readers have mastered the material in this appendix. I hope, however, that chapters 4 and 5 themselves do not contain any gratuitous technicality. One cannot reason clearly and correctly about logic without proper acquaintance with the technical development of the subject. Equally, neither can one develop the formal material usefully without embedding it in the appropriate philosophical analysis.

I start in chapter 1 by looking at the general nature of theories, and I approach this topic by considering the notion of knowledge. Does knowledge require an infallible and demonstrably certain proof, as is found only in logic and mathematics? I show that this conception of knowledge is mistaken. It follows that not only is scepticism refuted, but the true nature of all theories, in science, mathematics and logic, is one of essentially fallible explanatory models and generalisations. These can both be justifiably asserted on the basis of appropriate grounds, and equally rejected when countervailing evidence is discovered.

What then is the correct model and theory of validity? It is a plausible thought that an argument is valid if and only if it is impossible for the

[3] J.M. Dunn, 'Relevant Logic and Entailment', in *Handbook of Philosophical Logic*, vol. III: *Alternatives to Classical Logic*, edd. D. Guenthner and D. Gabbay (Dordrecht, 1986), pp. 117-224.

premises to be true and the conclusion false. However, this Classical Account of Validity entails that Ex Falso Quodlibet, that is, the inference from P and '$\sim P$' to Q, is valid. Also plausible is the Deduction Equivalence, that is, that a conditional follows from some other premises if and only if its consequent follows from them in conjunction with its antecedent. Adding this to the Account of Validity, it follows that the conditional is material, or truth-functional. Since such a view validates invalid arguments, we are forced to reject or revise the Classical Account, and in passing to conclude that disjunction is ambiguous.

Further reflection on conjunction shows that it exhibits an ambiguity matching that discovered for disjunction. The intensional sense of conjunction so discerned (we call it, 'fusion') allows a suitable revision to the Classical Account of Validity and also to the Deduction Equivalence, avoiding the unacceptable consequences of chapter 2. These principles—both plausible, and of great explanatory power—are, therefore, not simply rejected, but revised following proof-analysis. An argument is valid if and only if the fusion of its premises and the opposite of its conclusion cannot be true; and a conditional follows from some other propositions if and only if its consequent follows from the fusion of its antecedent and those other propositions.

The distinctions and logical notions informally mapped out in chapters 2 and 3 can now be elucidated in a formal treatment. First, the inferential structure of a non-truth-functional conditional together with negation and the various conjunctions and disjunctions is given. The basic inferential notion of a derivable sequent captures the idea of a conclusion following from a combination of premises, a combination making use of both types of conjunction as ways of putting premises together. Secondly, the logic is extended to embrace a 'necessity'-operator, and so is capable of expressing entailment itself.

The resulting formal system contains several non-truthfunctional (i.e. intensional) operators. When formal systems with such operators (modal or relevant) were first proposed, they were challenged to produce a semantics which would show them to be coherent and viable logics. I show how possible worlds, or indexed, semantics answered this challenge, and how this semantics can be adapted to provide a semantics for the systems of chapter 4. These systems are proved sound and, in an Appendix, complete with respect to their semantics.

What lies behind the challenge to the Classical Account of Validity in chapter 2 is the objection that that account validates inferences whose premises are in some way irrelevant to their conclusions. Many attempts have been made to explicate the notion of relevance needed here, in particular, to specify some meaning-connection between premises and conclusion which can act as a sieve on the classically valid inferences. The present approach, however, directly replaces the Classical Account with the Relevant Account, and extracts the notion of relevance from the new criterion for validity. For if the conclusion really does follow from the premises, then they must be (logically) relevant to it.

Recognising the Relevant Account of Validity allows us to see how the formal semantics of chapter 5 retained in its response to the classical challenge one disastrous classical belief, namely, acceptance of the Classical Account of Validity, or consequence. It was, therefore, forced to introduce strange formal notions in its analysis of the intensional operators. Among various attempts to give these formal semantic notions some intuitive feel, two have come to be known as the American and Australian plans, the former proposing two new 'truth-values', Both and Neither (that is, both and neither true nor false), the latter, a dual proposition, the content of not denying its dual. Following our earlier analysis, I urge adoption instead, of the Scottish plan, using the Relevant Account of Validity. Ex Falso Quodlibet is now invalidated, not by making both P and '$\sim P$' true (and false), but by making the fusion of 'P & $\sim P$' and '$\sim Q$' true.

Quine coupled his belief in the infallibility of logic (alluded to in the first chapter) to the dismissal of logical heretics as semantic deviants, poor fools who misguidedly ascribe strange meanings to common symbols. To see the error in this dismissal, we must consider counterfactuals in general, and the relationship between truth and meaning. When people question certain properties of implication, and propose a different theory, they propose it as a theory of implication, not of some other connective defined only by its (different) properties. The reference of the term is captured by its usual properties, which are then queried. In this way, we can make sense of disagreements and of rival theories about a common subject-matter. Indeed, it is possible to give a thoroughly homophonic semantics within a relevant metalanguage, in which the truth-conditions map propositions directly onto their translations in the metalanguage. Any possible equivocation in the sense of a connective is ruled out.

Although the Scottish plan dispenses with an index-relative truth-condition, worlds semantics has, nonetheless, a role to play. But that role should not be overestimated. It provides an extensional model for relevant logics, as it does for modal logics, allowing technical results to be proved within a classical (and non-modal) metalanguage. Nor should the exhibition of a homophonic semantics be misunderstood. It serves only to display the strength of the theory. Formal semantics cannot itself be a theory of meaning. It cannot explain the meaning of the terms in the logic, for it merely provides a mapping of the syntax into another formalism, which itself stands in need of interpretation. What does provide that explanation is the proof-theoretic analysis. For the rules of the formal theory of deduction state the conditions under which formulae may be asserted—it connects the formulae with their use. I present a general framework of proof-theoretic analysis, and in particular show how the theory of negation in chapter 4 can be fitted to this account of meaning.

In summary, a full formal theory, comprising both a deductive system and a sound and complete semantics, can be provided for the notion of relevant implication, respecting our intuitions about what can and cannot be inferred from what. The puzzles and paradoxes of the classical view are removed not by barring exceptions, or by restricting its application, but by a proof-analysis which discerns the true nature of consequence - of the notion of validity.

Chapter 1

Scepticism and Logic

It is a common thought that what distinguishes logic and mathematics from the sciences is their infallibility. In establishing a scientific theory, one is responsible to facts about the world, facts about what is actually the case. Any theory which is proposed can be shown false by recalcitrant experience, by empirical observation. In mathematics, and perhaps most particularly in logic, it appears that this is not so. One's brief is a conceptual study of how things will be in any world, how experiences must always be. One's topic is immediately to hand, in the study of a conceptual structure which is ours for the making. The possibility of error, of recalcitrance, cannot enter. 'If sheer logic is not conclusive', asks Quine, 'what is?'[1]

1.1 Scepticism

This familiar thought has a long history. We certainly find it in Plato. It lies behind his 'Argument from the Sciences'. Plato was disturbed and impressed by Heracleitus' enigmatic remark, 'Everything is in flux'. Everything is forever changing, and nothing stays the same from moment to moment.

> How can those things be real which are never in the same state? ... nor yet can they be known by anyone, for at the moment the observer approaches, they become other and of another nature, so that one cannot get any further in knowing their nature or state, for one cannot know that which has no state. ... Nor can we reasonably say that there is knowledge at all, if everything is in a state of flux and there is nothing abiding, for knowledge

[1] W.V. Quine, *Philosophy of Logic* (Englewood Cliffs, N.J., 1970), p. 81.

> cannot continue to be knowledge unless continuing always to abide and exist. (Plato, *Cratylus* 439e - 440a)[2]

Knowledge, however, is of what is true, and so unchanging. How, then, is knowledge possible? Plato seems poised for a sceptical conclusion. He escapes it by resort to the world of the Forms, what is truly real, stable and unchanging, the objects of intellectual apprehension.

> Then let us seek the true beauty, not asking whether a face is fair, or anything of that sort, for all such things appear to be in flux ... If that which knows and that which is known exist forever, and the beautiful and the good and every other thing also exist, then I do not think they can resemble a flux. (*Cratylus* 439d, 440b)[3]

But what is distinctive of the forms, and gives them their stable character, is that they deal in eternal verities, in matters of formal definition and of mathematical certainty.

> If a person says to me that the bloom of colour, or form, or any such thing is a source of beauty, I leave all that, which is only confusing to me, and simply and singly, and perhaps foolishly, hold and am assured in my own mind that nothing makes a thing beautiful but the presence and participation of beauty in whatever way or manner obtained; for as to the manner I am uncertain, but I stoutly contend that by beauty all beautiful things become beautiful. This appears to me the safest answer which I can give, either to myself or to another, and to this I cling, in the persuasion that this principle will never be overthrown, and that to myself or to anyone who asks the question, I may safely reply, that by beauty beautiful things become beautiful. (*Phaedo* 100d-e)[4]

> Socrates, we have reached a point at which the difference in clearness in different kinds of knowledge is enormous ... Those arts into which arithmetic and mensuration enter far surpass all others. (*Philebus* 57d)[5]

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[2] *The Dialogues of Plato*, tr. B. Jowett (Oxford, 1892), vol. I, p. 388.

[3] See also *Republic* 477-8, in *The Dialogues* . . . , vol. III, pp. 175-6.

[4] *The Dialogues* . . . , vol. II, p. 246. See also *Protagoras* 330d, *The Dialogues* . . . , vol. I, p. 152.

[5] *The Dialogues* . . . , vol. IV, p. 633.

Descartes followed the same path. Appalled by the deceptions to which his senses seemed prey, and by the fact that there was never proof that any perception was veridical, he avoided the sceptical conclusion by an *a priori* argument which ultimately allowed him to conclude that he was not always deceived.

Hume and Russell rejected this a priorism, and embraced the sceptical conclusion, that empirical knowledge is impossible.

> It seems to me that the only objects of the abstract sciences, or of demonstration, are quantity and number, and that all attempts to extend this more perfect species of knowedge beyond these bounds are mere sophistry and illusion ... All other inquiries of men regard only matter of fact and existence, and these are evidently incapable of demonstration. (Hume, *Inquiry concerning Human Understanding*, XII.iii)[6]

But this served only to point up the more the contrast between empirical observation and science on the one hand and logic and mathematics on the other.

Despite its pedigree, this conception of logic and mathematics rests on a mass of confusions. One is the conflation of the empirical and the contingent: even granted that what can be false cannot be certain, it only follows that empirical certainty is impossible if all empirical truths are contingent. But the categories *empirical* and *contingent* are conceptually distinct. If they are indeed coextensive, that needs proof.[7] Another confusion is that of the necessary and the infallible: necessity is an ontological notion, objectively pertaining to a proposition; infallibility connotes an epistemological relation to a proposition, an ideal or extreme of certainty in its truth. At the root of this idolisation of logic and mathematics is the sceptical challenge to epistemology based on what is often called the Cartesian model of knowledge (shared by Plato, Descartes, Hume and Russell *inter alia*):

1. what is known must be true, so
2. claims to knowledge must preclude error, but
3. we are rarely, if ever, in a position to rule out all possibility of error, hence

[6] David Hume, *Inquiry concerning Human Understanding*, ed. C.W. Hendel (New York 1955), p. 171.

[7] See, e.g., A. Sloman, ' "Necessary", "A Priori" and "Analytic" ', *Analysis*, 26 (1965-6), pp. 12-16; S. Kripke, *Naming and Necessity* (Oxford, 1980), esp. pp. 34-9.

4. most, if not all, of our claims to knowledge are mistaken.

The sceptic's challenge is paradoxical: (1) and (3) seem clearly true; accordingly, (4) appears to follow inevitably; and yet (4) is unacceptable. We must, most of the time, be right in our claims to knowledge. For if not, what is the practice which underpins the concept of knowledge used in the argument? (Some people think that rhetorical point is itself a rebuttal of scepticism. It is not. But it does give one reason to suspect that the sceptic's claim is false.)

1.2 Fallacy

The paradox rests on an illicit slide from (1) to (2). (1) records a feature of the concept of knowledge, that 'I know that P' entails P. But to infer from this that the claim to know that P must rule out any possibility that P is false puts the cart before the horse. The falsity of P can indeed undermine a claim to know it; any claim to knowledge is fallible. But the logical relation between 'I know that P' and P is independent of the methodological relation.

What is meant by 'justification' is here equivocal, in fact. In one sense, one can never say one is justified in making a false claim, simply because it is false. But that is not how we treat claims to knowledge. What we mean by a justified claim to knowledge is that appropriate evidence can be provided. The claim may be justified even when it is false. So whether a claim is true is in fact independent of whether one is justified in making it. (4) equivocates in this way: it is unclear whether the sceptic is claiming

(a) that we rarely know anything at all, or

(b) that we are rarely justified in claiming to know it.

(Is he an atheist or an agnostic?) If all the sceptic is claiming is (b), that we never have adequate grounds for our claims, then not only is he mistaken, but (1) is irrelevant to the issue. However good the evidence, claims to knowledge are fallible. Whether the claim is justified is a question of how good the evidence is, not of whether the claim is true. On the other hand, if the claim is a), that we rarely know anything, then (1) is indeed relevant to the argument, but insufficient for the conclusion drawn. For truths and falsehoods divide equally, and so, unless other reasons are adduced for supposing we score worse than chance, at least half our knowledge claims are justified.

The temptation to suppose that knowledge is infallible comes from reading the methodological relation as proceeding downwards from the knowledge-claim to the world. But we have no such power to make the world accord with our claims. Our ability is at most to attempt to make our claims accord with nature. The misunderstanding arises because 'I know that P' appears to describe a state of mind, which it is only too tempting to bedeck with occult powers. But the same logical behaviour is shown by such phrases as 'I realise that P', 'it is tragic that P', 'it is exciting that P', and so on.[8] That I have realised how the proof goes, or that it is exciting that every plane map can be coloured with at most four colours, do entail that my conjectured proof is right, and that the four colour theorem is true. But neither the realisation nor the excitement can guarantee that correctness. What wears the trousers is P, the contained sentence. If that is false, the whole claim is simply false.

In this respect, the phrases 'know', 'realise', 'exciting' and others differ from 'believe', 'likely', 'suppose' and so on. But the states of affairs described are not radically different. What is different is the mode of description, which sets up a fallible claim.

It might seem that such a fallibilist solution to the paradox concedes the sceptic's point. Is it not just to say that we are never justified in making claims to knowledge, for any such claim may prove false? Is the fallibilist's point not simply a reiteration of 3) above, that we are rarely - indeed worse: never - in a position to rule out all possibility of error?

Certainly not: for if the appropriate conditions on the gathering of evidence have been satisfied, then the claim is indeed justified. What that justification will lack is any power to guarantee its own indefeasibility. If P later proves false, the claim to have known P must be retracted; there was no knowledge. But if the evidence has been gathered, and in addition, P is, as a matter of fact, true, then the claim to know P was correct. In other words, those claims to know that P which are based on adequate evidence for P are, if P is true, quite correct. If P is true, in these circumstances, one does know that P; and if P is false, one does not. Hence, there is no challenge to the unacceptability of (4). Most of one's claims to knowledge may be perfectly justifiable; neither (1) nor (3) do anything to show that this is not so.

[8] See P. & C. Kiparsky, 'Fact', in *Semantics*, ed. D. Steinberg & L. Jakobovits (Cambridge, 1971), pp. 345-69.

1.3 Descartes' Test

Of course, there is something plausible about (2). If one makes a knowledge claim, one is claiming that certain possiblities can be ruled out. For example, a detective who, after weeks of patient searching for clues, announces that at last he knows that the robbers entered the bank with a key to the front door, is ruling out many other scenarios.

But if (2) is interpreted in this way, (3) immediately becomes implausible. For to try to justify (2) in this way, that is, independently of (1), one has to contrast cases where one does rule out error, and those where one does not. The detective would not be justified in making his claim if he had not undertaken any investigation of the doors and windows of the bank, or received reliable reports from his assistants, if he had not examined the lock to see if it was forced, had not interviewed the bank staff to ascertain if anyone had hidden in the bank after closing time, and so on. A claim to knowledge creates expectations not only about whether the claim is true, but also about the grounds on which it is made.

Thinking then of (2) as the sceptic's basic claim (without its background in (1)), his mistake is, in essence, to suppose that if knowledge requires inter alia the truth of the claim, then in order to justify one's claim to knowledge one must justify the claim that it is true. One finds this oft-repeated mistake in, for example, Barry Stroud's *The Philosophical Significance of Scepticism*.[9] In the first chapter, Stroud considers Descartes' familiar objection that he may not know he is sitting by the fire, for he may be dreaming. Is there, asks Stroud, any condition which may be added to his belief that he is sitting there, sufficient to entail that he truly knows he is there? Stroud writes: 'Of course, for that test or state of affairs to be of any use to him, Descartes would have to know of it ... [and it] would have to be something he could know obtains.'[10] Not so; the test itself is the condition for knowledge, not knowledge of the result of the test. If the test is satisfied, regardless of whether he knows it is satisfied, then Descartes knows he is sitting by the fire. If, moreover, he knows the test is satisfied, then he will also know that he knows it. But he may not. He may know it without even realising, and certainly without being at all sure, that he does, and so without knowing that he knows it.

It might, therefore, be thought that what I am saying is that one often knows that P, but can never know that one knows that P, for P may be

[9]B. Stroud, *The Philosophical Significance of Scepticism* (Oxford, 1984).

[10]Ibid., pp. 21-2. Cf. Plato, *Theaetetus* 209-10, in *Dialogues* ..., vol. IV, p. 279.

false. But that is to repeat the confusion at a higher level. If one has evidence for the belief that one knows that P; and if indeed one does know that P; then one knows that one knows that P. Of course, if P is false, then one does not know that P, and so certainly does not know that one knows it. But that can no more show that one never knows that one knows than it can show that one never knows it either.

1.4 Defeasibility

Thus knowledge is fallible: it can always be undercut by demonstrating the falsity of its claim. From this realisation follows an important methodological moral on the nature of logical and mathematical theories. Ever since the seventeenth century, it has been realised that the demonstrative method in science is mistaken. We cannot arrive at knowledge in the sciences by the intuition of self-evident truths, for example, as to the orbital paths of celestial bodies or the forces acting on a body in free fall. Theories in these fields are open to empirical refutation. The initial reaction to the demise of the demonstrative conception was to reject the deductive relation as itself at fault, and to seek an inductivist support for scientific theories. But that was a will o'the wisp, which waylaid the methodologist in a mire of confusion. The instructive insight was to separate the logical from the methodological relation, and realise that, although the inductivist was right in claiming that it is the observation statements, not intuited generalisations, that carry the weight in establishing a theory, the logical relation of evidence to theory is indeed deductive, not inductive. Hence, the important methodological phenomenon is the upwards transmission of falsity, in other words, Modus Tollens, not the upwards transmission of truth (induction), which is, as a matter of sheer logic, not available, nor the downwards transmission of truth (demonstrativism), which has no methodological authority. (We can leave till §1.6 certain complications in this picture.)

So much for science; but what of mathematics and logic? Within the Cartesian conception of knowledge, the following thought is tempting: since knowledge must preclude error, the most we can legitimately claim to know is what cannot possibly be false. Hence, the paradigm of knowledge is taken to be that where proof is in order, issuing in necessary truth, namely, mathematics and logic. This was the thought behind, for example, Descartes' attempt to circumvent scepticism by founding all knowledge on the 'Cogito' argument, and Russell's logicist enterprise for the foundation of mathematics.

It is not the whole story, of course. It was often coupled to a belief in the special incorrigible character of first person experiential reports. But think, for example, of the premise of the 'Cogito' argument. The point is that although it is contingent that I am thinking, it is not contingent that, if I am considering whether the 'Cogito' argument is sound, I am thinking. So at least in the cases of Plato and Descartes, the certainty of demonstrable propositions was contrasted with doubt about contingent propositions of all kinds.

But the infatuation with the demonstrable only serves to compound the confusion in supposing knowledge to be infallible in the first place. For the fact that a proof, if correct, will establish its conclusion as necessarily true cannot preclude the possibility that the supposed 'proof' is incorrect. Mistake is as likely in constructing proofs as it is in carrying out observation and experiment. That a discipline deals in necessities does not mean that it and it alone can preclude error.

1.5 Demonstrativism

None the less, demonstrativism deserves further elaboration - elaboration which will, I fear, do it no good. For the demonstrativist may object that what has been shown to fail is only a straw man, a caricature of the demonstrativist's proper position. How could any demonstrativist claim that a theory was unchallengeable, that it brooked no criticism? What is distinctive of logic and mathematics, he will say, and what makes the demonstrativist methodology correct for them, is that conjectures in these subjects are open to proof. Mathematics is totally disanalogous to science. Who could seriously suggest that group theory might be wrong, and should be abandoned? The situation is utterly unlike that in science, where a theory can be undercut by experimental discovery.

But that is not what the fallibilist is saying. We must compare like with like. It is a travesty of the history of mathematics to suppose that Galois simply presented the axioms for group theory, and all that mathematicians have done in that area since 1830 is deduce more and more consequences from them. What a dull exercise mathematics would then be! What Galois identified was an area of research, in which slowly over time certain ideas and principles were sifted out and abstracted as useful. But always there is an interplay in any living mathematics between the fundamental conceptions and the results consequential upon their elaboration. By now, of course, group theory has a certain stability as a received body of truth. But so has the theory of terrestrial dynamics, or of the periodic table.

The origins and history of group theory are in fact very interesting.[11] As the group-theoretic concept evolved in the work of Lagrange, Abel, Galois and others it was fundamentally tied to concrete application. When Galois used the word 'group', it was shorthand for 'group of permutations', and only slowly came to acquire its current abstract sense. The properties of the group were governed wholly by the interpretation and the application, as any set (or group) of permutations closed under multiplication (or composition). The associative property was unstated and assumed, the existence of an inverse guaranteed by the fact that only finite groups were considered. Results and conjectures were mixed, proofs forever revised and improved - a notable example being Ruffini's 'proof' that the quintic is insoluble in general by means of radicals (any finite arithmetical combination of integers and their roots), a proof which needed repeated additions of lemmata, and was only rigorously proved eventually by Abel. Similarly, much of the work of Gauss depended on concrete assumptions made obvious by geometric intuition, which for the general, abstract, case needed proof. Anything worthy of the names 'theory' or 'knowledge' was half a century in the gathering. The first formulation of the abstract group concept (by Cayley) was ignored for twenty years, and anything now recognisable as abstract group theory did not appear till a hundred years or more after the first results (by Lagrange and others) in which the group concept can be identified.

Unsophisticated demonstrativism has a conception of a trivial truth: if one objects to it, one clearly has not understood it. What the sophisticated demonstrativist realises is that allowance must somehow be made for a challenge not to the truth of the claim, but to its status as a trivial truth. But how can this be done? The obvious suggestion is self-evidence. All knowledge must be grounded on self-evident truth, and the steps by which it is grounded must themselves be self-evident and immediate. The eventual outcome of this line of thought is an axiomatic epistemology, placing faith in the deliverances of some formal system. But that hope was killed by Gdel's demonstration that no consistent recursively axiomatisable formal system for arithmetic was complete. Gdel's proof shows that no reasonably interesting theory can be recursively axiomatised. Moreover, axiomatic epistemology could never answer the underlying question, what is the ground for adopting one formal system rather than another. What proof does is to connect statements together, to make one realise what depends on what,

[11] See, e.g., H. Wussing, *The Genesis of the Abstract Group Concept* (Cambridge, Mass., 1984); B.L. van der Waerden, *A History of Algebra* (Berlin, 1985).

how different claims are logically related. How one uses the proof is a different matter, and depends on one's confidence in those different claims. Quine realised this many years before adopting the conservative position cited earlier: he noted that any claim can be given up if one is willing to make sufficient adjustments elsewhere. No one portion of the network is privileged over any other, not even the threads of the net itself.

> Total science is like a field of force whose boundary conditions are experience. A conflict with experience at the periphery occasions readjustments in the interior of the field. Truthvalues have to be redistributed over some of our statements. Reevaluation of some statements entails reevaluation of others - the logical laws being in turn simply certain further statements of the system, certain further elements of the field.[12]

It follows that the difference between demonstrativist, inductivist and fallibilist methodologies does not pertain to a difference between different disciplines. There are no doubt other differences of a methodological sort between, say, mathematics and science; but the correctness of a fallibilist methodology means that it is the correct description of the relation between evidence, theory and prediction in every discipline, including science, mathematics and logic, because it reflects a general fact about knowledge.

1.6 Fallibilism

This methodological analysis of course owes its inspiration to Imre Lakatos. However, Lakatos died before fully applying these ideas even to mathematics. Popper had preceded him in overthrowing the inductivist conception of science and setting up the falsificationist model. Popper retained a demonstrativist conception of mathematics.[13] In his early writings, Lakatos extended Popper's falsificationist ideas to mathematics.[14] Later, however, Lakatos' conception of science shifted, and he came to view the falsificationist model as too simplistic.[15] He died before he could work out in detail how his new

[12] W.V. Quine, 'Two Dogmas of Empiricism', in his *From a Logical Point of View* (New York, 1963), p. 42.

[13] K.R. Popper, *The Logic of Scientific Discovery* (London, 1959); 'Epistemology without a Knowing Subject', in his *Objective Knowledge* (Oxford, 1973), pp. 106-52.

[14] I. Lakatos, *Proofs and Refutations*, ed. J. Worrall & E. Zahar (Cambridge, 1976).

[15] I. Lakatos, 'Falsification and the Methodology of Scientific Research Programmes', in his *The Methodology of Scientific Research Programmes* (Philosophical Papers, vol. I), ed. J. Worrall & G. Currie (Cambridge, 1978), pp. 8-101.

ideas would apply to mathematics. He did, however, give a brief and cursory account.[16]

Logic is now the last preserve of the demonstrativist. In a brief passage printed in the Appendix to the posthumous edition of *Proofs and Refutations*, Lakatos makes a prophetic statement: 'To revise the whole infallibilist philosophy of mathematics ... one [has] to give up the idea that our deductive, inferential intuition is infallible.' His editors chide him in a footnote: 'This passage seems to us mistaken and we have no doubt that Lakatos, *who came to have the highest regard for formal deductive logic*, would himself have changed it. First order logic has arrived at a characterisation of the validity of an inference which (relative to a characterisation of the 'logical' terms of a language) does make valid inference essentially infallible.'[17] But Worrall and Zahar are wrong. Lakatos had correctly seen the consequence for logical theory of his methodological insights. We will discover in the following chapters that rigorous proof-analysis of the so-called paradoxes of strict implication will lead to a revision of this criterion of valid inference. The methodological moral is that nowhere is there infallibility.

What, then, is the fallibilist conception? In broad terms, it is that any claim to knowledge is defeasible. We can give grounds for justifiably making a claim to knowledge, but we cannot establish it categorically. Later discovery may show it to have been mistaken - it is fallible. But we have also to realise that, as Duhem put it, 'there are no crucial experiments'.[18] Few theoretical generalisations directly entail particular checkable predictions, that is, entail them without the additional support of auxiliary hypotheses. So if a particular - in the case of science, an observation - statement, entailed by a theory, is found to be false, there will almost always be a choice between rejecting one theoretical generalisation which was instrumental in making that prediction, and hanging onto that generalisation by giving up some auxiliary hypothesis. Duhem's example was Foucault's experiment, designed to decide once and for all whether light is corpuscular or wavelike. But the experiment, Duhem emphasises, judges not simply between these two hypotheses, but between two whole systems of optics.

Hence, an adequate description of a productive and useful methodology has to recognise a certain dialectic of revision and adaptation between the-

[16] I. Lakatos, 'A Renaissance of Empiricism in the Recent Philosophy of Mathematics?', in his *Mathematics, Science and Epistemology* (Philosophical Papers, vol. 2), ed. J. Worrall & G. Currie (Cambridge, 1976), pp. 24-42.

[17] I. Lakatos, *Proofs and Refutations*, Appendix I, p. 138.

[18] P. Duhem, *The Aim and Structure of Physical Theory* (Princeton, 1954), Ch. 8 sect. 3, pp. 188-90.

ory and evidence. When a theory meets an apparent counterexample, it is ridiculous to reject the whole theory immediately - as ridiculous as it was to reject the observation on the demonstrativist conception. What must happen is that the theory undergoes revision and improvement, so that at any one time there is scope for further adaptation, deeper explanation and greater understanding as a theory responds to refutation and counterexample. Moreover, even potential falsifiers depend for their weight on an interpretative theory. What makes a statement in science an observation statement is not anything inherent in it, but depends on a decision to prefer one theory as to its description to another. Hence, what makes a prediction a potential falsifier of one theory, is the adoption of an incompatible theory in its description. Fallibilism means that any statement, whether universal or particular, is defeasible.

Thus we arrive at a conception of theory and knowledge acquisition as involving a dialectic of hypothesis formation, empirical testing, revision of hypotheses in the light of the tests, further test, and so on, in an attempt to reach an explanation of phenomena by subsuming them under generalisations which are modelled to accommodate refutation. Observation serves to correct, modify and refute theoretical hypotheses.

Hence knowledge is not infallible. Moreover, not even our knowledge of necessary truths is infallible. Indeed, we might start to wonder what is left of the concept of necessity. If any claim may be given up, what sense is there in saying any claim is incapable of being false? The answer is that what is necessary is what we decide, at any given time, not to give up come what may. To this end, we formulate an abstract theory (as in the theory of groups), we connect axioms and theses by proof, and so on. But this can all change. In deriving empirical predictions from physical laws, mathematics and logic are auxiliary hypotheses. If their revision can provide a better account of recalcitrant experience, then they should be open to question, like any other field of knowledge. Necessity is a matter of convention.

It is not, therefore, in being alone open to knowledge, that mathematics and logic differ from science. Knowledge is possible in both, if the assertions are true and adequately grounded. Mathematics and logic do differ, in their degree of abstraction. But in methodology they are the same: fallible. The same basic structure must be found in any area yielding theoretical explanation, knowledge and understanding. This is a direct consequence of our earlier analysis of the concept of knowledge. In all fields of inquiry, the basic statements play the role of advancing development of the theory by criticism - by checking the consequences of the theory.

In particular, in logic too, the correct methodology is fallibilist. The logician's central concern is soundness or logical consequence - whether the conclusion of an argument follows from its premises. The logician sets out to give a criterion for the soundness of argument, coupled to a technique for applying the criterion to particular arguments. Initial inspection of certain arguments leads him to make hypotheses about what constitutes a sound argument, and how to apply that criterion. He must then examine how that criterion and recipe for application fares when tested on other arguments. Our task in the succeeding chapters is to explore the way in which such hypotheses come face to face with the evidence of particular inference.

Chapter 2

Classical Logic

The purpose of logical theory is to provide an explanation of the validity and invalidity of argument. The goal is to describe the relation which must hold between premises and conclusion for it to be correct to say that the premises entail the conclusion, that the conclusion follows from the premises, or that the inference from premises to conclusion is valid.

2.1 The Classical Account of Validity

A first thought is this: the point of a valid argument is to establish the truth of the conclusion by adducing premises from which it follows. That is, it is necessary to the validity of an argument that if the premises are true, so is the conclusion. Hence, if an argument has true premises and a false conclusion, or even if it is logically possible for its premises to be true and its conclusion false, then the truth of the premises is clearly inadequate to guarantee the truth of the conclusion, and so the argument cannot be valid. That is, a necessary condition of the validity of an argument is that it be impossible for the premises to be true and the conclusion false.

But more: suppose conversely that it is impossible for the premises to be true and the conclusion false. Then the truth of the premises would appear to rule out the falsity of the conclusion, and so, it would seem, we may validly infer the truth of the conclusion from that of the premises, and so the argument is valid. Hence, we seem to have shown that the impossibility of true premises and false conclusion is not only necessary, but also sufficient, for validity.

It is well known that such a theory of validity has some unintuitive consequences. For example, it follows that all arguments with a necessarily true conclusion, and all those with inconsistent premises, are valid. I shall, following normal practice, call this particular theory of validity, the Classical Account, or the Classical Theory.

I argued in Ch.1 that a theory, in any discipline, is but a conjectured set of generalisations postulated in explanation of the phenomena, and however strong the support may be for any particular such set, any claimed generalisation is fallible and open to revision in the light of countervailing evidence. This is not to deny a theorist the right to claim that he knows the correct explanation. But the rightness of his claim to knowledge can never make reexamination of his claims inappropriate. For example, it may have seemed clear that the claim in Newtonian theory that force is a product of mass and acceleration had offered a satisfactory and well-confirmed explanation of dynamical phenomena. The Newtonians were justified in claiming to know why bodies moved as they did. Nonetheless, they were wrong and did not know it. It was only when more extreme dynamical motions were considered that it was realised that this identity was strictly false, and only a limiting case within observational error of the more accurate identification of force with the derivative of momentum.

So too, in logic. It may seem clear that the Classical Theory is well supported. But, like any theory, it will need to be revised or rejected in the light of countervailing evidence; and its support will then need reexamination, to divine in what way its support has been suasive only because we have failed to appreciate subtleties, equivocations or untypical cases.

2.2 The Deduction Equivalence

A case in point are the so-called paradoxes of strict implication mentioned above. Undoubtedly, a contradiction strictly implies any proposition whatever, given the now technical sense of 'strictly implies'; and any proposition strictly implies a necessary truth. C.I. Lewis argued that intuition misled us if we took this to show that strict implication is an incorrect analysis of entailment.[1] Although intuition may incline us to suppose that these features do not pertain to entailment, an independent argument can be given to show that they do hold of it, and so, however paradoxical to intuition, they must be accepted.

There were two such independent arguments, one to show that a contradiction entails any proposition whatever; the other, that any proposition

[1]C.I. Lewis and C.H. Langford, *Symbolic Logic* (New York, 1932), p. 250; cf. C.I. Lewis, *A Survey of Symbolic Logic* (Berkeley, Calif., 1918), §5.5, pp. 324-39.

entails a necessary truth. What Lewis did was to separate each derivation into a succession of elementary steps, leading from—in the first case—an arbitrary contradiction, by seemingly valid principles of reasoning, to any other proposition we care to take. Hence, we can reject the principle Ex Falso Quodlibet (EFQ), that from P and 'not-P' we may infer Q, only on pain of denying either the validity of one of these elementary inference steps, or the transitivity of entailment (that a succession of valid inferences builds into a single valid step).

Even if we can show Lewis' derivations invalid—which eventually I shall do—to do so would only take us back to square one, to a position of stalemate between the intuitive implausibility of EFQ and its twin on the one hand, and the plausible considerations supporting the Classical Account on the other. So it is worth looking to see if there is some direct and categorical argument against EFQ. That there is.

Suppose B is deducible from A, together with some other assumptions $A_1, \ldots, A_n$ ($n \geq 0$). Then, provided $A_1, \ldots, A_n$ are true, if A is also true, so is B, and hence, 'if A then B' is true. That is, the truth of 'if A then B' follows from that of $A_1, \ldots, A_n$. Conversely, given that 'if A then B' follows from $A_1, \ldots, A_n$, then, provided $A_1, \ldots, A_n$ are true, so is 'if A then B', and so if A is true so is B. That is, if $A_1, \ldots, A_n$ are true, and so is A, then B is true too. Hence B follows from A together with the parametric assumptions $A_1, \ldots, A_n$. In other words,

B follows from $A_1, \ldots, A_n, A$
iff
'if A then B' follows from $A_1, \ldots, A_n$.

This result is usually referred to as the Deduction Equivalence or the Deduction Theorem. Later, we will find that it needs more careful statement. Indeed, the present application will fail when we make that revision. But since the revision will depend on the upshot of the present considerations, the line of argument is essentially a reductio of the classical position. What we can show is that the Classical Account, in conjunction with (the present form of) the Deduction Equivalence, entails that a clearly invalid argument is valid (indeed, that any number of invalid arguments are valid). The objection will be a familiar one to many: what we must analyse and clearly spell out are the consequences of surrender. Of course, we have here a clash between two opposed ways of evaluating the evidence. Any theory can be squared with the evidence by sufficient juggling with epicycles. But what must ultimately be accepted is that the better explanation relinquishes the belief that the impossibility of true premises and false conclusion suffices for validity.

2.3 Material Implication

We now put the Deduction Equivalence to work on the consequences of the Classical Account which we spelled out in §2.1. We saw that the Classical Account entails that Q follows from P and 'not-P'. Hence, by the Deduction Equivalence, 'if P then Q' follows from 'not-P', that is, 'if P then Q' is true if 'not-P' is true, i.e. if P is false. Furthermore, Q cannot be false if P and Q are each true. So, by the Classical Account, Q follows from P and Q, whence, again by the Deduction Equivalence, 'if P then Q' follows from Q, that is, if Q is true so is 'if P then Q'. Hence, simply in virtue of adopting the Classical Account in conjunction with the Deduction Equivalence, it follows that if P is false or Q is true, 'if P then Q' is true.

There is only one case left, that P is true and Q false. But if 'if P then Q' is true and P is true, then Q is true too. Since by hypothesis, Q is false, it follows that 'if P then Q' is false. Putting the three cases together, we see that the Classical Account coupled with the Deduction Equivalence entails that the truthvalue of 'if P then Q' is completely determined by the truthvalues of P and Q. We can set it out in a table:

P	Q	if P then Q
true	true	true
false	true	true
true	false	false
false	false	true

Accordingly, 'if' must be identified with the truth-function of material implication.

In other words, that 'if' is truth-functional follows from the acceptance as valid of the following inferences:

$$\text{not-}P \vdash \text{if } P \text{ then } Q$$
$$Q \vdash \text{if } P \text{ then } Q$$
$$P \text{ and not-}Q \vdash \text{not-(if } P \text{ then } Q\text{)},$$

together with the principle that A is false iff 'not-A' is true. Then, whether P and Q are true or false, their values completely determine the value of 'if P then Q'.

In general, a connective is truth-functional if its value (i.e., whether it is true or false) is determined by the values of its component formulae; and a logic is truth-functional if all its propositional connectives are truth-functional. There are problems with this characterisation, as we will see in Ch. 7. But enough is clear to see that classical logic is avowedly truth-functional. It rejects consideration of non-truth-functional connectives. Incidentally, note that the principle connecting truth and falsity builds in the stipulation that negation is truth-functional; also that on this account, intuitionistic logic is truth-functional. The truthvalue of any of its propositional formulae is completely determined by those of its component formulae. (Moreover, intuitionistic logic accepts the Classical Account of Validity.) Intuitionistic logic's difference from classical logic lies in not accepting Bivalence, that every proposition is true or false. The intuitionist does not deny that what the truth-table tells us. If the values of A and B are known, so is the value of '$A \supset B$'. What he will not assent to is the classical logician's assertion that the rows of the truth-table exhaust all possibilities. He will not assent to that; nor will he deny it.

We find, therefore, that adopting the Classical Account of Validity commits us to treating 'if' as truth-functional. But it has long been recognised that treating 'if' as material, or truth-functional, validates clearly invalid arguments. A particularly striking case, adapted from an example of Cooper's, is perhaps this.[2] Let us suppose that Roy Dyckhoff has claimed that John Slaney was in Edinburgh on a certain day, and that Crispin Wright has denied it. Consider the following three propositions as they describe this situation:

(1) If John was in Edinburgh, Roy was right.
This is clearly true: that's what Roy claimed.

[2]See W. Cooper, 'The Propositional Logic of Ordinary Discourse', *Inquiry*, 11 (1968), p. 297, example 2.1. Cooper's example is

> If John is in Paris, he is in France.
>
> If John is in Istanbul, he is in Turkey.
>
> So either if John is in Paris he is in Turkey
> or if John is in Istanbul he is in France.

The drawbacks of this example are first, that the name 'John' as used in it looks too like an arbitrary name, in which case the conclusion is false only when the name's scope is incorrectly construed narrowly, and will be true, as predicted by the Classical Account, when the name is given its correct wide scope, as essentially a free variable. Secondly, that what we are seeking is a counterexample to treating 'if' truth-functionally, and, as we have seen, intuitionistic logic is truth-functional. But Cooper's sequent is intuitionistically invalid. So his example builds in more than is really necessary, namely, acceptance of Excluded Middle.

(2) If Crispin was right, so was Roy.
That is equally obviously false, given the logic of denial.
(3) If John was in Edinburgh, Crispin was right.
That too is false, for Crispin denied it. Let us use these propositions to construct an argument, taking as premises (1) together with the denial of (2), and as conclusion (3):

> If John was in Edinburgh, then Roy was right.
> It's not the case that if Crispin was right, so was Roy.
> Hence, if John was in Edinburgh, Crispin was right.

Since (1) is true and (2) and (3) false, this argument, which takes the denial of (2) as its second premise, has true premises and a false conclusion. Hence it is invalid.

Classically, however, the argument is valid. For the sequent

$$P \supset Q, \sim(R \supset Q) \vdash P \supset R$$

which formalises the argument classically, using '$\supset$', representing material implication, to capture 'if', is (classically) valid. Hence, if the truth-conditions of 'if' were correctly captured by material implication, and the Classical Account of Validity were right, the argument would be valid. But it is not. So either treating 'if' as truth-functional, or the Classical Account of validity, is wrong. But if the truth-fuctional account of 'if' is abandoned, so too must be the Classical Account of Validity (since, with the Deduction Equivalence, it entails that 'if' is truth-functional). So, either way, the Classical Account is wrong. Moreover, the conclusion of the above argument is false: but it does not follow that John was in Edinburgh and Crispin was wrong - indeed, neither follows. John might well have been in Brisbane. Even so, although Crispin would then have been right, that would not make the conclusion true.

2.4 Truth and Assertibility

There are no crucial experiments, and there are many ways in which a classicist may respond to that analysis. Take the following.[3] It is quite clear, the reply goes, why the argument is deemed, classically, valid. For propositions (2) and (3), both conditionals, share a proposition, as antecedent of one and

[3]The objection was made by an anonymous referee. Note that it is not a plausible response for an intuitionist.

consequent of the other. So if it is indeed correct to treat these propositions as material conditionals, they cannot both be false, as was claimed. '$(P \supset Q) \vee (Q \supset R)$' is classically valid, and so one or other of (2) and (3) must be true. Hence, what has been described, a situation in which (2) and (3) are apparently false, is simply impossible. The suppositions on which the argument against the Classical Account is based are, therefore, inconsistent, and we can dismiss the argument without more ado.

This response is extremely puzzling. For the truthvalues attributed to (1) - (3) are not in any obvious way suppositions. Certainly, it was supposed that Roy and Crispin had made claims about John's whereabouts. But we can hardly conclude that such contradictory claims can never be made! The situation described is patently quite possible - indeed, contradictory claims of that sort are quite common (though not, perhaps, specifically about John's whereabouts!). Given the simple situation as described, propositions (1) - (3) would certainly appear to have the truthvalues ascribed to them. (1) is true because that is what Roy claimed; (2) is false because Crispin flatly denied what Roy said; and (3) is false because its antecedent states precisely what Crispin denied.

The present classical response, however, insists that one or other of (2) and (3) is true. Since that flies in the face of intuition, rather more must be said. The onus is now on the classical logician to justify this classical claim. Much more explanation is needed, to show how a proposition - (2) or (3) - may be true yet clearly not correctly assertible. For anyone asserting (2) or (3) in the situation described would be told he was wrong.

What this serves to bring out, in fact, is that the case is worse for the classical logician than it appeared. For the invalidity of the above argument followed only from the Classical Account in conjunction with the belief that 'if' is material. Since the Classical Account entails, given the Deduction Theorem, that 'if' is material, the options open, on recognising the argument to be invalid, are either to reject the Classical Account or to reject both. But we can now focus on the real problem as being the truth-valud assignments to 2) and 3). If 'if' were material, either 2) or its converse

> If Roy was right, so was Crispin

would be true, for $(R \supset Q) \vee (Q \supset R)$ is a classical tautology. But the logic of denial requires that they both be false, and that we recognise them to be false without needing to know who was right. It follows that the only option open is to reject both the account of 'if' as material and the Classical Account of Validity which entails that it is material.

One classical response to this analysis is simply to fly in the face of intuition, and claim that the truth-values of (2) and (3) do depend on John's actual whereabouts, and that is all there is to be said. We earlier assigned truth-values to them without knowing where John was. That was our mistake. If John was in Edinburgh, (2) is true, while if he was not, (3) is true.

That is so flatly counterintuitive that at this point the Classical Logician usually softens his stance by resorting to some sort of conversationalist doctrine, whereby propositions, though true, may not be correctly asserted in certain situations. For example, it may be misleading, and so wrong, to assert a disjunction, when the truth of either disjunct is also known. A classicist may claim, therefore, that although the truthconditions of conditionals are given by the truthtable for material implication (let us call this 'the Equivalence Thesis'),[4] their assertibility conditions are different.

One popular view is that the assertibility conditions of conditionals, 'if A then B', are given by the conditional probability, $\Pr(B/A)$.[5] So 'if A then B' will be highly assertible when the probability of B relative to that of A is high, that is, when $\Pr(B \;\&\; A)/\Pr(B)$ is high. Suppose A is false. On the equivalence thesis, 'if A then B' will be true. But it may well not be (highly) assertible, if in particular, $\Pr(B \;\&\; A)$ is very much less than $\Pr(B)$. For example, consider the answer the assertibility theory gives to the puzzle, which of (2) and (3) above is true. If John was indeed in Edinburgh, then, according to the Equivalence Thesis, (2) is true, while if John was not there, it says that (3) is true. But their conditional probabilities are each low, differing from zero only to the extent that the probability that Roy or Crispin did not make the assertion I have attributed to them itself differs from zero.

There is an enormous literature on the assertibility theory of conditionals, and an immense variety of basically similar theories. It is a many-headed Hydra, which grows ten new theories whenever any single one is shown wanting. My intention here is not to enter lengthily into the debate. My main purpose is to explore in full the consequences of rejecting the classical ac-

[4]The *locus classicus* for using a distinction between truth and assertibility to retain the Equivalence Thesis is H.P. Grice, 'Logic and Conversation', in *Syntax and Semantics*, vol. 3, ed. P. Cole and J.L. Morgan (New York, 1975), pp. 41-58, and 'Further notes on logic and conversation', in *Syntax and Semantics*, vol. 9, ed. P. Cole (New York, 1978), pp. 113-27. The phrase 'Equivalence Thesis' comes from F. Jackson, 'On Assertion and Indicative Conditionals', *Philosophical Review*, 88 (1979), pp. 565-89, expounded more fully in his *Conditionals* (Oxford, 1987), chapter 2.

[5]The idea stems from E. Adams, *The Logic of Conditionals* (Dordrecht, 1975). However, Adams himself did not subscribe to the Equivalence Thesis.

count. However, I must indicate briefly what I consider to be the essential weakness of the assertibility view.

The crucial question is, what does the claim that the truth-condition is extensional, or material, amount to? It is very arguable whether the conditional probability really does provide the correct account of when a conditional is assertible; but prescinding from that, what work does the other claim, the one about *truth*-conditions (the Equivalence Thesis, itself), do? Its only plausible role would seem to be to account for the behaviour of embedded conditionals. For assertibility only applies to whole utterances, to sentences. The semantic contribution of an embedded clause to a sentence of which it is only a part cannot be anything to do with assertibility, but only with truth.

But this is mistaken. Consider the disjunction of 2) and 3):

> Either if Crispin was right, so was Roy,
> or if John was in Edinburgh, Crispin was right.

On the Equivalence Thesis, one or other of these disjuncts is true, and so the whole disjunction is true. But the disjunction is not highly assertible.[6] Why not? The plausible answer is because neither disjunct is highly assertible. But that response is not open to the assertibility theorist, because there are many cases of highly assertible compound sentences whose parts are not highly assertible. For example,

> If the earth is flat then it has an edge.

So the assertibility thesis must add to the theory of assertibility for disjunctions an account of why a disjunction of true disjuncts is not highly assertible which does not depend simply on the fact that the disjuncts themselves are not highly assertible. But then it will clearly miss the point, because it is precisely because the disjuncts are not highly assertible that the whole is not to. In fact, the mistake is not to recognise that the disjuncts are not highly assertible because they are false, and it is its falsity which makes the disjunction not highly assertible.

The solution is to abandon the attempt to account for the behaviour of conditionals by retaining the Equivalence Thesis as truth-condition, and adding an epicycle of assertibility theory. What the Equivalence Theorist

[6] The extraordinary claim that such disjunctions are unintelligible is made in A. Appiah, *Assertion and Conditionals* (Cambridge, 1985), p. 208. He also floats the idea that sometimes $(P \to Q) \vee (R \to S)$ should be taken to mean $(P \vee R)$ & $(P \to Q)$ & $(R \to S)$, in other words, is reducible to some other form.

takes as the assertibility condition is in fact the conditional's truth-condition. When so recognised, it can take its proper role as primary semantic component, and feed into the recursion to give an adequate account of the truth-conditions of compounds of conditionals.

2.5 Three Morals

Consequently, it must be admitted that 'if A then B' can be false without A being true or B false. In other words, either A's being false, or B's being true, does not suffice for the truth of 'if A then B'. If we accept the validity of Contraposition, these two stand or fall together. Suppose we wish to hang onto the idea that the falsity of A suffices for the truth of the conditional, 'if A then B'. Then the truth of 'not-A' suffices for the truth of the conditional, 'if not-B then not-A'. So if we must reject one or the other, the sufficiency of false antecedent or of true consequent, we must reject both. Neither A's being false, nor B's being true, suffices for the truth of 'if A then B'. The truth-values of A and B do not alone completely determine the value of 'if A then B'. 'If' is not truth-functional.

From the realisation that 'if' is not truth-functional we can draw three morals. The first is that Ex Falso Quodlibet is invalid. We noted at the end of the last section that the theses

$$\begin{array}{rcl} \text{not-}P & \vdash & \text{if } P \text{ then } Q \\ Q & \vdash & \text{if } P \text{ then } Q \end{array}$$

stand or fall together, and so fall. Consequently, by the Deduction Equivalence, P and 'not-P' do not entail Q.

The second moral is the one spelled out at the end of 3, that the Classical Account of Validity is certainly mistaken - or the Deduction Equivalence fails. Let me note briefly that it would be mistaken to conclude, as some appear to have done, that 'if' therefore defies formal treatment. Formal treatment of non-truth-functional operators is, perhaps, more complex than the rather simple-minded theory of truth-functions. Nonetheless, there are many examples of perfectly respectable non-truth-functional formal theories. Modal logic is an obvious example.

Mistaken, too, is the suggestion that, even if a correct formal tratment of 'if' can be given, it is too complex, and even perhaps too parochial in its linguistic analysis of a particular English particle. The formal treatment to which we will turn later is certainly not semantically as straightforward as truth-tables. But then, neither is that for predicate logic, or even for intuitionistic propositional logic. Moreover, the formal theory is found exemplified not only in the English word 'if', but equally in German *wenn*, Latin *si*, and so on. I should also mention, if only to dismiss, the suggestion that even though material implication may not be the full story about 'if', which is much more complicated, at least it says nothing false about it. That idea we have seen to be wrong, since (at least in conjunction with its natural mate, the Classical Account) it wrongly predicts the above argument to be valid. The temptation arises, I think, because we have conceded that at least 'if' entails '$\supset$'. But that will only allow us to replace the first by the second in what are familiarly called positive occurrences or consequent-parts, not in negative ones, or antecedent-parts.[7]

The hypothesis that 'if' is a modal connective would be one way for the classical logician to reject the above argument that the Classical Account entails that 'if' is material. For if 'if' is modal, the Deduction Equivalence, as presently stated, fails, and so it would appear that the Classical Account can be sustained in the face of the apparent failed prediction (that, according to the Classical Theory, the above argument about John's whereabouts, is valid), by rejection of the auxiliary hypothesis, the Deduction Equivalence.

However, this move does not work. For if 'if' is modal, a revised Deduction Equivalence holds, where the parametric formulae are modalised. Consequently, we obtain a strict implicational form of the paradoxical consequence, namely, that 'if A then B' is true if A is necessarily false or B is necessarily true. We can show the incorrectness of this consequence by adapting the above example, replacing the contingent claim about John's whereabouts by a claim about, say, Goldbach's Conjecture, Roy claiming it is true, Crispin that it is false.

(1′) If every even number is the sum of two primes, then Roy was right.

(2) If Crispin was right, so was Roy.

(3′) If every even number is the sum of two primes, Crispin was right.

Again, the argument with, this time, (1′) and the negation of (2) as premises and (3′) as conclusion, is invalid, since it has true premises and a false conclusion. But to invalidate the corresponding classical modal sequent

$$P \mathrel{-\!\!3} Q,\ {\sim}(R \mathrel{-\!\!3} Q) \vdash P \mathrel{-\!\!3} R$$

P must be contingent. (The usual possible worlds model must contain a world where P is true and another where it is false.) But Goldbach's Conjecture, whatever its truthvalue, is not contingent. So if 'if' corresponds

[7] See Ch.6 3.

to strict implication, the particular argument given, focussing on a statement, Goldbach's Conjecture, which has the same value (whatever it is) in all worlds, is valid. Hence 'if' is not captured by strict implication, and this further attempt to hang onto the Classical Account of Validity fails.

One should not despair at this point. 'If' is not truth-functional, but strict implication is only one binary non-truth-functional connective. That 'if' cannot be identified with strict implication does not mean it defies analysis. In succeeding chapters, we will develop a non-modal theory of implication.

There is a third moral yet to be drawn from the above considerations. Suppose that Q followed validly from P and '$P \supset Q$'. Then, by the Deduction Equivalence, 'if P then Q' would follow from '$P \supset Q$'. But if '$P \supset Q$' is false then P is true and Q false, and we have seen that in that case 'if P then Q' is false. Hence, '$P \supset Q$' follows from 'if P then Q'. So, if Q did follow validly from P and '$P \supset Q$', '$P \supset Q$' and 'if P then Q' would be equivalent, and so 'if' would be truth-functional. But it is not, and so P and '$P \supset Q$' do not suffice to entail Q.

Indeed, not only is '$P \supset Q$' weaker than 'if P then Q'; the latter is the weakest additional premise which suffices with P to entail Q. For clearly it does suffice - by the Deduction Equivalence. Furthermore, anything which did suffice would, by the Deduction Equivalence, entail 'if P then Q'. Hence, 'if P then Q' is the weakest, or least, proposition R such that P and R entail Q. Accordingly, '$P \supset Q$', being strictly weaker, does not suffice.

Susan Haack appears to speak for many logicians of the classical persuasion when she fails to realise this point: "[Anderson and Belnap] don't deny that if '$P \vee Q$' (where '$\vee$' is truth-functional) is true, and '$\sim P$' is true, then necessarily, 'Q' is true."[8] In general, this inference fails. It is an interesting question, much studied in relevant logic, under what conditions it holds. The question is known as the γ-problem. γ holds for a theory T if whenever '$\sim P \vee Q$' and P belong to T, Q belongs to T. γ has been shown to hold for various relevant logics, such as **T**, **E** and **R** (see Ch. 4 below). It is an important, but open, question whether γ holds for relevant arithmetic (that is, Peano postulates added to relevant quantification theory).[9]

The present point is, however, that γ is a different issue from whether one can legitimately infer Q from P and '$\sim P \vee Q$'. One may not. That inference is invalid.

[8]Susan Haack, *Philosophy of Logics* (Cambridge, 1978), p. 201.

[9]It is now known that γ fails for relevant arithmetic.

2.6 The Lewis Argument

On that note, we can return to the famous derivation of Q from P and 'not-P' which Lewis gave. The derivation was not original to Lewis. The earliest presentation I know is given by Alexander Neckam in his De Naturis Rerum, written around 1200.[10] In Ch.173, Neckam retails many of the logical subtleties of the Parisian school of Adam of Balsham Parvipontanus (du Petit-Pont) where he studied in the 1170s. Included among these is the following derivation. The derivation proceeds rapidly by two applications of Simplification, one of Addition, and the notorious final step of Disjunctive Syllogism:

> I am amazed also at those criticising the claim that from the impossible in itself anything whatever follows. This may be established in many ways, but few will show more clearly. Is it not the case that if Socrates is a man and Socrates is not a man, then Socrates is a man? But if Socrates is a man, Socrates is a man or a stone. So if Socrates is a man and Socrates is not a man, Socrates is a man or a stone. But if Socrates is a man and Socrates is not a man, Socrates is not a man. So if Socrates is a man and Socrates is not a man, Socrates is a stone. By a similar deduction, it may be proved that if Socrates is a man and Socrates is not a man, Socrates is a goat, and so on for any other thing, such as a rose, a lily and so on. Don't you therefore see that in this way from this impossibility, that Socrates is a man and Socrates is not a man, anything follows?[11]

The structure of the argument is as follows:

Suppose	P and not-P	
Then	P	by Simplification,
whence	P or Q	by Addition,
	not-P	by Simplification again,
and finally	Q	by Disjunctive Syllogism.

We saw at the end of the last section that, reading 'not-P' for P, 'if not-P then Q' is the weakest additional premise which, with 'not-P', suffices correctly to infer Q. Hence, 'P or Q' used here to this effect in line 3, must entail 'if not-P then Q'. Yet that disjunction is in turn entailed by

[10] Alexander Neckam, *De Naturis Rerum*, ed. T. Wright (London, 1863).

[11] Ibid., ch. 173, pp. 288-9. C. Martin suggests the argument stems from William of Soissons: see his 'William's Machine', *Journal of Philosophy*, 83 (1986), p. 571.

P, and so, by the transitivity of entailment, P must entail 'if not-P then Q'. That, however, was one of the apparent consequences, by the Deduction Equivalence, of EFQ which led to the unacceptable conclusion that 'if' is truth-functional.

This argument will not satisfy the critic of §2.5 who detected a modal fallacy in the argument for the Deduction Equivalence. However, he had to concede that if the parametric sentences were appropriately modal, the argument went through. Hence, if 'P or Q' is modal, it is equivalent to 'if not-P then Q'. In other words, if both P and Q are non-contingent, the Lewis argument fails. It shows at most that if either P or Q is contingent then P and 'not-P' entail Q. This should be sufficient to discomfit the modal critic.

That complication aside, the situation is effectively this: A entails B, B entails C, but A does not entail C. P entails 'P or Q'; 'P or Q' entails 'if not-P then Q'; but P does not entail 'if not-P then Q'. Two solutions are possible: that entailment is not transitive, or that one (or more) of A, B or C is ambiguous. Excluding for the moment the former alternative, think of our usual grounds for ascribing ambiguity. It is because some assertion seems to be both true and false. To avoid contradiction, we postulate that what was asserted has two senses - perhaps arising from ambiguity in a word, sometimes from ambiguity in a syntactic construction. The present case is similar, but not quite the same: if A is true, so must be B, since A entails B. Moreover, if B were true, so too would be C, since B entails C. But it does not follow that C is true, since A does not entail C. From this, we cannot infer that it is possible that B is both true and false. Only on the Classical Account does it follow that if A does not entail C then it is possible that A is true and C false. Rather, it both follows from A that B is true, and does not follow from A that B is true. But the effect is the same: a plausible explanation is that B, that is, 'P or Q', is ambiguous. And the obvious location of that ambiguity is in the word 'or'.

2.7 Three Senses of Disjunction

In (at least) one sense, 'or' is indeed truth-functional: the truth of 'P or Q' follows immediately from that of P or of Q, and it is false if both are false. In other words,

$$P \vdash P \text{ or } Q$$
$$Q \vdash P \text{ or } Q$$
$$\text{not-}P, \text{not-}Q \vdash \text{not-}(P \text{ or } Q)$$

In this sense, the values of P and Q completely determine the value of 'P or Q'. But there must be another sense, since if 'P or Q' does serve to support the inference from 'not-P' to Q (and equivalently, from 'not-Q' to P), then 'P or Q', so occurring, does not follow from P or from Q. In its non-truth-functional sense, 'P or Q' is equivalent to 'if not-P then Q', taking over from 'if' its non-truth-functional character. (It's at least as strong as 'if not-P then Q'. It's implausible to suppose it stronger.) This sense of disjunction is often referred to familiarly as *fission*. I will denote it by '$P+Q$'. In contrast, I will use the familiar '$P \vee Q$' (vel) for the (familiar) truth-functional connective. These senses cannot be the same, for Addition, though invalid for fission, is clearly valid for vel. In contrast, Disjunctive Syllogism, used in Lewis' argument, though valid for fission, is invalid for vel. For if the ground for asserting 'P or Q' is P, say, then to learn that P is false (i.e. 'not-P' is true), far from enabling one to proceed to Q, undercuts the warrant for asserting 'P or Q' in the first place. When one uses Addition one has weakened one's position. There is no suggestion here that the original assertion (or assumption, or whatever) of P must be withdrawn. But from the weakened position neither P nor Q any longer follow. That is indeed what makes 'P or Q' (so inferred) weaker than P—its deductive closure is a proper subset of that of P.

By way of analogy, consider Simplification, the inference of Q from 'P & Q'. Q is a weaker assertion than 'P & Q', and accordingly entails less. In particular, it does not entail P, even though 'P & Q' does. Similarly, '$P \vee Q$' is weaker than P, and does not, even with the addition of 'not-Q', entail P, even though P clearly does.

The intensional sense of 'or' can also be seen in the earlier counterexample to the thesis that 'if' is truth-functional. Take the propositions about John's whereabouts, and express them using 'or'. Then the truth-values remain the same:

Either John was not in Edinburgh or Roy was right	True
Either Crispin was wrong or Roy was right	False
Either John was not in Edinburgh or Crispin was right	False

Treating 'or' here as truth-functional would permit one to argue as follows: from the contradictory of the second proposition, one could infer that Crispin was right, and so that the third proposition was true (by Addition). That would give us a valid argument with true premises and a false conclusion. Since that is impossible, 'or' is used here non-truth- functionally, to express the connection between the elements of these propositions, independently of where John actually was. We know the truth-values of the three propositions, given only what we know of Roy's and Crispin's claims, and knowing nothing of John's actual whereabouts.

The suggestion of equivocation in Lewis' argument should not seem extraordinary to anyone who reflects on another ambiguity in the use of 'or' which is more often recognised, between inclusive and exclusive senses, both truth-functional. All conference-attenders are familiar with the canteen injunction, 'Cereal or Fruit Juice', the implication of which is that, having taken cereal one may not then also take fruit juice. In other words

$$P \text{ or } Q,\ P \vdash \text{not-}Q$$

Let us call this mode of inference 'FJE' (short for 'Fruit Juice Elimination'). Combining Addition and FJE in a single argument, we can very quickly demonstrate the incompatibility of any pair of propositions whatever:

given	P	
then	P or Q	by Addition
whence	not-Q	by FJE.

Clearly the reasoning here is fallacious, consequent on the equivocation in the sense of 'or' at line 3. There is no single sense of 'or' in which 'P or Q' follows from P alone, and for which 'P or Q' with P entails 'not-Q'. Nonetheless, there is a sense of 'or' for which Addition is valid, and there is a sense of 'or' for which FJE is correct. The above argument confirms that they are different.

Lewis' argument similarly confirms that the intensional sense of 'or' is yet a third meaning to 'P or Q'. 'P or Q' can be taken to mean 'if not-P then Q'; it can also be taken to follow weakly from P alone. But 'if not-P then Q' does not follow from P. Hence these two interpretations of 'P or Q' cannot be identical.

2.8 Conclusion

To summarise: there seem to be good grounds for accepting that the impossibility of true premises and false conclusion is both necessary and sufficient for the validity of an argument, and that the validity of an argument and the truth of the conditional are tied by the Deduction Equivalence. However, these three theses cannot be upheld simultaneously, since together they entail that 'if' is truth-functional, which it is not.

The position for which I have argued is that the first and third theses can be retained while rejecting the second: just as falsity of antecedent and truth of consequent do not suffice for truth of the conditional, so too impossibility of true premises and false conclusion does not suffice for validity of argument. It follows that the Classical Account of Validity, and belief in the validity of detachment for the material conditional must be abandoned. Moreover, 'or', which is tied both to material implication, in that of its senses for which Addition holds, and to the conditional, in that of its senses for which Disjunctive Syllogism is valid, must be recognised to be ambiguous, once material implication and the conditional are seen to come apart.

In the next chapter I will proceed to look more closely at the Deduction Equivalence in order to state it with greater care, and so to ensure a firmer grip on the conditional through its connection thereby with the notion of validity. For we establish the truth of the conditional, via the Deduction Equivalence, by showing what follows from what, that is, by establishing validity.

Chapter 3

An Intensional Conjunction

We have seen that adopting the impossibility of true premises and a false conclusion as a sufficient condition of validity leads to unacceptable consequences - that the theory predicts an invalid argument to be valid. We therefore lack for the present any sufficient condition. The condition we have remains as a necessary condition only; certainly, if an argument is valid then it is impossible for the premises to be true and the conclusion false.

Lewis's purported proof that P and 'not-P' entail Q typified, however, an alternative approach to the question of validity, and in particular, to its sufficiency condition, namely, articulating an argument into a succession of elementary inference steps. Some (usually small) finite or at least recursive class of elementary inference-schemata are accepted as valid - as conjectures within the theory. The aim is thereby to characterise validity or consequence, in such a way that the recursive set fixes, and is a basis for, all valid argument. It is the possibility of such an alternative method which makes the option of denying transitivity of entailment unattractive, as we shall see below.

3.1 Conjunction

Among elementary inference-schemata which have commonly been thought to be valid, two concern conjunction, viz. Adjunction and Simplification:

> 'A and B' follows from A and B
> A and B follow from 'A and B'.

From these we can infer, by transitivity, that B follows from A and B, and so by the Deduction Equivalence, 'if A then B' follows from B. However, we saw in chapter 2 that this has the unacceptable consequence that 'if' is truth-functional. How can this consequence be avoided?

Many options are open: we can reject the transitivity of entailment, the Deduction Equivalence, Adjunction, or Simplification. Each has been tried; and each seems contrary to intuition. We are again in the paradoxical situation that each of these conceptions seems intuitively soundly based; yet their combination appears to lead to something unacceptable. Are we nonetheless forced to reject one of these plausible principles?

Fortunately, there is a fifth option. To understand it, we need to return to the argument, given at the beginning of chapter 2, which appeared to support the Classical Account of Validity, and which we have recently found wanting, that is, the argument purporting to show that the impossibility of true premises and false conclusion is sufficient for an argument to be valid. What, then, was wrong with that argument?

It went as follows: an argument is valid if the conclusion follows from the premises. But that is so if the conclusion must be true (cannot be false) whenever the premises are true. In other words, an argument is valid if it is impossible for the conclusion to be false and the premises true. But then, if the premises cannot be true, it is indeed impossible for the premises to be true and the conclusion false, and so we are immediately committed to *ex falso quodlibet* (EFQ), and so to its unacceptable consequences. Where is the error?

It lies again in an ambiguity. A entails B, and B entails C, but A does not entail C. In this case, that the premises cannot be true entails that it is impossible for the premises to be true and the conclusion false, and that the premises cannot be true and the conclusion false entails that the argument is valid; yet the necessary falsity of the premises does not entail that the argument is valid. Formally:

'It is impossible that P' entails 'It is impossible that P and not-Q'
'It is impossible that P and not-Q' entails 'P entails Q'

But

'It is impossible that P' does not entail 'P entails Q'.

The explanation is an ambiguity in the sense of 'and' exactly analogous to that which we discovered for 'or', giving rise to an ambiguity in the middle term, 'It is impossible that P and not-Q'. There is a familiar truth-functional conjunction, expressed by 'A and B', which entails each of A and B, and so for the falsity of which the falsity of either conjunct suffices, and the truth of both for the truth of the whole. But there is also a non-truth-functional conjunction, a sense of 'A and B' whose falsity supports the inference from A to 'not-B'. These senses cannot be the same, for if the ground for asserting 'not-(A and B)' is simply that A is false, then to learn that A is true, far from enabling one to proceed to 'not-B', undercuts the warrant for asserting 'not-(A and B)' in the first place. In this sense, 'not-(A and B)' is weaker than both 'not-A' and 'not-B', and does not, even with the addition of A, entail 'not-B', even though one possible ground for asserting 'not-(A and B)', viz 'not-B', clearly does.

Unsurprisingly, therefore, the two senses of 'or' and of 'and' are linked in the familiar De Morgan way. The intensional sense of 'and' is often referred to as *fusion*; I will use the symbol '$\times$' for it (others write '$\circ$'). This matches use of '+' in chapter 2 for fission. Let us use '$\rightarrow$' to represent the (non-truth-functional) conditional and '$\sim$' for negation. In the system R to be developed in chapters 4-5, the following equivalences hold:

$\sim(A \,\&\, B)$	is equivalent to	$\sim A \vee \sim B$
$\sim(A \times B)$		$\sim A + \sim B$
$\sim(A \,\&\, B)$		$A \supset \sim B$
$\sim(A \times B)$		$A \rightarrow \sim B$
$\sim A \vee B$		$A \supset B$
$\sim A + B$		$A \rightarrow B$

The fact that '+' and '$\times$' are not equivalent to '$\vee$' and '&' respectively follows from the fact that '$\rightarrow$' is not truth-functional.

3.2 The Deduction Equivalence

Given this ambiguity in the sense of 'and', the pair of rules given earlier by the names Adjunction and Simplification must be ambiguous, depending on the sense of 'and' employed in them. Taking a uniform reading of 'and' throughout, we have two pairs of rules,

(1) (a) '$A \,\&\, B$' follows from $A \,\&\, B$

(b) $A \,\&\, B$ follow from '$A \,\&\, B$' .

(2) (a) '$A \times B$' follows from $A \times B$

(b) $A \times B$ follow from '$A \times B$'.

Let us keep the names 'Adjunction' and 'Simplification' for the rules in (1). Then, since '&' is truth-functional, and the falsity of A (or B) suffices for the falsity of the whole, we know that each of A, B follows from 'A & B'. So, in particular, we can infer that B follows from 'A & B', which in turn follows from A and (&) B. Hence, by transitivity, B follows from A and (&) B. Suppose, however, it were the case that '$A \times B$' entailed A. Then, by contraposition, '$\sim A$' would entail '$\sim(A \times B)$'. But the latter entails (indeed, is equivalent to) '$A \rightarrow\ \sim B$'. So the falsity of A would suffice for the truth of 'if A then not-B', which, we have seen, it does not. Hence, '$A \times B$' does not entail A. Neither, for the same reason, does it entail B.

We are now ready to return to the paradoxical argument in 1,that if B is true, so is 'if A then B'. It relied on the application of the Deduction Equivalence to the consequence of Adjunction and Simplification, that B follows from A and (&) B. The conjunction of A and B here is extensional. Hence, if the conjunction used in the Deduction Equivalence were extensional, it would indeed follow that 'if' was truth-functional. (For, as we have seen, if we concede that, say, the truth of B suffices for that of 'if A then B', then so does the falsity of A.) In other words, when we pass to

> 'if A then B' follows from assumptions C

from

> B follows from A and C,

the occurrence of 'and' here must be intensional, on pain of making 'if' extensional.[1] It follows, therefore, that the correct statement of the Deduction Equivalence is:

> B follows from $A_1, \ldots, A_n$ and (fuse) A

iff

> 'if A then B' follows from $A_1, \ldots, A_n$.

Hence, in the paradoxical argument at the beginning of §3.1 for the truth-functionality of 'if', the fallacy arose from an equivocation over the sense of 'and'. To show that B follows from A and B, we need to take 'and' extensionally; but to infer from this that 'if A then B' follows from B, we need to take 'and' intensionally. There is no univocal sense of 'and' in which the argument goes through.

[1] This point was realised by Michael Clark in 'Ifs and Hooks: a Rejoinder', *Analysis*, 34 (1973-4), pp. 82-3; but used to opposite effect, namely, to argue from an extensional Deduction Equivalence to an extensional account of 'if'.

This diagnosis solves the outstanding issue from the last chapter. There we were able to infer that the Classical Account of Validity was mistaken, since it entailed the demonstrably unacceptable consequence that 'if' was truth-functional. But we did not at that point have any answer to the plausible considerations which appeared to suggest, at the start of that chapter, that the Classical Account was the right one. We can now understand the error. The conjunction in what we may call the Standard Account of Validity, is intensional. It is indeed true that an argument is valid if it is impossible for the premises to be true and (in its intensional sense, fusion) the conclusion false. But from the fact that the premises cannot be true, we cannot correctly infer that the premises cannot be true fuse the conclusion false: '$\sim\Diamond P$' does not entail '$\sim\Diamond(P \times Q)$' (where '$\Diamond$' represents possibility), since '$P \times Q$' does not entail P. Hence, interpreting the Standard Account properly, it does not follow that every argument with contradictory premises is valid—nor is every one with a necessarily true conclusion, for the same reason.

3.3 Bunches

It may be objected at this point that 'and' in

> B follows from A and C

is not a connective (whether it is here extensional or intensional), for it conjoins not two sentences but two names. But the properties of this conjunction are closely tied to those of the sentential connective. Let us call the string of sentence-names in the Deduction Equivalence, a bunch . Then to each bunch there naturally corresponds a single sentence(-name), as follows:

> B follows from $A_1, \ldots, A_n$

iff

> B follows from A_1 and $\ldots$ and A_n

iff

> B follows from 'A_1 and $\ldots$ and A_n'.

Whether the name-conjunction is extensional or intensional is determined by the nature of the sentence-conjunction to which it is naturally tied. In general, a bunch of sentences will consist of iterated extensional and intensional bunching of its constituent sentences:

1. Any formula is an I-bunch (an intensional bunch);
2. if X, Y are bunches, X and (fuse) Y is an I-bunch.
3. if X, Y are I-bunches, X and (&) Y is an E-bunch (an extensional bunch—more generally: any non-empty set of I-bunches is an E-bunch);
4. no I-bunch is an E-bunch.

Once we have these distinctions in mind, we can proceed to develop an appropriate formal theory of validity, based on the articulation of inferential steps into a succession of elementary inferences grounded in the senses of the connectives concerned. The distinction between the intensional and the extensional must be reflected, we have seen, not only in the formulae of the object language, but also in the combinations of those formulae in the premises of arguments. This led us to distinguish, therefore, intensional from extensional combinations of premises. In a natural deduction system, the validity of an argument is established by exhibiting a formal sequent corresponding to the argument as the final member of a sequence of sequents each of which is either an assumption-sequent, or follows by the rules of immediate consequence from earlier sequents. Accordingly, sequents will contain a bunch of premises, conjoined both extensionally and intensionally, and a formula asserted on the basis of those premises. That is, the building blocks of proofs will be sequents of the form

$$X : A$$

where X is a bunch of premises, and A is a formula.

3.4 Consequence

Tarski and Scott have given extensive analysis of the consequence relation.[2] Their difference is that for Scott the consequence relation allows multiple (that is, empty, singleton or many-membered) conclusions. What unites them is that the two terms of the consequence relation are taken to be sets—the second member in Tarski's case being a singleton.

[2]See, e.g., D. Gabbay, *Semantical Investigations in Heyting's Intuitionistic Logic* (Dordrecht, 1981), chapter 1; also D. Scott, 'Completeness and axiomatizability in many-valued logic', *American Mathematical Society, Proceedings of Symposia in Pure Mathematics*, vol. 25 (in honour of Tarski) (1974), pp. 411-35.

It is not only sets which have been taken as the components of the consequence relation, however. For example, in Gentzen's seminal investigations of the notion, both components were finite sequences (that is, ordered sets, again, empty, singleton or many-membered).[3] In Meyer and MacRobbie's analysis of relevant logic, the first component was taken to be a fireset or multiset (that is, like a sequence in being sensitive to repetitions of elements; like a set in being insensitive to order).[4] In Belnap's display logic, presenting a general theory of extensional, modal and relevant logics, they were sui generis indeed, different kinds of object for each different logic.[5]

Moreover, whatever the components, they need not be finite. Most commonly, indeed, consequence is taken as a relation between a set of premises, finite or infinite, and a conclusion, single or multiple. But what it is important to be clear about, whether the components be sets, sequences, firesets or bunches, are the generic properties of the consequence relation, which make it a relation of consequence between premises and conclusion. Let us, then, consider the various proposals which have been canvassed for generic and specific properties of the consequence relation.

I will concentrate here on consequence relations on the Tarski model, where the conclusion is a single formula; but the collection of premises I shall take to be some arbitrary type of collection of formulae, yet to be specified. '$\vdash$' will represent the consequence relation. Let X, A be whatever components '$\vdash$' relates. Then $X \vdash A$ reads 'A is a consequence of X'. In Gabbay's presentation, the Scott and Tarski consequence relations have three explicit properties. Combination of premises is marked by the comma:[6]

1. <u>Reflexivity</u>: $A \vdash A$
2. <u>Weakening</u> (or <u>Monotonicity</u>): if $X \vdash A$ then $B, X \vdash A$
3. <u>Cut</u>, or <u>transitivity</u>: if $X \vdash A$ and $A, Y \vdash B$ then $X, Y \vdash B$.

I shall distinguish such properties as <u>generic</u> properties of the consequence relation, contrasted with specific properties dependent on the <u>specific</u> nature of the components X and A. Gentzen calls the generic properties 'structural',

[3] G. Gentzen, 'Investigations into Logical Deduction', tr. in *The Collected Papers of Gerhard Gentzen*, ed. M. Szabo (Amsterdam, 1969), pp. 68-131.

[4] R. Meyer and M. MacRobbie, 'Multisets and Relevant Implication', *Australasian Journal of Philosophy*, 60 (1982), pp. 107-39, 265-81.

[5] N. Belnap, 'Display Logic', *Journal of Philosophical Logic*, 11 (1982), pp. 375-417.

[6] Reflexivity is often given in the form of an overlap rule: $X \vdash A$ if $A \in X$, or more generally, $X \vdash Y$ if X, Y have some member in common, where X, Y are sets. This, however, builds a form of Weakening into Reflexivity. We must keep them distinct.

the specific properties 'operational' . The operational properties are shown by citing operational inference rules, specifying properties of operations on formulae. The structural or generic properties are shown by the structural inference rules.

There are, however, other generic properties of the consequence relation on the Scott or Tarski model. They are not given explicitly by Gabbay, but are implicit in his use of sets for the components of the relation. In taking the components to be sequences, Gentzen was forced to make them explicit, but they are shared by both conceptions. As noted, sets are insensitive to associativity, permutation, contraction and multiplication of elements. Thus these are also generic or structural properties of the consequence relation on the Scott/Tarski model:

4. Associativity: $(X, A), Y \vdash C$ iff $X, (A, Y) \vdash C$
5. Permutation: if $X, A, B, Y \vdash C$ then $X, B, A, Y \vdash C$
6. Contraction: if $X, A, A, Y \vdash C$ then $X, A, Y \vdash C$
7. Premise repetition: if $X, A, Y \vdash C$ then $X, A, A, Y \vdash C$.

(7 is of course a special case of 2, given 5.)

There is yet another property implicit in Gabbay's presentation, stemming from his use of schematic variables, namely, substitution.[7] In other words,

8. Substitution (or Replacement): if $X \vdash A$, and X', A' result from X, A respectively by uniform substitution of B for C, then $X' \vdash A'$.

Finally note that, in admitting the empty set or sequence as possible value of X, Cut implicitly includes a notorious principle, namely, suppression:

9. Suppression: if $X, A \vdash B$ and $\vdash A$ then $X \vdash B$.[8]

It follows from our earlier considerations, that not all these properties are acceptable, in the form given. Indeed, a result of Pogorzelski's (cited by

[7] Tarski made the property explicit: A. Tarski, 'On the Concept of Logical Consequence', in his *Logic, Semantics, Metamathematics*, tr. J.H. Woodger (Oxford, 1956), p. 415; see also his 'On Some Fundamental Concepts of Metamathematics', *Logic, Semantics, ...*, pp. 30-37; and 'Fundamental Concepts of the Methodology of the Deductive Sciences', *Logic, Semantics, ...*, pp. 61-109.

[8] T. Smiley listed properties 1) - 3), 5) - 7), and 9), 4) and 8) being implicit, in his presentation: 'Entailment and Deducibility', *Proceedings of the Aristotelian Society*, 59 (1958-9), pp. 233-54.

Gabbay) puts the same point in different terms.[9] Pogorzelski showed that the intuitionistic implicational calculus is the weakest logic for which the Deduction Theorem holds. But of course, in that calculus $B \vdash A \to B$ holds, which we have found unacceptable. (Since negation is not expressible in the pure implicational calculus, EFQ and $\sim A \vdash A \to B$ are not expressible; but they will ensue in any standard addition of negation.)

3.5 Relevant Logic

In 1958, Anderson and Belnap took up ideas from Church and Ackermann, and started a research program into what in time became 'relevance (now often called, 'relevant') logic' .[10] Their chosen name picked up an informal use before that time of the epithet 'relevant' to characterise a consequence relation, and an implication, which was not paradoxical in the way material and strict implication were; in other words, for which EFQ fails. I will leave until chapter 6 consideration of whether this term is appropriate for the logic we are presently picking out. Moreover, although I will often use the strongest consequence relation compatible with rejection of EFQ and acceptance of the Deduction Equivalence and the Standard Account of Validity (suitably understood)—identifiable with Anderson and Belnap's calculus **R** of relevant implication—I will lay out a framework within which much weaker logics may be treated, and in which many of the above structural properties of the consequence relation fail to hold.

But it is unintuitive to maintain, as some have, that any of the generic properties given in the last section for the consequence relation *simply* fail. What our earlier considerations show is that proper appreciation of the intensional/extensional dichotomy enables us to preserve, as one might say, honour on both sides. There is something right about properties such as weakening and premise suppression; but they must be properly understood to avoid mistaken consequences.

Consider Pogorzelski's point about the Deduction Theorem (or Equivalence) and the intuitionistic calculus. By Reflexivity, $A \vdash A$. By Weakening,

[9] W. Pogorzelski, 'On the scope of the classical deduction theorem', *Journal of Symbolic Logic*, 33 (1968), pp. 77-81.

[10] A. Anderson and N. Belnap, 'A modification of Ackermann's "rigorous implication"' (abstract), *Journal of Symbolic Logic*, 23 (1958), pp. 457-8. See also A. Church, 'The weak theory of implication', *Kontrolliertes Denken, Untersuchungen zum Logikkalkl und der Einzelwissenschaften*, ed. A. Menne, A. Wilhelmy and H. Angell (Munich, 1951), pp. 22-37; W. Ackermann, 'Begründung einer strengen Implikation', *Journal of Symbolic Logic*, 21 (1956), pp. 113-28.

$B, A \vdash A$. By Permutation, $A, B \vdash A$. So by the Deduction Equivalence (the 'classical deduction theorem', as Pogorzelski explicitly calls it), $A \vdash B \to A$. Our earlier discussion pointed to the fact that there is here an equivocation over the way the premises are conjoined. Weakening is indeed valid, *if* the combination of premises is extensional, that is, if we are considering simply the set of premises $\{A, B\}$. But the Deduction Equivalence is correct only if A and B are more closely connected than this mere extensional joining signifies. We avoid this equivocation by distinguishing two ways in which premises may be conjoined, the familiar, extensional manner, and an intensional way independently needed to maintain the Standard Account of Validity.

Taking the results of these two manners of combination, the bunch, we must now ask which of the properties 1) - 9) are applicable, and if so, how, to the conception of a formula being a consequence of a certain bunch of formulae. Note then, that premise combination is not mentioned in 1) or 8). They stand unchanged, therefore. As regards 2), we saw earlier that Weakening is extensionally acceptable, intensionally not so. For 4), 5) and 6), Associativity, Permutation and Contraction are again clearly acceptable for the extensional, set-theoretic mode; whether they are intensionally acceptable we leave open (we admit them in **R**; weaker logics omit them). Clearly 7), Premise Repetition, is acceptable for the extensional combination, being a special case of weakening. If it were admitted for the intensional mode, however, it would lead to consequences such as $A \vdash A \to A$, and yield the logic **RM**, or **R**-mingle.

Finally, what of Cut (3), and its special case if X is empty, Suppression (9)? In itself, Cut is perfectly acceptable, for both intensional and extensional combination. Indeed, it is essential to the very idea of characterising consequence proof-theoretically, by demarcating a recursive set of elementary inferences in such a way that every consequence can be recursively enumerated. If we have established that $X \vdash A$ and that $A, Y \vdash B$, we want to combine the two demonstrations into a single proof that $X, Y \vdash B$. Note that 3) gives what Shoesmith and Smiley call 'Cut for formulae'.[11] They show that it is in general weaker than their preferred Cut, which allows infinitely many formulae to be 'cut out'. However, they also show that for compact logics the two versions are equivalent[12] and it is compact logics which we will consider below. (The proof requires Weakening, which we have seen is indeed acceptable for the extensional combination, where the

[11] D. Shoesmith and T. Smiley, *Multiple-conclusion Logic* (Cambridge, 1978), p. 17.

[12] Ibid., p. 18

proof uses it.)

But Suppression is not acceptable. Familiarly, it reintroduces the paradoxes. Suppose B is logically necessary. Then (as usually understood) $\vdash B$. By Reflexivity and Weakening, $A, B \vdash B$. So by Suppression, $A \vdash B$. That is, anything entails a necessary truth.

The obvious thought, following on our earlier reflections, is that Suppression must be restricted to the intensional combination. It could never then be combined with Weakening to give the paradoxical consequences. But how is this to be done, since it is simply a special case of Cut? The answer is to come clean about what '$\vdash A$' means, that is, what it means to say that A is a logical consequence of nothing. For there is here yet another equivocation. 'Nothing' can be taken either intensionally or extensionally. To say that A is logically true is to say that A is a consequence of the logic, that is, of nothing additional to what the logic can provide. Let us make this explicit by use of '$\mathtt{t}$' for the logic, that is, $\vdash A$ is shorthand for $\mathtt{t} \vdash A$. Then, given that $A, B \vdash B$ as above, we obtain $B, A \vdash B$ by Permutation, and so $t, A \vdash B$, by Cut. But that is no surprise, for $\mathtt{t} \vdash B$ yields $t, A \vdash B$ immediately, by Weakening.

Consider any logically true implication, $\vdash A \to B$. Then $\mathtt{t} \vdash A \to B$, by our convention for $\mathtt{t}$. Hence, by the Deduction Equivalence, $\mathtt{t}, A \vdash B$, where the combination of $\mathtt{t}$ and A is intensional, says no more nor less than $A \vdash B$. Thus $\mathtt{t}$ is intensionally invisible, or suppressible. Harm ensues only if we were to allow $\mathtt{t}$ to be suppressed extensionally, as in the deliverance of Weakening above, $\mathtt{t}, A \vdash B$.

Consequently, we may avoid the final paradox, of suppression, by requiring that in $X \vdash A$, the premise X be a non-empty bunch. The former role of the logically empty set of assumptions, that is, the logic, is taken not by the empty set (an extensional notion) but by $\mathtt{t}$ (an intensional constant—a left-identity for intensional bunching). We thus preserve the intuitively plausible properties 1) - 9) by proof analysis. For the extensional combination, 1) - 6) and 8) are acceptable. For the intensional, 3) is clearly acceptable, 2) and 7) not so; 4), 5) and 6) may be admitted as desired (see chapter 4). 9) now couples $\mathtt{t} \vdash A$ iff $\vdash A$ with what in chapter 4 will be written $\mathtt{t}; X \geq X$, that is, $\mathtt{t}$ may be suppressed when intensionally joined.

3.6 Distribution

We will develop the formal theory in chapter 4. Let us note one striking consequence of the present deliberations. Anderson and Belnap wish to deny that 'if A then B' follows from B. So, when confronted with the paradox of §3.1, they feel compelled to deny that Adjunction holds in its usual natural deduction form. They concede that if A and B have each been proven, we may conclude that 'A and B' is true: if $\vdash A$ and $\vdash B$ then $\vdash A \,\&\, B$. What they deny is that if A and B are simply true on assumption, then so is '$A \,\&\, B$'. In other words, they reject the validity of the sequent

$$A, B \vdash A \,\&\, B,$$

or spelt out in full,

if $X \vdash A$ and $Y \vdash B$ then $X, Y \vdash A \,\&\, B$.

(Essentially, they read the comma in sequents as intensional conjunction.) They insist that to be valid it read:

if $X \vdash A$ and $X \vdash B$ then $X \vdash A \,\&\, B$.

For they fear the following inference steps. Suppose $A, B \vdash A \,\&\, B$. Clearly $A \,\&\, B \vdash A$. So (as in 1 above) by transitivity, or Cut, $A, B \vdash A$. Hence $A \vdash B \to A$, by the Deduction Equivalence. The point, once again, is that A must have been extensionally conjoined at the start of that sequence of inference for the first conclusion $A \,\&\, B$ to follow; but the final step would require intensional conjoining. So once again, the fallacy is caused by an equivocation, between intensional and extensional combination of premises.

Anderson and Belnap impose a similar restriction on inferences concerning 'or'. In consequence, they find that their natural deduction rules no longer warrant the assertion of Distribution, that '$A \,\&\, (B \vee C)$' is equivalent to '$(A \,\&\, B) \vee (A \,\&\, C)$'. Their natural deduction formulation is, therefore, out of step with other criteria of validity they favour, whereby Distribution is a valid first degree entailment.

This unintuitive denial is no longer necessary. If A and B are both true, where 'and' is interpreted extensionally, so is '$A \,\&\, B$'; if A and B are both true, where 'and' is interpreted intensionally, so is '$A \times B$'. The proof of Distribution is immediate and straightforward (see Proposition 4.8). Anderson and Belnap's fear that, if the unconstrained moves of Adjunction were allowed, the Deduction Equivalence would render the conditional material, turns out to be unfounded. It depends on not distinguishing the intensional sense of 'and' needed for that part of the Deduction Equivalence constituting Conditional Proof from the extensional sense needed for Adjunction. Our care in finding the correct formulation of the Deduction Equivalence, and consequently of the rules of Conditional Proof and *Modus Ponens*, shows that this fear is unfounded.

3.7 Antilogism

In §2.6 we considered Disjunctive Syllogism. Perhaps its place in Lewis' argument is the explanation for its notoriety in the Classical/Relevant dispute. Nonetheless, it is surprising that so much ink has been used up on that particular inference, largely ignoring another bone of contention between Classicist and Relevantist. For Antilogism has surely held an even more central place in logical thinking. Yet its validity must (apparently) be denied by the Relevantist.

Antilogism is essentially a meta-rule (like the Deduction Equivalence), allowing one to infer the validity of one scheme of inference from that of another; namely,

> 'not-B' follows from A and 'not-C'

iff

> C follows from A and B.

The scheme is famous for the use made of it by Christine Ladd-Franklin.[13] But by another name it was equally famous in Aristotle's theory of the syllogism, for it constitutes the basis of his reductio per impossibile. For example, the validity of Baroco (aoo in figure II) follows from that of Barbara (the first of the four basic moods, aaa in figure I) by this scheme.[14]

Despite its pedigree, Antilogism has been rejected as invalid by relevant logicians. The argument is as follows: suppose Antilogism were valid. Clearly A follows from A and 'not-B', by Simplification. So by Double Negation (and transitivity), 'not-not-A' follows from A and 'not-B'. Hence, by Antilogism, B follows from A and 'not-A'. If, therefore, we are right to reject EFQ, one of Antilogism, Simplification or Double Negation must go. Relevantists have retained the other two, and so rejected Antilogism.

Following our earlier reflections, it will come as no surprise to suggest that we diagnose the problem here as an equivocation over the sense of 'and'. Thereby, honour will again be preserved on both sides. We can see this by considering how closely Antilogism is linked with the Deduction Equivalence:

> 'not-B' follows from A and 'not-C'

[13] See, e.g., A. Prior, *Formal Logic* (Oxford, 1956), p. 113. Strictly, in Ladd-Franklin's and Johnson's usage, an antilogism is an invalid form, consisting of the premises and the contradictory of the conclusion of a valid syllogism. In other words, they take an inconsistent triad of propositions, such that if two of them are true, the third must be false, to be the basis for all syllogistic inference.

[14] For explanation of this terminology, see, for example, ibid., p. 110ff.

iff

> 'if not-C then not-B' follows from A (by the Deduction Equivalence)

iff

> 'if B then C' follows from A (by Contraposition)

iff

> C follows from A and B (by the Deduction Equivalence).

Clearly, therefore, the conjunction used in Antilogism is intensional. However, the argument for EFQ in the last paragraph, which relied on Antilogism, applied the scheme to the result of Simplification, that 'not-not-A' follows from A and 'not-B'. It exemplifies, therefore, the same equivocation between intensional and extensional conjunction which we have diagnosed elsewhere.

Without that independent identification, simply postulating an equivocation would be open to the charge of being ad hoc . However, we now find the same equivocation underlying disparate problematic cases. Different aspects of the same essential fallacy are found in each. There is an underlying unity in the need to distinguish intensional from extensional modes of combination. Our fundamental objection to EFQ shows that some error has been made in the use of Antilogism. It turns out that the mistake was to confuse the invalid (extensional) principle

$$A \mathbin{\&} {\sim}C \vdash {\sim}B \text{ iff } A \mathbin{\&} B \vdash C$$

with the valid intensional principle

$$A \times {\sim}C \vdash {\sim}B \text{ iff } A \times B \vdash C.$$

This is equivalent, by the Deduction Equivalence, to

$$A \vdash {\sim}C \to {\sim}B \text{ iff } A \vdash B \to C,$$

whose correctness is immediate from Contraposition and Cut.

3.8 Conclusion

There is an intensional sense of conjunction linked to the intensional disjunction in the same way that extensional conjunction is linked to extensional disjunction. That intensional sense, fusion, provides the correct combination of true premises and false conclusion whose impossibility yields a necessary and sufficient condition for validity. What was in fact shown in those two arguments at the very beginning of chapter 2 was that an argument is valid if and only if it is impossible for the premises to be true fuse the conclusion false. From that it does not follow that a contradiction entails an arbitrary proposition. The correct formulation of the Standard Account uses the intensional notion of conjunction, fusion. The Classical error is to suppose the connection of premises and conclusion which must be ruled out is mere extensional conjunction.

Fusion is also needed for the correct statement of the Deduction Equivalence: briefly, B follows from C fuse A iff 'if A then B' follows from C. That does not enable us to infer that B entails 'if A then B', since B follows only from B and (&) A, not from B fuse A. The need for fusion in the statement of the Deduction Equivalence shows that we need fusion not only at the meta-level, to characterise validity correctly, but also at the object level, in the very combination of premises in argument. We turn in the next two chapters to give a formal specification, proof-theoretic and semantic, of a system of propositional logic respecting these distinctions. In chapter 6, we will proceed to use this notion of fusion to explicate the concept of relevance which many people have sought in order to provide the appropriate connection between premises and conclusion in a valid argument.

Chapter 4

Proof-Theory for Relevant Logic

We have seen that it is crucial to distinguish between intensional (non-truth-functional) and extensional (truth-functional) connectives. This distinction, and in particular, the need to give the formal treatment of an intensional implication distinct from material implication, was what motivated first C.I. Lewis, and later Ackermann and his followers Anderson and Belnap, to develop their formal systems. In Lewis' early work, he endeavoured to construct a system in which not only were the so-called paradoxes of material implication absent, but also *Ex Falso Quodlibet*. But later, in developing his famous formal systems of strict implication, he admitted this inference into the systems, and with it the paradoxes of strict implication, attempting to justify them by the ancient mode of argument which we looked at in chapter 2, and which has since come to bear his name. Later authors, including Moh, Ackermann and others, unconvinced by this argument, and undeterred by Lewis' failure to find a system free of paradox, went further in their explorations of an intensional implication.

These early formal systems were given, in the main, an axiomatic formulation. However, following Gentzen's investigations into the notion of consequence (or at least, after it became better known through the work of Prawitz and others), natural deduction and sequent calculus presentations have found more favour, in giving a better insight into the nature of deduction and inference. In particular, it forces one to concentrate on the consequence relations in a theory, not simply on the theorems obtainable. In most relevant logics (and all the main ones) all classical tautologies in $\sim$, & and $\vee$ (and so, by definition, in $\supset$) are derivable as theorems. It is the consequence relation—in what follows from what, and so by the Deduction Equivalence, in what are provable (relevant) implications and entailments—that the systems differ from classical logic.

Anderson and Belnap, in developing and explicating Ackermann's ideas, constructed a number of natural deduction systems of what became known as relevant (or relevance) logic. However, in their style of presentation, the crucial distinction between intensional and extensional connectives, well marked in the implied formula, is not mirrored to the left of the turnstile. This leads to considerable complications and tensions in their formulations, which come out particularly in their versions of the natural deduction rules &I and $\vee$E, and in the consequential need to add Distribution as an extra rule.

4.1 The Bunch Formulation

A much smoother presentation is achieved by introducing into the formal system the distinction between intensional and extensional combinations of premises which we explored in the last chapter. Intensional combination goes with the intensional connectives $\rightarrow, \times, \Rightarrow$ (see §4.5 below) and t, while extensional combination goes with $\&, \vee$ and T. (t and T will be further explained below.) The intensional and extensional combinations can be iterated, resulting in general in what will technically be known as a 'bunch' of wffs. Entailment is taken to be a relation between a bunch of wffs and a wff (premises and conclusion). The present aim is to characterise this relation proof-theoretically (i.e. syntactically); in chapter 5 we will proceed to give it a semantical characterisation. We consider three main systems of propositional relevant logic, **DW** (the basic system), **R** (the calculus of relevant implication), and $\mathbf{R}^{\Box}$ (**R** with an **S4**-style modal operator). Others, including the system **E** of entailment, will also be mentioned on the way.[1] It will also be shown how **S4** (Lewis' system) and **K** (classical truth-functional logic) fit into the general framework.

Vocabulary:

i) a denumerable list of propositional variables;

ii) connectives:

[1]For further details, see E.P. Martin & J.K. Slaney, *Contraction-Free Logics* (forthcoming), chapter 1. Note that the present system owes much to earlier presentations of relevant logic, notably Dunn's consecution (or sequent) formulation: see A. Anderson and N. Belnap, *Entailment*, vol. I (Princeton, 1975), §28.5, pp. 381-91.

nullary: t, T

unary: $\sim$

binary: $\&, \vee, \rightarrow, \times$.

The following recursion clauses define the notion of wff:

1. every propositional variable is an (atomic) wff;
2. if A and B are wffs, so are t, T, $\sim A, A \ \&\ B, A \vee B, A \rightarrow B, A \times B$.

Further connectives are introduced by definition:

$$\mathsf{f} =_{df} \sim\mathsf{t}, \mathsf{F} =_{df} \sim\mathsf{T}, A + B =_{df} \sim(\sim A \times \sim B).$$

Bunches are intensional (I-bunches) and extensional (E-bunches):

1. every wff is an I-bunch;
2. if X, Y are bunches, then $X \circ Y$ is an I-bunch,
3. any non-empty set of I-bunches is an E-bunch.
4. no I-bunch is an E-bunch.

$\circ$ is an algebraic operation. The structural rules below will spell out the properties of this operation, and will show that the systems differ really only in the conditions on $\circ$. Other structural rules will record combinatorial facts about set union as it affects the relation between an E-bunch and its members.

Lastly, we define subbunch:

1. every bunch is a subbunch of itself;
2. every subbunch of X is a subbunch of $X \circ Y$ and $Y \circ X$.
3. every subbunch of X is a subbunch of any E-bunch of which X is a member.

If X is a bunch and A a wff, then $\langle X, A\rangle$ is a sequent. In the specification of the systems and in laying out proofs in them certain conventions will apply. Bunches are written using two pairing operations symbolised by the comma and the semicolon: the comma stands for set union, and the semicolon stands for the intensional combination, $\circ$. Suppose I_1, I_2 are I-bunches, E_1, E_2 E-bunches. Then

$$
\begin{array}{lcl}
I_1; I_2 & = & I_1 \circ I_2 \\
E_1; I_2 & = & E_1 \circ I_2 \\
I_1, I_2 & = & \{I_1, I_2\} \\
E_1, I_2 & = & E_1 \cup I_2 \\
I_1; E_2 & = & I_1 \circ E_2 \\
E_1; E_2 & = & E_1 \circ E_2 \\
I_1, E_2 & = & \{I_1\} \cup E_2 \\
E_1, E_2 & = & E_1 \cup E_2.
\end{array}
$$

When this convention for the comma might lead to ambiguity, the braces are retained. The use of the comma for set union should be familiar from classical logic; one of the confusions for readers of Anderson and Belnap is that on the rare occasions when they use the comma in antecedents of sequents, it stands for intensional and not extensional combination. The colon is used to combine the representations of a bunch and a wff into the representation of a sequent.

It pays to be careful here. In many recent presentations of sequent and natural deduction calculi there is an ultimate irony. Frege introduced the content-stroke and the judgment-stroke, which combine into the assertion sign, $\vdash$, to avoid an important ambiguity, between merely considering a proposition (or thought), and actually asserting it.[2] '$\text{—}A$' signifies the thought that A (for Frege, this is its sense; it refers to the truthvalue of A.) '$\vdash\!\!\text{—}A$', which adds the judgment-stroke to the sign for A's content, actually asserts A. For example, it is crucial to distinguish between merely considering whether, say, B follows from A & $\sim A$, and actually asserting that it does. However, E.J. Lemmon, for example, in his influential elementary logic text, Beginning Logic,[3] uses the assertion sign both to construct sequents, that is, to divide the conclusion from the premises, whether valid (or assertible) or not, and to assert that some such sequents are correct, that is, derivable in his system, or valid. We shall not follow him in this conflation.

The distinction is made clear by Dag Prawitz, in his book *Natural Deduction*,[4] which strongly influenced Lemmon in constructing his system. He uses the notation touched on above, of presenting a sequent explicitly as an ordered pair (in his case, of a set of wffs and a wff). However, the convention is clumsy, and after using it to present the rules of immediate consequence,

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[2] G. Frege, *Begriffsschrift*, in *Translations from the Philosophical Writings of Gottlob Frege*, by P.T. Geach and M. Black (Oxford, 1966), §2, pp. 1-2.

[3] E.J. Lemmon, *Beginning Logic* (London, 1966).

[4] D. Prawitz, *Natural Deduction* (Stockholm, 1964).

Prawitz drops it in favour of simply presenting the consequent wffs of each derivation arranged in a tree. We shall, however, for explicitness, retain the sequent, as in Lemmon, but use the colon to separate antecedent and consequent, and keep the turnstile, or assertion sign, solely for the purpose of asserting (rightly or wrongly) that a sequent is derivable (in some system).

X, Y, Z etc. are used as metavariables over bunches, A, B etc as metavariables over wffs. $X(Y)$ is a bunch in which Y occurs at one particular place as a subbunch, and $X(Z)$ is the result of replacing that occurrence of Y by Z. Any other occurrence of Y in $X(Y)$ is unaffected by this replacement. The systems contain both structural and operational inference-rules. As noted, the difference between the systems is almost entirely a matter of variation in the structural rules governing intensional combination. Note that $X(Y)$ may be a null context (i.e. $X(Y)$ may just be Y); however, Y may not be empty. In other words, in a sequent $X : A$, neither X nor A may be empty. Various effects of empty antecedent, as familiar from, e.g., Gentzen,[5] are given by use of t and T; of consequent, by f and F.

4.2 A Deductive System for DW

I will argue later (in chapter 9) that the sense or meaning of the logical connectives is given by their use, in particular, their use in deduction. Throughout this book, it has been understood that the logician's primary concern is consequence, or its formal counterpart, deducibility. In proof-theory, we characterise deducibility by analysing it into a succession of elementary or immediate inference steps. What Gentzen and Prawitz realised was that these elementary inference steps pair off into a rule for introducing a connective, for asserting a formula, a rule which essentially gives the meaning of the connective, and a consequential elimination rule, that is, a rule for removing occurrences of the connective. The elimination rule uses an assertion and applies it to proceed further with the deduction. The elimination rule is justified by the nature of the introduction rule. The introduction rule states the appropriate conditions for asserting a formula of that form; the elimination rule follows from that assertion condition, taking it as necessary as well as sufficient.

Suppose that one has a formula A whose main connective is κ, of n arguments. (Usually, $n = 0, 1$ or 2.) The introduction rule for κ has the form

[5] G.Gentzen, 'Investigations into Logical Deduction', in *The Collected Papers of Gerhard Gentzen*, ed. M.E. Szabo (Amsterdam, 1969), p. 72.

$X : A$ is an immediate consequence of Π,

$1 \leq i \leq j$, where Π is a set of sequents. Π might, for example, be the single sequent $X; A : B$, and the introduction rule for '$\to$' allows one immediately to infer the sequent $X : A \to B$.

What general template for an elimination rule does this schema for the introduction rule justify? Suppose we have a derivation of $X : A$, where A has main connective κ . We know that what justified the assertion of $X : A$ (that is, of A on the basis of X) is the derivation of the set of sequents Π. So if we know that Π justify us in asserting $Y(A_i) : C$, then by replacing that justification of $Y(A_i) : C$ by the justification of $X : A$ adapted to X in place of A_i, we will have a legitimate deduction. In other words, the general schema for the elimination rule is

$Y(X) : C$ is an immediate consequence of $X : A$ and $Y(A_i) : C$,

discharging any assumptions A_i needed to obtain $Y(A_i) : C$.

This story is somewhat complicated by the fact that the format for the elimination rule so derived may often be simplified, and in the case of $\to$E, $\sim$E and (in §4.5) $\Box$E we will do this. Let us look at the operational inference rules, and see how the general theory applies to them.

Operational Inference Rules

$X, Y : A \,\&\, B$	is an immediate consequence of	$X : A$ and $Y : B$	(&I)
$Y(X) : C$		$X : A \,\&\, B$ and $Y(A, B) : C$	(&E)
$X : A \vee B$		$X : A$ (and of $X : B$)	($\vee$I)
$Y(X) : C$		$X : A \vee B$, $Y(A) : C$ and $Y(B) : C$	($\vee$E)
$X; Y : B$		$X : A \to B$ and $Y : A$	($\to$E)
$X : A \to B$		$X; A : B$	($\to$I)
$X; Y : \sim A$		$X; A : \sim B$ and $Y : B$	($\sim$I)
$X : A$		$X : \sim\sim A$	($\sim$E)
$X; Y : A \times B$		$X : A$ and $Y : B$	($\times$I)
$Y(X) : C$		$X : A \times B$ and $Y(A; B) : C$	($\times$E)

The elimination rules for $\&, \vee$ and $\times$ here correspond exactly to the general schema. Proposition 4.1 below will show that the perhaps unusual and general form of &E permits derivation of what we might call 'old-fashioned &E', that A (and B) follow from $A \,\&\, B$. Indeed, in the non-modal logics we could replace &E by its simpler form. However, this is not possible in $\mathbf{R}^{\Box}$ and so we retain the more general form throughout.

It may not, however, be too obvious how $\rightarrow$E and $\sim$E as given correspond to the general schema. To see how they fall under the general schema will have to wait until we develop the full theory of the relation between introduction and elimination rules in chapter 9. What is important is that in the full theory normal form proofs are possible, that is, ones in which no application of an introduction rule for a connective is ever immediately followed by an application of the elimination rule for the same connective. (The present form of the $\sim$-rules lacks this property.) Proofs in normal form have a simplicity and directness which is proof-theoretically attractive.

Basic Structural Rules

Certain of the structural rules follow immediately from our identification of extensional bunches with sets:

$X, Y = Y, X$	extensional commutativity	(EC)
$X, (Y, Z) = (X, Y), Z$	extensional associativity	(EA)
$X, X = X$	extensional contraction	(EW)

To express the others, we write $U \leq V$ to say that we may derive $W(V) : A$ from $W(U) : A$, and take $U = V$ as an abbreviation for $U \leq V$ and $V \leq U$. We then lay down the following derivability relations as cases of immediate consequence:

$X \leq X, Y$	extensional augmentation	(EK)
$\mathsf{T}, X \leq X$	extensional identity	($\mathsf{T}=$)
$\mathsf{t}; X = X$	intensional identity	($\mathsf{t}=$)

Of course, when we say that $X, X = X$, this is real identity of bunches; when we say that $\mathsf{T}, X = X$, we mean that the two (distinct) bunches have the same logical power (i.e., have the same consequences). $\mathsf{T}=$ and EK entail that $\mathsf{T}, X = X$, that is, $\mathsf{T} \cup X = X$, which characterises T as the empty set. To define derivability, we must first define an inclusion relation on bunches:

1. if W, Z are E-bunches and $W \subseteq Z$, then $Y(W) \angle' Y(Z)$;

2. $\angle$ is the reflexive, transitive closure of $\angle'$.

If $X \angle Y$ we say that X is a pruning of Y.[6] Note that $X \angle X$ for all X, and that, given EK, $X \angle Y$ entails $X \leq Y$. We can then define derivability as usual: $X \vdash A$ iff there is a finite sequence Π of sequents whose final member is $X' : A$, where X' is finite, $X' \angle X$, and every member of Π is either of

[6] Thanks to Mary and Max Cresswell for suggesting the term 'pruning'.

the form $A : A$, or is an immediate consequence, by one of the above rules, of earlier sequents in the sequence. We write $\vdash A$ as an abbreviation for $\mathsf{t} \vdash A$. t can be thought of as the conjunction of all logical truths; T has the (minimal) force of the disjunction of all propositions.

The intensional and extensional nature of semicolon and comma may be brought out by a recursive definition of the characteristic wff of a (finite) bunch.

1. $\mathsf{cf}(A) = A$;
2. (a) $\mathsf{cf}(X; Y) = \mathsf{cf}(X) \times \mathsf{cf}(Y)$;
   (b) $\mathsf{cf}(X, Y) = \mathsf{cf}(X)\ \&\ \mathsf{cf}(Y)$.

We show below that if X is finite, $X \vdash A$ iff $\mathsf{cf}(X) \vdash A$.

It may be useful to the reader briefly to give an example of a bunch, of its subbunches, and of its prunings. Suppose A, B, C, D are wffs (and so are I-bunches). Then the bunch $\{A, B\} \circ \{C, D\}$ has subbunches and prunings as follows:

Subbunches	Prunings
$\{A, B\} \circ \{C, D\}$	$\{A, B\} \circ \{C, D\}$
$\{A, B\}$	$A \circ \{C, D\}$
$\{C, D\}$	$B \circ \{C, D\}$
A	$A \circ C$
B	$B \circ C$
C	$A \circ D$
D	$B \circ D$
	$\{A, B\} \circ C$
	$\{A, B\} \circ D$

Note that the only bunch which is both a subbunch and a pruning of $\{A, B\} \circ \{C, D\}$ is the improper subbunch, the bunch itself. This is because to obtain proper subbunches we must omit the main operator, '$\circ$', whereas for prunings it is essential that this operation be retained.

4.3 Extensions of DW

The system comprising the operational and basic structural inference rules is the very weak logic **DW**. It is weak in the sense that its theory of implication is weak: the only theorems in its $\to$-fragment are identities of the form

$A \to A$. Stronger systems may be obtained by adding further structural rules, and strengthening the negation rules:

Optional operational rule

CM	$X : \sim A$	is an immediate consequence of	$X; A : \sim A$	Consequentia mirabilis

Optional Structural Rules

B	$X; (Y; Z) \leq (X; Y); Z$	Prefixing
B'	$X; (Y; Z) \leq (Y; X); Z$	Suffixing
C**	$X; \mathsf{t} \leq X$	
C*	$X; Y \leq Y; X$	Assertion
C	$(X; Y); Z \leq (X; Z); Y$	Permutation
W*	$X; X \leq X$	Conjunctive Assertion
W	$(X; Y); Y \leq X; Y$	Contraction
WB	$X; (X; Y) \leq X; Y$	Conjunctive Syllogism
S*	$Y; (X; Y) \leq X; Y$	
S	$(X; Z); (Y; Z) \leq (X; Y); Z$	
K*	$X \leq X; X$	Mingle, or Premise Repetition
K	$X \leq X; Y$	

Those familiar with combinators will recognise the designators in the left-hand column, but may be puzzled to see '$\leq$ ' rather than '$\geq$'. '$\leq$' is chosen here to match the ordering in the ternary semantics in chapter 5. Note that $\leq$ is reflexive and transitive. Relevant logics of particular (albeit in some cases historical) interest are:

TW = **DW** + B, B'	**DL** = **DW** + CM, WB
RW = **TW** + C	**TL** = **DW** + CM, WB (= **DL** + B, B')
T = **TL** + W	**E** = **T** + C**
R = **E** + C* (= **RW** + W)	**RM** = **R** + K*.

Classical logic **K** = **R** + K. The effect of K is to conflate semicolon and comma, and so to make $\times, \to$ and t extensional.

Note that such well-known systems as **T**, **E** and **R** do not usually contain $\times, \mathsf{T}$ or t, as the above systems do. What are here called **T**, **E** and **R** would usually be denoted $\mathbf{T}^{\times\mathsf{tT}}$, $\mathbf{E}^{\times\mathsf{tT}}$ etc. The same is true of the less well-known systems. The names for the weaker systems are taken from Brady.[7] He, and

[7]R.T. Brady, 'Natural Deduction Systems for some Quantified Relevant Logics', *Logique et Analyse*, 27 (1984), pp. 355-77.

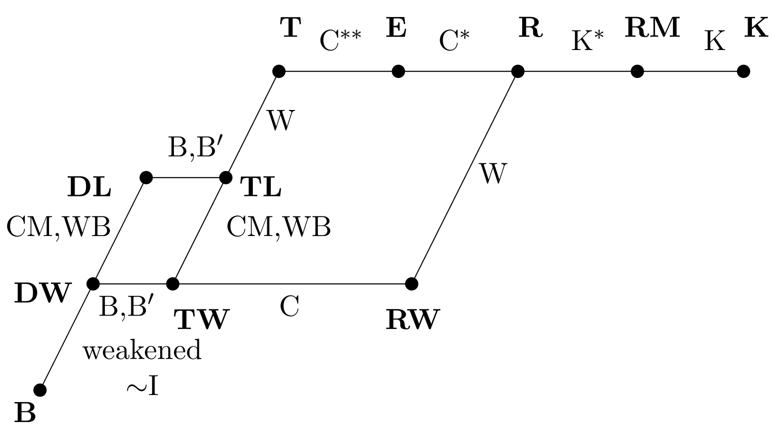
others,[8] include a yet weaker system, **B** (the Basic system). It results from DW by replacing the operational inference rule $\sim$I by the weaker rule:

$$X : \sim A \quad \text{is an immediate consequence of} \quad t; A : \sim B \text{ and } X : B.$$

DL was originally introduced as a possible Dialectical Logic to underpin Dialectical Set Theory (a simply inconsistent non-trivial set theory).[9]

In the context of **R**, CM is a special case of W. For we will find later that '$\sim$' can be defined, in the presence of rule C, as $A \to \mathsf{f}$ (with 'f' primitive). CM is then $(A \times A \to \mathsf{f}) \to (A \to \mathsf{f})$. Thus **TW** results from dropping both W and CM from **T**. CM is needed to obtain the Law of Excluded Middle, $A \vee \sim A$, which fails in **DW**, **TW** and **RW**. Every classical tautology is a theorem of **DL** and all stronger systems. Of course, the derivability relation is different—e.g., Disjunctive Syllogism (detachment for '$\supset$') fails in **RM** and all weaker systems. **TL** is named here on analogy with **DL**.

The systems may be set out as follows, where containment is indicated by the existence of a line upwards or to the right:



[8]E.g., F.R. Routley et al., *Relevant Logics and their Rivals* (Atascadero, Calif., 1982), p. 289.

[9]F.R. Routley and R.K. Meyer, 'Dialectical Logic, Classical Logic, and the Consistency of the World', *Studies in Soviet Thought*, 16 (1976), pp. 1-25.

4.4 Derivations in DW

We adopt Lemmon's and Guttenplan's notation[10] to present derivations. Note that frequently an assumption-line is used solely for the purpose of designating a wff with a numeral.

Proposition 4.1 *(Old-fashioned &E) If* $X \vdash A \mathbin{\&} B$ *then* $X \vdash A$ *and* $X \vdash B$.

Proof:

X	(1)	$A \mathbin{\&} B$	Given
2	(2)	A	A
3	(3)	B	A
2,3	(4)	A	2 EK
X	(5)	A	1,4 &E

Similarly for $X \vdash B$. □

Proposition 4.2 (i) $A; B \leq A \times B$;

(ii) $A, B \leq A \mathbin{\&} B$.

Proof:

(i)

$X(A;B)$	(1)	C	Given
2	(2)	$A \times B$	A
$X(2)$	(3)	C	1,2 ×E

(ii)

$X(A,B)$	(1)	C	Given
2	(2)	$A \mathbin{\&} B$	A
$X(2)$	(3)	C	1,2 &E

□

In leading up to Proposition 4.3 and its Corollaries, remember that we can speak of $\mathsf{cf}(X)$ only if we assume X to be finite.

Proposition 4.3 *If X is finite then*

(i) $X \vdash \mathsf{cf}(X)$,

(ii) *if* $Y(X) \vdash A$ *then* $Y(\mathsf{cf}(X)) \vdash A$.

Proof:

[10] E.J. Lemmon, *Beginning Logic*; S. Guttenplan, *The Languages of Logic* (Oxford, 1986), chapter 9.

(i) by induction on the complexity of X:

1. X is a wff: immediate.
2. (a) $X = Y \circ Z$: Then $Y \vdash \mathsf{cf}(Y)$ and $Z \vdash \mathsf{cf}(Z)$ by I.H., and so $Y; Z \vdash \mathsf{cf}(Y) \times \mathsf{cf}(Z)$ by $\times$I, i.e., $X \vdash \mathsf{cf}(X)$.
   (b) X is an E-bunch: as (i), using &I.

(ii) by induction on the complexity of X:

1. X is a wff: the result is immediate.
2. (a) $X = Y \circ Z$: suppose $Y(X) \vdash A$, i.e., $Y(W; Z) \vdash A$. Then $Y(\mathsf{cf}(W); \mathsf{cf}(Z)) \vdash A$, by I.H., and so $Y(\mathsf{cf}(X)) \vdash A$, by Proposition 4.2 i).
   (b) X is an extensional bunch: then $X = \{W\}$ or $X = \{W\} \cup Z$. If the former, the result is immediate by I.H. For the latter case, suppose $Y(X) \vdash A$, i.e., $Y(W, Z) \vdash A$. Then $Y(\mathsf{cf}(W), \mathsf{cf}(Z)) \vdash A$, by I.H., and so $Y(\mathsf{cf}(X)) \vdash A$, by Proposition 4.2 ii). □

Proposition 4.4 *(Cut) If $X \vdash A$ and $Y(A) \vdash B$, then $Y(X) \vdash B$.*

Proof: by cases:

1. $Y(A) = A$: immediate by $\mathsf{t}=$, $\to$I, $\to$E and $\mathsf{t}=$ again.
2. (a) $Y(A) = Y'(A; Z)$: if Z is infinite, then, by definition of derivability, $Y'(A; Z') \vdash B$, where $Z' \angle Z$ and Z' is finite. Since by hypothesis, $X \vdash A$, and $Z' \vdash \mathsf{cf}(Z')$ by Proposition 4.3 (i), $X; Z' \vdash A \times \mathsf{cf}(Z')$ by $\times$I. Moreover, since $Y'(A; Z') \vdash B, Y'(A; \mathsf{cf}(Z')) \vdash B$, by Proposition 4.3 (ii). So $Y'(X; Z') \vdash B$ by $\times$E, and so $Y(X) \vdash B$, since $Y'(X; Z') \angle Y(X)$. If $Y(A) = Y'(Z; A)$, the reasoning is similar.
   (b) $Y(A) = Y'(A, Z)$, or $Y'(Z, A)$: as i), using &I and &E. □

Corollary 4.5 $\vdash$ *is a generalised Tarski consequence relation, that is*

1. $A \vdash A$ (Reflexivity)

2. if $X \angle Y$ and $X \vdash A$ then $Y \vdash A$ (Monotonicity)

3. if $X \vdash A$ and $Y(A) \vdash B$ then $Y(X) \vdash B$ (Cut)

Proof: Proofs of 1) - 3) are immediate from the foregoing. That 1) - 3) warrant the title 'generalised Tarski consequence', note that if X and Y are E-bunches (i.e., sets) of wffs, Tarski's own conditions hold, that is,

($1'$) $X \vdash A$ if $A \in X$

($2'$) if $X \subseteq Y$ and $X \vdash A$ then $Y \vdash A$

($3'$) if $X \vdash A$ and $A, Y \vdash B$ then $X, Y \vdash B$.

($1'$) follows from the definition of $\vdash$, since if $A \in X$ then $A \angle X$. ($2'$) and ($3'$) follow from 2) and 3) given that X and Y are E-bunches of wffs. In other words, $\vdash$ is a conservative extension of Tarski's consequence relation, under which 1) - 3) capture the spirit of Tarski's own conditions, extended to the new vocabulary of bunches. □

Proposition 4.6 (i) *If X is finite, $Y(X) \vdash A$ iff $Y(\mathsf{cf}(X)) \vdash A$.*

(ii) *If X is finite, $X \vdash A$ iff $\mathsf{cf}(X) \vdash A$.*

(iii) *If $X \vdash B_i$ for all i, $1 \geq i \geq m$, and $B_1, \ldots, B_m \vdash C$ then $X \vdash C$.*

(iv) *If $X \vdash \mathsf{cf}(Y)$ then $Y \leq X$.*

Proof:

(i) Suppose $Y(\mathsf{cf}(X)) \vdash A$. By Proposition 4.3 (i), $X \vdash \mathsf{cf}(X)$. So $Y(X) \vdash A$ by Cut (Proposition 4.4). Combining with Proposition 4.3 (ii), it follows that $Y(X) \vdash A$ iff $Y(\mathsf{cf}(X)) \vdash A$.

(ii) Let Y be the empty context in i).

(iii) Apply Cut (Proposition 4.4) successively.

(iv) Suppose $X \vdash \mathsf{cf}(Y)$ and $Z(Y) \vdash B$. Then $Z(\mathsf{cf}(Y)) \vdash B$ by Proposition 4.3ii), and so $Z(X) \vdash B$ by Cut. So $Y \leq X$. □

Proposition 4.7 B, B$'$, C** *and* W* *are derivable in rule form in* **DW**. *That is,*

(i) *if* $\vdash A \to B$ *then* $\vdash (C \to A) \to (C \to B)$;

(ii) *if* $\vdash A \to B$ *then* $\vdash (B \to C) \to (A \to C)$;

(iii) *if* $\vdash (A \to A) \to B$ *then* $\vdash B$;

(iv) *if* $\vdash A \,\&\, (A \to B)$ *then* $\vdash B$.

We say these are the rule forms of these structural rules, since the structural rules correspond to $\to$-forms of rules i) - iv).

Proof:

(i)	t	(1)	$A \to B$	Given
	2	(2)	$C \to A$	A
	3	(3)	C	A
	2;3	(4)	A	2,3 $\to$E
	t;(2;3)	(5)	B	1,4 $\to$E
	2;3	(6)	B	5 t=
	2	(7)	$C \to B$	3,6 $\to$I
	t;2	(8)	$C \to B$	7 t=
	t	(9)	$(C \to A) \to (C \to B)$	2,8 $\to$I

(ii)	t	(1)	$A \to B$	Given
	2	(2)	$B \to C$	A
	3	(3)	A	A
	t;3	(4)	B	1,3 $\to$E
	2;(t;3)	(5)	C	2,4 $\to$E
	2;3	(6)	C	5 t=
	2	(7)	$A \to C$	3,6 $\to$I
	t;2	(8)	$A \to C$	7 t=
	t	(9)	$(B \to C) \to (A \to C)$	2,8 $\to$I

(iii)	t	(1)	$(A \to A) \to B$	Given
	2	(2)	A	A
	t;2	(3)	A	2 t=
	t	(4)	$A \to A$	2,3 $\to$I
	t; t	(5)	B	1,4 $\to$E
	t	(6)	B	5 t=

(iv)	t	(1)	$A \,\&\, (A \to B)$	Given
	t	(2)	A	1 Prop. 4.1
	t	(3)	$A \to B$	1 Prop. 4.1
	t; t	(4)	B	2,3 $\to$E
	t	(5)	B	4 t=

□

Proposition 4.8 *(Distribution)* $A \,\&\, (B \vee C) \vdash (A \,\&\, B) \vee C$.

Proof:	1	(1)	$A \,\&\, (B \vee C)$	A
	2	(2)	A	A
	3	(3)	B	A
	2,3	(4)	$A \,\&\, B$	2,3 &I
	2,3	(5)	$(A \,\&\, B) \vee C$	4 ∨I
	6	(6)	C	A
	2,6	(7)	C	6 EK
	2,6	(8)	$(A \,\&\, B) \vee C$	7 ∨I
	9	(9)	$B \vee C$	A
	2,9	(10)	$(A \,\&\, B) \vee C$	9,5,8 ∨E
	1	(11)	$(A \,\&\, B) \vee C$	10 4.2

□

Proposition 4.9 *(DN) If $X \vdash A$ then $X \vdash \sim\sim A$.*

Proof:	X	(1)	A	Given
	2	(2)	$\sim A$	A
	t;2	(3)	$\sim A$	2 t=
	t; X	(4)	$\sim\sim A$	1,3 ∼I
	X	(5)	$\sim\sim A$	4 t=

□

Proposition 4.10 *(Antilogism) $X; A \vdash B$ iff $X; \sim B \vdash \sim A$.*

Proof:	1	(1)	A	A
	2	(2)	$\sim B$	A
	$X; 1$	(3)	B	Given
	$X; 1$	(4)	$\sim\sim B$	3 DN (4.9)
	$X; 2$	(5)	$\sim A$	2,4 ∼I

Conversely,

$X; 2$	(6)	$\sim A$	Given
$X; 1$	(7)	$\sim\sim B$	1,6 ∼I
$X; 1$	(8)	B	7 ∼E

□

Corollary 4.11 *(Contraposition) If $X \vdash A \to B$ then $X \vdash \sim B \to \sim A$.*

Proof: immediate from Proposition 4.10 and the →-rules. □

Proposition 4.12 *(De Morgan) $\sim(\sim C \vee \sim D) \vdash C \,\&\, D$.*

Proof:

1	(1)	$\sim(\sim C \vee \sim D)$	A
2	(2)	$\sim C$	A
2	(3)	$\sim C \vee \sim D$	2 $\vee$I
1	(4)	$\sim\sim C$	3 Antilogism (4.11)
1	(5)	C	4 $\sim$E
6	(6)	$\sim D$	A
6	(7)	$\sim C \vee \sim D$	6 $\vee$I
1	(8)	$\sim\sim D$	7 Antilogism
1	(9)	D	8 $\sim$E
1,1	(10)	C & D	5,9 &I
1	(11)	C & D	10 EW

Conversely,

1	(1)	C & D	A
2	(2)	$\sim C \vee \sim D$	A
3	(3)	$\sim C$	A
1	(4)	C	1 &E
t;1	(5)	C	4 t=
t;1	(6)	$\sim\sim C$	5 DN
t;3	(7)	$\sim(C$ & $D)$	3,6 $\sim$I
8	(8)	$\sim D$	A
1	(9)	D	1 &E
t;1	(10)	D	9 t=
t;1	(11)	$\sim\sim D$	10 DN
t;8	(12)	$\sim(C$ & $D)$	8,11 $\sim$I
t;2	(13)	$\sim(C$ & $D)$	2,7,12 $\vee$E
t;1	(14)	$\sim(\sim C \vee \sim D)$	1,13 $\sim$I
1	(15)	$\sim(\sim C \vee \sim D)$	14 t=

□

That all the other standard cases of De Morgan laws are derivable is left as an exercise for the reader.

Corollary 4.13 *If* $X; C$ & $D \vdash B$ *then* $X; \sim B \vdash \sim C \vee \sim D$.

Proof: 1	(1)	$C \& D$	A
$X;1$	(2)	B	Given
X	(3)	$(C \& D) \to B$	$2 \to$ I
4	(4)	$\sim(\sim C \vee \sim D)$	A
4	(5)	$C \& D$	4 De Morgan (4.12)
X;4	(6)	B	3,5 $\to$E
7	(7)	$\sim B$	A
$X;7$	(8)	$\sim\sim(\sim C \vee \sim D)$	6 Antilogism
$X;7$	(9)	$\sim C \vee \sim D$	8 $\sim$E

□

Proposition 4.14 $\mathsf{T} \leq X$.

Proof: $\mathsf{T} \leq \mathsf{T}, X$ (EK)
$\leq X (\mathsf{T}=)$ □

Corollary 4.15 (i) *For all* $X, X \vdash \mathsf{T}$. *(*TI*)*

(ii) *For all* $A, \mathsf{F} \vdash A$. *(*FE*)*

(iii) $X \leq \mathsf{F}$.

(iv) $\vdash \sim\mathsf{F}$.

Proof:

(i) $\mathsf{T} \vdash \mathsf{T}$, so $X \vdash \mathsf{T}$, by 4.14.

(ii) from i), by Antilogism (4.10) (and DN and t= twice).

(iii) if $Y(X) \vdash A$ then $Y(X') \vdash A$ for some finite $X' \angle X$, whence $Y(\mathsf{cf}(X')) \vdash A$ by 4.3. But $\mathsf{F} \vdash \mathsf{cf}(X')$ by ii), and so $Y(\mathsf{F}) \vdash A$ by Cut, i.e. $X \leq \mathsf{F}$.

(iv) By i), $\mathsf{t} \vdash \mathsf{T}$, whence $\mathsf{t} \vdash \sim\mathsf{F}$ by DN. □

Corollary 4.16 *Suppose for some subbunch* Y *of* $X, Y \vdash \mathsf{F}$. *Then* $X \vdash \mathsf{F}$.

Proof: by induction on the complexity of X.

1. $X = Y$: immediate.

2. (a) i. $X = Z(Y); W$: then by I.H., $Z(Y) \vdash \mathsf{F}$. Let $W' \angle W$, W' finite. Then $(\mathsf{cf}(W') \to \mathsf{F}); \mathsf{cf}(W') \vdash \mathsf{F}$ by $\to$E, whence $(\mathsf{cf}(W') \to \mathsf{F}); W \vdash \mathsf{F}$ by EK and Proposition 4.3. But $\mathsf{F} \vdash \mathsf{cf}(W') \to \mathsf{F}$ by Corollary 4.15 (ii). Hence $\mathsf{F}; W \vdash \mathsf{F}$ by Cut. So by Cut again, applied to the I.H., $Z(Y); W \vdash \mathsf{F}$.

ii. $X = W; Z(Y)$: we show that $W \vdash \mathsf{F} \to \mathsf{F}$. Clearly, $\mathsf{F} \vdash \mathsf{F}$. But by 4.14, $\mathsf{T};\mathsf{T} \vdash \mathsf{T}$, so $\mathsf{T};\mathsf{T} \vdash \sim F$, by DN. Hence $\mathsf{T};\mathsf{F} \vdash \mathsf{F}$, by $\sim$I. But $W \vdash \mathsf{T}$, by 4.15i). So by $\to$I and Cut, $W \vdash \mathsf{F} \to \mathsf{F}$. But by I.H., $Z(Y) \vdash \mathsf{F}$. So $W; Z(Y) \vdash \mathsf{F}$, by $\to$E.

(b) X is an E-bunch, and $Z(Y) \in X$. By I.H., $Z(Y) \vdash \mathsf{F}$. So $X \vdash \mathsf{F}$ by EK. □

Corollary 4.17 *If $X \vdash \mathsf{F} \vee C$ then $X \vdash C$.*

Proof: By $\vee$E, from Corollary 4.15. □

F is the ultimate untruth. 4.17 shows that a version of extensional Disjunctive Syllogism holds of it.

We next prove two results which we will need in chapter 5. The first is a kind of Cut principle.

Proposition 4.18 *If $X \vdash A \vee B$ and $Y, B \vdash C$, then $Y, X \vdash A \vee C$.*

Proof:

X	(1)	$A \vee B$	Given
2	(2)	A	A
$Y, 2$	(3)	A	2 EK
$Y, 2$	(4)	$A \vee C$	3 $\vee$I
5	(5)	B	A
$Y, 5$	(6)	C	Given
$Y, 5$	(7)	$A \vee C$	6 $\vee$I
Y, X	(8)	$A \vee C$	1,4,7 $\vee$E

□

Proposition 4.19 *If $X \vdash (A \to C) \vee (B \to D)$, then $X \vdash (A \,\&\, B) \to (C \vee D)$.*

Proof:

X	(1)	$(A \to C) \vee (B \to D)$	A
2	(2)	$A \,\&\, B$	A
3	(3)	$A \to C$	A
2	(4)	A	&E
3;2	(5)	C	3,4 $\to$E
3;2	(6)	$C \vee D$	5 $\vee$I
7	(7)	$B \to D$	A
2	(8)	B	&E
7;2	(9)	D	7,8 $\to$E
7;2	(10)	$C \vee D$	9 $\vee$I
$X; 2$	(11)	$C \vee D$	1,6,10 $\vee$E
X	(12)	$(A \,\&\, B) \to (C \vee D)$	11 $\to$I

□

Proposition 4.20 *(Replacement Theorem) Suppose X and C contain any number, including zero, occurrences of A as subformula of C or of a formula occurring as a subbunch of X, and that X', C' result from X, C respectively by replacing any number of those occurrences of A by B. Then if $A \dashv\vdash B$ and $X \vdash C, X' \vdash C'$.*

Proof: We take the proof in two steps.

1. First, we show that if $X \vdash C$ then $X \vdash C'$, by showing that $C \dashv\vdash C'$: by induction on the complexity of C.

   Base: $C = A$. Then, unless $C' = A$ (when the result is immediate by Identity), $C' = B$, and the result is immediate by hypothesis.

   Induction Step:

   (i) $C = {\sim}D$, so $C' = {\sim}D'$. By Induction Hypothesis, $D \dashv\vdash D'$, whence $C \dashv\vdash C'$ by t=, DN, ${\sim}$I.

   (ii) $C = D \& E$. By I.H. $D \dashv\vdash D'$, $E \dashv\vdash E'$, so $C \dashv\vdash C'$ by &I and &E.

   (iii) $C = D \vee E$. By I.H. $D \dashv\vdash D'$, $E \dashv\vdash E'$, so $C \dashv\vdash C'$ by $\vee$I and $\vee$E.

   (iv) $C = D \to E$. By I.H. $D \dashv\vdash D'$, $E \dashv\vdash E'$, so $C \dashv\vdash C'$ by $\to$E, Cut (twice) and $\to$I.

   (v) $C = D \times E$. By I.H. $D \dashv\vdash D'$, $E \dashv\vdash E'$, so $C \dashv\vdash C'$ by $\times$I and $\times$E.

   So, by induction, $C \dashv\vdash C'$. Hence if $X \vdash C$, $X \vdash C'$, by Cut.

2. To deal with X we replace X by any finite bunch Y such that $Y \angle X$ and $Y \vdash C'$ (Y exists by the definition of derivation). Suppose the formula D is a subbunch of Y, and D' results from D in the usual way by substitution of B for A. Let Y' contain D' where Y contains D. Then $D \dashv\vdash D'$ by the argument in 1). Hence $\mathsf{cf}(Y) \dashv\vdash \mathsf{cf}(Y')$, by Proposition 4.5. The proof proceeds by induction on the complexity of Y. The base, $Y = D$, is immediate, and the two cases in the induction step are as in 1) ii) and v) above.

Hence $\mathsf{cf}(Y') \vdash C'$ by Cut, and so $Y' \vdash C'$ by 4.5. But $Y' \angle X'$, so $X' \vdash C'$, as required. □

Finally, we show that every classical tautology is derivable in **DL**.

Proposition 4.21 (i) $A \to B \vdash_{\mathbf{DL}} {\sim}A \vee B$

(ii) $\vdash_{\mathbf{DL}} A \vee \sim A$.

Proof:

(i)	1	(1)	$A \to B$	A
	2	(2)	$\sim(\sim A \vee B)$	A
	3	(3)	$\sim A$	A
	3	(4)	$\sim A \vee B$	3 $\vee$I
	t;3	(5)	$\sim A \vee B$	3 t=
	t;3	(6)	$\sim\sim(\sim A \vee B)$	5 DN
	t;2	(7)	A	6 Antilogism
	1;(t;2)	(8)	B	1,7 $\to$E
	1;(t;2)	(9)	$\sim A \vee B$	8 $\vee$I
	1;2	(10)	$\sim A \vee B$	9 t=
	1;2	(11)	$\sim\sim(\sim A \vee B)$	10 DN
	1	(12)	$\sim\sim(\sim A \vee B)$	11 CM
	1	(13)	$\sim A \vee B$	12 $\sim$E

(ii) immediate from i) since $\vdash_{\mathbf{DW}} A \to A$. □

Proposition 4.22 *Every classical tautology is derivable in* **DL** *(and all stronger logics).*

Proof: Let A be a classical tautology, i.e., containing only $\&, \vee, \supset$. Then A is classically equivalent to some formula A' in conjunctive normal form (CNF—i.e., a conjunction, every conjunct of which is a disjunction of variables or negated variables).[11] Since A' is a tautology, every disjunct B of A' contains both a variable p and its negation. But by Proposition 4.21, $\vdash_{\mathbf{DL}} p \vee \sim p$. Hence by $\vee$I, $\vdash_{\mathbf{DL}} B$. So, by &I and EW, $\vdash_{\mathbf{DL}} A'$. But A follows from A' by the application of De Morgan Laws, Distribution, $\sim$E and DN to the subformulae of A. All of these are rules or derived rules of **DL**, and the Replacement Theorem (4.20) permits successive replacements. Hence $\vdash_{\mathbf{DL}} A$. □

4.5 A Deductive System for R and $\mathbf{R}^{\Box}$

As we have seen, **DW** is a very weak relevant logic. In contrast, **R**, Anderson and Belnap's calculus of relevant implication, is very strong - some think

[11] See, e.g., S. Kleene, *Introduction to Metamathematics* (Amsterdam, 1952), pp. 134-5, Theorem 11.

too strong. It differs from **K** only in lacking the structural rule K, which collapses the relevant distinction between bunches and sets. In particular, **R** contains permutation (C). The presence of this principle enables us to give a very neat treatment of negation.

Besides implication, we are also interested in entailment. The distinction here is that entailment is a modal notion, expressing the logically true, or necessary, implications. C.I. Lewis' original motivation was to construct a calculus in which entailment could be expressed, in which some connective actually meant 'entails'. He rightly perceived that '$\supset$' could not sensibly be read in this way. Anderson and Belnap attempted to capture the notion directly, by restricting permutation to necessarily true wffs. This seemed entirely reasonable, but for one thing. It turned out that the resulting calculus, **E**, was not equivalent, under an appropriate translation, to the calculus $\mathbf{R}^{\Box}$, obtained by grafting a notion of necessity, expressed by $\Box$, onto **R**, and defining entailment, $A \Rightarrow B$, as $\Box(A \to B)$. This was entirely unexpected.[12] What we would expect is that $\vdash_{\mathbf{E}} A$ iff $\vdash_{\mathbf{R}^{\Box}} A'$, where A' results from A on replacing all occurrences of $B \to C$ in A by $B \Rightarrow C$. As we will see in chapter 5, $A \to (B \to B), B \to A \vee B \nvdash_{\mathbf{E}} B \to B$; but we will show below that $A \Rightarrow (B \Rightarrow B), B \Rightarrow A \vee B \vdash_{\mathbf{R}^{\Box}} B \Rightarrow B$. (In case of puzzlement here, remember that $\vdash A$ does not entail $X \vdash A$ for arbitrary X. In general, $\mathtt{t}\angle X$ does not hold.) We call this sequent, the Minc sequent. Minc showed it was provable in $\mathbf{R}^{\Box}$; Maximova that it was invalid in **E**.[13]

The theory of entailment contained in $\mathbf{R}^{\Box}$ is so natural, respecting the intuitions which have in the classical case favoured **S4**, while rejecting EFQ and its mates, that the discovery of this discrepancy between **E** and $\mathbf{R}^{\Box}$ sounded the deathknell of **E**. If one wishes to retain as much as possible of the classical theory of entailment, as developed by Lewis, leaving out only EFQ and its like, **E** is basically right, but clearly, in certain complex statements concerning entailment, it gets the answer wrong. $\mathbf{R}^{\Box}$ shows what, at this strength, one should assert.

Vocabulary:

(i) a denumerable list of propositional variables;

(ii) connectives:

[12]See, e.g., F.R. Routley & R.K. Meyer, 'The Semantics of Entailment' (II), *Journal of Philosophical Logic*, 1 (1972), p. 53; A.R. Anderson & N.D. Belnap, *Entailment*, pp. 351-2.

[13]L. Maximova, 'A Semantics for the Calculus E of Entailment', *Bulletin of the Section of Logic, Polish Academy of Sciences*, 2 (1973), pp. 18-21.

nullary: f, F
unary: $\Box$
binary: $\&, \vee, \rightarrow, \times$.

The following recursion clauses define the notion of wff:

(1) every propositional variable is an (atomic) wff;

(2) if A and B are wffs, so are $\mathsf{f}, \mathsf{F}, \Box A, A \mathbin{\&} B, A \vee B, A \rightarrow B, A \times B$.

Further connectives are introduced by definition:

$$\begin{array}{lll} \sim A =_{df} A \rightarrow \mathsf{f} & \mathsf{t} =_{df} \sim\mathsf{f} & \mathsf{T} =_{df} \sim\mathsf{F} \\ A + B =_{df} \sim A \rightarrow B & & \\ \Diamond A =_{df} \sim\Box\sim A & A \Rightarrow B =_{df} \Box(A \rightarrow B) & \end{array}$$

Bunches are defined as before. The operational rules are also as before, except that $\sim$I and $\sim$E are replaced by a single rule:

$X : A$ is an immediate consequence of $X; \sim A : \mathsf{f}$ (fK)

Note that fK is not contrary to our earlier ideas about the connection between introduction and elimination rules. What Prawitz does is set up an intuitionistic negation, by decreeing that f has no introduction rule (i.e., nothing counts as a derivation of f). He infers that the general template for deriving the elimination rule from the introduction rule then collapses to

$X : A$ is an immediate consequence of $X : \mathsf{f}$

(in our scheme), that is, from f, together with a derivation of A from whatever would ground a derivation of f, that is, no derivation of A at all, we may derive A. This rule, which we may call fJ (J for Jntuitionist),[14] together with the definition of $\sim A$ as $A \rightarrow \mathsf{f}$, then gives an intuitionist theory of negation. That is, it essentially amounts to EFQ. But we do not want this to hold of f, only of F. The confusion arises because of unclarity over what it means to say that nothing counts as a proof of f. Certainly nothing counts, in a classical sense, as a proof of F. Hence, by Prawitz' argument, Fsatisfies EFQ. $\mathsf{F}$is the ultimate absurdity (cf. Corollary 4.15). But f is distinct from F. fK is really (one half of) our earlier structural rule t= in a disguised form. (The other half is derivable using $\rightarrow$E and the definition of $\sim$.) $X : A$ is an immediate consequence of $\mathsf{t}; X : A$, and so of $X; \mathsf{t} : A$ (by C*); whence, by Antilogism (4.10), $X : A$ is an immediate consequence of $X; \sim A : \mathsf{f}$.

[14] See Gentzen, *Collected Papers*, p. 68

Finally, introduction and elimination rules are added for $\Box$:

$X : \Box A$	is an immediate consequence of provided X is S4-modal	$X : A$	($\Box$I)
$X : A$	is an immediate consequence of	$X : \Box A$	($\Box$E)

We define '**S4**-modal' as follows:

1. $\Box B$ and $\mathsf{t}$are **S4**-modal;
2. if B, C are **S4**-modal, so are $B \mathbin{\&} C, B \vee C, B \times C$.
3. X is **S4**-modal if every subbunch of X is **S4**-modal.

(If we wish, we can strengthen the theory of modality to **S5** by replacing the notion of **S4**-modality by that of **S5**-modality, which adds to clause 1) that f is also **S5**-modal, and to clause 2) that $B \to C$ is **S5**-modal if B and C are.)

The structural rules for $\mathbf{R}^{\Box}$ consist of EC, EA, EW, EK, $\mathsf{T}=, \mathsf{t} =$, B, W and C. (We could drop $\mathsf{t} =$ if we took C* as an additional structural rule. See Proposition 4.26 below.) Derivability is defined as in §4.2 above. Then $\mathbf{R}$ is the non-modal system resulting from dropping $\Box$ from the language and removing the rules $\Box$I and $\Box$E. $\mathbf{K} = \mathbf{R}$ + K, and $\mathbf{S4} = \mathbf{R}^{\Box}$ + K.

S4 can be thought of, not as an extension of **K**, but as a subsystem of it (as Anderson and Belnap consider **E** a subsystem of **R**) in this sense: every **S4**-valid sequent in $\mathsf{t}, \mathsf{T}, \sim, \mathbin{\&}, \vee, \times, \Rightarrow$ is **K**-valid, on replacing $\Rightarrow$ by $\to$. Similarly, $\mathbf{R}^{\Box}$ can be treated as a subsystem of **R**. It is in this sense that one might expect $\mathbf{R}^{\Box}$ and **E** to coincide. But, as remarked, they do not.

4.6 Derivations in $\mathbf{R}^{\Box}$

First we show that the theory of negation obtained by defining $\sim A$ as $A \to \mathsf{f}$ and taking fK as our sole rule for negation (essentially double negation elimination) gives us all we had before.

Proposition 4.23 *($\sim$I) If $X; A \vdash \sim B$ and $Y \vdash B$ then $X; Y \vdash \sim A$.*

Proof:			
1	(1)	A	A
$X; 1$	(2)	B	Given
Y	(3)	$\sim B$	Given
$(X; 1); Y$	(4)	f	2,3 $\to$E
$(X; Y); 1$	(5)	f	4 C
$X; Y$	(6)	$\sim A$	5 $\to$I

□

Proposition 4.24 *($\sim$E) If $X \vdash \sim\sim A$ then $X \vdash A$.*

Proof:	X	(1)	$\sim\sim A$	Given
	2	(2)	$\sim A$	A
	$X;2$	(3)	f	1,2 $\to$E
	X	(4)	A	3 fK

□

Proposition 4.25 *(CM) If $X;A \vdash \sim A$ then $X \vdash \sim A$.*

Proof:	1	(1)	A	A
	$X;1$	(2)	$\sim A$	Given
	$(X;1);1$	(3)	f	1,2 $\to$E
	$X;1$	(4)	f	3 W
	X	(5)	$\sim A$	4 $\to$I

□

Hence, every result we have proved for **DW** and **DL** also holds for $\mathbf{R}^{\Box}$. Inductions over the complexity of bunches proceed unaltered, and those over the complexity of wffs, as in Proposition 4.20, can be reworked as appropriate.

Proposition 4.26 $\mathsf{t}=$ *is derivable in* **R** *formulated with C^* and without* $\mathsf{t}=$.

Proof: We must show that $Y(X) : A$ iff $Y(\mathsf{t}; X) : A$, i.e., $Y((\mathsf{f} \to \mathsf{f}); X) : A$. By compactness, it suffices to treat finite X'. We show that $\mathsf{cf}(X') \dashv\vdash \mathsf{t} \times \mathsf{cf}(X')$, and use the Replacement Theorem (4.20) to infer their substitutability:

1	(1)	$(\mathsf{f} \to \mathsf{f}) \times \mathsf{cf}(X')$	A
2	(2)	$\mathsf{f} \to \mathsf{f}$	A
3	(3)	$\mathsf{cf}(X')$	A
4	(4)	$\sim\mathsf{cf}(X')$	A
4;3	(5)	f	3,4 $\to$E
2;(4;3)	(6)	f	2,5 $\to$E
(2;4);3	(7)	f	6 B
(2;3);4	(8)	f	7 C
2;3	(9)	$\mathsf{cf}(X')$	8 fK
1	(10)	$\mathsf{cf}(X')$	1,9 $\times$E

Conversely,

1	(1)	$\mathsf{cf}(X')$	A
2	(2)	$\mathsf{f} \to \mathsf{f}$	A
2;1	(3)	$(\mathsf{f} \to \mathsf{f}) \times \mathsf{cf}(X')$	1,2 $\times$I
4	(4)	$\sim\mathsf{cf}(X')$	A
4;(2;1)	(5)	f	3,4 $\to$E
(4;2);1	(6)	f	5 B
(4;1);2	(7)	f	6 C
4;1	(8)	f	7 fK
1;4	(9)	f	8 C*
1	(10)	$\mathsf{cf}(X')$	9 fK

The Replacement Theorem (Proposition 4.20) then shows that t;X and X may be interchanged in any context. □

Proposition 4.27 (i) *C*, C**, B‘, W*, WB, S and S* are all derivable in* **R**.

(ii) *The converses of B and B′ are derivable in* **R**.

Proof:

(i) C*: $X;Y = (\mathsf{t};X);Y$ (t=)
$\leq (\mathsf{t};Y);X$ (C)
$= Y;X$ (t=)

C**: $X;t \leq \mathsf{t};X$ (C*)
$= X$ (t=)

B′: $X;(Y;Z) \leq (X;Y);Z$ (B)
$\leq (Y;X);Z$ (C*)

W*: $X;X = (\mathsf{t};X);X$ (t=)
$\leq \mathsf{t};X$ (W)
$= X$ (t=)

WB: $X;(X;Y) = (X;X);Y$ (B′)
$\leq X;Y$ (W*)

S: $(X;Z);(Y;Z) \leq ((X;Z);Y);Z$ (B)
$\leq ((X;Y);Z);Z$ (C)
$\leq (X;Y);Z$ (W)

S*: $Y;(X;Y) \leq (X;Y);Y$ (C*)
$\leq X;Y$ (W)

(ii) the converse of B: $(X;Y);Z \leq (Y;X);Z$ (C*)
$\leq (Y;Z);X$ (C)
$\leq X;(Y;Z)$ (C*)

the converse of B′: $(Y;X);Z \leq (Y;Z);X$ (C)
$\leq X;(Y;Z)$ (C*) □

Proposition 4.28 *(RAA) If $X;A \vdash B$ and $Y;A \vdash {\sim}B$ then $X,Y \vdash {\sim}A$.*

Proof:

1	(1)	A	A
$X;1$	(2)	B	Given
$Y;1$	(3)	${\sim}B$	Given
$Y;(X;1)$	(4)	${\sim}A$	2,3 ∼I
$(Y;X);1$	(5)	${\sim}A$	4 B
$Y;X$	(6)	${\sim}A$	5 CM (4.25)
$(X,Y);X$	(7)	${\sim}A$	6 EK
$(X,Y);(X,Y)$	(8)	${\sim}A$	7 EK
$X;Y$	(9)	${\sim}A$	8 W*

□

Proposition 4.29 (i) *$B \Rightarrow C; \Box B \vdash \Box C$.*

(ii) *If X is* **S4***-modal, then $X;A \vdash B$ iff $X \vdash A \Rightarrow B$.*

Proof:

(i)

1	(1)	$B \Rightarrow C$	A
2	(2)	$\Box B$	A
1	(3)	$B \to C$	1 □E
2	(4)	B	2 □E
1;2	(5)	C	3,4 →E
1;2	(6)	$\Box C$	5 □I

(ii) suppose $X \vdash A \Rightarrow B$. Then $X \vdash A \to B$, by □E, and so $X;A \vdash B$, by →E. Conversely, suppose $X;A \vdash B$. Then $X \vdash A \to B$, by →I, whence $X \vdash A \Rightarrow B$, by □I, since X is **S4**-modal.

□

Lastly, we show that the Minc sequent is derivable in $\mathbf{R}^{\Box}$.

Proposition 4.30 *(The Minc sequent)*

$$A \Rightarrow (B \Rightarrow B), B \Rightarrow (A \vee B) \vdash B \Rightarrow B$$

Proof:

1	(1)	$A \Rightarrow (B \Rightarrow B)$	A
2	(2)	$B \Rightarrow (A \vee B)$	A
3	(3)	B	A
2	(4)	$B \rightarrow (A \vee B)$	2 $\Box$E
2;3	(5)	$A \vee B$	3,4 $\rightarrow$E
6	(6)	A	A
1	(7)	$A \rightarrow (B \Rightarrow B)$	1 $\Box$E
1;6	(8)	$B \Rightarrow B$	6,7 $\rightarrow$E
1;6	(9)	$B \rightarrow B$	8 $\Box$E
(1;6);3	(10)	B	3,9 $\rightarrow$E
(1;3);6	(11)	B	10 C
((1;3),6);6	(12)	B	11 EK
((1;3),6);((1;3),6)	(13)	B	12 EK + EC
(1;3),6	(14)	B	13 W*
(1;3),3	(15)	B	3 EK
(1;3),(2;3)	(16)	B	5,14,15 $\vee$E
((1,2);3),((1,2);3)	(17)	B	16 EK
(1,2);3	(18)	B	17 EW
1,2	(19)	$B \rightarrow B$	18 $\rightarrow$I
1,2	(20)	$B \Rightarrow B$	19 $\Box$I

$\Box$

Chapter 5

Worlds Semantics for Relevant Logics

Relevant logics have both extensional (truth-functional) and intensional (non-truth-functional) connectives. The truth-value of wffs formed by extensional connectives is determined simply by the truth-values of their constituents. But that of wffs formed by intensional connectives is not. Nonetheless, what we would like is a recursive definition of truth, whereby the truth-value of complex wffs is characterised in terms of some semantic property of their parts. The ultimate reason why we desire such an account of the truth-value of complex (or arbitrary) wffs is to characterise validity, or logical consequence. But in the presence of intensional connectives, this property of the parts cannot be simply their truth-value.

5.1 Worlds Semantics

A great deal of work has been put into the semantics of extensional logic, and the present semantics will conserve as much of that as possible. The definition of logical consequence in extensional semantics is as follows:

> A is a logical consequence of a set of wffs X (for short, $X \models A$) iff A is true whenever every member of X is true.

The notion 'whenever' is spelled out in terms of models. A model (or interpretation) in extensional propositional logic consists simply in a specification (or assignment) v of truth-values to the atomic wffs. Let $v(A) = \mathsf{T}$ abbreviate 'wff A is true under v' (= F if A is false under v). Then the condition becomes:

$X \models A$ iff for all $v, v(A) = \mathsf{T}$ if $v(B) = \mathsf{T}$ for all $B \in X$.

Much work has also been expended on certain intensional logics, notably the modal logics. Those who constructed the semantics of modal logic also wished to conserve as many as possible of the insights of extensional logic. But, as we noted, truth-value alone will not suffice. The insight here was to index truth-values. Thus in any modal logic, a wff A is not simply true or false, but true or false at some index a. The (indexed) truth-value of a complex wff is not simply a matter of the truth-values of its constituents, but of the *indexed* truth-values of its constituents. With appropriate constraints on the set of indices, W, we can then define logical consequence for any modal logic.

Formal semantics consists in a formal specification together with an intuitive backing. The formal work proceeds straightforwardly, in a recursive specification of indexed truth-values, leading through to the definition of logical consequence. But some intuitive motivation and explanation of, in particular, the idea of an indexed truth-value is welcome. The idea is, of course, that each index is, or encodes, in some sense, a possible world. In this context, possible worlds can be thought of as states of affairs or states of information. If the idea of such non-actual situations really existing is unwelcome, they can be construed nominalistically as the sets of statements themselves which could be taken as descriptive of those possible situations. So, that a wff A is true at a world w is identified with the condition that $A \in w$, that A is one of the propositions in, or constituting, w.

Suppose we have a classical modal propositional language built on propositional variables $p, q, r, \ldots$, and connectives $\sim, \&, \vee, \supset, \Box$, $\prec\!\!3$ in the usual way. $\langle 0, W, S\rangle$ is a model structure for such a language if W is a non-empty set (of indices), 0 is a distinguished member of W (truth simpliciter or the "real world"), and S is a binary relation on W (i.e., $S \subseteq W^2$). An interpretation $\mathcal{J}$ consists of a model structure $\langle 0, W, S\rangle$ together with a valuation v on $\langle 0, W, S\rangle$, that is, an assignment of truth-values to the wffs of the language relative to the set W of indices, as follows:

1. For all atomic wffs A and all $a \in W$, let $v(A, a) \in \{\mathsf{T}, \mathsf{F}\}$.

2. (i) $v({\sim}A, a) = \mathsf{T}$ if $v(A, a) = \mathsf{F}$.
   (ii) $v(A \mathbin{\&} B, a) = \mathsf{T}$ if $v(A, a) = \mathsf{T}$ and $v(B, a) = \mathsf{T}$.
   (iii) $v(A \vee B, a) = \mathsf{T}$ if $v(A, a) = \mathsf{T}$ or $v(B, a) = \mathsf{T}$.
   (iv) $v(A \supset B, a) = \mathsf{T}$ if $v(A, a) = \mathsf{F}$ or $v(B, a) = \mathsf{T}$.
   (v) $v(\Box A, a) = \mathsf{T}$ if for all b such that $Sab, v(A, b) = \mathsf{T}$.

(vi) $v(A \multimap B, a) = \mathsf{T}$ if for all b such that Sab, $v(A, b) = \mathsf{F}$ or $v(B, b) = \mathsf{T}$.

3. Otherwise $v(A) = \mathsf{F}$.

Then $X \models A$ iff for all interpretations $\mathcal{J}$ (model structures $\langle 0, W, S \rangle$ with valuation v), $v(A, a) = \mathsf{T}$ whenever $v(B, a) = \mathsf{T}$ for all $B \in X$. (Remember that classically, X is a set of wffs.) Intuitively, 'Sab' says that wffs are true at b only if it is possible that they are true at a—that is, b is (the set of statements) 'possible' relative to a. The differences between the various modal logics are then characterised semantically by different conditions on S (and on any additional structure to W). In particular, **S4** is captured by making S reflexive and transitive.

5.2 Relevant Semantics

In the above semantics, necessity and strict implication are given an intensional truth-condition. But the other connectives are treated extensionally. It does not, therefore, give a suitable semantics for relevant logic. To do that, we have to introduce three further complications. First, recall that the relevant logics are paraconsistent, that is,

$$A \;\&\; {\sim}A \not\models B.$$

Accordingly, we need to be able to make '$A \;\&\; {\sim}A$' true at some index and B false there too. But since '&' is truth-functional, that means we must somehow make both A and '${\sim}A$' true at the same index. But A cannot be both true and not true—or v will not be a function at all. So we must change the truth-conditions for negation. This is done by introducing a function $^* : W \to W$, requiring that

$$v({\sim}A, a) = \mathsf{T} \text{ if } v(A, a^*) = \mathsf{F},$$

and then laying down appropriate conditions on * (to obtain contraposition, double negation and so on).

What is the intuitive meaning of this *-operation? What we are here attempting to do is establish a framework in which both A and '${\sim}A$', for example, may be true at some world a. Thus a is an inconsistent world. Accordingly, by the formal truth-condition for '${\sim}$', just given, both A and '${\sim}A$' will be false at a^*. So a^* is (negation-)incomplete.

Traditionally, two notions have been associated with the conception of a possible world. First, that a possible world is, in some sense, maximal—it specifies in as much detail as possible how things are. That's what entitles it to be called a 'world'. Secondly, that its description satisfies all logical laws—that's what makes it a 'possible' world. The underlying thoughts here are good; but in the present context their manifestation must change. Worlds as now conceived will still be maximal, but in the following sense: every atomic wff will be either true or false. The valuation function will be total. Any further specification of the valuation would lose it its functional character. So a world is in this sense maximal, but not necessarily negation-complete. Indeed, the truth-definition as now applied to this atomic valuation will entail that at some worlds, not all logical laws will hold. (Those where they do will be called 'regular' worlds.) For example, at a^* as given in the last paragraph, neither '$A \vee \sim A$' nor '$A \mathbin{\&} \sim A$' (nor '$\sim(A \mathbin{\&} \sim A)$') is true. (The Laws of Excluded Middle and Non-Contradiction are not theorems of all the logics we consider—e.g., they are not provable in **DW**—but they are of most, of **DL** and stronger systems. Even so, the semantics for **DL** must admit worlds where '$A \vee \sim A$' fails to hold.) In contrast, at a both '$A \mathbin{\&} \sim A$' and '$\sim(A \mathbin{\&} \sim A)$', as well as A and '$\sim A$', are true.

The intuitive connection between a and a^* is this: what is asserted at a is what is not denied at a^*, and vice versa. If the story one of a and a^* tells about A is incomplete, the story told by the other will be inconsistent, and *vice versa*.

The second complication arises from the fact that the non-modal conditional '$\rightarrow$' of **DW** is intensional, and so the truth-value of '$A \rightarrow B$' at a cannot depend solely on the values of A and B at a. The solution is obtained by introducing a three-place relation R on W (i.e., $R \subseteq W^3$), and adopting the truth-condition

$$v(A \rightarrow B, a) = \mathsf{T} \text{ if } v(B, c) = \mathsf{T} \text{ whenever } v(A, b) = \mathsf{T} \text{ and } Rabc.$$

By laying down appropriate conditions on R we ensure whatever properties of '$\rightarrow$' we require (in **DW** or whichever relevant logic we seek). Intuitively, if $Rabc$ holds then wffs true at a and b are compossible at c—that is, a and b are 'compatible' relative to c. Thus, '$A \times B$', e.g., is true at c if there are indices a, b with $v(A, a) = \mathsf{T}$, $v(B, b) = \mathsf{T}$ and $Rabc$.

Thirdly, we wish to give a semantics for the version of relevant logic presented in chapter 4, in which X is a bunch, not (simply) a set. There are two ways we could proceed. First, recall that the comma and semicolon have the formal properties of extensional and intensional conjunction, respectively. So we could try replacing each bunch X by its characteristic formula, $\mathsf{cf}(X)$. (See §4.2.) However, this will only work for finite bunches.

For only finite sets can be obtained recursively by pairing. To cope with the infinite sets available on our earlier definition of bunch we need instead to provide a semantics for bunches directly. Recall the definition of bunch:

1. any wff A is an I-bunch;
2. if X, Y are bunches, $X \circ Y$ is an I-bunch;
3. any non-empty set of I-bunches is an E-bunch.
4. no I-bunch is an E-bunch.

We can define the satisfiability of a bunch by a valuation at an index a relative to a model structure by adding further clauses to the truth-definition, corresponding to 2) and 3) in the definition of bunch:

3. (i) $v(X \circ Y, a) = \mathsf{T}$ if $v(Y, c) = \mathsf{T}$ for some $c \in W$ for which there is a $b \in W$ with $v(X, b) = \mathsf{T}$ and $Rbca$;
   (ii) if X is an E-bunch, $v(X, a) = \mathsf{T}$ if $v(Y, a) = \mathsf{T}$ for all $Y \in X$.

The differences between the systems of relevant logic are then characterised by different conditions on $R, S, ^*$, and generally on the structure of W.

5.3 Formal Semantics for DW

We can now give the formal semantic characterisation of the propositional relevant logic **DW**. Let **L** be the language (of t, T, $\sim, \&, \vee, \to$ and $\times$). t is essentially the conjunction of all logical truths—i.e., the logic. To characterise it, we need a subset P of W, containing the real world 0, where the logic holds. Elsewhere it may not—incompleteness may strike so deep that not even all the logical truths hold. (For example, we require that

$$B \not\models A \to A,$$

and so there must be a world a for which $v(A \to A, a) = \mathsf{F}$ (while $v(B, a) = \mathsf{T}$).) An interpretation $\mathcal{J}$ for **L** consists of a model structure, namely, a quadruple $\langle P, W, R, ^* \rangle$, where P is a non-empty subset of W, the set of indices, $R \subseteq W^3$ (that is, a 3-place relation on W), and $^* : W \to W$; and a valuation function $v : L \times W \to \{\mathsf{T}, \mathsf{F}\}$, that is, a function mapping one of the values T, F to each wff and bunch relative to each index. First, we define

d1 $a \leq b$ if $(\exists x)(x \in P \;\&\; Rxab)$

d2 R^2abcd if $(\exists x)(Rabx \; \& \; Rxcd)$

d3 $R^2a(bc)d$ if $(\exists x)(Rbcx \; \& \; Raxd)$.

Recall that $Rabc$ says, roughly, that a and b are compossible relative to c; then R^2abcd says that first a and b, and then c are compossible relative to d, and $R^2a(bc)d$ that first a and then b and c are compossible relative to d. Our basic requirements on the model structure are:

p1: $\leq$ is a partial ordering:

(a) $a \leq a$

(b) $a \leq b \; \& \; b \leq c \Rightarrow a \leq c$

(c) $a \leq b \; \& \; b \leq a \Rightarrow a = b$

p2: R is monotonic:

(a) $a \leq x \; \& \; Rxbc \Rightarrow Rabc$

(b) $b \leq x \; \& \; Raxc \Rightarrow Rabc$

(c) $x \leq c \; \& \; Rabx \Rightarrow Rabc$

p3: $a^{**} = a$

p4: $Rabc \Rightarrow Rac^*b^*$

p5: $a \leq b \; \& \; a \in P \Rightarrow b \in P$

p1(c) is strictly unnecessary: a preordering will suffice. However, a partial ordering is easier to picture. The valuation v is a total function v: $\mathbf{L} \times W \rightarrow \{\mathsf{T}, \mathsf{F}\}$ such that:

V1 for all atomic wffs A, if $v(A, a) = \mathsf{T}$ and $a \leq b$, then $v(A, b) = \mathsf{T}$ (we call this the 'heredity' condition)

V2 (i) $v(\sim\!A, a) = \mathsf{T}$ if $v(A, a^*) = \mathsf{F}$.

(ii) $v(A \; \& \; B, a) = \mathsf{T}$ if v(A,a) $= \mathsf{T} and v(B, a) = \mathsf{T}$.

(iii) $v(A \vee B, a) = \mathsf{T}$ if $v(A, a) = \mathsf{T}$ or $v(B, a) = \mathsf{T}$.

(iv) $v(A \rightarrow B, a) = \mathsf{T}$ if $v(B, c) = \mathsf{T}$ whenever $v(A, b) = \mathsf{T}$ and $Rabc$.

(v) $v(A \times B, a) = \mathsf{T}$ if for some $b, c, v(A, b) = v(B, c) = \mathsf{T}$ and $Rbca$.

(vi) $v(\mathsf{t}, a) = \mathsf{T}$ if $a \in P$.

(vii) $v(\mathsf{T}, a) = \mathsf{T}$ for all a.

V3 (i) if Z is an I-bunch, $X \circ Y$, $v(Z,a) = \mathsf{T}$ if for some $b, c, v(X,b) = v(Y,c) = \mathsf{T}$ and $Rbca$.

(ii) if Z is an E-bunch, $v(Z,a) = \mathsf{T}$ if $v(Y,a) = \mathsf{T}$ for all $Y \in Z$.

V4 otherwise, $v(X,a) = \mathsf{F}$.

Then A is true under $\mathcal{J}$ if for every $e \in P$, $v(A,e) = \mathsf{T}$. $X \models A$ if for all $\mathcal{J}$, and all $a \in W$, if $v(X,a) = \mathsf{T}$ then $v(A,a) = \mathsf{T}$. $\models A$ if A is true under every $\mathcal{J}$.

5.4 Formal Results for DW-Semantics

The above semantics permits the demonstration of various general results as follows. First we show that the Heredity Condition (V1) extends to all wffs and bunches.

Proposition 5.1 *(The Heredity Lemma) If $a \leq b$ and $v(X,a) = \mathsf{T}$ then $v(X,b) = \mathsf{T}$.*

Proof: by induction on the degree of X. First, suppose that X is a single wff, A.

1. The base case is immediate, from V1).

2. (i) $A = {\sim}B$: suppose $v({\sim}B,a) = \mathsf{T}$ and $a \leq b$. Then $v(B,a^*) = \mathsf{F}$ by V2i), and $b^* \leq a^*$ by p4. So $v(B,b^*) = \mathsf{F}$ by Induction Hypothesis (I.H.), whence $v({\sim}B,b) = \mathsf{T}$ by V2i).

   (ii) $A = B \mathbin{\&} C$: suppose $v(B \mathbin{\&} C,a) = \mathsf{T}$ and $a \leq b$. Then $v(B,a) = \mathsf{T}$ and $v(C,a) = \mathsf{T}$ by V2ii), and so $v(B,b) = \mathsf{T}$ and $v(C,b) = \mathsf{T}$ by I.H. whence $v(B \mathbin{\&} C,b) = \mathsf{T}$ by V2ii).

   (iii) $A = B \vee C$: similar to ii).

   (iv) $A = B \to C$: suppose $v(B \to C,a) = \mathsf{T}$ and $a \leq b$. Then if $v(B,x) = \mathsf{T}$ and $Raxy, v(C,y) = \mathsf{T}$ for all x, y by V2iv). Suppose $Rbxy$. Then $Raxy$ by p2, so if $v(B,x) = \mathsf{T}, v(C,y) = \mathsf{T}$. Hence $v(B \to C,b) = \mathsf{T}$ by V2iv).

   (v) $A = B \times C$: suppose $v(B \times C,a) = \mathsf{T}$ and $a \leq b$. Then for some $x, y, v(B,x) = v(C,y) = \mathsf{T}$ and $Rxya$ by V2v). Then $Rxyb$ by p2 and so $v(B \times C,b) = \mathsf{T}$ by V2v).

   (vi) $A = \mathsf{t}$: suppose $v(\mathsf{t},a) = \mathsf{T}$ and $a \leq b$. Then $a \in P$ by V2vi), so $b \in P$ by p5. Hence, $v(\mathsf{t},b) = \mathsf{T}$ by V2vi).

(vii) $A = \mathsf{T}$: suppose $v(\mathsf{T}, a) = \mathsf{T}$ and $a \leq b$. Then $v(\mathsf{T}, b) = \mathsf{T}$ since $v(\mathsf{T}, b) = \mathsf{T}$ for all $b \in W$ by V2vii).

3. If X is not a single wff, then we use V3i) and ii):

   (i) $X = Y \circ Z$: suppose $v(Y \circ Z, a) = \mathsf{T}$ and $a \leq b$. Proof as 2v).

   (ii) X is an E-bunch: suppose $v(X, a) = \mathsf{T}$ and $a \leq b$. Then $v(Y, a) = \mathsf{T}$ for all $Y \in X$ by V3ii). So $v(Y, b) = \mathsf{T}$ by I.H., whence $v(X, b) = \mathsf{T}$ by V3ii). □

Proposition 5.2 *(The Deduction Equivalence)* $X; A \models B$ *iff* $X \models A \to B$.

Proof: Suppose $X; A \models B, V(X, a) = \mathsf{T}, v(A, b) = \mathsf{T}$ and *Rabc*. Then $v(X \circ A, c) = \mathsf{T}$ by V3i), and so $v(B, c) = \mathsf{T}$, whence $v(A \to B, a) = \mathsf{T}$ by V2iv), and so $X \models A \to B$. Conversely, suppose $X \models A \to B$, and $v(X \circ A, a) = \mathsf{T}$. Then, for some $b, c, v(X, b) = v(A, c) = \mathsf{T}$ and *Rbca*. So $v(A \to B, b) = \mathsf{T}$, whence $v(B, a) = \mathsf{T}$ by V2iv), and so $X; A \models B$. □

Proposition 5.3 *(t is the logic)* $\mathsf{t} \models A$ *iff* $\models A$.

Proof: suppose $\mathsf{t} \models A$, i.e., if $v(\mathsf{t}, a) = \mathsf{T}$ then $v(A, a) = \mathsf{T}$. Take $e \in P$. Then $e \leq e$ by p1a), and so $v(\mathsf{t}, e) = \mathsf{T}$ by V2vi). But P is non-empty. So $\models A$. Conversely, suppose $\models A$, i.e., for every $e \in P, v(A, e) = \mathsf{T}$, and suppose $v(\mathsf{t}, a) = \mathsf{T}$. Then $a \in P$ by V2vi). So $v(A, a) = \mathsf{T}$, whence $\mathsf{t} \models A$. □

Lemma 1 (i) *If* $v(Y(Z), a) = \mathsf{T}$ *and* $W \subseteq Z$, *then* $v(Y(W), a) = \mathsf{T}$;

(ii) *If* $X' \angle X$ *and* $X' \models A$ *then* $X \models A$.

Proof:

(i) by induction on the complexity of Y. The base case, where $Y = Z$ is immediate, by V3ii).

   Induction Step:

   (a) $Y(Z) = U(Z) \circ V$. Suppose $v(Y(Z), a) = \mathsf{T}$. Then there are b, c such that $v(U(Z), b) = v(V, c) = \mathsf{T}$ and *Rbca*, by V3i). By I.H. $v(U(W), b) = \mathsf{T}$. So by V3i) again, $v(Y(W), a) = \mathsf{T}$.

   (b) $U(Z) \in Y(Z)$. Suppose $v(Y(Z), a) = \mathsf{T}$. Then for all $\mathsf{T} \in Y(Z), v(\mathsf{T}, a) = \mathsf{T}$, by V3ii), in particular, $v(U(Z), a) = \mathsf{T}$. So by I.H. $v(U(W), a) = \mathsf{T}$. Hence, by V3ii) again, $v(Y(W), a) = \mathsf{T}$.

(ii) By induction on $\angle$: recall the definition:

1. if W, Z are E-bunches and $W \subseteq Z$, then $Y(W) \angle' Y(Z)$;
2. $\angle$ is the reflexive, transitive closure of $\angle'$.

If $X = X'$, the result is immediate. Otherwise, $X' = Y(W)$ and $W \subseteq Z$, where $Y(Z) \angle X$. By I.H., if $Y(Z) \models A$ then $X \models A$. Suppose $Y(W) \models A$, and suppose further that $v(Y(Z), a) = \mathsf{T}$. Then, by i), $v(Y(W), a) = \mathsf{T}$. So $v(A, a) = \mathsf{T}$, and so $Y(Z) \models A$. So by I.H. $X \models A$. Hence by induction, if $X' \angle X$ and $X' \models A$ then $X \models A$. □

Proposition 5.4 *(Soundness) If $X \vdash A$ then $X \models A$.*

Proof: Suppose $X \vdash A$. Then for some finite $X' \angle X, X' \vdash A$. We show, by induction on the length of proof, that $X' \models A$. The base case, where $X' = A$, is immediate. Otherwise, $X' : A$ results from earlier sequents in the proof by either operational or structural inferences.

(i) &I: By I.H. $Y \models A$ and $Z \models B$, where $X' = \{Y, Z\}$. Suppose $v(X', a) = \mathsf{T}$. Then $v(Y, a) = \mathsf{T}$ and $v(Z, a) = \mathsf{T}$ by V3ii), and so $v(A, a) = \mathsf{T}$ and $v(B, a) = \mathsf{T}$ by I.H., whence $v(A \,\&\, B, a) = \mathsf{T}$ by V2ii) and so $X' \models A \,\&\, B$.

(ii) &E: By I.H. $Z \models A \,\&\, B$ and $Y(A, B) \models C$, where $X' = Y(Z)$. Suppose $v(Y(Z), a) = \mathsf{T}$. We show by induction on the complexity of Y, that $v(Y(A, B), a) = \mathsf{T}$.

  (a) If $Y(Z) = Z$ then $v(A \,\&\, B, a) = \mathsf{T}$, so $v(A, a) = v(B, a) = \mathsf{T}$ by V2ii). Moreover, $Y(A, B) = \{A, B\}$, and so $v(Y(A, B), a) = \mathsf{T}$ by V3ii).

  (b) If $W(Z) \in Y(Z)$ then $v(U, a) = \mathsf{T}$ for all $U \in Y(Z)$ by V3ii), in particular, $v(W(Z), a) = \mathsf{T}$. So by I.H. on Y, $v(W(A, B), a) = \mathsf{T}$ and so $v(Y(A, B), a) = \mathsf{T}$ by V3ii).

  (c1) If $Y(Z) = U(Z) \circ W$ then $v(U(Z), b) = v(W, c) = \mathsf{T}$ and $Rbca$ by V3i). So by I.H. on Y, $v(U(A, B), b) = \mathsf{T}$ whence $v(U(A, B) \circ W, a) = \mathsf{T}$, i.e. $v(Y(A, B), a) = \mathsf{T}$ by V3i).

  (c2) $Y(X) = W \circ U(Z)$: similar to c1).

Hence $v(Y(A, B), a) = \mathsf{T}$ and so $v(C, a) = \mathsf{T}$ by I.H. on proofs. Hence $Y(Z) \models C$, as required.

(iii) $\vee$I: By I.H., w.l.g. $X' \models A$. Suppose $v(X', a) = \mathsf{T}$. Then $v(A, a) = \mathsf{T}$ by I.H., whence $v(A, a) = \mathsf{T}$ or $v(B, a) = \mathsf{T}$ and so $v(A \vee B, a) = \mathsf{T}$ by V2iii) whence $X' \models A \vee B$.

(iv) $\vee$E: By I.H. $Z \models A \vee B, Y(A) \models C$ and $Y(B) \models C$, where $Y' = Y(Z)$. Suppose $v(Y(Z), a) = \mathsf{T}$. We show by induction on the complexity of Y, that $v(Y(A), a) = \mathsf{T}$ or $v(Y(B), a) = \mathsf{T}$.

- (a) If $Y(Z) = Z$ then $v(A \vee B, a) = \mathsf{T}$, so $v(A, a) = \mathsf{T}$ or $v(B, a) = \mathsf{T}$. Moreover, $Y(A) = A$ and $Y(B) = B$, so $v(Y(A), a) = \mathsf{T}$ or $v(Y(B), a) = \mathsf{T}$.
- (b) If $W(Z) \in Y(Z)$, then $v(U, a) = \mathsf{T}$ for all $U \in Y(Z)$ by V3ii), in particular, $v(W(Z), a) = \mathsf{T}$, so $v(W(A), a) = \mathsf{T}$ or $v(W(B), a) = \mathsf{T}$ by I.H. on Y. Moreover, $W(A) \in Y(A)$ and $W(B) \in Y(B)$, so $v(Y(A), a) = \mathsf{T}$ or $v(Y(B), a) = \mathsf{T}$ by V3ii).
- (c1) If $Y(Z) = U(Z) \circ W$, then $v(U(Z), b) = v(W, c) = \mathsf{T}$ and *Rbca* by V3i). By I.H. on Y, $v(U(A), b) = \mathsf{T}$ or $v(U(B), b) = \mathsf{T}$, and so $v(Y(A), a) = \mathsf{T}$ or $v(Y(B), a) = \mathsf{T}$ by V3i).
- (c2) $Y(Z) = W \circ U(Z)$: similar to c1).

Hence $v(Y(A), a) = \mathsf{T}$ or $v(Y(B), a) = \mathsf{T}$, whence, either way, $v(C, a) = \mathsf{T}$ by I.H. on proofs. So $Y(Z) \models C$.

(v) $\to$I: by Proposition 5.2.

(vi) $\to$E: By I.H. $Y \models A \to B$ and $Z \models A$. Suppose $v(Y \circ Z, a) = \mathsf{T}$. Then $v(Y, b) = v(Z, c) = \mathsf{T}$ and *Rbca*. So $v(A \to B, b) = v(A, c) = \mathsf{T}$. Hence $v(B, a) = \mathsf{T}$ by V2iv) and so $X; Y \models B$.

(vii) $\times$I: By I.H. $Y \models A$ and $Z \models B$. Suppose $v(Y \circ Z, a) = \mathsf{T}$. Then $v(Y, b) = v(Z, c) = \mathsf{T}$ and *Rbca*. So $v(A, b) = v(B, c) = \mathsf{T}$ and *Rbca* whence $v(A \times B, a) = \mathsf{T}$ by V2v) and so $Y; Z \models A \times B$.

(viii) $\times$E: By I.H. $Z \models A \times B$ and $Y(A; B) \models C$. Suppose $v(Y(Z), a) = \mathsf{T}$. We show by induction on the complexity of Y, that $v(Y(A; B), a) = \mathsf{T}$.

- (a) If $Y(Z) = Z$ then $v(A \times B, a) = \mathsf{T}$, so $v(A, b) = v(B, c) = \mathsf{T}$ and *Rbca*. Moreover, $Y(A; B) = A \circ B$, so $v(Y(A; B), a) = \mathsf{T}$.
- (b) If $Y(Z) = \{U(Z), W\}$ then $v(U(Z), a) = v(W, a) = \mathsf{T}$ by V3ii). By I.H. on Y, $v(U(A; B), a) = \mathsf{T}$ and so $v(Y(A; B), a) = \mathsf{T}$ by V3ii).

(c1) If $Y(Z) = U(Z) \circ W$ then $v(Z(X), b) = v(W, c) = \mathsf{T}$ and $Rbca$ by V3i). By I.H. on Y, $v(U(A; B), b) = \mathsf{T}$ and so $v(Y(A; B), a) = \mathsf{T}$ by V3i).

(c2) $Y(Z) = W \circ U(Z)$: similar to c1).

Hence $v(Y(A; B), a) = \mathsf{T}$ and so $v(C, a) = \mathsf{T}$ by I.H. on proofs, whence $Y(X) \models C$.

(ix) $\sim$I: By I.H. $Y; A \models\ \sim B$ and $Z \models B$. Suppose $v(Y \circ Z, a) = \mathsf{T}$ and $v(\sim A, a) = \mathsf{F}$. Then $v(Y, b) = v(Z, c) = \mathsf{T}$, $Rbca$ and $v(A, a^*) = \mathsf{T}$. Then Rba^*c^* by p4, and so $v(Y \circ A, c^*) = \mathsf{T}$. So $v(\sim B, c^*) = \mathsf{T}$ by I.H. and so $v(B, c) = \mathsf{F}$. But also $v(B, c) = \mathsf{T}$ by I.H. This is impossible. Hence $v(\sim A, a) = \mathsf{T}$ and so $Y; Z \models\ \sim A$.

(x) $\sim$E: By I.H. $X' \models\ \sim\sim A$. Suppose $v(X', a) = \mathsf{T}$. Then $v(\sim\sim A, a) = \mathsf{T}$ and so $v(\sim A, a^*) = \mathsf{F}$, whence $v(A, a^{**}) = \mathsf{T}$ by V2i). But $a^{**} = a$, so $X' \models A$.

(xi) EC: Immediate, since $\{X, Y\} = \{Y, X\}$.

(xii) EW: Immediate again, since $\{X, X\} = \{X\}$.

(xiii) It remains to deal with EK, $\mathsf{T}=$ and $\mathsf{t}=$. In each case, we proceed by induction on the complexity of $W(U)$, where the rule has the form $U \leq V$. We suppose that $v(W(V), a) = \mathsf{T}$ and show that $v(W(U), a) = \mathsf{T}$:

(1) $W(V) = V$:

(a) EK: $W(V) = Y, Z$. Then $v(Y, a) = \mathsf{T}$ by V3ii).

(b) $\mathsf{T}=$: $W(V) = X'$. By V2i), $v(\mathsf{T}, a) = \mathsf{T}$, so $v(\mathsf{T}, X', a) = \mathsf{T}$ by V3ii). Conversely, if $v(\mathsf{T}, X, a) = \mathsf{T}$ then $v(X, a) = \mathsf{T}$.

(c) $\mathsf{t}=$: i) $W(V) = X$. By p1, $a \leq a$, so there is $e \in P$ (namely, a) such that $Reaa$. Hence $v(t \circ X', a) = \mathsf{T}$, by V3i).

Conversely, $W(V) = \mathsf{t} \circ Z = X'$. Then for some $b, c, v(\mathsf{t}, b) = v(Z, c) = \mathsf{T}$ and $Rbca$, by V3i). Then $b \in P$ by V2vi), whence $c \leq a$ by d1. So $v(Z, a) = \mathsf{T}$ by Proposition 5.1.

(2) w.l.g. $W(V) = T(V) \circ S$: then, for some $b, c, v(T(V), b) = v(S, c) = \mathsf{T}$ and $Rbca$, by V3i). By I.H. on Z, $v(T(U), b) = \mathsf{T}$, so $v(W(U), a) = \mathsf{T}$ by V3i).

(3) $W(V)$ is an E-bunch: then $v(S, a) = \mathsf{T}$ for all $S \in W(V)$, in particular, $v(S(V), a) = \mathsf{T}$. So $v(S(U), a) = \mathsf{T}$ by I.H., whence $v(W(U), a) = \mathsf{T}$.

So, by induction on $W, v(W(U), a) = \mathsf{T}$, and so $v(A, a) = \mathsf{T}$ by I.H. on proof. Hence $W(U) \models A$.

Hence by induction on length of proof, $X' \models A$. So, by Lemma 1, $X \models A$. □

Corollary 5.5 *$A \models B$ iff $\models A \to B$.*

Proof: immediate from Propositions 5.2, 5.3 and 5.4. □

Corollary 5.6 *$\not\vdash \mathsf{f}$.*

Proof: Take as m.s. $M_1 = \langle P, W, R, ^* \rangle$, where $P = \{0\}, W = \{0\}, R = \{000\}, 0^* = 0$. ($M_1$ is effectively just truth-tables.) Then M_1 is a (**DW**-)m.s. satisfying postulates p1-p5. $v(\mathsf{t}, 0) = \mathsf{T}$ by V2vi), since $0 \in P$. So $v(\mathsf{f}, 0) = v(\sim\mathsf{t}, 0) = \mathsf{F}$ by V2i). Hence $\mathsf{t} \not\models \mathsf{f}$, whence by Proposition 5.4 $\mathsf{t} \not\vdash \mathsf{f}$, and so $\not\vdash \mathsf{f}$. □

Proposition 5.7 *If $\vdash A$ and $a \in P$ then $v(A, a) = \mathsf{T}$.*

Proof: suppose $\vdash A$ and $a \in P$. Then $\models A$ by Proposition 5.5, so $v(A, a) = \mathsf{T}$ by V2vi). □

Proposition 5.8 (i) *$\not\models_{\mathbf{DW}} A \vee \sim A$.*

(ii) *$A, \sim A \not\models_{\mathbf{RM}} B$.*

(iii) *$A \vee B, \sim A \not\models_{\mathbf{RM}} B$.*

Proof:

(i) Let M_2 be the model structure $\langle P, W, R, ^* \rangle$, where $W = \{0, 1\}, P = \{0\}, 0^* = 1, 1^* = 0, R = \{000, 011\}$. Then M_2 is a **DW**-m.s. Let $v(A, 0) = \mathsf{F}$, $v(A, 1) = \mathsf{T}$. Then $v(A \vee \sim A, 0) = \mathsf{F}$. So $\not\models_{\mathbf{DW}} A \vee \sim A$.

(ii) Let M_3 be the model structure $\langle P, W, R, ^* \rangle$, where $W = \{0, 1\}, P = \{0\}, 0^* = 1, 1^* = 0, R = \{000, 010, 011, 100, 101, 110, 111\}$. Then M_3 is an **RM**-m.s. Let $v(A, 0) = \mathsf{T}, v(A, 1) = v(B, 0) = v(B, 1) = \mathsf{F}$. Then $v(\sim A, 0) = \mathsf{T}$. So $A, \sim A \not\models_{\mathbf{RM}} B$.

(iii) The same assignments to A and B in M_3 also give $v(A \vee B, 0) = \mathsf{T}$. So $A \vee B, \sim A \not\models_{\mathbf{RM}} B$. □

Proposition 5.9 *(Completeness for DW) If $X \models A$ then $X \vdash A$.*

Proof: The proof is given in the Appendix to this Chapter.

5.5 Extending the Semantics

To model the system **B**, with a weakened $\sim$I rule, we replace postulate p4 by the condition $a \leq b \Rightarrow b^* \leq a^*$. For the extensions of **DW**, we strengthen the conditions on the model structure, as follows:

Optional Operational Rule	Semantic Condition	Standard Name
CM	Raa^*a	Consequentia mirabilis

Combinator	Structural Rule	Semantic Condition	Standard Name
B	$X;(Y;Z) \leq (X;Y);Z$	$R^2abcd \Rightarrow R^2a(bc)d$	Prefixing
B'	$X;(Y;Z) \leq (Y;X);Z$	$R^2abcd \Rightarrow R^2b(ac)d$	Suffixing
C**	$X;t \leq X$	$(\exists x)(Px \,\&\, Raxa)$	
C*	$X;Y \leq Y;X$	$Rabc \Rightarrow Rbac$	Assertion
C	$(X;Y);Z \leq (X;Z);Y$	$R^2abcd \Rightarrow R^2acbd$	Permutation
W*	$X;X \leq X$	$Raaa$	Conjunctive Assertion
W	$(X;Y);Y \leq X;Y$	$Rabc \Rightarrow R^2abbc$	Contraction
WB	$X;(X;Y) \leq X;Y$	$Rabc \Rightarrow R^2a(ab)c$	Conjunctive Syllogism
S*	$Y;(X;Y) \leq X;Y$	$Rabc \Rightarrow R^2b(ab)c$	
S	$(X;Z);(Y;Z) \leq (X;Y);Z$	$R^2abcd \Rightarrow (\exists x)(R^2acxd \,\&\, Rbcx)$	
K*	$X \leq X;X$	$Rabc \Rightarrow a \leq c \vee b \leq c$	Mingle
K	$X \leq X;Y$	$Rabc \Rightarrow a \leq c$	

Proposition 5.10 *Each semantic condition in the above list shows the soundness of the conrresponding rule.*

Proof: We take each rule in turn, adding clauses to the induction on length of proof in Proposition 5.4. First we deal with the operational inference rules.

1. weakened $\sim$I: By I.H., $\mathsf{t}; A \models \sim B$ and $Z \models B$. Suppose $v(Z, a) = \mathsf{T}$ but that $v(\sim A, a) = \mathsf{F}$. Then $v(A, a^*) = \mathsf{T}$. By V2vi), $v(\mathsf{t}, a) = \mathsf{T}$ for all $e \in P$. Moreover, $a^* \leq a^*$ by p1a), i.e., Rea^*a^* for some $e \in P$. So $v(\mathsf{t} \circ A, a^*) = \mathsf{T}$ by V3i). So by I.H., $v(\sim B, a^*) = \mathsf{T}$, whence $v(B, a) = \mathsf{F}$. But by I.H., $v(B, a) = \mathsf{T}$, a contradiction. So $v(\sim A, a) = \mathsf{T}$. Thus $X \models \sim A$, as required. (Note that we have not used the weakened semantic condition explicitly here. But we needed it

to prove Proposition 5.1, the Heredity Lemma, which is used elsewhere in proving 5.4.)

2. CM: By I.H., $X'; A \models \sim A$. We must show that $X' \models \sim A$. Suppose $v(X', a) = \mathsf{T}$, but that $v(\sim A, a) = \mathsf{F}$. Then $v(A, a^*) = \mathsf{T}$. Given that Raa^*a, it follows by V3i) that $v(X \circ A, a) = \mathsf{T}$ and so by I.H. that $v(\sim A, a) = \mathsf{T}$, contradicting out hypothesis. So $v(\sim A, a) = \mathsf{T}$ and thus $X' \models \sim A$, as required.

   The other cases are structural rules, and are subsumed under case xiii) in the proof of 5.4. We need to extend the base case of the induction there to deal with each one in turn:

   (d) B: By I.H., $X; (Y; Z) \models A$. Suppose $v((X \circ Y) \circ Z, a) = \mathsf{T}$. Then there are b, c such that $v(X \circ Y, b) = v(Z, c) = \mathsf{T}$ and $Rbca$, and so there are d, e such that $v(X, d) = v(Y, e) = \mathsf{T}$ and $Rdeb$. Thus R^2deca, and so by the semantic condition, $R^2d(ec)a$, i.e., for some $x, Recx$ and $Rdxa$. Hence $v(Y \circ Z, x) = \mathsf{T}$, and so $v(X \circ (Y \circ Z), a) = \mathsf{T}$, by successive applications of V2v). Hence by I.H., $v(A, a) = \mathsf{T}$.

   (e) B′: By I.H., $X; (Y; Z) \models A$. Suppose $v((Y \circ X) \circ Z, a) = \mathsf{T}$. Then there are b, c such that $v(Y \circ X, b) = v(Z, c) = \mathsf{T}$ and $Rbca$, and so there are d, e such that $v(Y, d) = v(X, e) = \mathsf{T}$ and $Rdeb$. Then R^2deca, and so $R^2e(dc)a$ by the semantic condition, i.e., for some $x, Rdcx$ and $Rexa$. So $v(Y \circ Z, x) = \mathsf{T}$, whence $v(X \circ (Y \circ Z), a) = \mathsf{T}$ by successive applications of V3i). Hence by I.H. $v(A, a) = \mathsf{T}$.

   (f) C**: By I.H., $X; \mathsf{t} \models A$. Suppose $v(X, a) = \mathsf{T}$. By the semantic condition, there is an index $e \in P$ such that $Raea$, and by V2vi), $v(\mathsf{t}, e) = \mathsf{T}$. Thus $v(X \circ \mathsf{t}, a) = \mathsf{T}$ by V3i), and so $v(A, a) = \mathsf{T}$ by I.H.

   (g) C*: By I.H., $X; Y \models A$. Suppose $v(Y \circ X, a) = \mathsf{T}$. Then there are b, c such that $v(X, b) = v(Y, c) = \mathsf{T}$ and $Rbca$. Since $Rbca \Rightarrow Rabc$, it follows that $v(X \circ Y, a) = \mathsf{T}$. Hence by I.H., $v(A, a) = \mathsf{T}$.

   (h) C: By I.H., $(X; Y); Z \models A$. Suppose $v((X \circ Z) \circ Y, a) = \mathsf{T}$. Then there are b, c such that $v(X \circ Z, b) = v(Y, c) = \mathsf{T}$ and $Rbca$, and so there are d, e such that $v(X, d) = v(Z, e) = \mathsf{T}$ and $Rdeb$. Hence R^2deca and so R^2dcea, by the semantic condition, i.e., for some $x, Rdcx$ and $Rxea$. Thus $v(X \circ Y, x) = \mathsf{T}$, whence $v((X \circ Y) \circ Z, a) = \mathsf{T}$ by successive applications of V3i). Hence, as before, by induction, $W(V) \models A$.

(i) W*: By I.H., $X; X \models A$. Suppose $v(X, a) = \mathsf{T}$. Given that $Raaa$, it follows that $v(X \circ X, a) = \mathsf{T}$ and so $v(A, a) = \mathsf{T}$ by I.H.

(j) W: By I.H., $(X; Y); Y \models A$. Suppose $v(X \circ Y, a) = \mathsf{T}$. Then there are b, c such that $v(X, b) = v(Y, c) = \mathsf{T}$ and $Rbca$. Then R^2bcca, by the semantic condition, i.e., for some $x, Rbcx$ and $Rxca$. So $v(X \circ Y, a) = \mathsf{T}$, whence $v((X \circ Y) \circ Y, a) = \mathsf{T}$ by successive applications of V3i). Hence by I.H., $v(A, a) = \mathsf{T}$.

(k) WB: By I.H., $X; (X; Y) \models A$. Suppose $v(X \circ Y, a) = \mathsf{T}$. Then there are b, c such that $v(X, b) = v(Y, c) = \mathsf{T}$ and $Rbca$. Hence by the semantic condition, $R^2b(bc)a$, i.e., for some $x, Rbxa$ and $Rbcx$. Thus $v(X \circ Y, x) = \mathsf{T}$, and so $v(X \circ (X \circ Y), a) = \mathsf{T}$. Therefore, $v(A, a) = \mathsf{T}$ by I.H.

(l) S*: By I.H., $Y; (X; Y) \models A$. Suppose $v(X \circ Y, a) = \mathsf{T}$. Then there are b, c such that $v(X, b) = v(Y, c)$ and $Rbca$. So by the semantic condition, $R^2c(bc)a$, i.e., for some $x, Rcxa$ and $Rbcx$. Hence $v(X \circ Y, x) = \mathsf{T}$ and so $v(Y \circ (X \circ Y), a) = \mathsf{T}$, whence $v(A, a) = \mathsf{T}$ by I.H.

(m) S: By I.H., $(X; Z); (Y; Z) \models A$. Suppose $v((X \circ Y) \circ Z, a) = \mathsf{T}$. Then there are b, c such that $v(X \circ Y, b) = v(Z, c) = \mathsf{T}$ and $Rbca$, and so there are d, e such that $v(X, d) = v(Y, e) = \mathsf{T}$ and $Rdeb$. Thus R^2deca, and so by the semantic condition, for some x, R^2dcxa and $Recx$, i.e., for some $x, y, Rdcy, Ryxa$ and $Recx$. Hence $v(X \circ Z, y) = v(Y \circ Z, x) = \mathsf{T}$, and so $v((X \circ Z) \circ (Y \circ Z), a) = \mathsf{T}$. So $v(A, a) = \mathsf{T}$ by I.H.

(n) K*: By I.H., $X \models A$. Suppose $v(X \circ X, a) = \mathsf{T}$. Then there are b, c such that $v(X, b) = v(X, c) = \mathsf{T}$ and $Rbca$. So by the semantic condition either $b \leq a$ or $c \leq a$, whence by Heredity, $v(X, a) = \mathsf{T}$. Therefore $v(A, a) = \mathsf{T}$ by I.H.

(o) K: By I.H., $X \models A$. Suppose $v(X \circ Y, a) = \mathsf{T}$. Then there are b, c such that $v(X, b) = v(Y, c) = \mathsf{T}$ and $Rbca$. So by the semantic condition, $b \leq a$, and so by Heredity $v(X, a) = \mathsf{T}$. Thus $v(A, a) = \mathsf{T}$ by I.H.

That completes the proof, showing that each semantic condition is adequate to show the corresponding rule sound. In the proof of the completeness theorem (Appendix to chapter 5) the conditions will be shown also to be necessary. □

The presentation in §§5.3-5.4 treated the semantics in full generality. When we concentrate on particular systems, we can simplify the full model structure, by eliminating reference to * (for positive logics), P (for many of the logics), and even W itself for **K**, of course. However, Maximova showed[1] that at least a two-element logical area P is necessary for an adequate semantics for **E**, on the ground that if a model structure for **E** contained only a singleton P (i.e., 0), it could not invalidate the **E**-invalid

$$\sim p \vee ((p \to p) \to p).$$

The problem arises from the interaction of the postulates **CM**, i.e., consequentia mirabilis, and C**, i.e., $X; \mathsf{t} \leq X$, in their semantic form as

$$Raa^*a$$

and $$(\exists x)(x \in P \ \&\ Raxa).$$

The argument is as follows: suppose $P = \{0\}$. If $v(\sim p \vee ((p \to p) \to p), 0) = \mathsf{F}$, then $v(\sim p, 0) = \mathsf{F}$ and $v(((p \to p) \to p, 0) = \mathsf{F}$. So $v(p, 0^*) = \mathsf{T}$, and for some $a, b, R0ab, v(p \to p, a) = \mathsf{T}$ and

(1) $$v(p, b) = \mathsf{F}$$

Since **E** contains CM, that is, $Raa^*a, 0^* \leq 0$. Hence $v(p, 0) = \mathsf{T}$, by Heredity. By C**, since $P = \{0\}, Ra0a$, and so $Ra0b$, by p2, since $a \leq b$. Since $v(p \to p, a) = \mathsf{T}$, for all x, y for which $v(p, x) = \mathsf{T}$ and $Raxy, v(p, y) = \mathsf{T}$. In particular, since $v(p, 0) = \mathsf{T}$ and $Ra0b$, we have

(2) $$v(p, b) = \mathsf{T}$$

(2) contradicts (1). Thus, if $P = \{0\}, v(\sim p \vee ((p \to p) \to p), 0) \neq \mathsf{F}$.

But $\not\models_{\mathbf{E}} \sim p \vee ((p \to p) \to p)$. Let M_4 be the m.s.$\langle P, W, R, ^* \rangle$, where $P = W = \{0, 1\}, 0^* = 0, 1^* = 1, R000, R011, R111$. Then M_4 is an **E**-m.s. Put $v(p, 0) = \mathsf{T}$, $v(p, 1) = \mathsf{F}$. Then $v(\sim p \vee ((p \to p) \to p), 0) = \mathsf{F}$. So $\not\models_{\mathbf{E}} \sim p \vee ((p \to p) \to p)$.

Hence in the semantics for **E**, we must allow for non-singleton P.

Recall Proposition 4.30, in which we showed that the Minc sequent was provable in $\mathbf{R}^{\Box}$. We now show it is invalid in **E**.

Proposition 5.11 *(The Minc sequent)*

$$A \to (B \to B), (B \to A \vee B) \not\models_{\mathbf{E}} (B \to B)$$

[1] L.Maximova, 'A Semantics for the Calculus E of Entailment', *Bulletin of the Section of Logic, Polish Academy of Sciences*, 2 (1973), pp. 18-21.

Proof: Consider the following countermodel, M_5: $W = \{0,1,2,3,4\}, P = \{0,1\}, 0^* = 0, 1^* = 4, 2^* = 3$, in which R holds of the following triples:

000 011 022 033 041 044

100 111 122 133 141 144

200 201 202 211 222 230 232 233
240 241 244

300 303 311 320 322 323 333 341
344

400 401 402 411 414 422 430 432
433 440 441 444

Then M_5 is an **E**-m.s.[2] Let $v(A,1) = \mathsf{T}$, $v(B,0) = v(B,2) = \mathsf{T}$, and A, B otherwise false. Then the antecedent of the Minc sequent is true in M_5 at 2, while the consequent is false there. Hence $A \to (B \to B), (B \to A \vee B) \not\models_{\mathbf{E}} (B \to B)$. $\Box$

Hence the conjecture that $\mathbf{E} = \mathbf{R}^{\Box}$, in the sense that $X \vdash_{\mathbf{E}} A$ iff $X' \vdash_{\mathbf{R}^{\Box}} A'$, where X', A' result from X, A by replacing $\to$ by $\Rightarrow$, is false. $\mathbf{E} \neq \mathbf{R}^{\Box}$.

5.6 Semantics for R and R$^{\Box}$

In our proof-theoretic formulation of **R** and $\mathbf{R}^{\Box}$ in chapter 4, we treated $\sim$ by definition, in terms of the primitive connectives $\mathsf{f}$and $\to$, and introduced a new primitive $\Box$, for necessity. It is now time to show how to give a suitable semantics for $\mathsf{f}$and $\Box$. The primitives of $\mathbf{L}'$ are $\{\mathsf{f}, \mathsf{F}, \&, \vee, \to, \times, \Box\}$. An interpretation $\mathcal{J}$ for $\mathbf{R}^{\Box}$ consists of a model structure, namely, a quintuple $\langle 0, W, R, S, ^* \rangle$, where $0 \in W, R \subseteq W^3, S \subseteq W^2$, and $^* : W \to W$; and a valuation function $v : \mathbf{L}' \times W \to \{\mathsf{T}, \mathsf{F}\}$; subject to the following conditions:

d1 $S^\frown Rabc$ if $(\exists x)(Sax \;\&\; Rxbc)$

d2 $R^\frown Sabc$ if $(\exists x)(Rabx \;\&\; Sxc)$

d3 $a \leq b$ if $S^\frown R0ab$ (i.e., $(\exists x)(S0x \;\&\; Rxab)$)

d4 S^2ab if $(\exists x)(Sax \;\&\; Sxb)$

[2] I am here indebted to A.V. LeBlanc's computer program MASTER to establish that M_5 is indeed an (**E**-)model structure.

d5 R^2abcd if $(\exists x)(Rabx \;\&\; Rxcd)$

d6 $R^2a(bc)d$ if $(\exists x)(Rbcx \;\&\; Raxd)$

r1 $\leq$ is a partial order:

(a) $a \leq a$
(b) $a \leq b \;\&\; b \leq a \Rightarrow a \leq c$
(c) $a \leq b \;\&\; b \leq a \Rightarrow a = b$ (again, r1c is optional)

r2 R is monotonic:

(a) $a \leq x \;\&\; Rxbc \Rightarrow Rabc$
(b) $b \leq x \;\&\; Raxc \Rightarrow Rabc$
(c) $x \leq c \;\&\; Rabx \Rightarrow Rabc$

r3 * is an involution:

(a) $a^{**} = a$
(b) $Rabc \Rightarrow Rac^*b^*$

r4 S is reflexive and transitive (making $\Box$ an **S4**-connective):

(a) Saa
(b) $S^2ab \Rightarrow Sab$

r5 S is monotonic:

(a) $a \leq x \;\&\; Sxb \Rightarrow Sab$
(b) $x \leq b \;\&\; Sax \Rightarrow Sab$

r6 $Raaa$ (W*, which with r7 entails W)

r7 $R^2abcd \Rightarrow R^2acbd$ (the Pasch postulate, combining B and C)

r8 $R^\frown Sabc \Rightarrow (\exists x)(\exists y)(Sax \;\&\; Sby \;\&\; Rxyc)$

We require of v:

V1) for all atomic wffs A, if $v(A, a) = \mathsf{T}$ and $a \leq b$, then $v(A, b) = \mathsf{T}$ (the Heredity condition);

V2) (i) $v(\mathsf{f}, a) = \mathsf{F}$ if $S0a^*$;
(ii) $v(A \;\&\; B, a) = \mathsf{T}$ if $v(A, a) = \mathsf{T}$ and $v(B, a) = \mathsf{T}$;

(iii) $v(A \vee B, a) = \mathsf{T}$ if $v(A, a) = \mathsf{T}$ or $v(B, a) = \mathsf{T}$;

(iv) $v(A \to B, a) = \mathsf{T}$ if $v(B, c) = \mathsf{T}$ whenever $v(A, b) = \mathsf{T}$ and $Rabc$;

(v) $v(A \times B, a) = \mathsf{T}$ if for some $b, c, v(A, b) = v(B, c) = \mathsf{T}$ and $Rbca$;

(vi) $v(\Box A, a) = \mathsf{T}$ if $v(A, b) = \mathsf{T}$ whenever Sab;

(vii) $v(\mathsf{F}, a) = \mathsf{F}$ for all a;

V3) (i) if Z is an I-bunch, $X \circ Y, v(Z, a) = \mathsf{T}$ if for some $b, c, v(X, b) = v(Y, c) = \mathsf{T}$ and $Rbca$;

(ii) if Z is an E-bunch, $v(Z, a) = \mathsf{T}$ if $v(Y, a) = \mathsf{T}$ for all $Y \in Z$.

A is true under $\mathcal{J}$ if $v(A, a) = \mathsf{T}$ whenever $S0a$. $X \models A$ if for all $\mathcal{J}$ and all $a \in W$, if $v(X, a) = \mathsf{T}$ then $v(A, a) = \mathsf{T}$. $\models A$ if A is true under every $\mathcal{J}$.

We can now rework the results of §5.4, adapted to $\mathbf{R}^{\Box}$. First we need a lemma.

Lemma 2 (i) $a \leq b \Rightarrow b^* \leq a^*$

(ii) $Rabc \Rightarrow Rbac$ *(*C**)*

(iii) $R^2abcd \Rightarrow R^2b(ac)d$ *(*B$'$*)*

(iv) $R^2abcd \Rightarrow R^2a(bc)d$ *(*B*)*

(v) $Rabc \Rightarrow R^2abbc$ *(*W*)*

Proof:

(i) Suppose $a \leq b$. Then for some $x, S0x$ and $Rxab$. So by r3b), Rxb^*a^*, whence $b^* \leq a^*$.

(ii) Suppose $Rabc$. By r1a) for some $x, S0x$ and $Rxaa$. So R^2xabc, whence R^2xbac by r7), i.e., for some $x, y, S0x, Rxby$ and $Ryac$. Hence $b \leq y$, and so $Rbac$.

(iii) Suppose R^2abcd. then R^2acbd by r7), i.e., for some $x, Racx$ and $Rxbd$, whence $Rbxd$ by ii) above, i.e., $R^2b(ac)d$.

(iv) Suppose R^2abcd, i.e., for some $x, Rabx$ and $Rxcd$. Then $Rbcx$ by ii), whence R^2bacd, and so $R^2a(bc)d$ by iii).

(v) Suppose $Rabc$. Then $Rbac$ by ii) and $Rbbb$ by r6). Hence R^2bbac and so R^2babc by r7), i.e., for some $x, Rbax$ and $Rxbc$, and so $Rabx$ by ii) again, giving R^2abbc. $\Box$

Proposition 5.12 *(Heredity) If $a \leq b$ and $v(X, a) = \mathsf{T}$ then $v(X, b) = \mathsf{T}$.*

Proof: by induction on the degree of X, as before.

1. immediate.

2. (i) $A = \mathsf{f}$: suppose $a \leq b$ and $v(\mathsf{f}, b) = \mathsf{F}$. Then $S0b^*$ and, by Lemma 2, $b^* \leq a^*$. Hence $S0a^*$ by r5b), and so $v(\mathsf{f}, a) = \mathsf{F}$. Contraposing, if $v(\mathsf{f}, a) = \mathsf{T}$ and $a \leq b, v(\mathsf{f}, b) = \mathsf{T}$.

   (ii) $A = B \,\&\, C$: as before (Proposition 5.1ii).

   (iii) $A = B \vee C$: as 5.1iii).

   (iv) $A = B \to C$: as 5.1iv), using r2a).

   (v) $A = B \times C$: as 5.1v), using r2c).

   vi) $A = \Box A$: suppose $a \leq b$ and $v(\Box A, a) = \mathsf{T}$. Then, for all $x, Sax \Rightarrow v(A, x) = \mathsf{T}$. Suppose Sbc. Then Sac by r5), so $v(A, c) = \mathsf{T}$. Hence $v(\Box A, b) = \mathsf{T}$ by V2vi).

   vii) $A = \mathsf{F}$: $v(\mathsf{F}, a) \neq \mathsf{T}$ for any a, so we do not require $v(\mathsf{F}, b) = \mathsf{T}$ for any $b, a \leq b$.

3. i), ii) as 5.1. □

Proposition 5.13 *$X; A \models B$ iff $X \models A \to B$.*

Proof: as for Proposition 5.2. □

Lemma 3 (i) *$v(\sim A, a) = \mathsf{T}$ iff $v(A, a^*) = \mathsf{F}$;*

(ii) *$v(t, a) = \mathsf{T}$ iff $S0a$;*

(iii) *$v(\mathsf{T}, a) = \mathsf{T}$ for all a.*

Proof:

(i) Suppose $v(\sim A, a) = \mathsf{T}$, i.e., $v(A \to \mathsf{f}, a) = \mathsf{T}$. Then for all x, y, if $v(A, x) = \mathsf{T}$ and $Raxy$ then $v(\mathsf{f}, y) = \mathsf{T}$, i.e., not $S0y^*$; contraposing, and taking y into the antecedent, for all x, if for some $y, S0y^*$ and $Raxy$ then $v(A, x) = \mathsf{F}$. But by r3b) and Lemma 2 ii), Ry^*ax^* iff $Raxy$, and by d3, $S0y^*$ and Ry^*ax^* iff $a \leq x^*$. So if $a \leq x^*$ then $v(A, x) = \mathsf{F}$. But $a \leq a^{**}$ by r3a) and r1a). So $v(A, a^*) = \mathsf{F}$.

Conversely, suppose $v(A, b) = \mathsf{T}, Rabc$ and $v(\mathsf{f}, c) = \mathsf{F}$, i.e., $S0c^*$. Then by r3 and Lemma 2 ii), Rc^*ba^*, so $b \leq a^*$, by d3. Hence $v(A, a^*) = \mathsf{T}$. Contraposing, if $v(A, a^*) = \mathsf{F}$, then for all b, c, if $v(A, b) = \mathsf{T}$ and $Rabc, v(\mathsf{f}, c) = \mathsf{T}$, i.e., $v(A \to \mathsf{f}, a) = \mathsf{T}$. That is, if $v(A, a^*) = \mathsf{F}$ then $v(\sim A, a) = \mathsf{T}$.

(ii) $v(\mathsf{t}, a) = \mathsf{T}$ iff $v({\sim}\mathsf{f}, a) = \mathsf{T}$ iff $v(\mathsf{f}, a^*) = \mathsf{F}$, by i) iff $S0a$, by r3 and V3i).

(iii) By V2vii), $v(\mathsf{F}, a^*) = \mathsf{F}$ for all a. So $v({\sim}\mathsf{F}, a) = \mathsf{T}$, by i), i.e., $v(\mathsf{T}, a) = \mathsf{T}$ for all a. □

Proposition 5.14 $\mathsf{t} \models A$ *iff* $\models A$.

Proof: Suppose $\mathsf{t} \models A$, i.e., if $v(\mathsf{t}, a) = \mathsf{T}$ then $v(A, a) = \mathsf{T}$. Suppose $S0a$. Then, by Lemma 3, $v(\mathsf{t}, a) = \mathsf{T}$, so $v(A, a) = \mathsf{T}$, whence $\models A$.

Conversely, suppose $\models A$, and suppose $v(\mathsf{t}, a) = \mathsf{T}$. Then $S0a$ by Lemma 3, so $v(A, a) = \mathsf{T}$. Hence $\mathsf{t} \models A$. □

Lemma 4 (i) *if* $v(Y(Z), a) = \mathsf{T}$ *and* $W \subseteq Z$, *then* $v(Y(W), a) = \mathsf{T}$*;*

(ii) *if* $X_1 \angle\, X$ *and* $X_1 \models A$, *then* $X \models A$.

Proof: as before (Lemma 1 of §5.4). □

Proposition 5.15 *If* $X \vdash A$ *then* $X \models A$.

Proof: as before (Proposition 5.4), with revisions to ix), x) and xiii).

(ix′) fK: by I.H. $X'; {\sim}A \models \mathsf{f}$. Suppose $v(X', a) = \mathsf{T}$. By Proposition 5.13, $X' \models {\sim}A \to \mathsf{f}$, so by definition, $X' \models {\sim}{\sim}A$. Hence $v({\sim}{\sim}A, a) = \mathsf{T}$. So by Lemma 3, $v(A, a) = \mathsf{T}$. Hence $X' \models A$.

(x′) (a) □I: by I.H. $X' \models A$, where X' is **S4**-modal. Suppose $v(X', a) = \mathsf{T}$, and suppose further that Sab. We show by induction on the complexity of X' , that $v(X', b) = \mathsf{T}$.

1. Base: X' is a wff.
   (i) $X' = \Box B$: Then for all x, if Sax then $v(B, x) = \mathsf{T}$. Suppose Sbc. Then S^2ac (since, by hypothesis, Sab), and so Sac by r4b). So $v(B, c) = \mathsf{T}$. Hence $v(\Box B, b) = \mathsf{T}$.
   (ii) $X' = \mathsf{t}$: Then $S0a$, and so S^20b, whence $S0b$ by r4b). Hence $v(\mathsf{t}, b) = \mathsf{T}$, i.e., $v(X', b) = \mathsf{T}$.
2. (i) $X' = B \,\&\, C$, where B and C are **S4**-modal. Then $v(B, a) = \mathsf{T}$, $v(C, a) = \mathsf{T}$, and so by I.H., $v(B, b) = v(C, b) = \mathsf{T}$ and so $v(B \,\&\, C, b) = \mathsf{T}$.
   (ii) $X' = B \vee C$, where B and C are **S4**-modal. As i).

(iii) $X' = B \times C$, where B and C are **S4**-modal. Then there are u, v such that $v(B, u) = v(C, v) = \mathsf{T}$ and $Ruva$. By r8, if $Ruva$ and Sab then there are x, y such that Sux, Svy and $Rxyb$. By I.H. on X' , given $v(B, u) = \mathsf{T}$ and Sux it follows that $v(B, x) = \mathsf{T}$, and given $v(C, v) = \mathsf{T}$ and Svy it follows that $v(C, y) = \mathsf{T}$. Hence we have x, y such that $v(B, x) = v(C, y) = \mathsf{T}$ and $Rxyb$. Thus $v(B \times C, b) = \mathsf{T}$, i.e., $v(X', b) = \mathsf{T}$.

3. (i) $X' = Y \circ Z$, where Y, Z are **S4**-modal. As 2iii) above.

   (ii) X' is an E-bunch. Then $v(W, a) = \mathsf{T}$ for all $W \in X'$. So by I.H. on $W, v(W, b) = \mathsf{T}$, whence $v(X', b) = \mathsf{T}$.

So, by induction, $v(X', b) = \mathsf{T}$. Hence by I.H. on proofs, $v(A, b) = \mathsf{T}$. Thus for all b, if Sab then $v(A, b) = \mathsf{T}$. Hence $v(\Box A, a) = \mathsf{T}$, and so $X' \models \Box A$.

(b) $\Box$E: by I.H. $X' \models \Box A$. Suppose $v(X', a) = \mathsf{T}$. Then $v(\Box A, a) = \mathsf{T}$, and so for all x, if Sax then $v(A, x) = \mathsf{T}$. by r4a), Saa. So $v(A, a) = \mathsf{T}$. Hence $X' \models A$.

(xiii) We must reconsider $\mathsf{T}=$ and $\mathsf{t}=$, and add base cases for B, B$'$, W and C. Recall that we are performing an induction on the complexity of $W(U)$ in the rule $U \leq V$. We suppose $v(W(V), a) = \mathsf{T}$.

(b) $\mathsf{T}=$: as before, using Lemma 3.

(c) $\mathsf{t}=$: i) $W(V) = Z$. By r1, for some $x, S0x$ and $Rxaa$. By Lemma 3, $v(\mathsf{t}, x) = \mathsf{T}$. So $v(\mathsf{t}; Z, a) = \mathsf{T}$ by V3i).

ii) conversely, $W(V) = \mathsf{t} \circ X'$. Then for some $x, y, v(\mathsf{t}, x) = v(X', y) = \mathsf{T}$ and $Rxya$, in particular, by Lemma 3, $S0x$ and $Rxya$, i.e., $y \leq a$. So $v(X', a) = \mathsf{T}$ by Proposition 5.12.

(d)—f): B, C and W were dealt with in Proposition 5.10. We use Lemma 2 to show that the appropriate semantic conditions hold.

That completes the proof of Proposition 5.15. $\Box$

Corollary 5.16 *$A \models B$ iff $\models A \to B$.*

Proof: as Proposition 5.5, this time from Propositions 5.13, 5.14 and 5.15. $\Box$

Corollary 5.17 $\not\models \mathsf{f}$.

Proof: Take as m.s. $M_5 = \langle 0, W, R, S, ^* \rangle$, as M_1, but dropping P and adding $S = \{00\}$. Then M_5 is an $\mathbf{R}^{\Box}$-m.s., and $v(\mathsf{f}, 0) = \mathsf{F}$. Consequently, $\not\models \mathsf{f}$, and so $\not\vdash \mathsf{f}$ by Proposition 5.15. $\Box$

Proposition 5.18 *If* $\vdash A$ *and* $S0a$ *then* $v(A, a) = \mathsf{T}$.

Proof: as Proposition 5.7, with $S0a$ for $a \in P$. $\Box$

Proposition 5.19 *(Completeness for* $\mathbf{R}^{\Box}$*) If* $X \models A$ *then* $X \vdash A$.

Proof: The proof is given in the Appendix to this Chapter.

Appendix

We give here the proofs of Propositions 5.9 and 5.19 (Completeness for **DW** and $\mathbf{R}^{\Box}$).[3] The way the proof works should be familiar from Henkin proofs of completeness for classical non-modal and modal logics.[4] It proceeds by assuming that some wff A is not derivable from (in the present case) the bunch X, and exhibits a model in which A is not true (at some index) while X is. As in the Henkin construction in modal logic, the model is a whole collection of (in this case) prime deductively closed sets of wffs, that is, prime theories (the indices or worlds). Particular complications arise in the present case because of three factors:

1. The proof of 5.9 establishes the completeness of a very weak logic (at first glance, if one thinks about it, an extremely unlikely result). There is very little power in the logic, and so many of the slick features of a more usual Henkin proof are unavailable.

2. The systems concerned are relevant logics, and so in particular, full weakening of the antecedent is not available (it is restricted to extensional weakening, EK). So even though $X \not\vdash A$, A may still be a theorem, whence the index at which X is true and A false may not be the real world 0 (or in the case of unreduced modelling, any member of the set of regular worlds, P), where all theorems are true, but some other index, a. Our procedure is to build a model independent of X and A, and then locate an index where X is true and A false.

3. The antecedent, X, is a bunch, not simply a set of wffs. Hence the value of X at a may depend not simply on the value of subbunches of X at a, but at other indices too. This complicates the recursion showing that X is true at

[3] The proofs draw heavily on F.R. Routley et al., *Relevant Logics and their Rivals*, vol.1 (Atascadero, 1982), ch. 4, and F.R. Routley and R.K. Meyer, 'The Semantics of Entailment' (II), *Journal of Philosophical Logic*, 1 (1972), pp. 53-73.

[4] See, e.g., G. Hunter, *Metalogic* (London, 1971), §32, pp. 105-16; M. Bergmann, J. Moor and J. Nelson, *The Logic Book* (New York, 1980), §6.4, pp. 220-8.

a. In particular, since Tmust be true at all worlds, and Fat none, the null index (the empty set of wffs) and the universal index (the set of all wffs) must be excluded. Consequently, if X contained Fas a subbunch, the recursion, which proceeds via indices at which subbunches of X are true, would break down, for $\mathsf{F}$is true at no index. Fortunately, Corollary 4.16 shows that $\mathsf{F}$is not a subbunch of X, for if it were, $X \vdash \mathsf{F}$, and so by Corollary 4.15 and Cut, $X \vdash A$. Since ex hypothesi $X \nvdash A$, the recursion can proceed.

5A.1 Completeness for DW

Suppose $X \nvdash A$. We show that there is a (**DW**-)model structure $\langle P, W, R, ^* \rangle$ and a valuation v such that for some $a, v(X, a) = \mathsf{T}$ and $v(A, a) = \mathsf{F}$.
The deductive closure of a bunch X is the E-bunch y of all wffs derivable from X, i.e., $C \in y$ iff $X \vdash C$.
A (**DW**-) theory is an E-bunch of wffs a which is deductively closed, i.e., $a \vdash B$ iff $B \in a$. Note that $\mathsf{T} \in a$ for all a.
A theory a is regular if $\mathsf{t} \in a$;
a is prime if whenever $B \vee C \in a$, either $B \in a$ or $C \in a$;
a is consistent if $\mathsf{f} \notin a$.
a is null if $\mathsf{T} \notin a$;
a is universal if $\mathsf{F} \in a$.
a is improper if a is null or universal, otherwise proper.
'a', 'b', 'x', 'y', etc. will be metavariables over theories (E-bunches of wffs); 'X', 'Y' etc. remain as metavariables over bunches in general, including theories.

Lemma 1 *The deductive closure of a bunch is a theory.*

Proof: Let b be the deductive closure of X, i.e., $B \in b$ iff $X \vdash B$. Suppose $b \vdash B$. Then, since proofs are finite, there are $B_1, \ldots, B_m \in b$ such that $B_1, \ldots, B_m \vdash B$. But for all $i, 1 \leq i \leq m, X \vdash B_i$. Hence, by Corollary 4.6,$X \vdash B$. So $B \in b$. $\square$

Let Y, Z be two E-bunches of wffs (not necessarily theories). We say that Z is derivable from Y (written $Y \Vdash Z$) if $Y \vdash Z_1$, where Z_1 is some disjunction of wffs in Z. A pair $\langle Y, Z \rangle$ is maximal if $Y \nVdash Z$ and for all $B, B \in Y \cup Z$.

Lemma 2 (i) *$\langle Y, Z \rangle$ is maximal only if Y is a prime theory.*

(ii) *Cut holds for $\Vdash$. That is, suppose $B \in Y \cup Z$, $X \Vdash Y$ and $Z \Vdash W$. Then $X, Z - \{B\} \Vdash Y - \{B\}, W$.*

Proof:

(i) Suppose $Y \Vdash B$ but $B \notin Y$. Then $B \in Z$, and so $Y \Vdash Z$. So if $\langle Y, Z \rangle$ is maximal, $B \in Y$. So Y is a theory. Moreover, suppose $B \vee C \in Y$, but $B \notin Y, C \notin Y$. Then $B \in Z, C \in Z$, so $Y \Vdash Z$. Hence, if $\langle Y, Z \rangle$ is maximal, Y is prime.

(ii) We use Proposition 4.18. Since $X \Vdash Y$, it follows that $X \vdash B \vee A$, for some disjunction A, not containing B, of members of Y, and since $Z \Vdash W$, we have that $Z - \{B\}, B \vdash C$ for some disjunction C of members of W. Hence

$$X, Z - \{B\} \vdash A \vee C,$$

and so

$$X, Z - \{B\} \Vdash Y - \{B\}, W,$$

since $A \vee C$ is a disjunction of members of $Y - \{B\} \cup W$. □

Lemma 3 *(Lindenbaum's Lemma) Suppose $Y \not\Vdash Z$. Then there are Y', Z' with $Y \subseteq Y', Z \subseteq Z'$ and $\langle Y', Z' \rangle$ maximal.*

Proof: Take an enumeration $\{A_n : n \in w\}$ of the wffs of the language. We will define sets Y_n, Z_n recursively, and take their unions for Y' and Z'. The idea of the construction is this: assuming at stage n that so far we have ensured that Z_n is not derivable from Y_n, we consider whether adding A_n to Y_n would make Z_n derivable from it. If so, we add A_n to Z_n instead; if not, we may safely add A_n to Y_n.

	$Y_0 = Y$	$Z_0 = Z$	
If	$Y_n \cup \{A_n\} \not\Vdash Z_n$ then	$Y_{n+1} = Y_n \cup \{A_n\}$	$Z_{n+1} = Z_n$
	else	$Y_{n+1} = Y_n$	$Z_{n+1} = Z_n \cup \{A_n\}$
	$Y' = \cup_{n \in w} Y_n$	$Z' = \cup_{n \in w} Z_n$	

Clearly, $Y \subseteq Y'$ and $Z \subseteq Z'$. We must show that $\langle Y', Z' \rangle$ is maximal.

By construction, for all $B, B \in Y' \cup Z'$. We must show that $Y' \not\Vdash Z'$.

Suppose, for reductio, that $Y' \Vdash Z'$. Then, by the finiteness of proof, $Y_{n+1} \Vdash Z_{n+1}$ for some n. But we can show that if $Y_{n+1} \Vdash Z_{n+1}$ then $Y_n \Vdash Z_n$. For the idea of the construction is that if including a wff on the left would make Z_{n+1} derivable from Y_{n+1}, it is included instead on the right. We must now show that this move will not lead to Z_{n+1}'s being derivable from Y_{n+1}.

Suppose, then, that $Y_{n+1} \Vdash Z_{n+1}$. If $Y_n \cup A_n \not\Vdash Z_n$ then by the definition of Y_{n+1} and Z_{n+1}, $Y_{n+1} = Y_n \cup \{A_n\}$ and $Z_{n+1} = Z_n$, and so $Y_n \cup A_n \Vdash Z_n$, a contradiction. Hence $Y_n \cup A_n \Vdash Z_n$. That is,

(I)
$$Y_n, A_n \Vdash Z_n$$

So, by the definition of Y_{n+1}, $Y_{n+1} = Y_n$ and $Z_{n+1} = Z_n \cup \{A_n\}$, in which case

(II)
$$Y_{n+1} = Y_n \Vdash Z_n \cup A_n = Z_{n+1}$$

Putting (I) and (II) together, it follows by Cut, Lemma 2ii), that $Y_n, Y_n \Vdash Z_n, Z_n$, and so $Y_n \Vdash Z_n$.

So if $Y' \Vdash Z'$, it follows by induction that $Y_0 \Vdash Z_0$. But by hypothesis, $Y = Y_0 \not\Vdash Z_0 = Z$. Hence $Y' \not\Vdash Z'$, as required.

Thus $\langle Y', Z' \rangle$ is maximal. □

Lemma 4 *Suppose a is a theory and Y a set of wffs disjoint from a, but closed under disjunction, i.e., if $B, C \in Y$, then $B \vee C \in Y$. Then there is a theory a' such that $a \subseteq a'$, a' is disjoint from Y, and a' is prime.*

Proof: Suppose $a \Vdash Y$. Then $a \vdash D$ (where D is some disjunction of wffs in Y). Then $D \in a$, since a is a theory. But $D \in Y$, since Y is closed under disjunction. So a and Y are not disjoint, contrary to hypothesis. So $a \nVdash Y$. Hence, by Lemma 3, there are a', Z such that $a \subseteq a', Y \subseteq Z, \langle a', Z\rangle$ maximal. So, by Lemma 2, a' is a theory and prime. Moreover, since $\langle a', Z\rangle$ is maximal, $a' \nVdash Z$. Since $Y \subseteq Z$, it follows that $a' \nVdash Y$. $\square$

Let V be the set of proper theories. The idea now is to construct a model, the canonical model, out of elements of V. Recall from chapter 4 the definition of $X \leq Y$, that we may derive $Z(Y) : A$ from $Z(X) : A$. For $a, b, c \in \mathsf{V}$, let $Rabc$ hold if $a; b \leq c$. Let W be the set of prime theories in V, and P the set of prime regular theories in W. For $a \in \mathsf{W}$, let $a^* = \{B : \sim B \notin a\}$.

We must show that P (and hence W) is non-empty. Let $\tau = \{B : \mathsf{t} \vdash B\}$, as above. Then τ is a theory, by Lemma 1, and by Corollary 5.6, $\mathsf{f} \notin \tau$. So by Lemma 4, there is a prime theory $\mathbf{0}$, $\tau \subseteq \mathbf{0}$, $\mathsf{f} \notin \mathbf{0}$. But clearly $\mathsf{t} \in \mathbf{0}$. So there is a prime regular consistent theory $\mathbf{0} \in \mathsf{P} \subseteq \mathbf{W}$.

We define the canonical m.s. $\mathbf{M}$ as the structure $\langle \mathsf{P}, \mathsf{W}, R, ^* \rangle$.

Lemma 5 W *is closed under *, i.e., if $b \in \mathsf{W}$, then $b^* \in \mathsf{W}$.*

Proof: suppose $b \in \mathsf{W}$, i.e., b is a prime theory. First, we must show that b^* is a theory. Suppose $b^* \vdash B$, but $B \notin b^*$. Then $\sim B \in b$. Since proofs are finite, there are $C_1, \ldots, C_m \in b^*$ such that $C_1 \ \& \ \ldots \ \& \ C_m \vdash B$. Hence by Antilogism and De Morgan (Propositions 4.10 and 4.12), $\sim B \vdash \sim C_1 \vee \ldots \vee \sim C_m$. But $\sim B \in b$. So $\sim C_1 \vee \ldots \vee \sim C_m \in b$. But b is prime. So for some $i, 1 \leq i \leq m$, $\sim C_i \in b$. So $C_i \notin b^*$. But by hypothesis, $C_i \in b^*$ for all i. Hence $B \in b^*$.

Next, we must show that b^* is prime. Suppose $B \notin b^*, C \notin b^*$. Then $\sim B \in b$, $\sim C \in b$, so $b \vdash \sim B, b \vdash \sim C$. Hence $b \vdash \sim B \ \& \ \sim C$, and so $b \vdash \sim(B \vee C)$ by De Morgan, and so $\sim(B \vee C) \in b$, whence $B \vee C \notin b^*$. Contraposing, if $B \vee C \in b^*$, then either $B \in b^* or C \in b^*$. Lastly, note that b^* is proper, since b is. $\square$

Lemma 6 (i) *$a \leq b$ iff for some $x \in \mathsf{P}, Rxab$.*

(ii) *$a \leq b$ iff $a \subseteq b$, where $\subseteq$ is set inclusion.*

(iii) *$Rabc$ iff whenever $A \to B \in a$ and $A \in b, B \in c$.*

(iv) *Let $x = \{A : X \vdash A\}$ for an arbitrary bunch X. Then $x \leq X$ and $X \leq x$ (i.e., $x = X$ in the sense of chapter 4).*

Proof:

(i) Suppose $a \leq b$. Then $\mathsf{t}; a \leq b$ by t=. But $\mathsf{t} \in \mathsf{0}$, so $\mathsf{0}; a \leq b$. Since $\mathsf{0} \in \mathsf{P}$, it follows that for some $x \in \mathsf{P}, Rxab$.

Conversely, suppose that for some $x \in \mathsf{P}, Rxab$, i.e. $x; a \leq b$, and suppose $Z(a) \vdash A$. Then $Z(\mathsf{t}; a) \vdash A$, by t=, and so $Z(x; a) \vdash A$, since $\mathsf{t} \angle x$ (since x is regular). Hence $Z(b) \vdash A$. Thus $a \leq b$ as required.

(ii) Suppose $a \leq b$, and suppose $A \in a$. Then $a \vdash A$, so $b \vdash A$, whence $A \in b$, since b is a theory. Thus $a \subseteq b$.

Conversely, suppose $a \subseteq b$ and $Z(a) \vdash A$. Then $a \angle b$ and so $Z(b) \vdash A$ by definition of derivation. Thus $a \leq b$.

(iii) Suppose $Rabc$, i.e. $a; b \leq c$, and $A \to B \in a$ and $A \in b$. Then $a \vdash A \to B$ and $b \vdash A$, so $a; b \vdash B$, whence $c \vdash B$, and so $B \in c$, since c is a theory.

Conversely, suppose that whenever $A \to B \in a$ and $A \in b, B \in c$, and suppose $Z(a; b) \vdash A$. Then for some finite $a' \angle a, b' \angle b, Z(a'; b') \vdash A$, and so $Z(\mathsf{cf}(a') \times \mathsf{cf}(b')) \vdash A$, by Proposition 4.3ii). By 4.3i) and $\times$I, a'; $cf(b') \vdash \mathsf{cf}(a') \times \mathsf{cf}(b')$, so $a' \vdash \mathsf{cf}(b') \to \mathsf{cf}(a') \times \mathsf{cf}(b')$, whence $\mathsf{cf}(b') \to \mathsf{cf}(a') \times \mathsf{cf}(b') \in a$. Moreover, $\mathsf{cf}(b') \in b$. Hence $\mathsf{cf}(a') \times \mathsf{cf}(b') \in c$. Hence $c \vdash \mathsf{cf}(a') \times \mathsf{cf}(b')$, and so $Z(c) \vdash A$, by Cut. Thus $a; b \leq c$, as required.

(iv) Suppose $Z(x) \vdash A$. Then for some finite $x' \subseteq x, Z(x') \vdash A$, and so $Z(\mathsf{cf}(x')) \vdash A$, by Proposition 4.3. Moreover, for all $A \in x', X \vdash A$, so $X \vdash \mathsf{cf}(x')$ by successive applications of &I and EW. Hence $Z(X) \vdash A$, by Cut, i.e., $x \leq X$.

Conversely, suppose $Z(X) \vdash A$. Then for some $X' \sim X, Z(X') \vdash A$. By 4.3i), $X' \vdash \mathsf{cf}(X')$, so $X \vdash \mathsf{cf}(X')$, whence $\mathsf{cf}(X') \in x$. Hence, $x \vdash \mathsf{cf}(X')$ and so by Cut, $Z(x) \vdash A$. Thus $X \leq x$. □

We must now deduce some rather special corollaries of Lemma 4, whose full import will not be appreciated until they are used and fitted into place. We will then show that the set of theories is a model structure and with its attendant valuation constitutes an interpretation for DW.

Lemma 7 (i) *If $x, b, c \in \mathsf{V}, c \in \mathsf{W}$ and $Rxbc$, then there is $a \in \mathsf{W}$, such that $x \subseteq a$ and $Rabc$;*

(ii) *If $a, x, c \in \mathsf{V}, c \in \mathsf{W}$ and $Raxc$, then there is $b \in \mathsf{W}$, such that $x \subseteq b$ and $Rabc$;*

(iii) *If $a, b, x \in \mathsf{V}, C \notin x$ and $Rabx$, then there is $c \in \mathsf{W}$, such that $x \subseteq c, C \notin c$ and $Rabc$.*

Proof:

(i) Let $Y = \{E$: for some $G, H, E \vdash G \to H, G \in b$ and $H \notin c\}$. Then Y is non-empty, since b, c are proper, and Y is closed under disjunction: for suppose $E_1, E_2 \in Y$. Then there are $G_1, G_2 \in b$ and $H_1, H_2 \notin c$ such that $E_1 \vdash G_1 \to H_1$ and $E_2 \vdash G_2 \to H_2$. Since c is prime, $H_1 \vee H_2 \notin c$. By $\vee$E, $E_1 \vee E_2 \vdash (G_1 \to H_1) \vee (G_2 \to H_2)$, and so by Proposition 4.19, $E_1 \vee E_2 \vdash (G_1 \mathbin{\&} G_2) \to (H_1 \vee H_2)$. So $E_1 \vee E_2 \in Y$. Moreover, x is disjoint from Y, by the definition of Y and the fact that $Rxbc$. So by Lemma 4, there is a prime theory a disjoint from Y extending x. We need to show that $Rabc$. Suppose $G \to H \in a$ and $G \in b$. Then $G \to H \notin Y$, since a is disjoint from Y. But if $H \notin c, G \to H \in Y$. So $H \notin c$, whence $Rabc$ by Lemma 6ii). Since $x \subseteq a$ and Y is non-empty, a is proper.

(ii) Let $Y = \{E : \text{for some } G, a; E \vdash G \text{ and } G \notin c\}$. Then Y is non-empty, since a, c are proper, and Y is closed under disjunction: for suppose $E_1, E_2 \in Y$. Then there are $G_1, G_2 \notin c$ such that $a; E_1 \vdash G_1$ and $a; E_2 \vdash G_2$. Since c is prime, $G_1 \vee G_2 \notin c$. But by $\vee$E, $a; (E_1 \vee E_2) \vdash G_1 \vee G_2$. So $E_1 \vee E_2 \in Y$. Moreover, x is a theory disjoint from Y, by the definition of Y and the fact that $Raxc$. So by Lemma 4, there is a prime theory b disjoint from Y extending x. We need to show that $Rabc$. Suppose $E \to G \in a$ and $E \in b$. Then $a \vdash E \to G$, so by $\to$E, $a; E \vdash G$, and since b is disjoint from $Y, E \notin Y$. Hence $G \in c$, and so $Rabc$, by Lemma 6ii). Since $x \subseteq b$ and Y is non-empty, b is proper.

(iii) Let Y be the closure of $\{C\}$ under disjunction. Then x and Y are disjoint, so by Lemma 4, there is a prime theory c disjoint from Y extending x. Since $C \in Y, C \notin c$, and so, since $x \subseteq c$, it follows that $Rabc$. Since $x \subseteq c$ and $C \in Y, c$ is proper. □

Lemma 8 M *is a* **DW**-*m.s.*

Proof: We showed earlier that $\mathsf{t} \in \mathsf{0} \in \mathsf{P}$, so P and W are non-empty. R has been defined on V and hence on W, and by Lemma 5, * is an operation on W. We must show that p1 - p5 hold. d1 is satisfied, by Lemma 6i).

p1 that $\leq$ is a partial ordering follows from Lemma 6ii) since $\subseteq$, set inclusion, is a partial ordering on theories.

p2 monotonicity:

(a) Suppose $a \leq x$ and $Rxbc$. Then $a; b \leq x; b \leq c$, i.e. $Rabc$.

(b) Suppose $b \leq x$ and $Raxc$. Then $a; b \leq a; x \leq c$, i.e.$Rabc$.

(c) Suppose $x \leq c$ and $Rabx$. Then $a; b \leq x \leq c$, i.e. $Rabc$.

p3 $a^{**} = \{A : {\sim}A \notin a^*\} = \{A : {\sim}{\sim}A \in a\}$
$= \{A : A \in a\}$ since $A \dashv\vdash {\sim}{\sim}A$ by DN and ${\sim}$E,
$= a$.

p4 Suppose $Rabc$, and suppose $B \to C \in a$ and $B \in c^*$. Then ${\sim}B \notin c$ and ${\sim}C \to {\sim}B \in a$, by Corollary 4.11, so ${\sim}C \notin b$, whence $C \in b^*$. Hence Rac^*b^* by Lemma 6iii).

p5 Suppose $a \leq b$ and $a \in \mathsf{P}$. Then $\mathsf{t} \in a$ and $a \subseteq b$, by Lemma 6ii). So $\mathsf{t} \in b$, whence $b \in \mathsf{P}$. □

We define the canonical valuation v on M by setting $v(p, a) = \mathsf{T}$ iff $p \in a$, for every variable p and every $a \in \mathsf{W}$.

Lemma 9 *For all* $a \in \mathsf{W}$*, and for all wffs* $B, v(B, a) = \mathsf{T}$ *iff* $B \in a$.

Proof: by induction on the degree of B.

Base: suppose $v(p,a) = \mathsf{T}$ and $a \leq b$. Then $p \in a$ and $a \subseteq b$, by Lemma 6ii). So $p \in b$, whence $v(p,b) = \mathsf{T}$.

Induction step:

(i) $B = \sim C$: then $v(B,a) = \mathsf{T}$ iff $v(C,a^*) = \mathsf{F}$
iff $C \notin a^*$, by I.H.
iff $\sim C \in a$ iff $B \in a$.

(ii) $B = C \mathbin{\&} D$: then $v(B,a) = \mathsf{T}$ iff $v(C,a) = v(D,a) = \mathsf{T}$
iff $C \in a$ and $D \in a$, by I.H.
iff $C \mathbin{\&} D \in a$, by &I, &E, since a is a theory,
iff $B \in a$.

(iii) $B = C \vee D$: then $v(B,a) = \mathsf{T}$ iff $v(C,a) = \mathsf{T}$ or $v(D,a) = \mathsf{T}$
iff $C \in a$ or $D \in a$, by I.H.
iff $C \vee D \in a$, by $\vee$I and primeness,
iff $B \in a$.

(iv) $B = C \to D$: suppose $C \to D \in a, v(C,b) = \mathsf{T}$ and $Rabc$. Then $C \in b$ by I.H., and so $D \in c$, whence $v(D,c) = \mathsf{T}$ by I.H. again. Hence $v(C \to D, a) = \mathsf{T}$.

The converse is hard. Suppose $C \to D \notin a$. We must show that there are $b', c' \in \mathsf{W}$ such that $C \in b', D \notin c'$ and $Rab'c'$. Let b be the deductive closure of $\{C\}$ (i.e., $E \in b$ iff $C \vdash E$). Then b is a theory, by Lemma 1. Moreover, b is proper, since if $\mathsf{F} \in b$, then $C \vdash \mathsf{F}$ and so $C \to D \in a$ by FE (Corollary 4.15). Let c be the deductive closure of b under a-derivability (i.e., $E \in c$ iff $a; b \vdash E$). Then c is a theory, by Lemma 1. Clearly, $Rabc$, and $C \in b$. We show that $D \notin c$: for suppose $D \in c$. Then $a; b \vdash D$, and so for some finite (extensional) conjunction B' of members of $b, a; B' \vdash D$. But $C \vdash B'$, whence by Cut, $a; C \vdash D$, and so $a \vdash C \to D$, that is, $C \to D \in a$. But $C \to D \notin a$, by hypothesis. So $D \notin c$. So $c \in \mathsf{V}$.

We now extend b, c to prime theories b' and c' such that $Rab'c', C \in b'$ and $D \notin c'$. First, by Lemma 7iii), there is a prime theory $c', c \subseteq c', D \notin c'$ and $Rabc'$. So by Lemma 7ii), there is a prime theory $b', b \subseteq b', C \in b'$ and $Rab'c'$. Hence, by induction hypothesis, $v(C, b') = \mathsf{T}$ and $v(D, c') = \mathsf{F}$. So $v(C \to D, a) = \mathsf{F}$.

Contraposing, if $v(C \to D, a) = \mathsf{T}$ then $C \to D \in a$.

(v) $B = C \times D$: suppose $v(C \times D, a) = \mathsf{T}$. Then there are $b, c \in \mathsf{W}$ such that $v(C,b) = v(D,c) = \mathsf{T}$ and $Rbca$. So by I.H., $C \in b$ and $D \in c$. Hence $C \times D \in a$.

Conversely, suppose $C \times D \in a$. We must show that there are b, c such that $v(C,b) = v(D,c) = \mathsf{T}$ and $Rbca$. The construction is similar to that in iv) above. Let b, c be the deductive closures of $\{C\}, \{D\}$ respectively. Then b and c are theories, by Lemma 1. Moreover, b, c are proper. For if $\mathsf{F} \in b$, then $C \vdash \mathsf{F}$, and so $C \vdash D \to \mathsf{F}$, by FE and Cut. But $a \vdash C \times D$, and so $a \vdash \mathsf{F}$. But a is proper. Hence so is b. Similarly for c. Suppose now that $E \to G \in b$

and $E \in c$. Then $c \vdash E \to G$ and $D \vdash E$, whence $C; D \vdash G$ by $\to$E, and so $G \in a$ since a is a theory and $C \times D \in a$. Hence $Rbca$, by Lemma 6iii). We now extend b, c to prime theories b', c' by Lemma 7: first, since a is prime, there is a prime theory b' extending b such that $Rb'ca$, and so there is a prime theory c' extending c such that $Rb'c'a$. Clearly, $C \in b'$ and $D \in c'$, so by I.H., $v(C, b') = \mathsf{T}$ and $v(D, c') = \mathsf{T}$. Hence $v(C \times D, a) = \mathsf{T}$.

(vi) $B = \mathsf{t}$: then $v(B, a) = \mathsf{T}$ iff $v(\mathsf{t}, a) = \mathsf{T}$ iff $a \in \mathsf{P}$ iff $\mathsf{t} \in a$.

(vii) $B = \mathsf{T}$: then $v(B, a) = \mathsf{T}$ for all a. We must show that for all $a, \mathsf{T} \in a$. This is immediate from TI (Corollary 4.15), since $a \in \mathsf{W}$, and so is a non-null theory. □

Lemma 10 *For all bunches Y and prime theories y' such that $\{B : Y \vdash B\} \subseteq y'$, $v(Y, y') = \mathsf{T}$.*

Proof: by induction on the complexity of Y.

Base: Y is a wff, then $Y \in y'$, and so $v(Y, y') = \mathsf{T}$ by Lemma 9.

Induction step:

(i) $Y = Z \circ W$. Suppose z is the deductive closure of Z, and w of W. Then z, w are theories, by Lemma 1. Suppose $B \to C \in z$ and $B \in w$. Then $Z \vdash B \to C$ and $W \vdash B$. So $Z; W \vdash C$ by $\to$E, i.e. $Y \vdash C$. Hence $C \in y'$ and so $Rzwy'$ by Lemma 6iii). Therefore, by Lemma 7i) there is a prime theory $z', z \subseteq z'$ such that $Rz'wy'$, and so by Lemma 7ii), there is a prime theory $w', w \subseteq w'$ such that $Rz'w'y'$. By I.H., $v(Z, z') = \mathsf{T}$ and $v(W, w') = \mathsf{T}$. Hence$v(Y, y') = \mathsf{T}$ by V3i).

(ii) Y is an E-bunch. Take $Z \in Y$. Then $Z \leq Y$, so $\{B : Z \vdash B\} \subseteq \{B : Y \vdash B\} \subseteq y'$. So by I.H. $v(Z, y') = \mathsf{T}$, whence $v(Y, y') = \mathsf{T}$ by V3ii). □

We are now ready to put the Completeness Proof together. Take X, A as given, with $X \nvdash A$. Let x be the deductive closure of X. Clearly $\mathsf{T} \in x$, since $X \vdash \mathsf{T}$ by Corollary 4.15. $X \nvdash \mathsf{F}$, since $X \nvdash A$, and so $\mathsf{F} \notin x$. So x is proper. Moreover, $A \notin x$. Let Y be the disjunction closure of $\{A\}$. Then by Lemma 4, there is a prime theory a with $x \subseteq a$ and a disjoint from Y. So a is proper and $a \in W$. By Lemma 10, $v(X, a) = \mathsf{T}$ in the canonical model. Moreover, $A \notin$a, by construction, since a is disjoint from the disjunction closure of $\{A\}$. So, by Lemma 9, $v(A, a) = \mathsf{F}$. Hence $X \not\models A$.

Contraposing, if $X \models A$ then $X \vdash A$.

5A.2 Extending DW

Logics stronger than **DW** were characterised proof-theoretically in chapter 4, and semantically in chapter 5. We showed in Proposition 5.9 that these semantic conditions were sufficient to capture the deductive power of the inference rules. It remains here to show that the conditions are also necessary; that is, that the canonical model of theories incorporating such deductive power also validates these semantic conditions.

Recall the definitions:

d2 R^2abcd iff for some $x \in W, Rabx$ and $Rxcd$.

d3 $R^2a(bc)d$ iff for some $x \in W, Rbcx$ and $Raxd$.

Lemma 11 (i) R^2abcd *iff* $(a;b);c \leq d$.

(ii) $R^2a(bc)d$ *iff* $a;(b;c) \leq d$.

Proof:

(i) Suppose R^2abcd. Then for some $x, Rabx$ and $Rxcd$, i.e., $a;b \leq x$ and $x;c \leq d$. Then $(a;b);c \leq x;c \leq d$.

Conversely, suppose $(a;b);c \leq d$. Let $x = \{A : a;b \vdash A\}$. Then by Lemma 6iv), $a;b \leq x$ and so $x;c \leq d$. Hence, by Lemma 7i), for some $x' \in W, x \subseteq x'$ and $x';c \leq d$. So $a;b \leq x \leq x'$. Thus $Rabx'$ and $Rx'cd$, i.e. R^2abcd, as required.

(ii) Suppose $R^2a(bc)d$. Then for some $x, Rbcx$ and $Raxd$, i.e., $b;c \leq x$ and $a;x \leq d$. Then $a;(b;c) \leq a;x \leq d$.

Conversely, suppose $a;(b;c) \leq d$. Let $x = \{A : b;c \vdash A\}$. Then $b;c \leq x$ and so $a;x \leq d$. Hence, by Lemma 7ii), for some $x' \in W, x \subseteq x'$ and $a;x' \leq d$. So $b;c \leq x \leq x'$. Thus $Rbcx'$ and $Rax'd$, i.e. $R^2a(bc)d$, as required. □

We can now establish the correctness of the semantic postulates:

1. weakened $\sim$I: suppose $a \leq b$, and suppose $B \in b^*$. Then $\sim B \notin b$. Moreover, $a \subseteq b$, by Lemma 6ii). So $\sim B \notin a$, whence $B \in a^*$. Thus $b^* \subseteq a^*$, as required.
2. CM: suppose $A \to B \in a$ and $A \in a^*$. Then $\sim A \vee B \in a$, by Proposition 4.21. Moreover, since $A \in a^*$, $\sim A \notin a$. But a is prime, so $B \in a$. Hence Raa^*a, by Lemma 6iii), as required.
3. B: suppose R^2abcd. Then by Lemma 11i), $(a;b);c \leq d$. So $a;(b;c) \leq (a;b);c)$ (by B) $\leq d$, whence by Lemma 11ii), $R^2a(bc)d$.
4. B′: suppose R^2abcd. Then by Lemma 11i), $(a;b);c \leq d$. So $b;(a;c)) \leq a;(b;c)$ (by B’) $\leq d$, whence by Lemma 11ii), $R^2b(ac)d$.
5. C**: by C**, $a;\mathsf{t} \leq a$ for all bunches a. But $\mathsf{t} \angle \mathsf{0}$, so $a;\mathsf{0} \leq a$. Hence for some $x \in \mathsf{P}$ (viz $\mathsf{0}$), $Raxa$, as required.
6. C*: suppose $Rabc$, i.e., $a;b \leq c$. By C*, $b;a \leq a;b$. Hence $b;a \leq c$, i.e., $Rbac$, as required.
7. C: suppose R^2abcd. Then by Lemma 11i), $(a;b);c \leq d$. So $(a;c);b) \leq (a;b);c$ (by C) $\leq d$, whence by Lemma 11i), R^2acbd.

8. W*: immediate, since W* asserts $a; a \leq a$, i.e., $Raaa$.
9. W: suppose $Rabc$, i.e., $a; b \leq c$. So $(a; b); b) \leq a; b$ (by W) $\leq c$, whence by Lemma 11i), R^2abbc.
10. WB: suppose $Rabc$, i.e., $a; b \leq c$. So $a; (a; b)) \leq a; b$ (by WB) $\leq c$, whence by Lemma 11i), $R^2a; (a; b); c$.
11. S*: suppose $Rabc$, i.e., $a; b \leq c$. So $b; (a; b)) \leq a; b$ (by S*) $\leq c$, whence by Lemma 11i), $R^2b; (a; b); c$.
12. S: suppose R^2abcd. Then by Lemma 11i), $(a; b); c \leq d$. Let $x = \{A : b; c \vdash A\}$. By Lemma 6iv), $b; c \leq x$. So $(a; c); x \leq (a; c); (b; c) \leq (a; b); c$ (by S) $\leq d$. So there is $x \in \mathsf{V}$ such that R^2abxd and $Rbcx$, i.e., there is $x \in \mathsf{V}$ and $y \in \mathsf{W}$ such that $Racy, Ryxd$ and $Rbcx$. However, we must show that there is $x' \in \mathsf{W}$ with this property. By Lemma 7ii), since $Ryxb$, there is $x' \in \mathsf{W}$ such that $Ryx'b$ and $x \subseteq x'$. Since $Rbcx$ and $x \subseteq x'$, it follows that $Rbcx'$. Hence there is $x' \in \mathsf{W}$ such that $R^2abx'd$ and $Rbcx'$, as required.
13. K*: suppose $Rabc$, but neither $a \leq c$ nor $b \leq c$. Then by Lemma 6ii), neither $a \subseteq c$ nor $b \subseteq c$, and so there are $A \in a, B \in b$ for which $A \notin c$ and $B \notin c$. Because $a \vdash A$ and $b \vdash B, a; b \vdash A \vee B$ by the following derivation:

a	(1)	A	Given
b	(2)	B	Given
a	(3)	$A \vee B$	1 ∨I
b	(4)	$A \vee B$	2 ∨I
$a; b$	(5)	$(A \vee B) \times (A \vee B)$	3,4×I
6	(6)	$A \vee B$	A
6; 6	(7)	$A \vee B$	6 K*
$a; b$	(8)	$A \vee B$	5,7 ×E

    Therefore, $c \vdash A \vee B$, since $a; b \leq c$, and so $A \vee B \in c$, whence either $A \in c$ or $B \in c$ since c is prime, contradicting the assumption that neither $a \leq c$ nor $b \leq c$. Hence either $a \leq c$ or $b \leq c$.
14. K: suppose $Rabc$, i.e., $a; b \leq c$. Then by K, $a \leq a; b \leq c$.

That completes the proof. □

5A.3 Completeness for $\mathbf{R}^{\Box}$

The completeness proof for $\mathbf{R}^{\Box}$ follows a similar pattern to that for **DW**. In this context, of course, deductive closure means closure under the resources of $\mathbf{R}^{\Box}$. Note that, if $a \vdash B$, then there is a finite pruning $Y \angle\, a$ such that $Y \vdash B$, whence by Corollary 4.5, $\mathsf{cf}(Y) \vdash B$, and so by Proposition 4.26 $\vdash \mathsf{cf}(Y) \Rightarrow B$. So deductive closure is closure under entailment, not simply implication.

However, there is one major change we must make in the procedure followed, which is to use (prime) a-theories rather than (prime) theories alone in the construction of the model. An a-theory is an E-bunch of wffs b closed under a-derivability, i.e., $C \in b$ iff $a; b \vdash C$. An a-theory b is proper provided b is neither null nor universal and a is regular, i.e., $\mathsf{t} \in a$.

We need a-theories because of the reduced modelling given for $\mathbf{R}^{\Box}$, that is, the use of a single world $\mathsf{0}$ for the definition of truth simpliciter. As we saw in Lemma 2ii) of §5.6, we require that $\mathsf{t}$be true in a region accessible from $\mathsf{0}$ (viz $\{a : S\mathsf{0}a\}$). Because $\mathsf{0}$ is a prime theory (and so may contain more than simply the theorems), there can be no guarantee that a is accessible from $\mathsf{0}$ whenever $\mathsf{t} \in a$ unless a is a $\mathsf{0}$-theory, that is, unless $\mathsf{0}$, and not simply the logic, t, lays down the parameters for accessibility.

We must now rework the completeness proof for $\mathsf{0}$-theories. First, we briefly restate Lemmas 1 - 4 for a-theories in general.

Lemma 1 *The deductive closure of a bunch under a-derivability is an a-theory.*

Proof: as before, with $a; X \vdash B$ replacing $X \vdash B$. $\Box$

Let Y, Z be two E-bunches of wffs. Z is a-derivable from Y (written $Y \Vdash_a Z$) if $a; Y \vdash Z_1$, for some disjunction Z_1 of wffs of Z. A pair $\langle Y, Z\rangle$ is a-maximal if $Y \nVdash_a Z$ and for all $B \in \mathbf{L}'$, $B \in Y \cup Z$.

Lemma 2 (i) *If $\langle Y, Z\rangle$ is a-maximal, Y is a prime a-theory.*

(ii) *If a is regular, Cut holds for $\Vdash_a$. That is, suppose $B \in Y \cup Z, X \Vdash_a Y$ and $Z \Vdash_a W$. Then $X, Z - \{B\} \Vdash_a Y - \{B\}, W$.*

Proof:

(i) as before, with $Y \Vdash_a Z$ replacing $Y \Vdash Z$.

(ii) In place of Proposition 4.18 we appeal to the fact that, if Z is an E-bunch, $\mathsf{t} \in Z$ and if $Z; X \vdash A \vee B$ and $Z; (Y, B) \vdash C$, then $Z; (Y, X) \vdash A \vee C$. (The proof adds steps of t= and EK to that of 4.18.) The fact that a is regular then permits one to handle the parameter a. $\Box$

Lemma 3 *Suppose $Y \nVdash_a Z$, where a is regular. Then there are Y', Z' with $Y \subseteq Y', Z \subseteq Z', \langle Y', Z'\rangle$a-maximal.*

Proof: as before. Y_{n+1} and Z_{n+1} are defined relative to a-derivability:

$$\begin{array}{llll} \text{If} \quad Y_n \cup A_n \nVdash_a Z_n & \text{then} & Y_{n+1} = Y_n \cup A_n & Z_{n+1} = Z_n \\ & \text{else} & Y_{n+1} = Y_n & Z_{n+1} = Z_n \cup \{A_n\} \end{array}$$

The proof then proceeds relative to a-derivability. $\Box$

Lemma 4 *Let b be a proper a-theory and Y a set of wffs disjoint from a but closed under disjunction. Then there is a prime a-theory b' such that $b \subseteq b'$ and b' is disjoint from Y.*

Proof: as before, relativised to a-theories. Note that Lemma 3 applies, since b is proper, and so a is regular. □

As before, we let $\tau = \{B : \mathsf{t} \vdash B\}$, and extend τ by the original Lemma 4 to a prime theory $\mathsf{0}$, $\tau \subseteq \mathsf{0}, \mathsf{f} \notin \mathsf{0}$. Clearly $\mathsf{0}$ is regular. The proof now proceeds relative to $\mathsf{0}$-theories. Note that, since $\mathsf{t} \in \mathsf{0}$, $\mathsf{0}$ is a $\mathsf{0}$-theory, by t= and EK.

Let V be the set of proper $\mathsf{0}$-theories. For $a, b, c \in \mathsf{V}$, let $Rabc$ hold iff $a; b \leq c$, and let Sab hold if $B \in b$ whenever $\Box B \in a$. Let W be the set of prime $\mathsf{0}$-theories in V, and for each $a \in \mathsf{W}$, let $a^* = \{B : \sim B \notin a\}$.

We define the canonical ($\mathbf{R}\Box$-)m.s. M as the structure $\langle \mathsf{0}, \mathsf{W}, R, S, ^* \rangle$.

Lemma 5 W *is closed under* *.

Proof: as before, with $\mathsf{0}; a \vdash B$ in place of $a \vdash B$, for $a = b, b^*$. □

Lemma 6 (i) *$a \leq b$ iff $(\exists x)(S\mathsf{0}x$ & $Rxab)$.*

(ii) *$a \leq b$ iff $a \subseteq b$.*

(iii) *$Rabc$ iff whenever $A \to B \in a$ and $A \in b, B \in c$.*

(iv) *Let $x = \{A : X \vdash A\}$ for an arbitrary bunch X. Then $x \leq X$ and $X \leq x$ (i.e., $x = X$ in the sense of chapter 4).*

Proof:

(i) suppose $a \leq b$. As before, $R\mathsf{0}ab$. By $\Box$E, Sxx for all x, in particular, $S\mathsf{00}$. So for some x, viz $\mathsf{0}$, $S\mathsf{0}x$ and $Rxab$.

Conversely, suppose that for some $x, S\mathsf{0}x$ and $Rxab$. Then $x; a \leq b$. Suppose $Z(a) \vdash A$. Then $Z(\mathsf{t}; a) \vdash A$. By $\Box$I, $\mathsf{t} \vdash \Box\mathsf{t}$, so $\Box\mathsf{t} \in \mathsf{0}$, so $\mathsf{t} \in x$. Hence $\mathsf{t} \angle x$, and so $Z(x; a) \vdash A$, whence $Z(a) \vdash A$. Thus $a \leq b$.

The proofs of ii) - iv) are unchanged. □

Lemma 7 (i) *If $x, b \in \mathsf{V}, c \in \mathsf{W}$ and $Rxbc$, then there is $a \in \mathsf{W}$ such that $x \subseteq a$ and $Rabc$.*

(ii) *If $a, x \in \mathsf{V}, c \in \mathsf{W}$ and $Raxc$, then there is $b \in \mathsf{W}$ such that $x \subseteq b$ and $Rabc$.*

(iii) *If $a, b, x \in \mathsf{V}, C \notin x$ and $Rabx$, then there is $c \in \mathsf{W}$ such that $x \subseteq c, C \notin c$ and $Rabc$. We add a fourth clause:*

(iv) *If $a, x \in \mathsf{V}, C \notin x$ and Sax, then there is $b \in \mathsf{W}$ such that $x \subseteq b, C \notin b$ and Sab.*

Proof:

(i), ii), iii) as before, with proofs relativised to $\mathsf{0}$.

(iv) Let Y be the closure of $\{C\}$ under disjunction. Then x and Y are disjoint, and so by Lemma 4 there is a prime $\mathsf{0}$-theory b disjoint from Y extending x. Since $C \in Y, C \notin b$, and since $x \subseteq b$, it follows that Sab. □

Lemma 8 M *is an* $\mathbf{R}\Box$*-m.s.*

Proof: Clearly W is non-empty, as $\mathsf{0} \in \mathsf{W}$. We must show that r1 - r8 hold.

r1 partial ordering: immediate from Lemma 6.

r2 R is monotonic: as before.

r3 as before.

r4 (a) S is reflexive: immediate by $\Box$E.

(b) S is transitive: suppose Sab and Sbc, that is, whenever $\Box B \in a, B \in b$, and whenever $\Box B \in b, B \in c$. Suppose $\Box B \in a$. Then $a \vdash \Box B$. By $\Box$I, $\Box B \vdash \Box\Box B$. So by Cut (Proposition 4.4) $a \vdash \Box\Box B$. So $\Box B \in b$, and so $B \in c$. Hence Sac.

r5 S is monotonic:

(a) suppose $a \leq x$ and Sxb, and suppose $\Box B \in a$. then by Lemma 6, $\Box B \in x$, so $B \in b$. Hence Sab.

(b) suppose $x \leq b$ and Sax, and suppose $\Box B \in a$. Then $B \in x$, and so $B \in b$ by Lemma 6. Hence Sab.

r6 immediate by W*, which holds by Proposition 4.26.

r7 suppose R^2abcd, i.e., $(a;b);c \leq d$, and suppose $Z((a;c);b) \vdash A$. Then $Z((a;b);c) \vdash A$, by C, and so $Z(d) \vdash A$. Thus $(a;c);b \leq d$, i.e., R^2acbd.

r8 suppose $R^\frown Sabc$, i.e., $(\exists x)(Rabx \,\&\, Sxc)$. Define $y = \{B : \Box B \in a\}, z = \{B : \Box B \in b\}$. Then $y, z \in W$ and Say, Sbz. Suppose $B \to C \in y, B \in z$. Then $B \Rightarrow C \in a$, and $\Box B \in z$. By Proposition 4.28, $B \Rightarrow C; \Box B \vdash \Box C$. Hence $a; b \vdash \Box C$, whence, since $Rabx, \Box C \in x$, and so $C \in c$. Hence $Ryzc$. So by Lemma 7i), there is $y' \in \mathsf{W}$ such that $y \subseteq y'$ and $Ry'zc$. Clearly, since Say and $y \subseteq y', Say'$. Moreover, by Lemma 7ii), there is $z' \in \mathsf{W}$ such that $z \subseteq z'$ and $Ry'z'c$. Again, since Sbz and $z \subseteq z', Sbz'$. Hence Say' and Sbz' and $Ry'z'c$, as required. $\Box$

Lemma 9 *For all* $a \in \mathsf{W}$ *and all* $B \in \mathbf{L}'$, $v(B, a) = \mathsf{T}$ *iff* $B \in a$.

Proof: by induction on the degree of B. The base is immediate, by definition. Induction step:

(i) $B = \mathsf{f}$: first we show that $\mathsf{t} \in a$ iff $S\mathsf{0}a$. Suppose $\mathsf{t} \in a$, and $\Box B \in \mathsf{0}$. Since $\mathsf{t} \vdash \Box B \to B$, $\Box B \to B \in a$, and so, since $\Box B; (\Box B \to B) \vdash B$ by $\to$E and C*, $\mathsf{0}; a \vdash B$, whence $B \in a$. Hence $S\mathsf{0}a$.

Conversely, suppose $S\mathsf{0}a$. Clearly $\Box\mathsf{t} \in \mathsf{0}$, by $\Box$I, since $\mathsf{0}$ is regular. Hence $\mathsf{t} \in a$.

Consequently, $v(\mathsf{f}, a) = \mathsf{F}$ iff $S\mathsf{0}a^*$ by V2i)
iff $\mathsf{t} \in a^*$ (just proven)
iff $\mathsf{f} \notin a$.

Contraposing, $v(\mathsf{f}, a) = \mathsf{T}$ iff $\mathsf{f} \in a$.

(ii), iii) $B = C \mathbin{\&} D, C \vee D$: as before.

(iv), v) $B = C \to D, C \times D$: as before, relativised to $\mathsf{0}$-theories.

(vi) $B = \Box C$: suppose $\Box C \in a$, and Sab. Then $C \in b$, so by I.H., $v(C, b) = \mathsf{T}$. Hence $\mathrm{v}(\Box C, a) = \mathsf{T}$.

Conversely, suppose $\Box C \notin a$. We must show that there is $x' \in \mathsf{W}$ such that Sax' and $v(C, x') = \mathsf{F}$. Let $x = \{D : \Box D \in a\}$. Then $D \in x$ iff $\Box D \in a$ iff $\mathsf{0}; a \vdash \Box D$ iff $\mathsf{0}; x \vdash D$. So x is a $\mathsf{0}$-theory. Clearly, Sax and $C \notin x$. Hence by Lemma 7iv), there is $x' \in \mathsf{W}$ such that $C \notin x'$ and Sax' . By I.H., $v(C, x') = \mathsf{F}$, and so $\mathrm{v}(\Box C, a) = \mathsf{F}$. Contraposing, if $v(\Box C, a) = \mathsf{T}$ then $\Box C \in a$.

(vii) $B = \mathsf{F}$: then $v(B, a) = \mathsf{F}$ for all a. We must show that for all $a, \mathsf{F} \notin a$. This is immediate by Corollary 5.6, since a is proper, and so not universal. □

Lemma 10 *For all bunches Y and prime $\mathsf{0}$-theories y' such that $\{B : \mathsf{0}; Y \vdash B\} \subseteq y', v(Y, y') = \mathsf{T}$.*

Proof: by induction on the complexity of Y.

Base: Y is a wff, then $Y \in y'$, and so $v(Y, y') = \mathsf{T}$ by Lemma 9.

Induction step:

1. $Y = Z \circ W$. Suppose z is the deductive closure of Z, and w of W under $\mathsf{0}$-derivability. Then z, w are $\mathsf{0}$-theories, by Lemma 1. Suppose $B \to C \in z$ and $B \in w$. Then $\mathsf{0}; Z \vdash B \to C$ and $\mathsf{0}; W \vdash B$. So $\mathsf{0}; (Z; W) \vdash C$ by $\to$E, B, C and W, i.e. $\mathsf{0}; Y \vdash C$. Hence $C \in y'$ and so $Rzwy'$ by Lemma 6iii). Therefore, by Lemma 7i) there is a prime $\mathsf{0}$-theory $z', z \subseteq z'$ such that $Rz'wy'$, and so by Lemma 7ii), there is a prime $\mathsf{0}$-theory $w', w \subseteq w'$ such that $Rz'w'y'$. By I.H., $v(Z, z') = \mathsf{T}$ and $v(W, w') = \mathsf{T}$. Hence $v(Y, y') = \mathsf{T}$ by V3i).

2. Y is an E-bunch. Take $Z \in Y$. Then $Z \leq Y$, so $\{B : \mathsf{0}; Z \vdash B\} \subseteq \{B : \mathsf{0}; Y \vdash B\} \subseteq y'$. So by I.H. $v(Z, y') = \mathsf{T}$, whence $v(Y, y') = \mathsf{T}$ by V3ii). □

Once again, we are ready to assemble the Completeness Proof. Take X, A such that $X \nvdash A$. Let x be the deductive closure of X under $\mathsf{0}$-derivability. Then x is a $\mathsf{0}$-theory, by Lemma 1. As before, x is proper by Corollary 4.16. Let Y be the disjunction closure of $\{A\}$ under $\mathsf{0}$-derivability. Then, by Lemma 4, there is a prime $\mathsf{0}$-theory a such that $x \subseteq a$ and a is disjoint from Y. So $a \in W$. By Lemma 10, $v(X, a) = \mathsf{T}$ in the canonical model. Moreover, since $A \notin a, v(A, a) = \mathsf{F}$. So there is a proper $\mathsf{0}$-theory $a \in W$ such that $v(X, a) = \mathsf{T}$ and $v(A, a) = \mathsf{F}$. Hence $X \not\models A$.

Contraposing, if $X \models A$ then $X \vdash A$. □

Chapter 6

Relevance

Much of Anderson and Belnap's argument for rejecting *ex falso quodlibet* and setting up a logical system in which implication is non-truth-functional depends on such claims as that 'A and not-A' is not relevant to B, and that relevance is a necessary condition for validity. In so doing, they have attracted a great deal of criticism, largely on the ground that the appropriate notion of relevance here has never been properly explained.

In fact, in their arguments in favour of the relevantist revision of the Classical Account of Validity, Anderson and Belnap refer to relevance in two rather different senses. On the one hand, there is an idea of connection of meanings, and on the other, of derivational utility. The former suggests that a necessary but not in itself sufficient condition for an entailment to hold between A and B is that there be some connection of meaning between A and B. The latter, derivational utility, gives rise to the idea of a condition which is both necessary and sufficient for entailment, namely, that there be a derivation of B from A in which A is actually used.

These informal notions in the motivational preliminaries are then transformed into formal constraints in the formalised theory.

6.1 The Pre-History of Relevance

The idea that validity requires a relevant connection between premises and conclusion has a long history. It certainly featured in Greek discussion on the nature of conditionals, since Sextus Empiricus, in his history of Pyrrhonism, speaks of 'those who introduce connection or coherence assert that it is a valid hypothetical whenever the opposite of its consequent contradicts its antecedent' (*Pyrrhoneiae Hypotyposes*, ii 111).[1] He does not say who held this view. Earlier in the same chapter, Sextus explicitly attributes the material view of conditionals to the Stoics: 'A valid hypothetical syllogism is one which does not begin with truth and end with a false consequent ... Only that which begins with truth and ends with falsehood is invalid, and the rest valid' (ii. 104-5).

However, Cicero certainly appears at one point in his *De Fato* (vi.12) to associate the idea of necessary conection with Chrysippus, third head of the Stoic school.[2] If conditionals express a necessary connection between antecedent and consequent, which he takes to be Chrysippus' view, then the necessary truth of a past tense statement may be transferred by conditionals to the necessary truth of statements about the future, a notion he thinks would be unpalatable to Chrysippus. However, at viii 15, Cicero suggests that Chrysippus would rephrase the conditional as a negated conjunction, and in the context that strongly implies a material view of the conditional, as attributed by Sextus. It is also conceivable, as comes out in Diogenes Laertius' discussion of the Stoic view, that 'connection' here merely connotes strict implication.[3]

It seems to be only in the Middle Ages that it was explicitly realised that the Classical Account of Validity entails that a contradiction entails any proposition whatever. The argument of the school of Petit Pont for EFQ which we cited earlier (from Neckam - §2.6), was certainly accepted by such much quoted figures as Peter of Spain and John Buridan (who was probably responsible for rescuing Peter from a century of obscurity and placing his work as that logic book most often reproduced and commented upon). But the argument was equally often rejected. Consider two cases, those of Ralph Strode, writing in Oxford in the 1360s, and Domingo de Soto, writing in Paris in the 1520s. Strode accepted EFQ as a materially valid inference, but not as formally valid. He wrote:

> The eleventh rule is this: in no formally valid inference whose premise is purely negative is the conclusion affirmative... The twelfth is a corollary to that. If the conclusion is purely affirmative, so is the premise or some part of it ... A possible counterexample is this:

[1] Sextus Empiricus, *Pyrrhoneiae Hypotyposes*, ed. & tr. R.G.Bury (London, 1933), p. 223.

[2] Cicero, *De Fato*, ed. & tr. H. Rackham (London, 1948).

[3] Diogenes Laertius, *Lives of the Eminent Philosophers*, vii 73, ed. & tr. R.D.Hicks (London, 1925), p.181.

> God is not, so God is,
>
> which is a valid argument by rules given earlier, namely, that anything follows from a contradiction. But it has both an impossible premise and a necessary conclusion ...
>
> The solution is that although this argument is materially valid, it is not in fact formally valid ... There are arguments with the same form, e.g.,
>
> Socrates is not, so Socrates is;
> an animal is not, so an animal is,
>
> which are not valid.(Ralph Strode, *Consequentie* (Venice, 1517), fo. 12^{rb-va})[4]

His rejection of EFQ depends, therefore, on his distinction between formal and material validity. This reads: 'Valid inference is of two sorts, formally valid and only materially valid. ... In a formally valid inference the consequent is formally understood [*de formali intellectu*] in the premise' (fo. 2^{rb-va})

Satisfactorily to interpret this idea of 'being formally understood in' involved later commentators in a difficult task. For on the one hand, the argument from a patent or explicit contradiction to an arbitrary proposition does appear to obtain its validity, if such it has, as a matter of form; and on the other, the concept of 'understanding one proposition in another' is badly in need of explication.

One idea which repays investigation is mentioned by Domingo do Soto, writing in Paris in 1529. He too considered the inference:

> God is not, so God is.

The basis for his rejection of it was in fact acceptance of 'Aristotle's Principle', as it is often known,

$$\sim(\sim A \to A)$$

that no proposition is entailed by its own contradictory (or entails it).[5] The fault with the traditional argument for EFQ is this:

[4] Ralph Strode, *Consequentie* (Venice, 1517). My translation.

[5] Aristotle, *Analytica Priora*, ed. & tr. A.J.Jenkinson (Oxford, 1928), ii ch.4, 57b3-16. Aristotle's principle is not valid in **R** (or **K** of course), but is accepted in connexive logics: see, e.g., S. McCall, 'Connexive implication', *Journal of Symbolic Logic*, 31 (1966), pp. 415-53.

> Note that a pair of contradictories can be taken in two ways, one way absolutely and without any supposition, and then the one is destructive of the other, that is, the truth of the one takes away the truth of the other. In the other way, they are taken as conceded by someone from some assumption for the sake of argument to see what follows. Taken this way, neither destroys the other, for conceding that the one is true, it does not follow that the other is false, since both are conceded to be true. So the inference
>
> Peter disputes or a man is a stone
> and Peter does not dispute,
> hence a man is a stone,
>
> if those contradictories are taken absolutely, is valid. But if they are taken as conceded in some initial premise, it is not valid, for from the fact that a negative is true it does not follow by the assumption given that the affirmative is false, since in the cause of disputation both were conceded. (*Summule*, or *Introductiones Dialecticae* (Burgis, 1529), fo. 74^{va})[6]

In Neckam's argument the contradiction is assumed merely for the sake of argument, rather than taken absolutely.

This is a distinction of which we can make sense. Consider the rule of substitution for variables in propositional logic. Suppose A' results from A by substituting 'q' for 'p' throughout. Then although A' does not follow from A, we can infer A' from a proof of A. In other words, if $\vdash A$ then $\vdash A'$, although $A \nvdash A'$. This distinction seems to have been what de Soto was driving at in his talk of taking a proposition absolutely or for the sake of argument. He was rejecting the validity of Disjunctive Syllogism:

$$A \vee B,\ {\sim}A \nvdash B$$

but accepting what has come to be known as γ (gamma):[7]

$$\vdash A \vee B,\ \vdash {\sim}A \Rightarrow\ \vdash B.$$

[6]Domingo de Soto, *Summule*, or *Introductiones Dialecticae* (Burgis, 1529). My translation.

[7]Anderson and Belnap obtained their system **E** by dropping γ from Ackermann's system Π'.

Disjunctive Syllogism fails in all the relevant logics we have considered. But γ has been shown to hold for **TW**, **T**, **E** and **R**.[8]

6.2 The Later History of Relevance

C.I. Lewis' main aim was to find a satisfactory theory of implication. Material implication was unsatisfactory properly to capture implication, for it involves the claim that a false proposition entails any proposition whatever, and a true proposition is entailed by any such. In 1917 he wrote:

> A relation which does not indicate relevance of content is merely a connection of 'truthvalues', not what we mean by a 'logical' relation or 'inference' ... [Proof] requires that a connection of content or meaning or logical significance be established ... True postulates may materially imply theorems which are quite irrelevant.[9]

But it was relatively late in his development of the theory of strict implication that he settled categorically in favour of EFQ. In an earlier article he had given a set of postulates for strict implication which entailed that 'p & $\sim p$' strictly imply q. But if one dislikes this consequence, he wrote, one should replace one of the postulates (S9) by a weaker postulate. This will eliminate these theorems.[10] The postulate is given as one of three optional additional postulates (Postulate 19) in his presentation of the theory a year earlier.[11] The twelve postulates there given for the main system, and the optional Postulate 17, satisfy variable-sharing (see §6.3 below). Postulates 18 and 19 do not.

He goes on to remark, however, that some of these implausible consequences 'are examplified in our everyday reasoning.' In a footnote he gives an example:

> Suppose $p \supset r$ and q is irrelevant. Then $pq \supset r, p - r$ is impossible, and $-q$ might be any irrelevant proposition.

[8]See R. Routley et al., *Relevant Logics and their Rivals* (Atascadero, Calif., 1984), p. 391; J.M. Dunn, 'Relevance Logic and Entailment', in *Handbook of Philosophical Logic*, vol. III: *Alternatives to Classical Logic*, ed. D. Guenthner and D. Gabbay (Dordrecht, 1986), §2, pp. 150-65.

[9]C.I. Lewis, 'The Issues concerning Material Implication', *Journal of Philosophy*, 14 (1917), pp. 356, 355, 354.

[10]C.I. Lewis, 'The Calculus of Strict Implication', *Mind*, 23 (1914), p 246.

[11]C.I. Lewis, 'A New Algebra of Implications and some consequences', *Journal of Philosophy*, 10 (1913), pp. 428-38.

In other words, if p strictly implies r, then 'p & $\sim r$' is impossible. But if p strictly implies r, a fortiori 'p & q' strictly implies r, whence, by Antilogism, 'p & $\sim r$', an impossiblity, strictly implies '$\sim q$', an irrelevant proposition. He replaced this argument for EFQ by the one stemming from the Middle Ages which bears his name in later presentations of the theory. But as we saw in chapters 2 and 3, both Disjunctive Syllogism and Antilogism need careful consideration to understand in what circumstances (i.e., for what connectives) they hold, and when they do not.

The immediate origins of relevant logic lie, however, in the 1950s. The main cause was the publication of Ackermann's ideas about 'strong' or 'rigorous' implication.[12] But later developments were anticipated and influenced by Church's theory of weak implication and Moh's investigation of weak systems in the context of discussion of the deduction theorem.[13] Moh found EFQ and its like paradoxical, and developed systems which for that reason lacked them. But Church had no quarrel with EFQ. His idea was to continue a plan started by Russell and Whitehead: in order to make explicit the uses of an axiom A, for example, the Axioms of Choice or of Infinity, we remove it from the axiom set and replace all theorems T in the old calculus by theorems of the form

$$A \supset T$$

in the new one. By this procedure Church arrived at a pure implicational system equivalent (on replacing '$\supset$' by '$\to$') to the implicational part of **RW**.

Our consideration of **DW** in chapter 4 takes this procedure to its logical conclusion. The only axiom of Church's which is retained is Identity, '$A \to A$'. His first three axioms, which amount to W, B and C, are identified as structural rules to be added if required; so too for K, which Church himself had eliminated. Church believed that he needed to retain W, B and C as axioms, in order to preserve the Deduction Theorem. But we saw in §3.2 how properly to state the Deduction Theorem so that it holds for **DW** (Proposition 5.2). Moreover, other systems of implication, such as Church

[12] W. Ackermann, 'Begründung einer strengen Implikation', *Journal of Symbolic Logic*, 21 (1956), pp. 113-28.

[13] A. Church, 'The Weak Theory of Implication', *Kontrolliertes Denken, Untersuchungen zum Lokikkalkül und zur der Einzelwissenschaften* (Munich, 1951), pp. 22-37; see also Church, 'The weak positive implicational propositional calculus' (abstract), *Journal of Symbolic Logic*, 16 (1951), p. 237, and Church, 'Minimal Logic' (abstract), *Journal of Symbolic Logic*, 16 (1951), p. 238. Moh Shaw-Kwei, 'The Deduction Theorems and two new logical systems', *Methodos*, 2 (1950), pp. 56-75.

hints at (e.g., Abelian logic[14]) can be explored by adding other postulates (as structural rules or as additional axioms). The advantage of exploring these weak systems and their interrelationships is that one comes to understand what exactly depends on what: that is, inference. It is not my aim in this book to recommend **DW** as the right theory of inference. **R** and $\mathbf{R}^{\Box}$ are the most plausible candidates. But considering the whole range of systems serves to bring out the connection between the various constituents of **R**.

It was on 4 December 1959 that the Journal of Symbolic Logic received Belnap's paper 'Entailment and Relevance'.[15] It was not the first time that logicians had identified fallacies of relevance. Indeed, it was the use of that phrase in the traditional identification of fallacy which led Belnap to describe the system he and Anderson had extracted from Ackermann's paper in these terms. The first use of the word 'relevance' in the literature of relevant logic itself, however, seems to have been by Saul Kripke. In the abstract of a paper written under the influence of Anderson and Belnap presented to the meeting of the Association of Symbolic Logic on 28 December 1959, Kripke wrote: 'The rule K was dropped because it allowed the introduction of 'irrelevant' constituents.' Timothy Smiley had made a similar remark earlier the same year: 'Acceptable entailment-statements should be strictly relevant.'[16] But although they doubtless had an influence on its development, Smiley's ideas no more belong to the research programme which has come to be known as relevant logic than did Lewis's.

Belnap's and Kripke's conception of relevance picked up and extended an element in Ackermann's motivation. Ackermann had written:

> Rigorous implication, which we write as $A \to B$, should express the fact that a logical connection holds between A and B, that the content of B is part of that of A ... That has nothing to do with the truth or falsity of A or B. Thus one would reject the validity of the formula $A \to (B \to A)$, since it permits the inference from A of $B \to A$, and since the truth of A has nothing to do with whether a logical connection holds between B and A.[17]

[14] Church, 'The weak positive implicational propositional calculus'; R.K. Meyer and J.K. Slaney, *Abelian Logic (from A to Z)*, Australian National University Research School of Social Sciences Logic Group Research Paper no. 7 (Canberra, 1980).

[15] N. Belnap, 'Entailment and Relevance', *Journal of Symbolic Logic*, 25 (1959), pp. 144-6.

[16] T. Smiley, 'Entailment and Deducibility', *Proceedings of the Aristotelian Society*, 59 (1959), p. 246.

[17] Ackermann, in *Journal of Symbolic Logic*, 21, p. 113

He went on to point out that his concept of implication differs from Lewis's strict implication in that he rejects the validity of 'A & $\sim A \rightarrow B$', 'since the existence of a proposition which is implied by all or implies all others does not fit the concept of implication as a logical connection between two propositions'.[18]

We can explicate this idea of a logical connection in two ways: as connoting a meaning-connection, and as connoting an inferential connection.

6.3 Variable-Sharing

Variable-sharing is a formal notion designed to reflect the idea that there be a meaning-connection between A and B when A entails B. Certain theories can then be shown to satisfy the demand of variable-sharing, that is, that if A entails B, for propositional formulae A and B, then A and B share a variable.

It is plausible to suppose that if A is connected in meaning with B, then so too is B with A. A is certainly connected in meaning with itself. But that A is meaning-connected with B and B in turn with C, does not seem to warrant concluding that A is connected in meaning with C. Meaning-connection is, therefore, a reflexive, symmetric and non-transitive relation. Variable-sharing also has these formal properties. It is a necessary but not a sufficient condition for entailment. Note that the condition cannot be sufficient, for $A \vdash A \,\&\, B$ and $A \vee B, \sim A \vdash B$ satisfy the variable-sharing condition, but are invalid.

Let us show that $\mathbf{R}^{\Box}$ satisfies the variable-sharing condition.[19] Let M_0 be the $\mathbf{R}^{\Box}$-model structure $\langle 0, W, R, S, ^* \rangle$, where $W = \{0, 1, 2\}, 0^* = 0, 1^* = 2, R = \{000, 011, 022, 122, 101, 111, 120, 121, 122, 202, 210, 211, 212, 222\}, S = \{00, 11, 22\}$.[20] Definition of <u>antecedent part</u> (a-part) and <u>consequent part</u> (c-part) of a bunch or a formula: let X, Y, Z range over wffs and bunches; B, C over wffs:

1. X is a c-part of X.
2. if $\sim B$ is an $\left\{\begin{matrix}\text{a-part}\\ \text{c-part}\end{matrix}\right\}$ of X then B is a $\left\{\begin{matrix}\text{c-part}\\ \text{a-part}\end{matrix}\right\}$ of X.

[18] Ibid.

[19] See A. Anderson & N. Belnap, *Entailment*, vol. I (Princeton, NJ, 1975), §§5.1.2, 22.1.3.

[20] The m.s. is called M_0 following the standard designation for the underlying lattice structure in e.g., *Entailment* and *Relevant Logics and their Rivals*.

3. if $B \mathbin{\&} C$ or $B \vee C$ are $\left\{\begin{matrix}\text{a-parts}\\ \text{c-parts}\end{matrix}\right\}$ of X then B and C are $\left\{\begin{matrix}\text{a-parts}\\ \text{c-parts}\end{matrix}\right\}$ of X.
4. if $B \to C$ is an $\left\{\begin{matrix}\text{a-part}\\ \text{c-part}\end{matrix}\right\}$ of X then B is a $\left\{\begin{matrix}\text{c-part}\\ \text{a-part}\end{matrix}\right\}$ and C is an $\left\{\begin{matrix}\text{a-part}\\ \text{c-part}\end{matrix}\right\}$ of X.
5. if $B \times C$ is an $\left\{\begin{matrix}\text{a-part}\\ \text{c-part}\end{matrix}\right\}$ of X then so are B and C.
6. if $\Box B$ is an $\left\{\begin{matrix}\text{a-part}\\ \text{c-part}\end{matrix}\right\}$ of X then so is B.
7. if $Y \circ Z$ is an $\left\{\begin{matrix}\text{a-part}\\ \text{c-part}\end{matrix}\right\}$ of X then so are Y and Z.
8. if an E-bunch Y is an $\left\{\begin{matrix}\text{a-part}\\ \text{c-part}\end{matrix}\right\}$ of X then so are the members of Y.

Suppose W contains worlds $a_1, \ldots, a_n$, in some preassigned order. If $v(A, a_1) = \delta_1, \ldots, v(A, a_n) = \delta_n$, $\delta_i \in \{\mathsf{T}, \mathsf{F}\}$, we write $v(A) = \delta_1 \ldots \delta_n$. For example, if W contains only two worlds, 0,1, and $v(A, 0) = \mathsf{T}, v(A, 1) = \mathsf{F}$, we write $v(A) = \mathsf{TF}$.[21]

Proposition 6.1 *Suppose that when expressed in primitive terms,* F *does not occur in* X *or* A *and* f *occurs only in the context* $\sim B$*. If* $X \vdash_{\mathbf{R}\Box} A$*, then some variable occurs as an a-part of both* X *and* A*, or as a c-part of both.*

Proof: Suppose X and A do not share a variable as a-part or c-part of each. Then we interpret each variable p of X and A in M_0 as follows:

	X	A	$v(p)$
p	c-part	\	FTF
	a-part	\	TTF
	\	a-part	FFT
	\	c-part	TFT
	a-part	c-part	FFF
	c-part	a-part	TTT

We now show by induction on the degree of X and A that

1. every a-part of X takes a value in $\{\mathsf{FFF}, \mathsf{FTF}, \mathsf{TTF}\}$ and every c-part a value in $\{\mathsf{FTF}, \mathsf{TTF}, \mathsf{TTT}\}$; and

[21] Those who are familiar with matrix techniques will recognise the n-tuples $\delta_1 \ldots \delta_n$ as matrix values.

2. every a-part of A takes a value in $\{\mathsf{FFT}, \mathsf{TFT}, \mathsf{TTT}\}$ and every c-part a value in $\{\mathsf{FFF}, \mathsf{FFT}, \mathsf{TFT}\}$.

Ad (1): Take an $\left\{\begin{array}{c}\text{a-part}\\ \text{c-part}\end{array}\right\}$ Y of X.

Base: Y is a variable. Immediate from the definition of the assignment.

Induction Step:

(i) $Y = {\sim}B$. Then B is a $\left\{\begin{array}{c}\text{c-part}\\ \text{a-part}\end{array}\right\}$ of X. So by Induction Hypothesis, $v(B) \in \{\mathsf{FTF}, \mathsf{TTF}, \mathsf{TTT}\}/\{\mathsf{FFF}, \mathsf{FTF}, \mathsf{TTF}\}$ respectively. $v(\mathsf{f}) = \mathsf{FTT}$ in accord with the truth-definition. So by Lemma 3i) of §5.6,

$v(Y, 0) = \mathsf{T}$ if $v(B, 0) = \mathsf{F}$
$v(Y, 1) = \mathsf{T}$ if $v(B, 2) = \mathsf{F}$
and so on,

whence $v(Y) \in \{\mathsf{FFF}, \mathsf{FTF}, \mathsf{TTF}\}/\{\mathsf{FTF}, \mathsf{TTF}, \mathsf{TTT}\}$ respectively, as required.

(ii) $Y = B \mathbin{\&} C$ or $B \vee C$. Then B and C are $\left\{\begin{array}{c}\text{a-parts}\\ \text{c-parts}\end{array}\right\}$ of X. So by I.H., $v(B) \in \{\mathsf{FFF}, \mathsf{FTF}, \mathsf{TTF}\}/\{\mathsf{FTF}, \mathsf{TTF}, \mathsf{TTT}\}$ respectively, and similarly for C. Since

$v(B \mathbin{\&} C, 0) = \mathsf{T}$ if $v(B, 0) = v(C, 0) = \mathsf{T}$
and so on,

it follows that $v(Y) \in \{\mathsf{FFF}, \mathsf{FTF}, \mathsf{TTF}\}/\{\mathsf{FTF}, \mathsf{TTF}, \mathsf{TTT}\}$ respectively, as required.

(iii) $Y = B \to C$. Then B is a $\left\{\begin{array}{c}\text{c-part}\\ \text{a-part}\end{array}\right\}$ and C is an $\left\{\begin{array}{c}\text{a-part}\\ \text{c-part}\end{array}\right\}$ of Y. So by I.H., $v(B) \in \{\mathsf{FTF}, \mathsf{TTF}, \mathsf{TTT}\}/\{\mathsf{FFF}, \mathsf{FTF}, \mathsf{TTF}\}$ and $v(C) \in \{\mathsf{FFF}, \mathsf{FTF}, \mathsf{TTF}\}/\{\mathsf{FTF}, \mathsf{TTF}, \mathsf{TTT}\}$. By the truth-definition

$v(Y, 0) = \mathsf{T}$ if whenever $v(B, x) = \mathsf{T}$ and $R0xy, v(C, y) = \mathsf{T}$
and so on for 1 and 2.

From the R-triples which hold in M_0 it follows that $v(Y) \in \{\mathsf{FFF}, \mathsf{FTF}, \mathsf{TTF}\}/\{\mathsf{FTF}, \mathsf{TTF}, \mathsf{TTT}\}$ respectively.

(iv) $Y = B \times C$. Then B and C are $\left\{\begin{array}{c}\text{a-parts}\\ \text{c-parts}\end{array}\right\}$ of X. So by I.H., $v(B), v(C) \in \{\mathsf{FFF}, \mathsf{FTF}, \mathsf{TTF}\}/\{\mathsf{TTF}, \mathsf{FTF}, \mathsf{TTT}\}$. By the truth-definition,

$v(Y, 0) = \mathsf{T}$ if there are x, y such that $v(B, x) = \mathsf{T}$, $v(C, y) = \mathsf{T}$ and $Rxy0$,
and so on for 1 and 2.

From the R -triples which hold in M_0 it follows that $v(Y) \in \{\mathsf{FFF}, \mathsf{FTF}, \mathsf{TTF}\}/\{\mathsf{FTF}, \mathsf{TTF}, \mathsf{TTT}\}$.

(v) $Y = \Box B$. Then B is an $\left\{ \begin{array}{c} \text{a-part} \\ \text{c-part} \end{array} \right\}$ of X, and so by I.H. $v(B) \in \{\mathsf{FFF}, \mathsf{FTF}, \mathsf{TTF}\}/\{\mathsf{FTF}, \mathsf{TTF}, \mathsf{TTT}\}$. Since Sxy holds if and only if $x = y$, it follows that $v(Y) \in \{\mathsf{FFF}, \mathsf{FTF}, \mathsf{TTF}\}/\{\mathsf{TTF}, \mathsf{FTF}, \mathsf{TTT}\}$ also.

(vi) $Y = Z \circ W$. Proof similar to iv) above.

(vii) Y is an E-bunch. Proof similar to ii) above.

That completes the proof of (1).

Ad (2): A similar induction on degree establishes the result for A.

Since X and A are c-parts of themselves, it follows that $v(X) \in \{\mathsf{TTF}, \mathsf{FTF}, \mathsf{TTT}\}$ and $v(A) \in \{\mathsf{FFF}, \mathsf{TFT}, \mathsf{FFT}\}$. Hence, whatever values X and A take, there is always a world a in M_0 such that $v(X, a) = \mathsf{T}$ and $v(A, a) = \mathsf{F}$. Hence $X \not\models A$, and so by Proposition 5.13, $X \nvdash_{\mathbf{R}^\Box} A$. Contraposing, if $X \vdash_{\mathbf{R}^\Box} A$, then X and A share a variable as a-part or c-part of each. $\Box$

Corollary 6.2 *Suppose A and B do not contain F, and contain f only in the context $\sim C$. If $\vdash_{\mathbf{R}^\Box} A \to B$ then A and B share a variable.*

Proof: Take A, B as given, with $\vdash_{\mathbf{R}^\Box} A \to B$. Then $A \vdash_{\mathbf{R}^\Box} B$, by Proposition 5.11, and so by Proposition 6.1, A and B share a variable (as a-part or c-part of each). $\Box$

6.4 Derivational Utility

Derivational utility is also given a formal analogue in the natural deduction formulation of relevant logics. It first appeared in Anderson and Belnap's abstract of a paper presented to the meeting of the Association of Symbolic Logic in Philadelphia in January 1959.[22] The aim is once again to maintain a form of the Deduction Theorem, and the method is to track the premises to ensure that in a derivation of A from some set of assumptions X, it is

[22] A.Anderson and N.Belnap, 'A modification of Ackermann's "rigorous implication"' (abstract), *Journal of Symbolic Logic*, 23 (1958), pp. 457-8.

clear which members of X have actually been used in the derivation. Anderson and Belnap state explicitly that the method is designed to explicate Ackermann's notion of logical connection. The basic method consists in attaching a star to each premise, and tracking its essential use through the rules. We then obtain a Deduction Theorem of this kind:

> If there is a derivation of A from a set of assumptions $A_1, \ldots, A_n$ in which all are used (starred), then there is a similar derivation of $A_n \to A$ from $A_1, \ldots, A_{n-1}$.

Anderson and Belnap point out, however, that what one would really like is a notion of consequence that both permits a Deduction Theorem but at the same time does not require that A be derivable from all of X for A to be a consequence of X.[23] To gain this, their solution is to track each hypothesis separately, by its own numerical flag, the so-called 'subscripting' technique.

What is a proof, or derivation? It is a finite partial ordering of proof-objects. But what the proof-objects themselves are, and whether the partial ordering is in fact total (i.e. a finite sequence) is not generally agreed. The second question is trivial, since proofs are finite. For any finite partial ordering can be mapped to a linear ordering and vice versa without loss of information. What of the proof-objects? Often they are taken to be formulae. In an axiomatic framework, this is of no consequence. But in a natual deduction framework, where the correctness of application of a rule may depend on the nature of some assumption, taking proof-objects to be formulae requires interpreting the rules of inference as global conditions on the proof. When we say that B follows from (is an immediate consequence of) proof-objects $A_1, \ldots, A_n$ we often need to state some constraint. A local constraint refers only to $A_1, \ldots, A_n$ or B; a global constraint refers to elements of the proof other than those proof-objects. For example, we may need to discharge some assumption on which $A_1, \ldots, A_n$ depend.

One can avoid the need for global rules (that is, global constraints on rules) by using different proof-objects. For example, in Fitch's logical system, the proof-objects consist of both formulae and sub-proofs.[24] A rule has a finite number of formulae and subproofs as premises and a formula as conclusion. An alternative, stemming ultimately from Gentzen,[25] is to carry the assumptions with the formula, in other words, to construe the

[23] *Entailment*, p. 21.

[24] F.B. Fitch, *Symbolic Logic* (New Haven, Conn., 1952).

[25] G. Gentzen, 'The Consistency of Elementary Number Theory', in *Collected Papers of Gerhard Gentzen*, ed. & tr. M. Szabo (Amsterdam, 1969), pp. 132-213, §5.

proof-objects as sequents. This is the conception of derivation which we drew on in chapter 4 above.

Whether an assumption has been used to derive a formula commutes with the question whether the formula 'really' depends on the assumption. For formulae, therefore, it is a global condition. Anderson and Belnap's tracking idea tries to convert it to a local condition. Fitch's solution to the global condition problem, however, cannot track global conditions outwith subproofs. Working with Fitch's proof-architecture, therefore, Anderson and Belnap were forced to supplement his resources, and subscript their formulae with references to the assumptions on which they 'really' depended. Subscripted formulae, however, are simply a rebarbative, or at least unusual, notation for sequents. In the proof-architecture adopted by Lemmon, Guttenplan[26] and ourselves, the relevance of asumptions to the conclusion drawn is tracked by the assumptions tied to each formula in its sequent.

Of course, this tracking can go awry, as Prawitz nicely shows.[27] Lax statement of & I and &E can result in an unintended admission of Weakening, which destroys the relevance measure. Without our distinction of intensional from extensional bunching, or Anderson and Belnap's subscripting restrictions, we can argue: 'A & B' depends on A and B (by &I) and so A depends on A and B too (by &E). (Cf. §3.1.) Anderson and Belnap avoid the problem by implausible constraints on the conjunction (and disjunction) rules, as we saw. But the seeds of our solution were there in their early statements. We must be careful, they say, not 'to confound $(A_1 \wedge \ldots \wedge A_n) \to B$ with $A_1 \to .A_2 \to \ldots .A_n \to B$'.[28] They write '$\wedge$' for our '&'.) The second of these formulae is equivalent (in **RW** and **R**) to

$$(A_1 \times A_2 \times \ldots \times A_n) \to B$$

and the importance of the distinction between '&' and '$\times$' takes off from there.

6.5 Logical Relevance

Of course, Anderson and Belnap's intention in showing that their theory satisfies the formal constraints was that it should reinforce the impression that

[26] E.J. Lemmon, *Beginning Logic* (London, 1965); S. Guttenplan, *The Languages of Logic* (Oxford, 1986).

[27] D. Prawitz, *Natural Deduction* (Stockholm, 1965), p. 84.

[28] A. Anderson and N. Belnap, 'The Pure Calculus of Entailment', *Journal of Symbolic Logic*, 27 (1962), p. 36. See also Moh's first Deduction Theorem in 'The Deduction Theorems and two new Logical Systems', in *Methodos*, 2, p. 61.

the system satisfies the informal intuitions which prompted its construction. However, it does nothing to answer the critics who feel either that the formal analogues are in fact false to the original intuitions or who wish to reject the suggestion of such constraints altogether. Iseminger, for example, claims that there is no precisely formulable sense of 'meaning' in the phrase 'meaning-connection' which can act as a sieve on classical entailments to give the account of entailment endorsed by Anderson and Belnap.[29] Indeed, it seems to put the cart before the horse, in this manner: sound inferences are often characterised as those which simply spell out what is implicitly in the premises—deductive argument cannot be extrapolative. Hence, in a sound argument, the conclusion must be contained in the meaning of the premises. It is, therefore, possible to claim, as Prior did, that what the Lewis argument shows, if it is sound, is that the meaning of a contradiction contains that of every other proposition.[30]

The same goes for derivational utility. In the Lewis proof, the contradictory premises do appear to be used to obtain an arbitrary proposition. It is then incumbent on us somehow to decide on the validity of the derivation before we can tell whether the contradiction has indeed, in the relevant sense, been used to obtain the conclusion. Derivational utility cannot, therefore, be used to dismiss the Lewis argument, or any other such, as unsound.

However, our examination of Lewis' argument has pointed us towards an ambiguity in the sense of certain connectives, and building on that, has enabled us to state the Deduction Theorem in its correct form. Aiming to prove Propositions 5.2 and 5.13 led us to formulate **DW** and $\mathbf{R}^{\Box}$ as derivational systems in the way we did. The correct form of the Deduction Theorem uses the sense of conjunction we have labelled 'fusion'. It is fusion which provides a suitable vehicle for us to state the appropriate sense of 'relevance'.

Recall the Classical Account of Validity: A entails B iff A cannot be true and B false. We found, in chapter 3, that what distinguishes the Classical Account is taking 'and' here to be extensional. Only on such a reading does it follow that EFQ is valid. Since it is not—which follows from the fact that 'if' is non-truth-functional—the correct account of validity is given by taking 'and' in the account as intensional. In this way we obtain the Relevant Account of validity:

[29] G. Iseminger, 'Is Relevance Necessary for Validity?', *Mind*, 89 (1980), pp. 196-213.

[30] A. Prior, *Formal Logic* (Oxford, 1962), p. 196. See also R.K. Meyer, *Why I am not a Relevantist*, Australian National University Research School of Social Sciences Logic Group Research Paper no. 1 (Canberra, 1978), §1.

> A entails B iff A cannot be true and (fuse) B false.

Recall that relevance, or meaning-connection, is symmetric. Accordingly, let us say that

> A is *logically relevant* to B if A fuse 'not-B' and (&) B fuse 'not-A' cannot be true.

It follows that A entails B only if A is logically relevant to B, whereby relevance is the necessary condition of entailment which was sought. Logical relevance, so defined, is symmetric, reflexive and non-transitive, as required. We can also weaken the definition to obtain a material notion of relevance, whereby

> A is *relevant* to B iff A fuse 'not-B' and (&) B fuse 'not-A' are not true.

Then A (relevantly) implies B only if A is relevant to B.

Prior's point is conceded: the test of whether two propositions are logically relevant is whether either entails the other. Hence, relevance cannot be picked out prior (if you'll excuse the pun) to establishing validity or entailment. What is important is that, having picked out the correct entailments by careful analysis of the arguments involved—not by a general definition of a meaning-connection sieve—it is possible to extract from that examination a notion of relevance which accords with the intuitive recognition that A is (in general) relevant to 'A and (&) B', and that in turn relevant to B, although A is not (in general) relevant to B.

6.6 Fusion

The use of fusion rather than ampersand in the definition of relevance is crucial. For fusion binds two propositions together in a way that extensional conjunction does not. What follows from '$A \times B$' requires both A and B in its derivation. Contrast this with Lewis's consistency connective, '$A \circ B$' $=_{\mathrm{df}}$ '$\Diamond(A \mathbin{\&} B)$'. On his account of consistency, any impossible proposition is inconsistent with any other proposition whatever, because '&' allows Augmentation, that is, 'A & B' entails A: whatever A entails is entailed by 'A & B'. But fusion does not. Hence, by the account of relevance in §6.5, a contradiction is not deemed relevant to an arbitrary proposition. This accords with Anderson and Belnap's intuitions. If we had used '&' in place of fusion in the definition of relevance, we would immediately obtain the unintuitive traditional answers.

It may of course be objected that this suggested explication of relevance is entirely circular and unilluminating, since it amounts to saying no more than that two propositions are logically relevant if either entails the other. Hence, the claim that relevance is a necessary condition of entailment is trivial and empty. But this objection misses the point of the definition. It was conceded that the notion of relevance has no prior role to play in picking out true entailments. What its definition does is highlight what is distinctive of the relevant account of entailment, namely, the important role played by fusion in its proper analysis. Fusion binds two propositions together in such a way that one is assured of their mutual relevance in deriving any consequence from the fused conjunction, as in the Deduction Theorem or in the Relevant Account of Validity. Whatever '$A \times B$' entails needs both A and B in its derivation. Hence, the new definition has these three features:

1. it characterises a logical relation between two propositions with the correct formal properties;

2. it provides an appropriate sense of derivational utility; and

3. it respects the intuition that a contradiction is not relevant to every proposition.

But even if an arbitrary contradiction is not relevant, on this account, to any other proposition whatever, might there not still be some proposition with this feature, and would that not also be counterintuitive? Indeed, *pace* Ackermann, there is such a proposition: the Church constant 'F' obeys the principle

$$\mathsf{F} \vdash A$$

for all A, and similarly

$$A \vdash \mathsf{T}.$$[31]

Hence, 'T' and 'F' are logically relevant to every proposition. But this is unsurprising. For 'F' has the formal properties of an (extensional) conjunction of all propositions, and we have admitted that A is relevant to 'A & B' for any A. Similarly, 'T' behaves like the (extensional) disjunction of all propositions, and so its universal relevance accords with the intuition that, for any A, A is relevant to '$A \vee B$'.

[31] See Corollary 4.15.

6.7 Conclusion

The main of point of this chapter has been to explain why relevant logic is so called, and so to characterise a logically appropriate notion of relevance. That characterisation uses the intensional notion of conjunction (fusion) which, we discovered in chapter 3, was necessary correctly to state a relevantly acceptable necessary and sufficient condition for validity. Two propositions are logically relevant if the fusion of one with the contradictory of the other cannot both be true. Substituting extensional conjunction for fusion here (i.e. reading 'and' extensionally), there is nothing from which the classical logician could wish to dissent. So the difference centres on the characterisation of fusion, distinct from '&'. One way that can be done is by tying it to implication. In **R**, '$A \times B$' is equivalent to '$\sim(A \to \sim B)$'. But that is not so in systems without C; and is unhelpfully circular. The better solution is to characterise fusion, as we have done, relative to an algebraic operation on premises, of premise-combination, satisfying certain plausible structural conditions.

Chapter 7

Logic on the Scottish Plan

In recent years, two approaches to the semantics of relevant logic have been affectionately dubbed the Australian and the American plans.[1] Their authors are anxious to point out that these are not in themselves semantics, but recipes for particular way of doing semantics.[2] The semantics of chapter 5 above was constructed on the Australian plan. In this chapter I intend to show that both the American and the Australian plans are flawed, based on the same fundamental mistake and misconception. I wish to urge a radically different conception of relevant semantics. Without being excessively nationalistic, it seemed appropriate to christen this approach, 'Logic on the Scottish plan'. The Scottish plan could also be termed, but perhaps here I am being more than a little presumptuous, 'Logic on the relevant plan', or perhaps, 'on the relevantistic plan'. For it takes seriously the suggestion by Belnap and Dunn, that relevant logic is the One True Logic,[3] and so should underpin its own semantics as much as any other inferential behaviour.

7.1 Geach's Challenge

In his review of the first volume of Anderson and Belnap's *Entailment*, Geach derided the claim that certain classically valid inferences, for exam-

[1] R.K. Meyer & E.P. Martin, 'Logic on the Australian Plan', *Journal of Philosophical Logic*, 15 (1986), pp. 305-32.

[2] R.K. Meyer, *A Boolean-Valued Semantics for* **R**, Australian National University Research School of Social Sciences Logic Group Research Paper no. 4 (Canberra, 1979), p. 4.

[3] See N. Belnap & J.M. Dunn, 'Entailment and Disjunctive Syllogism', in *Contemporary Philosophy: A New Survey*, vol. 1, ed. G. Floistad & G.H. von Wright (The Hague, 1981), pp. 337-66.

ple, modus tollendo ponens , or Disjunctive Syllogism (DS), $A \vee B$, $\sim A \vdash B$, were invalid, on the ground that not only had no counterexample to their validity been offered, but that it was clear that none could be forthcoming: 'It would be difficult to describe even a possible set-up in which ... "$A \vee B$" should be true but both its disjuncts false.'[4] If A and B are both false, then '$A \vee B$' is false too. So contraposing, if '$A \vee B$' is true, at least one of A, B is true. Hence if one, say A, is false, the other, B, must be true. 'And,' one can hear Geach saying, 'there's an end on't.'[5]

It is unfortunately not uncommon for heterodox logic to be dismissed in such a cavalier fashion. Nonetheless, it makes (unwittingly, I fear) a good point. For it focuses our attention on the important issue, the semantics of consequence. Relevant logicians may come along with all sorts of formal calculi—Hilbert-style axiomatic systems, Fitch-style natural deduction formulations, Gentzen-style consecution (or sequent) calculi, the bunch formulations of chapter 4 above and all sorts of informal connections between them and natural language (that '$\sim$' captures 'not', etc.); but what it all comes down to is this: is Disjunctive Syllogism (or detachment for '$\supset$') valid? Since they do not include the inference in their systems, relevant logicians presumably think it invalid. But does that not mean they think it possible for its premises to be true and its conclusion false? Surely they cannot.

7.2 The Dialetheic Response

The relevant logician seems to be caught in a bind. But there are many ways of wriggling. Wriggling, however, is not the right thing to do, as we will see. I shall cut the Gordian knot in good time.

What certainly will not do is in any way to deny that 'or', as it occurs in Geach's challenge, is truth-functional. Of course, if 'or' were not truth-functional, that is, the value of 'A or B' was not completely determined by the values of A and B, there would be a simple solution. For it is only the fact that vel is that truth-function which is true if and only if one of its disjuncts is true which compels '$A \vee B$' to be false when A and B are both false. So if 'or' were here non-truth-functional, it would be quite feasible for A and B to be both false and yet 'A or B' not false, but true. For

[4] A.R. Anderson & N.D. Belnap, *Entailment*, vol. I (Princeton, 1975), reviewed by P.T. Geach in *Philosophy*, 52 (1977), p. 495.

[5] Not literally; you may recognise the words as Dr Johnson's, refusing to hear Boswell's pleas on behalf of Lady Diana Beauclerk: 'the woman's a whore, and there's an end on't.' J. Boswell, *Life of Johnson*, ed. G.B. Hill (Oxford, 1934), entry for 7 May 1773, vol. II, pp. 246-7.

example, suppose—counterfactually—that 'A or B' was true if and only if it was merely possible for one of A, B to be true. Then 'A or B' would, under suitable circumstances, be true; so too would 'not-A' (since A was supposed, under those circumstances, to be false); but B would be false, and we would have the desired counterexample to Disjunctive Syllogism.

But this is irrelevant. Even if 'or' has a non-truth-functional sense, that would be ignoratio elenchi. For the starting point is the relevant logician's denial that Disjunctive Syllogism, or detachment for '$\supset$', is in the classical logician's sense of '$\vee$' and '$\supset$', valid. It would show nothing as far as the dispute between relevantist and classicist is concerned to exhibit a non-truth-functional sense of 'or', or of '$\supset$', in which the inference was invalid. What is needed is for the relevant logician to defend his claim that the classical logician is wrong about the validity of good old-fashioned familiar extensional Disjunctive Syllogism. Geach's challenge that he cannot do that still stands.

More to the point is the following idea. Suppose 'A or B' were true, and both A and B were false. Then, since 'or' is (as concerns us here) truth-functional, at least one of A and B would be true. Suppose it is A. Then A would be both true and false.

This is the dialetheic response. The dialetheist believes (as his name should reveal) that some propositions are both true and false. Suppose A is both true and false, and B is simply false. Then 'A or B' is also both true and false—true because A is true, false because both A and B are false. Moreover, since A is false, 'not-A' is true. (It's also false, since A is true.) So both 'A or B' and 'not-A' are true, while B is false. Thus a counterexample to Disjunctive Syllogism is possible after all, keeping 'or' strictly truth-functional. All that is needed is to concede that some proposition, A, can be both true and false.[6]

Of course, Geach did not conceive such a response. There are at least two reasons why he should not. First, because, I am sure he believed, no propositions are both true and false. Secondly, and more importantly, because it is no part of the relevantist challenge to classical logic, and to the validity of Disjunctive Syllogism, to claim that some propositions are both true and false. There may be some people who believe both that the relevantist challenge is correct and that there are propositions with two values.[7] There are others who believe one but not the other,[8] and there are some

[6]See, e.g., F. R. Routley et al., *Relevant Logics and their Rivals* (Atascadero, 1982), p. 25.

[7]E.g., ibid., §1.7.

[8]I believe the relevantist challenge is correct, but that propositions are single-valued.

who believe neither (Geach, for example). Of course, if one does believe both, the existence of double-valued propositions provides a very neat counterexample. Even then, one needs independent evidence for the existence of propositions which are both true and false. It will not do simply to say they are needed as counterexamples to DS. Such propositions, a dialetheist might say, are plentifully supplied by, for example, the many and varied semantic paradoxes.

Nonetheless, I think it has muddied the waters greatly in respect of proselytising the relevant challenge, for the dialetheist counterexample to be used. For it has made it less clear what that challenge is, and it has laid the relevantist open to irrelevant classical accusations and objections to positions to which, qua relevantist, he is not committed.

7.3 The Australian Plan

The dialetheist idea lies at the heart of the worlds semantics for relevant logic which we examined in chapter 5. That semantics was modelled on the Australian plan, a way of understanding the semantics of propositions which does indeed provide a counterexample to Disjunctive Syllogism, a response to Geach's challenge.

The reader will recall the method from §5.2. To provide this counterexample, there must be some situation, not necessarily actual, in which 'A or B' and 'not-A' are true and B false. If 'A or B' is to be true, one of A or B must be true. We want B to be false, so it had best be A which is true. But we also want 'not-A' to be true. So both A and 'not-A' must be true.

Can 'not-A' be true without A being false? Yes, comes the reply. The trick is pulled by use of the so-called 'star-operation'. Indeed, recognition of the star-operation was the original insight which led to the first plausible semantics for relevant logic. The usual semantics for 'not' is that 'not-A' is true if (and only if) A is not true. Our fairly straightforward dialetheic semantics—the one we looked at in §7.2—retains this truth-condition. In consequence, the contradiction we require (that both A and 'not-A' be true) permeates to the metalanguage: that A be both true and not true (in particular, false). In §7.2, A was both true and false, the latter being at least one way for A not to be true.

The present plan wishes to keep the metalanguage consistent. The solu-

Graham Priest appears to believe that there are many true contradictions, but that the relevantist challenge is incorrect, at least in detail—see his *In Contradiction* (The Hague, 1987), §6.6.

tion is to abandon the straightforward truth-condition for 'not'. The idea is to let the value of 'not-A' depend not simply on the truth-value of A, but on something more detailed—on some less gross property of A. The traditional way to do this has been to index the truth-values of A, and we can readily do that here, since we are already considering various alternative situations in looking for a (perhaps not actual) situation to provide a counterexample to DS.

What we require, then, is some situation, call it x, where A is true, and another situation, y, the value of A in which makes 'not-A' true at x. (In special cases, x might be identical with y. But in general we need to allow them to be different.) The star-operation makes this arrangement systematic. Let 'not-A' be true at x if and only if A is false at $y = x^*$. Then if certain familiar conditions are laid down on the star-operation, for example, functionality ($x = y \Rightarrow x^* = y^*$), and that $x^{**} = x$, the semantics can be shaped to yield desirable logical laws.

The upshot is that a counterexample to Disjunctive Syllogism can now be fashioned in which it is no longer required that in the situation in question A be both true and not true, but that in one situation A be true and in its star-mate A be false. Consequently, 'not-A' will be true at the first, and so too will 'A or B' (whose truth-condition is unaffected by star), while B may be arbitrarily selected to be false. Geach's challenge has, it seems, been met.

7.4 Truth-functionality

Nonetheless, this all sounds like making (what is hoped is) a virtue out of (what is thought to be) a necessity. First, let us see why it is not a virtue. We can then proceed to question its necessity. Is there such a thing as a truth-functional connective? That may seem a surprising question. Of course there are truth-functional connectives: $\vee, \&, \supset$ are just three. One reason we say this is that if someone were to come along with a theory of '&', say, which was not truth-functional, we would reply along roughly these lines: that may or may not be a very interesting theory about something—'and', say—but, since it is not truth-functional, it is not a theory of '&'. '&' is by definition truth-functional.

I have two worries about this answer. The first is that however the referent of '&' may be fixed—by talk of truth-functions, for example—there seems just as good sense in asking of that connective whether it might really not be truth-functional, as it makes sense to ask whether Aristotle—whom we pick out as the chap who wrote the *Metaphysics*—might not actually have written that book. We will pursue that question in chapter 8. My second worry is this. It prompted my original question. It is not that I wonder whether certain connectives do or do not have the appropriate properties to count as truth-functional. What I doubt is whether any single clear sense makes such a decision possible. The basic idea is, of course, as we used it in §7.2 above, that the main connective of a compound wff is truth-functional if its truth-value is completely determined by the truth-values of its immediate constituents, and of them alone. The puzzle is that this presupposes that the logic, and the connective, is presented semantically. However, many formal systems appear, at least initially, as purely syntactic devices; and the natural language which they may formalise by informal correlations does not come with a single formal semantics ready specified. So although there may be a semantics for a system in which a connective is deterministically assigned such a truth-function, there may be another for the calculus in which it has a different treatment.

Consider, for example, the following semantics for a propositional logic **C** containing as primitives '$\sim$' and '$\supset$'. A model structure for **C** is a non-empty set W of indices. A valuation for **C** on a model structure W consists of a total function $v : \mathbf{C} \times W \to 2$ such that

$$v(\sim A, w) = \begin{cases} 1 & \text{if for some } w' \in W, v(A, w') = 0 \\ 0 & \text{otherwise;} \end{cases}$$

$$v(A \supset B, w) = \begin{cases} 1 & \text{if either for some } w' \in W, v(A, w') = 0, \\ & \quad \text{or for all } w' \in W, v(B, w') = 1 \\ 0 & \text{otherwise.} \end{cases}$$

A wff A of **C** is valid if for every model structure W for **C**, $v(A, w) = 1$ for all $w \in W$. Neither '$\sim$' nor '$\supset$' is here treated truth-functionally. The value of '$\sim A$' and '$A \supset B$' (at an index) depends on the values of A (and B) at other indices. Nonetheless, those who recognise Gödel's translation of **C** into the semantics of **S5** will realise that A is valid in **C** if and only if A is classically valid.[9]

One might, therefore, try to capture the notion of truth-functionality, as we did in §2.3, purely syntactically. To do so one has to find some syntactic analogue of truth-value. For example, the following inferences could be taken to capture the idea that material implication, '$\supset$', is truth-functional:

[9]K. Gödel, 'An Interpretation of the Intuitionistic Sentential Logic', reprinted in *The Philosophy of Mathematics*, ed. J. Hintikka (Oxford, 1969), pp. 128-9.

$Q \vdash P \supset Q$
$\sim P \vdash P \supset Q$
$P, \sim Q \vdash \sim(P \supset Q)$

In other words, if Q is true or P is false, '$P \supset Q$' is true; while if P is true and Q false, '$P \supset Q$' is false.

However, this builds in implicitly that negation is truth-functional. For we are assuming that 'not-P' is true if and only if P is false. Is there no way out? Here is one suggestion. What we might do is first characterise the relative notion 'truth-functional with respect to a certain semantics', and then if required say that a connective is truth-functional *simpliciter* if it is truth-functional with respect to some semantics. Accordingly, '$\vee$', '&' and '$\supset$' are truth-functional *simpliciter* in virtue of the traditional Boolean semantics.

The Australian semantics for relevant logic certainly treats 'not' as non-truth-functional. For the value of 'not-A' at one index, x, depends on the value of A at another index (its star-relative, x^*), and not at x alone. So it does not depend simply on the truth-value of A. It does not follow that 'not' is non-truth-functional *simpliciter* in relevant logic. We will see in the next section that there is a semantics which does treat it truth-functionally.

7.5 The American Objection

The Australian plan has two significant features. First, relativising the truth-value of a wff to an index, it retains the dialetheic idea that there may be true contradictions, that both A and 'not-A' may be true at the same index. Of course, in doing so it has dropped the other dialetheic idea, that a wff may have more than one value. Secondly, the Australian plan retains the classical, or traditional, view that there are only two values, true and false. Relative to an index, a wff has one, and only one, of two possible truth-values.

Indices differ radically, therefore, from the familiar notion of a possible world. According to Plantinga, for example, possible worlds may be given an informal elucidation and justification as maximal consistent states of affairs, as situations which are exhaustively descriptive and yet, by being consistent, are possible.[10] Indices on the Australian plan may be both inconsistent and incomplete. In other words, they may be neither possible nor worlds. As we have seen, both A and 'not-A' may be true at some index, x. Correspond-

[10] A. Plantinga, *The Nature of Necessity* (Oxford, 1974), ch. 4.

ingly, thinking how the star-operation works, neither A nor 'not-A' may be true at another index—in particular, at x^*.

According to what has become known as the American plan, the fact that even with the star-semantics, a proposition and its negation may both be false (or both true) shows that the complication of star is unnecessary. For we can avoid the lack of functionality in simple-minded dialetheism more easily. Why not simply admit that propositions may be true, false, both true and false or, denying Bivalence, neither? Functionality in the assignment relation may be preserved by interpreting it as the assignment of one of four values, rather than the over- or under-assignment of two. The four values are affectionately known as True, False, Both and Neither. The four values are the elements of the power set of the two usual values.

How does the American plan provide the required counterexample to DS? Once again, what is needed is a model in which the premises, 'A or B' and 'not-A' are true and B false. Let B be False (that is, false only), while A is Both. Then both 'A or B' and 'not-A' take the value Both, and so DS is not truth-preserving. It is crucial here to observe the distinction between True (big True, i.e. (little) true only) and true (little true). The recursion on the connectives is governed by the normal rules for (little) truth and falsity. 'not-A' is (at least) (little) true if and only if A is (at least) (little) false. Hence 'not-A' is (big) True if and only if A is (big) False, and vice versa. 'Not-A' is Both if and only if A is Both, since 'not-A' is (little) true if and only if A is (little) false and (little) false if and only if A is (little) true. Finally, 'not-A' is Neither if and only if A is Neither, since 'not-A' is not true or false if A is not respectively false or true. Thus:

	A			$\sim A$	
t		f	t		f
1	T	0	0	F	1
1	B	1	1	B	1
0	N	0	0	N	0
0	F	1	1	T	0

Thus truth-preservation, required for validity, is just that, (little) truth-preservation, and not (big) Truth-preservation. So A follows from some hypotheses if and only if A is neither False nor Neither when all the hypotheses are either True or Both.

The dialetheic account relies on the counterintuitive idea that some propositions are both true and false. The Australian plan interprets this as saying that a proposition and its negation might both be true; and the

American plan, as saying that Bivalence fails, and there are four possible semantic values. They thus give to relevant logic a semantic strangeness of which neither its original philosophical motivation nor its syntactic formulation (in particular, its natural deduction presentation) would give any hint. Its original philosophical motivation was to construct a formal model of reasoning which lacks the spread principle, A, $\sim A \vdash B$, or *ex falso quodlibet*, and in which simple and absolute consistency are distinct. This unfortunate connection between relevant ideals and dialetheism can be traced in each case to another feature which the three semantic plans mistakenly share: the Classical Account of Validity.

7.6 Classical Relevant Logic

There is an explanation. Bob Meyer has recently observed that the development of this semantics was a response to criticism from classical logicians that the relevant systems lacked a semantics, and so in constructing the semantics, its authors firmly set out 'to preach to the Gentiles in their own tongue'.[11] Accordingly, its criterion for validity, or logical consequence, was quite consciously the classical one, in which the conjunction of true premises and false conclusion, whose impossibility entails validity, was firmly extensional.[12]

It is important, therefore, to distinguish the Standard account of validity from the Classical Account of Validity, in which the conjunction is taken unequivocally to be extensional; similarly, to distinguish the Standard Deduction Equivalence, connecting validity and the conditional, from the Classical Deduction Equivalence, in which both the conjunction and the conditional are extensional. There is something right in the standard accounts. But the Classical Accounts are wrong, for they have false consequences. Indeed, the same countermodel will refute the Classical Deduction Equivalence as refutes EFQ and its contrapositive.

Once the Australian plan, with its non-homophonic truth-condition for 'not', was up and running, Bob Meyer and Richard Sylvan had an intriguing thought. Let us add a new connective '$\neg$' to relevant logic, initially charac-

[11] R.K. Meyer, *Proving Semantical Completeness 'Relevantly' for* **R**, Australian National University Research School of Social Sciences Logic Group Research Paper no. 23 (Canberra, 1985), p. 1. See also F.R. Routley, 'Relevantism, Material Detachment, and the Disjunctive Syllogism Argument', *Canadian Journal of Philosophy*, 14 (1984), p. 175.

[12] See, e.g., R. & V. Routley, 'The Semantics of First Degree Entailment', *Nous*, 6 (1972), p. 336: 'only when ... the antecedent of [an] implication belongs and the consequent does not belong to the relevant set-up—is the implication false'.

terising it semantically within the Australian plan semantics by the directly truth-functional condition that '$\neg A$' is true at x iff A is not true (i.e., false) at x.[13] Recall that the whole point of the Australian treatment was that '$\sim A$' is true at x iff A is not true at some index x^* possibly different from x. They called the new connective 'Boolean negation', distinct from so-called De Morgan negation, '$\sim$'. Substituting '$\neg$' for '$\sim$' in DS and EFQ, the resulting theses were then found to be valid:

$$A \vee B, \neg A \vdash B$$
$$A \mathbin{\&} \neg A \vdash B.$$

What should we conclude from this? One might be led to claim that Boolean negation is the truly truth-functional (or 'extensional') connective, and '$\sim$' belongs more closely with '$\rightarrow$' and '$\times$' as a non-truth-functional connective.[14] But this is a mistake. Our previous reflections should show this. One semantics treats '$\neg$' truth-functionally and '$\sim$' not—the value of '$\sim$ A' at x depends on its value at x^*. But as we noted in §7.4, and as Meyer observed[15] this gives one no reason to say that '$\sim$' is non-truth-functional *simplciter*. Meyer goes further, to claim that the notion of truth-functionality has no meaning at all independent of a particular semantics.

There can be no surprise that the homophonic truth-condition, coupled to the Classical Account of Validity, validates DS and EFQ. Certainly Boolean negation ($\neg$) is truth-functional, and De Morgan negation ($\sim$) is non-truth-functional, with respect to the Australian semantics. But this does not show that De Morgan negation is non-truth-functional *simpliciter*. Another semantics may treat De Morgan negation truth-functionally. It will not incorporate the Classical Account of Validity.

7.7 Local Consistency

A certain sort of eclecticism has of late become common among relevant logicians, namely, to claim that DS and EFQ are not invalid tout court, but have a restricted validity, provisional on local consistency.[16] The idea is this: the counterexample to these two principles relies on finding instances

[13] R.K. Meyer and F.R. Routley, 'Classical Relevant Logics' I and II, *Studia Logica*, 32 (1973), pp. 51-68; 33 (1974), pp. 183-94.

[14] Meyer, *A Boolean-Valued Semantics for* **R**, p. 98.

[15] Ibid., pp. 102-3

[16] See, e.g., Routley, 'Relevantism, ...' (see note 11), p. 176; C. Mortensen, 'The Validity of Disjunctive Syllogism is not so easily proved', *Notre Dame Journal of Formal Logic*, 24 (1983), pp. 35-40.

in which, on the American plan, one of the propositions in question is Both (true and false), or, on the Australian version, one proposition and its contradictory are both true. Hence, if it can be shown that this situation does not pertain, that is, that the context or theory in question is consistent, it is justifiable to use DS and EFQ. DS and EFQ are valid principles in consistent situations.

As I have noted, if there were such counterexamples, they would indeed provide excellent reason for dismissing DS and EFQ as invalid. However, the reasons for giving a formal theory containing a non-truth-functional conditional, namely, that the Classical Account of Validity and the Classical Deduction Equivalence entail the identification of 'if' with material implication and so validate invalid arguments, give no reason to suppose there are such contradictory propositions. This should make one suspect that some sleight of hand has gone on the above reasoning towards eclecticism. For even if the known counterexamples have some quality ϕ, that is no reason to infer that in the absence of ϕ the argument is reliable. For there may be, yet unknown, some other quality ψ which is equally damning. In fact, in a nutshell, there are no propositions with quality ϕ (viz self-contradictory); but there are propositions with quality ψ—to be explained below in the final section: in brief, irrelevant propositions.

Thus local consistency, and the idea that DS and EFQ are at least valid in such a situation, is another myth consequent on the implication that the semantics of relevant logic shows that we must suppose there to be, at least in principle, even if not in the real world itself, self-contradictory propositions. The distinctive element of the Scottish plan is, therefore, to adopt the Relevant Account of Validity, and provide a semantics for relevant logic on relevant principles. It will not then be necessary to depart from (relevantly interpreted but) homophonic truth-conditions.

7.8 The Scottish Plan

Let us then return to the problem with which we began, Geach's challenge. Iseminger noted that Geach was presupposing the correctness of the Classical Account. 'To insist that the showing of invalidity (or "formal incorrectness") requires the production of actual (or possible) counter-examples is just to assume the classical account of validity (or equivalence of validity and "formal correctness") which is in question.'[17] If 'and' is interpreted truth-functionally, and not intensionally, it is indeed impossible for, say, A

[17] G. Iseminger, 'Is Relevance Necessary for Validity?', *Mind*, 89 (1980), p. 198.

and '$A \supset B$' to be true and B false. Hence, on the Classical Account, it follows that A and '$A \supset B$' entail B. But the Classical Account is wrong. That it is impossible for A and '$A \supset B$' to be true and (&) B false does not show that A and '$A \supset B$' entail B.

If Geach insists that he meant 'and' extensionally (that is, in such a way that the falsity of its conjuncts suffices for the falsity of the conjunction) then we can, nay must, agree that no counterexample is possible. But we must also deny that that shows that DS is valid. This move would be as irrelevant to the validity of DS as the earlier consideration (in §7.2) of intensional disjunction for its major premise. The impossibility of true premises and (&) false conclusion does not suffice for validity. What is needed is the impossibility of true premises fuse false conclusion.

How then can we show the invalidity of DS and EFQ? It follows from our considerations that we cannot do it merely by instancing the truth-values of their component propositions, A, B, '$\sim A$', etc. What we must show is that, for example, '$A \ \& \ \sim A$' can be true fuse B false, not simply their extensional conjunction. Fusion is not truth-functional, in particular, the fusion of two propositions may be true even though both are false. Certainly, to repeat the point again, it would suffice to instance circumstances in which each component was true, since, at least in the strong relevant logic **R**, the truth of each component entails the truth of their fusion. But '$A \ \& \ \sim A$' is false, whatever A may be, as even the dialetheist will admit; and we require B to be false too. So both components of the counterexemplary fusion to EFQ are false. That does not preclude the truth of their intensional combination.

The answer to Geach's challenge is, therefore, to instance A, B such that, for example, '$(A \vee B) \ \& \ \sim A$' fuse '$\sim B$' is true, that is, for which B does not follow from '$A \vee B$' and '$\sim A$'.

How do we know that there are A, B for which this is true? We might similarly ask, of the dialetheist, Australian or American plans, how we know that there are A, B for which '$A \ \& \ \sim A$' is true and B false. Geach, of course, would refuse to go any further: we know, he would say, that '$A \ \& \ \sim A$' is never true. The advocate of the dialetheic plan can only reply by instancing suitable A: for example, 'This sentence is false' or 'The Russell set belongs to itself'.

Similarly, on the Scottish plan, we instance A, B for which '$(A \vee B) \ \& \ \sim A$' fuse '$\sim B$' are true, that is, for which '$((A \vee B) \ \& \ \sim A) \times \sim B$' is true—in other words, for which B does not follow from '$A \vee B$' and '$\sim A$'. For example, let A be 'Socrates was a man' and B be 'Socrates was a stone' (cf. §2.6 above). It follows from the fact that Socrates was a man that Socrates was a man or a stone. So '$A \vee B$' is true. But '$(A \vee B) \ \& \ \sim A$' is not true, since '$\sim A$' is false. Nonetheless, '$((A \vee B)\ \&\ \sim A) \times \sim B$' is true for this interpretation of A and B, even though '$((A \vee B)\ \&\ \sim A)\ \&\ \sim B$' is false, since '$(A \vee B)\ \&\ \sim A$' is false (though '$\sim B$' is true). That (extensional, truth-functional) conjunction is a stronger statement than the corresponding fusion '$((A \vee B)\ \&\ \sim A) \times \sim B$', so it is false even though the weaker "fusion"-statement is true. Why is the "fusion"-statement true? Because Socrates being a man or a stone and not a man is consistent with him not being a stone. That is, from Socrates being a man or a stone and not being a man it doesn't follow that he is a stone. Hence Geach's challenge is blocked. But note that this is not a proof that B does not follow from '$A \vee B$' and '$\sim A$', nor from A and its negation, but an explanation of why Geach's challenge is misguided, given that the Classical Account of Validity (see §6.5) should be replaced with the Relevant Account.[18]

[18]This final paragraph replaces the original one, in response to the observation (personal communication) by Dr Indrek Lõbus that the response by the Scottish Plan to Geach's challenge had not been properly spelled out.

Chapter 8

Deviant Logic and the Sense of the Connectives

We noted in the last chapter that the development of the worlds semantics for relevant logic was a response to criticism from classical logicians that the relevant systems lacked a proper semantics. So it firmly set out 'to preach to the Gentiles in their own tongue' (see §7.6). Accordingly, its criterion for validity, or logical consequence, was quite consciously the classical one, in which the conjunction of true premises and false conclusion, whose impossibility entails validity, was firmly extensional.

8.1 The Deviant's Predicament

It followed by the reasoning given in chapter 7 above that the semantics could not be homophonic. For given the connection in the Deduction Equivalence between the conditional and consequence, and that the conditional is not material, a contradiction must not entail an arbitrary proposition. Hence, it must be possible for a contradiction to be true and that arbitrary proposition false. In particular, since classical consequence requires taking that conjunction extensionally, it must be possible for a contradiction to be true, and so for each of the conjuncts of that contradiction, namely, a proposition and its contradictory, to be true together. On pain of making the value assignment not functional (that is, the contradiction's permeating to the metalanguage), the idea that the contradictory of a proposition is true if and only if that proposition is not true had to be rejected. Instead, as we saw, resort was had to the standard manner of resolving an apparent inconsistency, ambiguity. Truth cannot be univocal, but must be relative to an index. The contradictory of a proposition is true at an index if and only if that proposition is false at some other index. A similar index-relative truth-condition for the conditional was given. By placing suitable constraints on the interrelation of these indices, we gave soundness and completeness proofs for **R** and other logics in chapter 5.

Ironically, but not unnaturally, the presentation of this semantics brought the criticism down on its original authors that either the semantics was entirely formal, and of no philosophical import, or it showed that relevant logic was after all no rival to classical logic, since the sense of negation (and of the conditional) in relevant logic was shown by the semantics to differ from the classical one.[1] Or in Quine's famous and artful words: 'Here ... is the deviant logician's predicament: when he tries to deny the doctrine he only changes the subject.'[2]

The charge was ironic, since the very attempt to speak in the classicist's tongue elicited the criticism of having changed the meaning of the terms. There was, it was said, no real disagreement about whether an arbitrary proposition follows from a contradiction, since the relevantist does not deny that it follows from a classical contradiction, only from his relevant contradiction, whose sense is given by his queer truth-condition for negation. As Copeland put it: 'The failure of the formula $A \vee B \ \& \ {\sim}A \rightarrow B$ in a semantics which gives a non-classical meaning to any of the connectives $\&, \vee, \sim$ will have no bearing on the key question of whether ... the disjunctive syllogism does or does not commit a fallacy of relevance.'[3] Copeland spends much time railling at the semantics of relevant logic for giving non-classical meanings to the connectives. However, his final claim appears to be that the question of what sense is actually given by the famous star-condition is unclear, and that the fault with the semantics is its being empty, purely formal, and yielding no philosophical illumination.

That charge is only apposite if the relevant logician tries to work within the classical framework. But the whole point of the relevant venture is that that framework is misconceived. The conditional is not material, and entailment, or validity, is not the modalisation of the material conditional. As we saw in the last two chapters, the proper relevant account of validity is that an argument is valid if and only if it is impossible for the premises to be true fuse the conclusion false, that is, if and only if the fusion of its

[1]B.J. Copeland, 'On When a Semantics is not a Semantics', *Journal of Philosophical Logic*, 8 (1979), pp. 399-413; 'Pure Semantics and Applied Semantics', *Topoi*, 2 (1983), pp. 197-204.

[2]W.V. Quine, *Philosophy of Logic* (Englewood Cliffs, 1970), p. 81.

[3]Copeland, in *Journal of Philosophical Logic*, 8, p. 403.

premises and the negation of its conclusion cannot be true. In fact, in this dispute, two questions have been conflated:

1. Is there disagreement over the sense of the word 'not'?
2. What principles should negation satisfy?—e.g., is it possible for both A and 'not-A' to be true together?

These questions can become confused. For if we build into the identification of a connective what its logical properties are, disagreement will be impossible, as Quine realised. He correctly observed that one can disagree about whether one proposition entails another only if one agrees about the identity of the two propositions.[4] His mistake was to add to this observation a claim that identification relies on the presence of a certain set of properties.

8.2 Rigid Designation

It is often thought that the reference of a name is determined by some properties or set of properties. For example, Russell suggested that the name 'Homer' necessarily refers to whomever wrote the Iliad and the Odyssey.[5] John Searle, and others, noted that if the reference of, e.g., 'Cicero', were determined as, say, whoever denounced Catiline, then the proposition

> Cicero might not have denounced Catiline

would be self-contradictory. Surely whoever denounced Catiline denounced Catiline. The supposition was, therefore, made that it is not a single property, or even any (large) group of properties which determines the reference of a name, but rather some weighted cluster of properties.[6] No one property is essential to the reference, so that the reference of 'Cicero' in the above proposition is determined not by his denouncing Catiline, nor by a single group of properties, but by a sufficient number of sufficiently important properties—writing certain books, being consul in a particular year, and so on. This fixes the reference, relative to which one or more of those properties, such as that of denouncing Catiline, can vary.

Saul Kripke criticised such accounts of the reference of names.[7] The behaviour of names in modal contexts such as that of 'Cicero' in the above

[4] Quine, *Philosophy of Logic*, p. 82.
[5] B. Russell, *Introduction to Mathematical Philosophy* (London, 1919), ch. 16, p. 179.
[6] J.Searle, 'Proper Names', *Mind*, 67 (1958), pp. 166-73.
[7] S.Kripke, *Naming and Necessity* (Oxford, 1980).

is not a peripheral problem: such behaviour is crucial to understanding the notion of naming, and of seeing how name-reference is different from the relation between other types of expression and what they are true of. (Unfortunately, Kripke uses 'refer' for both relations.) It is distinctive of names that in modal contexts they keep their reference fixed—they there refer to the same object as they refer to in non-modal contexts. In this respect they contrast, he said, with definite descriptions, whose reference is non-rigid. Definite descriptions denote, in any context, whatever has the property encapsulated in them. Unless this property is essential, and is true of the same object in any situation, the reference of a definite description varies from one situation to another. Descriptions are in general non-rigid; in special cases where the object does in fact remain the same, they are *de facto* rigid. But names are rigid *de jure*. Rigidity is essential to their being names.

Kripke is wrong about descriptions, as Donnellan recognised.[8] Descriptions can be used rigidly or non-rigidly. Indeed, Kripke made the very distinction necessary to appreciate this point.[9] Logicians may indeed propose theories of designation in which certain types of expression are, by definition, designatory of whatever satisfies them. But they are wrong about what those descriptions in fact refer to, in any everyday sense of 'refer', and they accordingly assign propositions containing those expressions the wrong truth-conditions. Such propositions are, in the abstract and out of context, ambiguous. The context may serve to disambiguate them, and show whether the description is used rigidly or non-rigidly, and so what are the proposition's truth-conditions.

What is important for our purposes, however, is Kripke's notion of rigidity, and so his discussion of names. Here too, he is wrong to claim that rigidity and scope are two separate issues.[10] His argument is that questions of rigidity apply to simple sentences such as

Aristotle was fond of dogs,

whereas questions of scope cannot, since there is no modal or other context to which such distinctions can apply, the propositions being by definition simple ones.

But the whole point about rigidity is that rigid designators are identified by their behaviour in modal contexts: we 'call something a *rigid designator*

[8]K. Donnellan, 'Reference and Definite Descriptions', *Philosophical Review*, 75 (1966), pp. 281-304.

[9]Kripke, *Naming and Necessity*, p. 4.

[10]Ibid., pp. 10ff.

if in every possible world it designates the same object'.[11] We can only find out whether that is the case by considering the truth-conditions of modal propositions containing the term. We have no other epistemic access to possible worlds. Kripke concedes that rigidity is equivalent to the thesis that wide and narrow readings of modalisations of simple sentences are equivalent.[12] What he in fact criticises is the cogency of the particular theses about scope which were claimed equivalent to his own criterion, supposedly not a matter of scope. Once the distinction of scope is correctly stated, it is seen to be precisely one of scope: it turns on the behaviour of names in contexts where scope distinctions are usually important. What is distinctive of rigid designators is that they do not admit scope distinctions in positions in which other expressions, e.g., descriptions, do (or, following our earlier remarks about the possibility of ambiguity, may).

Kripke's idea, then, is that it is mistaken to suppose that 'Aristotle' refers to whomever had various properties—teacher of Alexander, born at Stagira, author of the *Organon*, etc. If so, it would be an analytic truth that Aristotle taught Alexander. But it is not. Hence 'Aristotle' cannot obtain its meaning by connoting certain properties. It is a rigid designator, whose reference remains fixed in contexts such as

> Aristotle might not have taught Alexander.

That reference may indeed have been originally fixed by use of certain descriptions. But once its use is fixed, one can proceed to question whether that individual really did satisfy the description.

8.3 Rigid Connectives

Kripke's primary thesis is that the proper names of ordinary language are rigid designators. But are these the only rigid designators? In particular, is it possible to conceive of the logical connectives as rigid, or as non-rigid, designators? Quine claims that once one supposes that some propositions of the form 'A & $\sim A$' are true, and that they do not entail all other propositions, the symbol '$\sim$' has 'ceased to be recognizable as negation'.[13] But even if these were the features used to fix its reference, we could still ask whether that connective, dealt with in this way in classical logic, does indeed have these properties. Moreover, it is questionable at least whether the

[11] Ibid., p. 48.

[12] Ibid., p. 12 n. 15.

[13] Quine, *Philosophy of Logic*, p. 81.

property of forming propositions which entail all others is central to picking our negation. What marks a negation connective is that, attached to one proposition, it forms another proposition which is false if the first is true, and true if the first is false. That is consistent with their both being true, if one is willing to accept that some propositions are both true and false. But relevantism is not committed to dialetheism (as we saw in chapter 7).

Colin McGinn explored the issue, whether expressions other than names and descriptions might be rigid designators.[14] He extracted from Kripke as a test of whether an expression t is a rigid designator, whether the proposition

t might not have been t

is false.[15] If so, t is rigid; if not, t is non-rigid (McGinn calls such terms 'flexible'). At first, following Kripke's practice, t is a term. McGinn subsequently generalises this test to propositions, whereby a proposition A is rigid if

A but possibly not-A

is false; to other types of expression—predicates, adverbs, and so on; and to the connectives.

This 'Kripkean intuitive' test for the rigidity of propositions cannot be right. It makes all falsehoods rigid. What McGinn needs is the pair of conditions:

that possibly 'not-A' be true if A is true
and that possibly A be true otherwise,

that is, that the semantic value (truth) of A might have been different from what it is. If so, A is not rigid.

Nor is McGinn's test for the rigidity of negation correct. His test of whether 'not' is rigid is whether

Negation might not have been negation

is false.[16] He thinks it is. However, that is to test whether 'negation' is a rigid designator, not the connective 'not'. There will be any number of non-rigid designators of the appropriate functors: e.g., 'the truth-function first introduced in *Principia Mathematica*', as McGinn himself points out. What

[14] C. McGinn, 'Rigid Designation and Semantic Value', *The Philosophical Quarterly*, 32 (1982), pp. 97-115.

[15] Ibid., p. 98.

[16] Ibid., p. 104.

we wish to know is whether 'not' is rigid, and to do that we need to consider the behaviour of 'not' itself in modal contexts. How is that to be done? One way is to consider whether, to whatever sentences (or propositions) we attach it, the result might not have had the value it does, i.e., whether

possibly 'not-not-A' be true if 'not-A' is true
and possibly 'not-A' be true otherwise,

for all A. In other words, substituting 'not-A' for A in the criterion for rigidity of sentences, might the semantic value of 'not-A' have been different? Although true for some, these conditions are not met for all A. Hence 'not' is rigid. So too for the other connectives.

The logical deviant attributes different properties to a logical connective from the orthodox. Is logical deviance possible? It is possible in just the same way as disagreement in any other area. When historians disagree about the provenance of the works of Homer, one placing him in the tenth century BC, another in the eighth, yet a third claiming the *Iliad* and the *Odyssey* were written, or compiled, by two different bardic poets, only one, or neither, of whom was Homer, their dispute is set against a background with some fixed points, some fulcrum. The reference of the term is fixed by reference to certain provisional beliefs. But once fixed, any or all of those beliefs can be questioned.

Similarly for negation. The sense of 'not' is, let us suppose, fixed by reference to classical beliefs about the inferential properties of that particle. But once that fixed point is settled, any of those properties may be questioned. We may wonder whether 'not', that particle fixed as the flip-flop truth-function, does satisfy *reductio*, contraposition, double negation, excluded middle, non-contradiction, and so on, for any or all of those orthodox features. The sense of negation, how its meaning is fixed, may be one of what inferential properties it has (as I shall argue in chapter 9). We may use classical beliefs, or any other, about those properties to fix that meaning. But once that sense is fixed, the logical debate can start.

8.4 The Homophonic Principle

Once we have fixed what we are talking about, it is plausible to claim that the truth-conditions for a connective should be homophonic. That is

$$\text{`}\delta A_1 \ldots\ A_n'\text{' is true iff } \delta(A_1 \text{ is true, } \ldots, A_n \text{ is true}) \qquad \text{(T)}$$

should hold, where $A_1, \ldots, A_n$ are the sentential arguments of the connective 'δ'. If the metalanguage is distinct from the object language, then 'δ' on the right should translate 'δ' on the left. But if object and metalanguage share their logics, then 'δ' must not equivocate. The metalanguage should in this case be homophonic—same sound, same meaning.

Clearly, Principle (T) ties object language and metalanguage together, so as to allow any contradiction to spread from the object language to the metalanguage. We saw how the Australian plan avoided this unfortunate consequence by adopting a valuation principle for negation contrary to Principle (T), and how its authors, therefore, laid themselves open to the charge, from Copeland and others, that their disagreement with classical logic is only superficial, consisting not in a disagreement about whether a contradiction entails an arbitrary proposition, but in a difference as to the sense of the sign '$\sim$'.

We can avoid this charge by constructing a semantics for relevant logic which respects two features. First, it should be elaborated in a relevant metalanguage, in which the account of soundness, or logical consequence, is relevant - that it be impossible for the premises to be true fuse the conclusion false. Secondly, it should respect the homophonic account of the meaning of the connectives, that is, the homophonic Principle (T). For example, that

'if A then B' is true iff if A is true then B is true
and 'not-A' is true iff A is not true.

Such a semantics can easily be adapted from the formal metatheory of **R** which Meyer has recently presented.[17] Notwithstanding his conviction that such an enterprise is a 'harmful misdirection of effort',[18] Meyer sets out to develop a 'relevant metalogic' for **R**. This metatheory RELMODEL retains all the apparatus of worlds, R, and * that were used in the classical semantics (cf. chapter 5 above) to explicate the connectives $\sim, \rightarrow$ and $\times$ of **R**. But Meyer recognises that once we adopt a relevant metalogic which incorporates the Relevant Account of Validity, his principle SemEnt,[19] we can dispense with these notions. They were needed, as we saw in chapter 7, only to bridge the gap between the relevant object-theory and the classical metatheory. We remove mention of the *-operation, and adopt the above homophonic truth-definition for 'not', or in Meyer's formal style of presentation,[20]

$$\text{RT}\sim\text{:}\ [\sim A]x =_{\text{df}} \sim[A]x$$

[17]R.K. Meyer, *Proving Semantical Completeness 'Relevantly' for* **R**, Australian National University Research School of Social Sciences Logic Group Research Paper no. 23 (Canberra, 1985).

[18]Ibid., p. 4

[19]Ibid., p. 14.

[20]Ibid., p. 19.

where '$[A]x$' means 'A is true at x'. A similar simplification of the truth-condition for '$\to$' removes the only other 'complication'[21] in the proof, and results in the homophonic truth-condition

$$\text{RT}\to\text{: } [A \to B]x =_{\text{df}} [A]x \to [B]x$$

Accordingly, the semantics becomes wholly homophonic. Nonetheless, the completeness proof given by Meyer,[22] still goes through, since, as is said there, '* and all its works are among the subtleties wiped out' by the method of translation at the heart of the proof. Hence, whenever A is valid, that is, whenever we can prove $[A]0$ (that A is true at 0), that proof in RELMODEL can be reduced to a proof in **R** of A itself. Moreover, removing * from the Soundness proof only serves to make it that much more immediate. The reason all this is possible is that SemEnt[23] encapsulates the Relevant Account of Soundness, that A entails B iff A cannot be true fuse B false. Accordingly, we obtain the truly relevant completeness result, that

B is a logical consequence of A (i.e., A entails B)
iff B is derivable from A (in **R**).

In the next two sections we give the details of this homophonic semantics.

8.5 A Metalanguage for R

The object language **R** is as in chapter 4 above, in particular, with 'f' as primitive. In other words, it is the theory of §4.4, adopting as structural rules for '$\circ$' all of t=, B, C and W.

The vocabulary of our metalanguage **MR** consists of

individual constants	$\overline{p}, \overline{q}, \overline{r}, \ldots$	(to 'name' the atomic wffs of **R**)
propositional letters	$P, Q, R, \ldots$	(to translate the atomic wffs of **R**)
1-place predicate	$\mathcal{T}$	(the truth predicate)
logical constants	$\mathsf{f}, \mathsf{F}, \&, \vee, \times, \to$	
parentheses		

A propositional letter, or the predicate '$\mathcal{T}$' followed by an individual constant, is an (atomic) wff; if A, B are wffs, so are $\mathsf{f}, \mathsf{F}, (A \mathbin{\&} B), (A \vee B), (A \to B), (A \times B)$. We define $\sim A$ as $A \to \mathsf{f}$, $A \leftrightarrow B$ as $(A \to B) \mathbin{\&} (B \to A)$, t as $\sim\mathsf{f}$ and T as $\sim\mathsf{F}$. We drop brackets by the previous conventions.

[21]Ibid., p. 20.
[22]Ibid., pp. 19-20.
[23]Ibid., pp. 15-16

'Bunch' is defined as in chapter 4. Following Meyer (in *Proving Semantical Completeness*), we give the truth-definition as an abbreviatory definition, built up on a specification of the translation of **R** into **MR** : where A_i is the ith propositional letter of **MR**, and $\overline{a_i}$ the ith individual constant, we specify as semantic axioms of **MR**:

$$\mathcal{T}\overline{a_i} \leftrightarrow A_i$$

The truth definition is then:

the False is false:	$\mathcal{T}(\bar{\mathsf{f}}) =_{\mathrm{df}} \mathsf{f}$ and $\mathcal{T}(\bar{\mathsf{F}}) =_{\mathrm{df}} \mathsf{F}$
and truth distributes over the connectives:	$\mathcal{T}(\overline{a \;\&\; b}) =_{\mathrm{df}} \mathcal{T}\overline{a} \;\&\; \mathcal{T}\overline{b}$ $\mathcal{T}(\overline{a \vee b}) =_{\mathrm{df}} \mathcal{T}\overline{a} \vee \mathcal{T}\overline{b}$ $\mathcal{T}(\overline{a \to b}) =_{\mathrm{df}} \mathcal{T}\overline{a} \to \mathcal{T}\overline{b}$ $\mathcal{T}(\overline{a \times b}) =_{\mathrm{df}} \mathcal{T}\overline{a} \times \mathcal{T}\overline{b}$

where a, b are wffs of **R**.

A derivation in **MR** is a sequence of sequents each of which is either of the form '$A : A$' or is a semantic axiom, or is an immediate consequence of earlier sequents in the sequence by one of the operational or structural rules given in chapter 4. $X \vdash_{\mathbf{MR}} A$ iff there is a derivation in **MR** whose last member is $X' : A$, where $X' \angle\, X$. $\vdash_{\mathbf{MR}} A$ iff $\mathsf{t} \vdash_{\mathbf{MR}} A$.

Proposition 8.1 *For every wff a of* **R** *containing lower case propositional letters, there is a corresponding wff A of* **MR** *(its 'translation') containing corresponding upper case propositional letters. Accordingly,* $\vdash_{\mathbf{MR}} \mathcal{T}\overline{a} \leftrightarrow A$.

Proof: Induction on degree of a. Base: suppose a is a propositional letter. Then $\mathcal{T}\overline{a} \leftrightarrow A$ is an axiom of **MR**, whence immediately, $\vdash_{\mathbf{MR}} \mathcal{T}\overline{a} \leftrightarrow A$.

Induction Step: there are six cases:

(i) $a = \mathsf{f}$: by definition $\mathcal{T}(\mathsf{f})$ reads f. Clearly $\vdash_{\mathbf{MR}} \mathsf{f} \leftrightarrow \mathsf{f}$:

1	(1)	f	A
$\mathsf{t}; 1$	(2)	f	1 t=
t	(3)	$\mathsf{f} \to \mathsf{f}$	1,2 $\to$I
t, t	(4)	$\mathsf{f} \leftrightarrow \mathsf{f}$	3,3 &I
t	(5)	$\mathsf{f} \leftrightarrow \mathsf{f}$	4 EW

(ii) $a = \mathsf{F}$: similarly, $\vdash_{\mathbf{MR}} \mathsf{F} \leftrightarrow \mathsf{F}$.

(iii) $a = b \;\&\; c$: we show that $\vdash_{\mathbf{MR}} \mathcal{T}(\overline{b \;\&\; c}) \leftrightarrow B \;\&\; C$, where B, C translate b and c. By I.H., $\vdash_{\mathbf{MR}} \mathcal{T}(\overline{b}) \leftrightarrow B$ and $\vdash_{\mathbf{MR}} \mathcal{T}(\overline{c}) \leftrightarrow C$.

1	(1)	$\mathcal{T}(\bar{b})$ & $\mathcal{T}(\bar{c})$	A
1	(2)	$\mathcal{T}(\bar{b})$	1 4.2
t	(3)	$\mathcal{T}(\bar{b}) \leftrightarrow B$	I.H.
t	(4)	$\mathcal{T}(\bar{b}) \rightarrow B$	3 4.2
$\mathsf{t}; 1$	(5)	B	2,4$\rightarrow$E
1	(6)	$\mathcal{T}(\bar{c})$	1 4.2
t	(7)	$\mathcal{T}(\bar{c}) \leftrightarrow C$	I.H.
t	(8)	$\mathcal{T}(\bar{c}) \rightarrow C$	7 4.2
$\mathsf{t}; 1$	(9)	C	6,8 $\rightarrow$E
$(\mathsf{t}; 1), (\mathsf{t}; 1)$	(10)	B & C	5,9 &I
$\mathsf{t}; 1$	(11)	B & C	10 EW
t	(12)	$\mathcal{T}(\bar{b})$ & $\mathcal{T}(\bar{c}) \rightarrow B$ & C	11 $\rightarrow$I

The converse is similar, as are cases iv) - vi), where $a = b \vee c$, $b \rightarrow c$ and $b \times c$. □

Proposition 8.1 shows that **MR** satisfies Tarski's material adequacy constraint on any theory of truth.[24] Each $\mathcal{T}$-sentence is provable in **MR** . By inspection, **MR** also satisfies the formal constraints, namely, in containing translations of all the sentences of **R**, and the means to refer to those sentences, in being formally specifiable, and in '$\mathcal{T}$' being well-defined. We have succeeded, therefore, in the first task of providing an adequate homophonic theory of truth for **R**.

8.6 Logical Consequence

The next task is to define logical consequence for **R**. Recall the Relevant Account of Soundness from chapter 6, that a proposition is a logical consequence of other propositions if and only if it is impossible for the other propositions to be true and (fuse) the original proposition false. To formalise this conception, we need to add a modal connective '$\Diamond$', for possibility, to **MR** , obtaining thereby an augmented language **MR**$^{\Diamond}$. First we define '**S4**-modal' and '**Br**-modal':

1. the wffs $\sim\Diamond A$ and t are each **S4**-modal and $\Diamond A$ and f are **Br**-modal;
2. if A, B are **S4**-(resp. **Br**-)modal, so are A & $B, A \vee B, A \times B$;
3. X is **S4**-(resp. **Br**-)modal if every subbunch of X is **S4**-(resp. **Br**-)modal.

[24] A. Tarski, 'The Concept of Truth in Formalised Languages', in *Logic, Semantics, Metamathematics*, tr. J.H. Woodger (Oxford, 1956), pp. 152-278.

Then we can extend the criterion for immediate consequence by two further cases:

$X : \Diamond A$	is an immediate consequence of	$X : A$
$Y(X) : B$	is an immediate consequence of provided $Y(\mathsf{t})$ is **S4**-modal and B is **Br**-modal.	$X : \Diamond A$ and $Y(A) : B$,

The definitions of $X \vdash_{\mathbf{MR}^\Diamond} A$ and $\vdash$ $\mathbf{MR}\Diamond A$ are as for **R**. For wffs of **R** and $\mathbf{MR}^\Diamond$ we can define the characteristic wff of a bunch in the usual way. We also define $A \Rightarrow B$ as $\sim\Diamond(A \times \sim B)$ for wffs of $\mathbf{MR}^\Diamond$, and a two-place predicate $\mathcal{C}(\overline{a,b})$ as $\mathcal{T}(\overline{a}) \Rightarrow \mathcal{T}(\overline{b})$. Then the relation $\mathcal{C}$ captures the notion of one proposition's being a logical consequence of (the characteristic wff of a bunch of) other propositions: it is impossible for those propositions to be true and (fuse) the other propositions false (not true).

Proposition 8.2 $\vdash_{\mathbf{MR}^\Diamond} A \to B$ *iff* $\vdash_{\mathbf{MR}^\Diamond} A \Rightarrow B$.

Proof:

t	(1)	$A \to B$	Given
2	(2)	$\Diamond(A \times \sim B)$	A
3	(3)	$A \times \sim B$	A
4	(4)	A	A
5	(5)	$\sim B$	A
$\mathsf{t};4$	(6)	B	1,4 MPP
$4;\mathsf{t}$	(7)	B	4 C*
$4;5$	(8)	f	5,7 MTT
3	(9)	f	3,8 ×E
2	(10)	f	2,9 ◊E
t	(11)	t	A
t	(12)	$\sim\mathsf{f}$	11 DN
t	(13)	$\sim\Diamond(A \times \sim B)$	10,12 MTT

Conversely,

t	(1)	$\sim\Diamond(A \times \sim B)$	Given
2	(2)	A	A
3	(3)	$\sim B$	A
$2;3$	(4)	$A \times \sim B$	2,3 ×I
$2;3$	(5)	$\Diamond(A \times \sim B)$	4 ◊I
$2;\mathsf{t}$	(6)	$\sim\sim B$	1,5 MTT
$2;\mathsf{t}$	(7)	B	6 DN
$\mathsf{t};2$	(8)	B	7 C*
t	(9)	$A \to B$	8 CP

□

Proposition 8.3 $\mathbf{MR}^\Diamond$ *is a conservative extension of* **MR**.

Proof: Take any non-theorem A of **MR**. We need to show that it is not a theorem of $\mathbf{MR}^{\Diamond}$ either. **MR** adds to **R** only the analysis of atomic propositions into, in addition to propositional variables, the application of '$\mathcal{T}$' to individual variables. We add to the model structures $\langle \mathbf{0}, \mathbf{W}, R, ^* \rangle$ for **R** a domain **D**, and extend any valuation v on **R** by putting $v(\overline{p}) \in D$ and $v(\mathcal{T}, w) = \{v(\overline{p}) \in D : v(P, w) = \mathsf{T}\}$. Since $\nvdash_{\mathbf{MR}} A$, there is an interpretation $\mathcal{J}$ with an associated valuation v such that $v(A, 0) = \mathsf{F}$. We extend $\mathcal{J}$ to an interpretation $\mathcal{J}'$ for $\mathbf{MR}^{\Diamond}$ by setting $v'(B, w) = v(B, w)$ for all propositional variables B, and $v'(\Diamond B, w) = \mathsf{T}$ if $v(B, w')$ for some $w' \in W$. Then $\mathcal{J}'$ is an interpretation for $\mathbf{MR}^{\Diamond}$. But $v'(A, w) = v(A, w) = \mathsf{F}$. So $\mathcal{J}'$ shows that $\nvdash_{\mathbf{MR}^{\Diamond}} A$. Hence $\mathbf{MR}^{\Diamond}$ conservatively extends **MR** . □

Proposition 8.4 *(Soundness and Completeness) Suppose $X \vdash_{\mathbf{R}} a$. Then there is a finite $X' \angle\, X$ such that $X' \vdash_{\mathbf{R}} a$. Conversely, if $X' \vdash_{\mathbf{R}} a$ then $X \vdash_{\mathbf{R}} a$ for all X such that $X' \angle\, X$. Let b be the lower case wff $\mathsf{cf}(X')$. Then, for some such b, $X \vdash_{\mathbf{R}} a$ iff $\vdash_{\mathbf{MR}^{\Diamond}} \mathcal{C}(\overline{b}, \overline{a})$.*

Proof: $X \vdash_{\mathbf{R}} a$

iff for some finite $X' \angle\, X, X' \vdash_{\mathbf{R}} a$	
iff $\mathsf{cf}(X) \vdash_{\mathbf{R}} a$	by Corollary 4.6,
iff $b \vdash_{\mathbf{R}} a$	
iff $B \vdash_{\mathbf{MR}} A$	(where B is the upper case 'translation' of b, i.e., $\mathsf{cf}(X)$), since the same derivation is acceptable in **MR**
iff $B \vdash_{\mathbf{MR}^{\Diamond}} A$	since $\mathbf{MR}^{\Diamond}$ is a conservative extension of **MR**, by Proposition 8.3
iff $\mathsf{t}; B \vdash_{\mathbf{MR}^{\Diamond}} A$	by t=
iff $\mathsf{t} \vdash_{\mathbf{MR}^{\Diamond}} B \to A$	by $\to$I and $\to$E
iff $\mathsf{t} \vdash_{\mathbf{MR}^{\Diamond}} \mathcal{T}(\overline{b}) \to \mathcal{T}(\overline{a})$	by Propositions 8.1 and 4.20
iff $\mathsf{t} \vdash_{\mathbf{MR}^{\Diamond}} \mathcal{T}(\overline{b}) \Rightarrow \mathcal{T}(\overline{a})$	by Proposition 8.2
iff $\vdash_{\mathbf{MR}^{\Diamond}} \mathcal{C}(\overline{b}, \overline{a})$	as required. □

It is important to be clear about quite what we have shown here. The statement of Proposition 8.4 belongs to the metametalanguage. It might be objecte, therefore, that it shows only that any derivation in the object-language can be mimicked in the metalanguage. The result hardly merits the title 'Completeness', since if the object-language were incomplete, the result would still hold, for the metalanguage would lack the resources to pinpoint that incompleteness. The fault lies, it might be said, in the poverty of the metalanguage, $\mathbf{MR}^\Diamond$, containing only modality, designation and truth. What it importantly lacks is any means of referring to interpretations, and of expressing truth-preservation under interpretation. It cannot, therefore, express true consequence and so is inadequate as a metalanguage for **R**.

But the objection is mistaken. Consequence, or truth-preservation, is captured by the Relevant Account of Validity without need to refer to interpretations, and is captured in $\mathbf{MR}^\Diamond$ by $\mathcal{C}$. This notion can be explicated by reference to models, and to truth-preservation under interpretation. But it is equally well captured by modality. Reference to models and interpretation cannot guarantee that one will never overlook incompleteness. Consider our 'proof' in chapter 5 that **DW** is complete. **DW** clearly lacks many valid inference-patterns—valid by any plausible lights. The model theory is merely fashioned to accord with this weak logical viewpoint. It is no different with $\mathbf{MR}^\Diamond$. Its logical resources have been moulded to accord with **R**.

What, then, does Proposition 8.4 tell us? For one whose logical preferences are captured by **R**, it spells out in formal terms the theory of truth and consequence. It is a non-starter as proselytising **R** itself. Seen as a defence of the theory of consequence in **R** it is clearly circular. Its interest lies in its making clear in formal terms what the notions of truth and consequence are.

8.7 Conclusion

The deviant's predicament was that when he tried to query the doctrine he only changed the subject. When the relevant logician challenges the classical logician's most sacred beliefs—the Classical Account of Validity, say, or that a contradiction entails any proposition - how should he be understood? To suppose he is really challenging the doctrine, as differing in belief, we must surely at least interpret him as sharing some meaning with the classical logician. Is it really a contradiction which he believes lacks the permeation property, or is it some other proposition, only superficially the same? If he means the same by his words, must he not use them the same way? But his logical practice is, by definition, different: he rejects certain classical beliefs.

This conundrum is a special case of the holism of meaning and belief, a holism which led Quine at one time himself to challenge the whole apparatus of meaning-theory. How can we tell what a man believes until we can interpret what he says; but how can we interpret what he says without seeing what he believes? Quine inferred: '[I]t is misleading to speak of the empirical content of an individual statement.'[25] But we should neither succumb to this challenge so easily, nor reject it cavalierly as requiring firmness and fixity of belief to ground meaning—as we saw Quine does in his later writing. We will see in the next chapter how the concept of meaning, as it applies to logical particles, can be explicated proof-theoretically. In the present context, however, we can respond to Quine's 'blank stare' from two complementary directions.

First, it is typical of the use of certain phrases to provide just such a pivot in modal contexts, both alethic and epistemic, for consideration of alternative possibilities. Proper names and certain uses of descriptions are common examples of such terms and phrases. But generalising Kripke's intuitive test for such particles to other parts of speech shows that the usual terms referring to logical constants (though not, of course, descriptive phrases) play just such a role. In particular, 'not' is a rigid designator, and can be used fixedly to reflect on the logical properties of negation, and the properties of contradictions. Once its meaning is fixed by reference to certain provisionally shared beliefs, those beliefs can be queried, challenged and rejected if necessary.

But, secondly, how is that meaning to be fixed? We shall provide two answers to that question. They are not in conflict. We saw in this chapter how, within truth-conditional semantics, the Relevant Account of Validity as part of the Scottish plan allows us to construct a semantics which respects the homophonic truth-conditions for the connectives. On the Classical Account, such homophonic clauses would require a contradictory truth-value assignment, both assigning truth and simultaneously assigning falsity (and not assigning truth) to a wff. No such unacceptable consequence follows from combining homophony with the Relevant Account. Its virtue is that no charge of 'funny business' can now be directed at the valuation clauses. It may still be directed at the metalanguage itself, but at least that is where it should be directed, and where the proof-conditional account is designed to meet it. To that we now turn.

[25]W.V. Quine, 'Two Dogmas of Empiricism', in *From a Logical Point of View* (New York, 1953), p. 42.

Chapter 9

Semantics and the Theory of Meaning

In the last chapter, we explored two responses to Quine's challenge to the deviant logician. We argued that logical constants such as 'not' behave, as more familiarly do other parts of speech, as rigid designators, permitting us to retain their reference while exploring other possibilities as to their logical character. We also saw how to develop a homophonic semantics, in which the sense of the connectives is mapped directly from the metalanguage. Nonetheless, neither of these responses gets to the heart of the matter. We must ask, what is semantics for, and what can we do with a semantics? In particular, we must consider, what is the correct way to provide a semantics for relevant logic, or for any logic?

9.1 Semantics

In constructing logical theory, we start with a discourse in natural language—English, or Gaelic, or Tagalog, say—in which there are various arguments. We want to know which arguments are valid, which invalid, and why. (Even this is an idealisation, part of the theory to come, which will give content to an abstract notion of validity.) There is another program, inconsistent with this picture. It sets out to erect logical theory *a priori*, finding the simplest, most elegant theory of pure inference. To the limited extent to which it is willing to consider inference in ordinary discourse, such inference is made to fit the antecedent Procrustean bed. Such a Euclidean view is, of course, incompatible with the quasi-empirical methodology urged in chapter 1. Logical theory, from that perspective, is an attempt to explain and justify an already existing pre-theoretical practice of judging validity.

Our theory says, let us suppose, that at least certain arguments are valid in virtue of form, and proceeds to exhibit certain forms of argument, providing first a way of mapping natural language arguments to their forms, and secondly a division of forms into the valid and the invalid. As is familiar, arguments are valid (sometimes called 'sound') when they are instances of a valid form. Invalid forms (such as DS and *ex falso quodlibet*, say) may have valid instances, but only, on such a view, by virtue of those instances being either materially valid or being instances of some other, valid, form.

Where does semantics fit into this picture? Semantics covers two rather different activities. Recently these have acquired the honorifics 'pure' versus 'applied' semantics, or 'merely formal' or 'merely algebraic' versus 'illuminating' semantics or 'semantics properly so-called', and so on.[1] This distinction is in fact rather misleading. For it suggests that there is essentially one single activity, semantics, which prosecuted in slightly different ways can yield different results. I prefer to distinguish formal semantics from the theory of meaning. They are equally illuminating, provided one realises what illumination is forthcoming.

Formal semantics will give important technical results, such as soundness, decidability, independence, compactness, variable-sharing and so on. What it will not show is what any of the expressions of the formal language, the language in which the forms of argument are expressed, means. It is often tempting to suppose it can, since the motivating idea for many formal semantics has arisen from consideration of the informal mapping of natural language onto its form in concert with the underlying meaning of the natural language. An example is possible worlds semantics for modal logic, relying on intuitions about the sense of 'necessary' and 'possible'. But we must not think that Kripke semantics tells us anything we did not know before, for example, what 'necessary' and 'possible' mean. One can only read that off the formal semantics because it was read onto it in the first place.

Theory of meaning is an utterly different activity, and there is one very good reason why it has to be. Formal semantics, as Meyer and Sylvan have so elegantly put it (they attribute the insight to Urquhart), is a homomorphism.[2] Commonly, formal semantics maps formulae onto truth-values, or onto 'propositions' (i.e., functions from worlds to truth-values). For example, truth-conditional semantics takes a formula and evaluates it at an

[1] See, e.g., B.J. Copeland, 'What is a Semantics for Classical Negation?', *Mind*, 95 (1986), p. 480.

[2] R.K. Meyer and F.R. Routley, 'Extensional Reduction' II (typescript, Canberra, n.d. [revised c. 1987]), §3 and *passim*.

index. In other words, it maps the formula to a function from indices to truth-values. $2^{W\times F} = (2^W)^F$. But 2^W is (effectively) the power set of W, and subsets of W are just propositions.

The problem is that such 'propositions' themselves stand in need of explanation. A translation from one language into another does not tell one the meaning of the first language, unless the meaning of the second is known. How then is the sense of propositions (maps from worlds to truth-values) to be given? If by a further map, the problem is merely postponed. If by some other means, then the theory is, as Blackburn has so well put it, dog-legged.[3] Whatever (we know not what, yet) can now be done for the proposition, can be done for the formula directly. So at some point in this chain, meaning must be given differently than by such a translation. That, or at least the theory of how that is done, is the task of the theory of meaning, and it cannot be given by formal semantics.

How it is done is to relate language to its context of use. But that is a slogan which only points in the right direction. It tells us, however, that to understand the meaning of logical connectives one must look to see in what contexts they are used.

9.2 Tonk

The idea that the senses of the logical connectives depend on their use in inference has been variously described as the analytical validity view or as the synthetic view.[4] I prefer to call it a holistic view. However one describes it, Arthur Prior believed it put the cart before the horse. For, he said, one could not thereby discriminate between valid and invalid inference. The meaning of the connectives must be given by their truth-conditions independently of their use in inference, prior (again!) to inferential practice.

> [N]o one can *make* an expression consisting of two significant sentences joined by 'and' or 'or' or 'tonk' into a significant sentence just by (a) saying it is one, and (b) giving us 'permission' to utter certain other sentences before or after it and insert a few 'therefores' … [Such] a definition implies that the sentence formed by placing a conjunction-forming sign between two other sentences already *has* a meaning.[5]

[3] S. Blackburn, *Spreading the Word* (Oxford, 1984), ch. 2.

[4] A.N. Prior, 'The Runabout Inference Ticket', *Analysis*, 21 (1960), pp. 38-9; N.D. Belnap, 'Tonk, Plonk and Plink', *Analysis*, 22 (1962), pp. 130-4.

[5] A.N. Prior, 'Conjunction and Contonktion Revisited', *Analysis*, 24 (1964), pp. 194,

Much of the discussion of Prior's argument has been vitiated by a narrow concentration on truth-functional connectives. But nothing in his argument essentially depends on their being truth-functional. Indeed, it would suggest that his point was almost trivial, for it relates the connective to a semantic condition, truth-functionality, separate from the rules of inference whose efficacy in providing meaning for the connective is what is at issue.[6] Prior's point has been totally misunderstood if one argues as follows: Prior's connective 'tonk' cannot be identified with any truth-function. Hence 'tonk' lacks sense (or reference, or whatever): 'tonk' is empty (meaningless). Prior's argument applies just as well to quantifiers, modal connectives, indeed, any connective, whether extensional (truth-functional) or not.

Prior's argument is this: suppose the 'analytical validity' view, the claim that the meaning of (at least some) logical connectives depends holistically on their use in inference, were correct. Let the connective 'tonk' be defined by the following rules of inference:

A tonk B	is an immediate consequence of	A	(tonk-I)
B	is an immediate consequence of	A tonk B	(tonk-E)

tonk-I serves to introduce occurrences of 'tonk'; tonk-E serves to remove such occurrences. But now consider the following speedy argument:

	A	
so	A tonk B	by tonk-I
so	B	by tonk-E

That is, using tonk we can show that every proposition entails, indeed, is equivalent to, every other. But that is absurd.

What is wrong? Prior infers that the 'analytical validity' view is at fault. The meaning of connectives must be given independently of their inferential properties. Those inferential properties are consequences of their meaning, not constitutive of it. The independently established meaning can then be used to decide whether the inference is valid or not. One cannot lay down arbitrarily which inferences are valid. Validity can only be decided given an independent assignment of meaning to the propositions, and so to their constituents, in the inference.

Prior does not commit himself as to how the independent assignment of meaning is to be made. His comments hint, however, at a truth-conditional approach—indeed, as noted above, some commentators have understood

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[6] S. Wagner, 'Tonk', *Notre Dame Journal of Formal Logic*, 22 (1981), pp. 293, 296.

him to be referring to an assignment of truth-functions to the connectives.[7] A truth-conditional approach, specifying the conditions under which a formula is true, need not be truth-functional. The truth-value of a compound formula may depend on—may be conditional on—other semantic values besides simply the truth-value of its constituents, for example, the indexed value. Or it may depend on that value non-truth-functionally, in, for example, 'James knows that p', whose value is given by whether James knows the truth-value of the constituent, p.

The holistic view is not so easily refuted, however. Prior's argument certainly shows that such a view, expressed as he expresses it, is untenable. But his argument serves not to refute it, but to refine it. There is undoubtedly something wrong with the introduction of connectives like 'tonk'. What is wrong, however, is not the very attempt to explain the meaning of a connective in terms of its use, but the particularly simple and careless way in which that connection was expressed.

9.3 Conservative Extension

In an early reaction to Prior's challenge, Belnap tried to revise the holistic, or as he called it, the synthetic view.[8] It is a general fact about definitions, he recalled, that one cannot define things into existence. Definition requires a proof that what one is defining both exists and is unique. But what can an existence condition mean for a logical connective? Belnap claims that it amounts to a conservative extension condition. Suppose one has specified a logical language: a vocabulary and a definition of wff; and has defined derivability on this language: conditions have been laid down for $X \vdash A$. Subsequently, the language is extended by new vocabulary, and further conditions on derivability are admitted to accommodate the new vocabulary. The consequence relation is thereby extended to a new consequence relation, $X \vdash' A$. The new consequence relation is a conservative extension of the old one if whenever $X \vdash' A$, for X, A exclusively in the old vocabulary, $X \vdash A$. That is, no cases of derivability in the old vocabulary are added by the extension to the consequence relation. All new sequents admitted contain some new vocabulary.

We considered an example in chapter 8: $\mathbf{MR}^{\Diamond}$ is a conservative extension of **MR**. For if X, A contain no occurrence of '$\Diamond$', and if $X \vdash_{\mathbf{MR}^{\Diamond}} A$, then

[7] See also J.T. Stevenson, 'Roundabout the Runabout Inference Ticket', *Analysis*, 21 (1961), pp. 126-7.

[8] Belnap, 'Tonk, Plonk and Plink', *Analysis*, 22.

$X \vdash_{\mathbf{MR}} A$ (Proposition 8.3). By Belnap's criterion, therefore, the rules for '$\Diamond$' are acceptable. But adding the rules for 'tonk' to any non-trivial logic (that is, any logic for which $X \vdash A$ does not hold for all X and A) will result in triviality. So these rules will not in general yield a conservative extension. Hence they are unacceptable. For it is an assumption, he says, that before extending the language, all valid consequences in the old vocabulary were defined. Breaking the requirement of conservativeness will contravene that assumption.

But Belnap's criterion will not do. There are at least two objections to it. For one thing, Belnap does not clearly relate it to his rationale concerning definitions. What exactly is supposed to be the connection between an existence claim and the conservative extension criterion? That the new rules give a conservative extension of the old consequence relation certainly does not mean that the new relation is definable in terms of the old. Indeed, that would not then be an extension at all. It is not definability we are after—that is at best an analogy.

That Belnap's condition is not right is shown by the following example. Consider the positive fragment of classical logic given by our system of chapter 4 in which '$\sim$' is dropped from the vocabulary, and the rules for negation are removed. Take, for example, the I- and E-rules for '&', '$\vee$' and '$\rightarrow$', and all the structural rules (equivalently, take all bunches to be sets, and forget about $\circ$). Call this system $\mathbf{K}_+$. Many negation-free theorems of $\mathbf{K}$ are not provable in $\mathbf{K}_+$; for example, Peirce's Law:

$$((A \rightarrow B) \rightarrow A) \rightarrow A$$

So adding '$\sim$' to the language together with the rules $\sim$I, $\sim$E and CM does not always give a conservative extension. $\mathbf{K}$ is not a conservative extension of $\mathbf{K}_+$. But the negation rules, $\sim$I, $\sim$E and CM, do not appear to suffer from the same defect as those for 'tonk'. Belnap would needs produce further argument, very good argument, to persuade us that these rules are suspect.

In particular, the example shows that we cannot assume that all valid consequences in the positive language are characterised by the positive rules. The classical logician clearly believes that certain valid positive laws can be proved only by an excursus through negation. It cannot be simply that 'tonk' is not conservative which is its fault.

9.4 Assertion-conditions

That rejection of Belnap's idea may be too swift, however. It is certainly adequate as an *ad hominem* objection, for ∼I, ∼E and CM are rules Belnap himself elsewhere accepts.[9] To be sure, in the context in which Belnap presents them, they do constitute a conservative extension of the underlying logic, viz when added to **T**, **E** and **R**.[10] But, although Belnap considers adding 'tonk' only to a bare definition of consequence with no theory of connectives at all, the criterion appears to require a conservative extension on any non-trivial underlying logic. We are, Belnap says, defining connectives not '*ab initio*, but rather in terms of an antecedently given context of deducibility', 'some assumptions' but none in particular.[11]

It may be, however, that despite Belnap's acceptance of these rules for negation, the unconservativeness of ∼I, ∼E and CM in the context of $\mathbf{K}_+$ and **K** may indeed tell against them. Not, however, deep down for that reason. What is wrong, rather, with collections of rules such as tonk-I and tonk-E, or ∼I, ∼E and CM is that they are mere collections of rules, and not equivalent to any set of rules in the canonical form. What, then, is the canonical form?

Dag Prawitz claims that assertibility is the core semantic notion.[12] The meaning of a statement is constituted by the conditions under which it may warrantedly be asserted. In brief, belief in elements of meaning other than what are exhibited in the assertibility conditions of propositions is myth and superstition. Meaning is what any speaker must know who understands an expression, and there is no more to language than what can be shared and communicated. What the speaker knows must be capable of being manifested and acquired; and manifestation and acquisition require publicity of meaning. The core notion is, therefore, a publicly inspectable phenomenon, what is asserted. Truth cannot outrun assertibility, for no substance can be attached to any component of meaning which might go beyond assertibility.

The notion of assertibility in play here must not be confused with that in §2.4 above. There the idea was to distinguish truth from assertiblity—what is true might not be assertible, and indeed, within that whole conversa-

[9]See, e.g., A. Anderson and N. Belnap, *Entailment*, vol. I (Princeton, 1975), ch. 2.

[10]Ibid., §§24.4.2 and 28.3.1.

[11]Belnap, in *Analysis*, 22, p. 131.

[12]E.g., D. Prawitz, 'Meaning and Proofs: on the conflict between classical and intuitionistic logic', *Theoria*, 43 (1977), pp. 2-40; 'Proofs and the meaning and completeness of the logical constants', in *Essays on Mathematical and Philosophical Logic*, ed. J. Hintikka et al. (Dordrecht, 1978), pp. 25-40.

tionalist doctrine, what is assertible might not be true. (Grice's Maxim of Truthfulness suggests he would not accept that what is assertible might not be true. But examples such as the acceptability in context of saying 'France is hexagonal' show that truth is not a necessary condition for assertibility.) Assertibility, so construed, and truth come apart.

In the present sense, however, assertibility and truth are intimately linked. There is no possibility that some proposition might be true and yet not assertible and vice versa. For if it were true yet not assertible, there would needs be some component of the meaning of the proposition which contributed to its truth-condition and yet was not revealed in the occasions of use, in the assertion-conditions. Meaning would transcend what could be manifested on occasions of utterance (assertion). The proposition would be true in virtue of the obtaining of some element of meaning, but that element could not be made public in assertion. Such an element cannot exist, for it could be neither manifested nor acquired. Truth cannot, therefore, extend beyond assertibility.

9.5 Proof-conditions

When applied to the logical particles, the claim that assertibility is the primary semantic notion, and that the meaning of an expression is given by the conditions for its use in assertion, becomes what we may call a 'proof-conditional' theory of meaning. For in distinguishing a particle as logical, we mean that it plays a substantive role in matters of consequence. By consequence, is meant formal logical consequence, whether one proposition follows from others in virtue of its form. In obtaining that form, we abstract away all aspects of the constituent propositions which do not have logical content, that is, do not affect the question whether one proposition logically follows from others in virtue of that form. The substantive components left, besides the formal place-holders, are the logical particles. Of course, so described, that explanation is circular. It obtains content by setting up a particular theory of consequence, and testing it against the practice it sets out to describe and regulate.

But what is consequence? In the course of this book, we have not only subjected the pre-formal intuitive notion of consequence or validity to analysis, but have also introduced two technical senses of 'consequence': in chapter 4, derivability, and in chapter 5, what is usually called logical consequence, or semantic entailment. But each of these technical notions is erected and developed in order to explicate the pre-formal notion of proof or argument.

On the one hand, the technical notions are responsible to our actual logical practice as that of which they are the formal theory. On the other, our concept of valid argument or proof is a normative one, one which already contains a conception of error or invalidity. In erecting our theory of validity we are not responsible to a practice which simply consists of actual argument, all valid by fiat. The theory is responsible to a practice which includes assessments, right or wrong, of those arguments, and it attempts to produce the best theory to explain that practice of assessment.

The logical particles appear then as the sole substantive components in a dialectical practice of argumentation and proof. Their meaning can reside, therefore, in nothing else than their role in that practice.

Uses of the logical particles divide into two aspects. We can look at the grounds for assertion of a formula containing the particle; and we can look at the consequences we draw from such an assertion. Both aspects are aspects of the use of the expression. But the former is that aspect which encapsulates the grounds for assertion. Indeed, we can see that there must be what Dummett calls a harmony between the two aspects, a dependency of the latter aspect on the former.[13] Use made of an assertion must not go beyond the grounds for its assertion. This is to repeat the point made in the last section, that all aspects of an expression's meaning must be manifested in its assertion-conditions, and to add to it the observation that valid inference, though it may be psychologically extrapolative, must not be logically so. The conclusion of a valid inference is contained in its premises. One can correctly infer from a formula no more than its meaning allows.

Suppose that a proof Π has justified assertion of a sequent $X : A$,

$$\frac{\Pi}{X : A}$$

Then, whatever may be inferred from $X : A$ may, provided truth does not outrun assertion-conditions, and provided Π is the only ground for assertion of $X : A$, be equally well inferred from Π. That is, if a proof has the form

$$\cfrac{\cfrac{X : A}{\Pi'}}{Y : B}$$

it may be replaced by

$$\cfrac{\cfrac{\Pi}{\Pi'}}{Y : B}$$

[13] M. Dummett, *Frege: Philosophy of Language* (London, 1973), p. 396.

Put this way, we can see that the principle that, for the logical connectives, meaning is proof-conditions, is the principle of transitivity of proof. Moreover, transitivity of proof comes out in such systems as ours, whose building blocks are sequents, as Cut or normalisation. Cut and normalisation are two aspects of the same phenomenon.

9.6 Cut

Consider, for example, the rule of $\vee$E:

$Y(X) : C$ is an immediate consequence of $X : A \vee B$, $Y(A) : C$ and $Y(B) : C$

From the introduction rule, we know that what has justified assertion of $X : A \vee B$ is a derivation either of $X : A$ or of $X : B$. Suppose it was $X : A$. Then we may put that derivation together with that of $Y(A) : C$, and apply Cut:

$$\frac{X : A \quad Y(A) : C}{Y(X) : C}$$

On the other hand, if what justified the assertion of $X : A \vee B$ was a proof of $X : B$, we may put that proof together with that of $Y(B) : C$, and obtain the same conclusion:

$$\frac{X : B \quad Y(B) : C}{Y(X) : C}$$

Either way, the elimination rule permits no more than would be permitted by an aplication of Cut directly to the ground for the assertion of $X : A \vee B$, namely, $X : A$ or $X : B$.

Cut rules are usually associated with sequent calculi, that is, calculi all of whose operational rules introduce occurrences of connectives, sometimes in wffs in the antecedent of sequents, sometimes in wffs in their succedents. Natural deduction systems, on the other hand, usually express transitivity by the admission of a normalisation theorem, asserting that proofs may be put into normal form, that is, a form in which no formula occurrence is maximal in the sense of being introduced into the proof by an application of an introduction rule, only to be removed from the proof by an application of an elimination rule. Our formulation, however, allows both aspects of transitivity to be revealed. For the proof-objects (in the sense of §6.4) are sequents, while the operational inference rules pair off into introduction rules, giving (one or more) sufficient grounds for assertion of a wff, and elimination rules, showing that those proof-grounds are also (jointly) necessary.

Consider, as an example, how the above correspondence between the introduction and elimination rules of '$\vee$' expresses itself in the possibility of removal of maximal occurrences of '$A \vee B$'. An occurrence of '$A \vee B$' is maximal if it occurs both as the consequent of the conclusion of an application of $\vee$I and as the consequent of the major premise (that is, the premise containing the connective occurrence in question) of an application of $\vee$E. Without loss of generality, suppose $X : A \vee B$ has been derived from $X : A$. (The argument proceeds relative to B and Σ_2 if $X : A \vee B$ came from $X : B$.)

$$\dfrac{\dfrac{\dfrac{\Pi}{X : A}}{X : A \vee B} \quad \dfrac{\Sigma_1(A)}{Y(A) : C} \quad \dfrac{\Sigma_2(B)}{Y(B) : C}}{Y(X) : C}$$

This schema can be simplified as follows. We trace occurrences of A in the antecedent of bunches in Σ_1 back from $Y(A)$ in a natural way. These precursors, as we might call them, must contain a first member—a line at which A arises in one of three ways: either

1. A is assumed, or
2. A is introduced by EK, or
3. A is '$\mathtt{t}$', and arises by $\mathtt{t}$=.

(If A results at any point from contraction, extensional or intensional, we must trace both precursors. So there may be several initial precursors.) However A arose, we replace those occurrences of A in the antecedent of sequents by X. If (1) A arose by assumption, the line containing X in place of A now reads $X : A$, and is accordingly justified by Π. If (2) A arose by weakening, X may be equally justified by the same rule. If 3) $A = \mathtt{t}$, and was introduced by $\mathtt{t}$=, then we may replace '$\mathtt{t}$' by X since Π tells us that $X \vdash \mathtt{t}$ and so, given $Z(\mathtt{t}) : D$, we may infer $Z(X) : D$ by Cut. Thus, that family of occurrences of A related as precursors of the distinguished occurrence of A in $Y(A)$, may each and every one be replaced by X. In other words, the proof reduces to

$$\dfrac{\Sigma_1(X)}{Y(X) : C}$$

and we have a derivation of $Y(X) : C$ without the particular maximal occurrence of '$A \vee B$'.

Admission of such a normalisation result depends on the 'harmony' we mentioned obtaining between the introduction and the elimination rules. In particular, it depends in a way the Cut Theorem does not, on the harmony holding for all the connectives. Inspection of the proof of Proposition 4.4 (The Cut Theorem), and reflection on the rules for negation in **DW** and its extensions, immediately reveals that normalisation fails for our calculus as presented in §§4.2 and 4.3, while Cut holds in virtue of the harmony between the introduction and elimination rules for the connectives '&' and '×' whose logic matches the two modes of combination of bunches. (They stand proxy for those operations in the characteristic wffs of finite bunches.)

The general case of harmony between the introduction and the elimination rules is as follows: we have a wff A, whose assertion is grounded on X. We wish to proceed to use A to obtain further information—to 'eliminate' A. The general template is this:

$$\frac{X : A \quad Y(\text{bunch}(A)) : C}{Y(X) : C}\ \text{E}$$

where bunch(A) expresses the grounds on which $X : A$ was asserted. In the case of '&' and '×', we can express bunch(A) in our formalism. Bunch(A & B) is $\{A, B\}$; bunch($A \times B$) is $A \circ B$. (E) is then valid because $A = \mathsf{cf}(\text{bunch}(A))$, and so (E) follows from Cut because $\text{bunch}(A) \leq \mathsf{cf}(\text{bunch}(A))$ by Proposition 4.6i).

But for the other connectives, '→', '∨', '∼', we have no corresponding bunching operations. We could add them, as Schroeder-Heister proposes.[14] But we can avoid that complexity by the following reasoning. Take '→' as an example. What justifies assertion of '$A \to B$' is a derivation of B from A (with parameters, of course). Let us signify that B is derivable from A by $A \Supset B$, pretending that '$\Supset$' is some appropriate bunching operation. $A \to B = \mathsf{cf}(A \Supset B)$. The elimination rule now reads:

$$\frac{X : A \to B \quad A \Supset B}{Y(X) : C} \to \text{E}'$$

We cannot use →E′ in our calculus, for we have not set out the theory of '$\Supset$'. But →E′ is justified, since $A \Supset B \leq \mathsf{cf}(A \Supset B)$ by Proposition 4.6, i.e., $A \Supset B \leq A \to B$, and so →E′ is a consequence of Cut—or, put differently, its inclusion would be needed to allow the new clauses in the proof of Proposition 4.4 (Cut) to be given.

I deferred from chapter 4 an explanation of how →E is indeed justified by the introduction rule, →I. We are now ready to see how it works. Clearly,

[14] P. Schroeder-Heister, 'Structural Frameworks with Higher-Level Rules', Habilitationsschrift: University of Konstanz, 1987, ch. 5.

$A \supseteq B \vdash A \rightarrow B$, whence by the Deduction Equivalence, $(A \supseteq B); A \vdash B$. We can then reason as follows:

$$\dfrac{X : A \rightarrow B \qquad \dfrac{Y : A \quad (A \supseteq B); A : B}{(A \supseteq B); Y : B}\,\text{Cut}}{X; Y : B}\rightarrow \text{E}'$$

Hence our rule $\rightarrow$E is justified. It follows from $\rightarrow$E$'$ and Cut, and$\rightarrow$E$'$ is directly justified by its 'harmonious' relation to $\rightarrow$I.

$$\dfrac{X : A \rightarrow B \quad Y : A}{X; Y : B}\rightarrow \text{E}$$

Everything obtainable by $\rightarrow$E (and the other rules) is obtainable by$\rightarrow$E$'$ and Cut, together with those other rules. Conversely, everything obtainable by $\rightarrow$E$'$ (and the other rules) can be obtained by the other rules (and Cut), since as we saw, $A \supseteq B \leq A \rightarrow B$.

The problematic connective is '$\sim$': as formulated in chapter 4, the elimination rule for '$\sim$' frequently immediately follows an application of the introduction rule. Consider, for example, the proof of Proposition 4.10, Antilogism. To derive B from A we first derive '$\sim\sim B$' by $\sim$I, and then use $\sim$E to obtain B. '$\sim\sim B$' would appear to be a maximal wff in that derivation. The rules $\sim$E and $\sim$I (and subsequently CM) do not, therefore, stand in the appropriate harmony. $\sim$E is not justified by $\sim$I, consequential on interpreting $\sim$I as a necessary condition for assertion of '$\sim$A'. In other words, $\sim$I does not encapsulate in proof-conditional terms the necessary conditions for assertion of '$\sim A$'.

9.7 Negation

Can we bring our theory of negation into the appropriate 'harmonious' form, properly to appreciate the correct proof-conditions for '$\sim A$'? In this section, we will consider '$\sim$' as primitive, as described in the formal systems for **DW**, **R** and **K**. We leave to the next section consideration of '$\sim$' as defined in terms of '$\rightarrow$' and 'f'.

Dag Prawitz infers from the lack of harmony in the rules for '$\sim$' that classical negation is ill-founded. The proof-conditional theory which he adopts leads one to reject classical negation, he says, in favour of intuitionistic negation, in which the inference of A from '$\sim\ \sim$ A' (equivalently, in his context, inferring A on finding that '$\sim$ A' leads to contradiction) is invalid:

> There is some well-established deductive practice which is rejected by the intuitionistic meaning theory; in particular, this is the case with the classical rule of indirect proof (i.e., the rule of inferring A given a contradiction $\neg A$, or what amounts to the same, the rule of double negation elimination), for there is no known procedure that justifies it in the sense of the previous sections.[15]

We can see what he means by considering our rule for the assertion of '$\sim A$':

$X;Y : \sim A$	is an immediate consequence of	$X;A : \sim B$ and $Y : B$	($\sim$I)

In the stronger logics there is an additional ground for assertion of '$\sim A$':

$X : \sim A$	is an immediate consequence of	$X;A : \sim A$	(CM)

Both these rules are intuitionistically acceptable, while $\sim$E is not. Hence these rules cannot justify $\sim$E.

The fault is clear; for one does not get out more than one has put in. That is, if we wish our introduction rule(s) for '$\sim A$' to justify double negation elimination, we must build it into the introduction rule in the first place.

The clue for how to do this lies in Wittgenstein's *Tractatus*. Among the several insights which at various times Wittgenstein picked out as crucial was this: 'My fundamental idea is that the "logical constants" are not representatives; that there can be no representatives of the logic of facts.'[16] His idea is this: negation is an operation which outputs as value a proposition contradictory to its input. So, for example, '$\sim A$' is a proposition which results from A by the operation of negation. But more than that: A is also a proposition which results from '$\sim A$' by the operation of negation. 'An operation can vanish (e.g. negation in '$\sim\sim p$': $\sim\sim p = p$).[17]

Our usual logical notation obscures this. For we write what for Wittgenstein are identical propositions in different ways. For example, we write the negation of 'p' as '$\sim p$', '$\sim\sim\sim p$', '$\sim p \vee \sim p$', '$\sim p\ \&\ \sim p$' and so on.[18] What we need is a notation for the operation itself, something which '$\sim$' obscures. Wittgenstein chose the letter 'N' to stand for the operation of negation. 'N'

[15] Prawitz, in *Theoria*, 43, p. 34.

[16] L. Wittgenstein, *Tractatus Logico-Philosophicus*, tr. D.F. Pears and B.F. McGuinness (London, 1961), S 4.0312.

[17] Ibid., §5.254.

[18] Cf. ibid., §5.512.

is not a representative, he says. It has no meaning. If it represents anything, then it 'represents' a logical, a formal, relation. But of course, in Wittgenstein's theory, formal relations cannot be represented. They show themselves. Frank Ramsey suggested that we might 'express negation not by inserting the word 'not', but by writing what we negate upside down.'[19] Inverting would then be the operation of negation, and would invite no danger of supposing negation was a constituent of a proposition.

Wittgenstein's reasons for adopting this difficult doctrine were deeply embedded in an underlying metaphysic, limning the structure of reality by analysing the true logical form of language. We need not follow him in this in order to use the particular insight about negation. For the *Tractatus* is the most thorough and coherent analysis of the classical realist world-view; and so it is unsurprising that it gives us the tool we need to give a satisfactory proof-conditional account of the meaning of classical (i.e., realist) negation.

We can now set out the realist account of negation. As introduction rule, we take this:

$X;Y : \mathsf{N}A$ is an immediate consequence of $X;A : \mathsf{N}B$ and $Y : B$ (NI)

where N is Wittgenstein's N-operation. We define NA (metalinguistically) as follows: NA stands (equivocally) for C and '$\sim\ \sim$ C' if A has the form '$\sim$ C' for some C, and (univocally) for '$\sim$ A' otherwise. The consequential elimination rule (NE') is therefore:

$Y(X) : C$ is an immediate consequence of $X : \mathsf{N}A$ and $Y(\text{bunch}(\mathsf{N}A)) : C$ (NE')

where $\mathsf{N}A = \mathsf{cf}(\text{bunch}(\mathsf{N}A))$. Can we simplify this to obtain a suitable elimination rule within the present expressibility of our calculus, without needing to introduce a new bunching operation to express bunch($\mathsf{N}A$)?

Consider the following rule:

$Y(X) : C$ is an immediate consequence of $X : \mathsf{N}A$ and $Y(\mathsf{N}A) : C$ (NE)

Clearly this rule is valid, by Cut. Conversely, since $\mathsf{N}A = \mathsf{cf}(\text{bunch}(\mathsf{N}A)) \leq \text{bunch}(\mathsf{N}A)$, by Proposition 4.6i), anything proven with (NE') may be proven with NE.

Recall that our aim in formulating (NI) and (NE) is to capture a realist

[19]F.P. Ramsey, 'Facts and Propositions', in his *Foundations*, ed. D.H. Mellor (London, 1978), p. 48. I was reminded of Ramsey's remark by A.T. Fuhrmann, 'Gentzen Semantics for De Morgan Lattice Logic', *Bulletin of the Section of Logic, Polish Academy of Sciences*, 15 (1986), pp. 96-101.

theory of negation. That we have done; for the following derivation is an instance of (NE):

$$\frac{X : \sim\sim A \quad A : A}{X : A}\ \mathsf{NE}$$

For both A and '$\sim\sim A$' have the form '$\mathsf{N}\sim A$'. Since $A : A$ needs no justification, we may infer $X : A$ from $X : \sim\sim A$, as $\sim$E says. $\sim$E is (entailed by) an instance of (NE).

Seen in this way, $\sim$I and $\sim$E do not contravene our proof-conditional theory of meaning. It appeared in chapter 4 that proofs might contain maximal occurrences of wffs of the form '$\sim\sim A$', introduced by $\sim$I and removed by $\sim$E in ways which could not be reduced. That is, certain proofs appeared not to be reducible to normal form, to proofs with no maximal formula occurrences. But this was a superficial consequence of our simplified rules. Taken in the context of the meaning-grounding rules (NI) and (NE), proofs with genuinely maximal occurrences of '$\sim\sim A$' are reducible as follows:

$$\frac{\dfrac{X; \sim A : \mathsf{N}B \quad Y : B}{X; Y : \sim\sim A}\ \mathsf{NI} \qquad A : A}{X; Y : A}\ \mathsf{NE}$$

simplifies to

$$\frac{X; \sim A : \mathsf{N}B \quad Y : B}{X; Y : A}\ \mathsf{NI}$$

and the maximal occurrence of the wff '$\sim\sim A$' has been removed.

CM gives an additional assertion-condition for '$\sim A$'. The situation becomes similar to that for disjunction, where there are two different grounds for asserting '$A \vee B$', a proof of A or equally a proof of B (from X). So here we have two different grounds for asserting '$\sim A$', a derivation of contradiction from A, or a derivation of '$\sim A$' itself from A.

Since it is part of our exercise to build double negation elimination into the assertion-condition for '$\sim$', CM can be seen as a special case of an acceptable rule which replaces '$\sim A$' in CM by '$\mathsf{N}A$':

$X : \mathsf{N}A$	is an immediate consequence of	$X; A : \mathsf{N}A$	

(NI'), the appropriate generalisation of (NI) to incorporate the new assertion condition, now becomes (on analogy with $\vee$I):

$X; Y : \mathsf{N}A$	is an immediate consequence of	$X; A : \mathsf{N}B$ and $Y : B$	
	or equally of	$(X; Y); A : \mathsf{N}A$	(NI')

Hence the elimination rule is

$Y(X) : C$	is an immediate consequence of	$X : \mathsf{N}A, Y(\mathsf{N}A) : C$	
	and	$Y(A \to \mathsf{N}A) : C$	(NE'')

(replacing '$\Supset$' by '$\to$' as in $\to$E). Recall that $\mathsf{N}\sim A$ (ambiguously) designates A. Hence $\sim$E is now a consequence of an instance of (NE''):

$$\dfrac{X :\ \sim\sim A \quad A : A \qquad \dfrac{\dfrac{(\sim A \to A) :\ \sim A \to A \quad \sim A :\ \sim A}{(\sim A \to A);\ \sim A : A}\,\to\mathrm{E}}{(\sim A \to A) : A}\,\mathsf{NI}''}{X : A}\,\mathsf{NE}''$$

Since all other nodes are initial, we may infer $X : A$ from $X :\ \sim\sim A$, as $\sim$E says.

9.8 Normalisation

However, setting '$\sim$' in the context of the (NI) and (NE) rules introduces the possibility of radical changes in the assertion-conditions of the other connectives. For '$A \vee B$', for example, may now be asserted not only on the basis of a proof of A (or of B), but equally on the basis of a proof of contradiction from '$\sim(A \vee B)$'. Does this extension in fact produce any significant increase in the logical power of the other connectives?

In one sense, yes: formulae not previously assertible are so. For example, in the presence of (CM), '$A \vee \sim A$' is assertible, and indeed it no longer follows that if '$A \vee B$' is assertible, so too is one of A or B. But in another sense, it does not produce the dramatic increase one might expect. For the extension does not occasion any change in the elimination rules for the other connectives.

The important case to consider is $\mathbf{K}$, for as we noted, $\mathbf{K}$ is not a conservative extension of $\mathbf{K}_+$, and so the addition of negation rules to $\mathbf{K}_+$ does serve to engender a radical change in the notion of derivability. However, $\mathbf{K}$, formulated with the rules (NI) and (NE), admits a normal form theorem, that maximal formulae may be removed from proofs.

Recall that a maximal formula is a formula occurrence which is the consequent of a sequent standing as conclusion of an introduction rule and at the same time the major premise of an elimination rule. Normally these introduction and elimination rules will be rules for the same connective—&I and &E, $\vee$I and $\vee$E and so on. Clearly, maximal formulae so occurring can be removed in the usual way. Moreover, occurrence standing between (NI) and (NE) can be removed as in §9.7.

The case to consider is that of formulae standing as consequent of the conclusion of an application of (NI) and as major premise of &E, ∨E and →E (×E drops out as no different from &E in the context of **K**). We take them in turn. First &E; recall that we are operating within **K**. Hence the difference between intensional and extensional premise combination may be ignored, and bunches may be treated as sets. The maximal formula 'A & B' occurs as follows:

$$\dfrac{\dfrac{\dfrac{\Pi(\mathsf{N}(A \mathbin{\&} B))}{X, \mathsf{N}(A \mathbin{\&} B) : \mathsf{N}C} \quad Y : C}{X, Y : A \mathbin{\&} B}\,\mathsf{NI} \qquad Z, A, B : D}{Z, X, Y : D}\,\&\mathrm{E}$$

where Π is the derivation of $X, \mathsf{N}(A \mathbin{\&} B) : \mathsf{N}C$, marking the precursors of $\mathsf{N}(A \mathbin{\&} B)$.

The maximal formula 'A & B' may now be removed, by tracing $\mathsf{N}(A \mathbin{\&} B)$ to its initial precursors, and replacing them all by $Z, \mathsf{N}D$. (NI) then operates directly on $\mathsf{N}D$, making the subsequent use of &E impossible and unnecessary:

$$\dfrac{\dfrac{\Pi(Z, \mathsf{N}D)}{Z, X, \mathsf{N}D : \mathsf{N}C} \quad Y : C}{Z, X, Y : D}\,\mathsf{NI}$$

The initial precursors of $\mathsf{N}(A \mathbin{\&} B)$ are of two types:

1. occurrences of $\mathsf{N}(A \mathbin{\&} B)$ as antecedents of axioms, and

2. results of Weakening.

(The third type we considered in §9.5 was of occurrences of '$\mathtt{t}$', but $\mathsf{N}(A \mathbin{\&} B)$ does not have that form.) Clearly $\mathsf{N}D$ may be as well introduced by EK as $\mathsf{N}(A \mathbin{\&} B)$, so we concentrate on case 1).

The axiom has the form $\mathsf{N}(A \mathbin{\&} B) : \mathsf{N}(A \mathbin{\&} B)$. We need to show that we may derive $Z, \mathsf{N}D : \mathsf{N}(A \mathbin{\&} B)$ in place of this. Recall that the minor premise of the problematic &E is $Z, A, B : D$. We can then derive $Z, \mathsf{N}D : \mathsf{N}(A \mathbin{\&} B)$ as follows:

$$\dfrac{\dfrac{Z, A, B : D}{Z, A \mathbin{\&} B : D}\,4.2 \qquad \mathsf{N}D : \mathsf{N}D}{Z, \mathsf{N}D : \mathsf{N}(A \mathbin{\&} B)}\,\mathsf{NI}$$

The occurrences of Z are then transmitted to the end-sequent of Π. More than one occurrence of Z consequent on there being several initial precursors of $\mathsf{N}(A \mathbin{\&} B)$ are contracted to one, by virtue of the bunches being sets.

A similar analysis and reduction applies to maximal formulae of the forms '$A \vee B$' and '$A \to B$'. We require only to derive:

$$\begin{array}{r}
Z, \mathsf{N}D : \mathsf{N}(A \vee B) \\
\text{and } Z, \mathsf{N}D : \mathsf{N}(A \to B)
\end{array}$$

in each case (the conclusion is B in the latter case). First we take '$\vee$'. We have, as minor premise of the problematic occurrence of $\vee$E, derivations of $Z, A : D$ and $Z, B : D$. We can derive $Z, \mathsf{N}D : \mathsf{N}(A \vee B)$ as follows:

$$\dfrac{\dfrac{A \vee B : A \vee B \quad Z, A : D \quad Z, B : D}{Z, A \vee B : D}\,\vee\mathrm{E} \qquad \mathsf{N}D : \mathsf{N}D}{Z, \mathsf{N}D : \mathsf{N}(A \vee B)}\,\mathsf{NI}$$

Occurrences of $\mathsf{N}(A \vee B)$ as initial precursors in axioms may then be replaced by $Z, \mathsf{N}D : \mathsf{N}(A \vee B)$.

Finally, we consider '$\to$'. We have, as minor premise of the problematic occurrence of $\to$E, a derivation of $Z : A$.

$$\dfrac{\dfrac{X, \mathsf{N}(A \to B) : \mathsf{N}C \quad Y : C}{X, Y : A \to B}\,\mathsf{NI} \qquad Z : A}{X, Y, Z : B}\,\to\mathrm{E}$$

We replace initial precursors of $\mathsf{N}(A \to B)$ by $Z, \mathsf{N}B$, obtaining thereby a derivation of $X, Y, Z : B$ without the maximal formula '$A \to B$'. This replacement is justified by the following derivation of $Z, \mathsf{N}B : \mathsf{N}(A \to B)$.

$$\dfrac{\dfrac{A \to B : A \to B \quad Z : A}{Z, A \to B : B}\,\to\mathrm{E} \qquad \mathsf{N}B : \mathsf{N}B}{Z, \mathsf{N}B : \mathsf{N}(A \to B)}\,\mathsf{NI}$$

Proofs in **K** can, therefore, be placed in normal form, without occurrence of maximal formulae. Hence, the rules (NI) and (NE) stand in the appropriate harmony.

It remains to show how to adapt this analysis to the case where '$\sim$' is defined as '$A \to \mathsf{f}$', and a rule is given for 'f' only.

't' and 'f' are structural constants. What this means is that their logical behaviour is governed by rules for introducing and eliminating them in antecedent positions, as parts of bunches. That is, the rules concern their introduction and elimination as I-bunches, not as wffs. In chapter 4 we followed usual practice, as in Prawitz, of giving a rule for the elimination of '$\sim$' in '$\sim A$' if, in context, it entailed 'f'. But this disguises the real nature of the rule, which, as we said there, is at root a structural rule.

The structural rules serve to induce a preordering on bunches. $\leq$ is reflexive (clearly, $W(X) : A$ is derivable if $W(X) : A$ is) and transitive (if $W(Y) : A$ is derivable from $W(X) : A$, and $W(Z) : A$ in turn from $W(Y) : A$, then $W(Z) : A$ is derivable from $W(X) : A$ by concatenating the derivations). Our notational choice of $X = Y$ as an abbreviation for $X \leq Y$ and $Y \leq X$ suggests converting the preordering $\leq$ to a partial ordering in the usual way, by identifying bunches X, Y such that $X = Y$. (Or, what comes to the same, replacing bunches as antecedents of sequents by their equivalence classes under $=$.) Seen in this way, T, X is the very same bunch as X, as was already the case, given our definition of E-bunches as sets, when we said that $X, Y = Y, X$. Similarly, as equivalence classes, or as canonical members of equivalence classes, $\mathsf{t}; X$ and X are the same bunch. (Note that, so conceived, different logics, such as **DW** and **R**, or **R** and **K**, actually contain different sequents. Their theory of what a sequent is, is different.)

We showed in Proposition 4.26 how $\mathsf{t}=$ follows from fK given the definition of '$\sim A$' as '$A \to \mathsf{f}$', 't' as '$\sim\mathsf{f}$' and the structural rules B,C and C*. Conversely, fK follows from $\mathsf{t}=$, with '$\sim$' as primitive, given NI and NE:

$$\dfrac{\dfrac{\dfrac{X;\ \sim A : \mathsf{f} \quad \mathsf{t} : \mathsf{t}}{X; \mathsf{t} : A}\ \mathsf{NI}}{\mathsf{t}; X : A}\ \mathrm{C}*}{X : A}\ \mathsf{t}=$$

fK is, therefore, a classically justifiable rule. The reason it does not appear to fit easily into the framework of the proof-conditional theory of meaning is that its true nature, as a structural rule, is disguised. It is really a rule for the manipulation of I-bunches.

9.9 Tonk Revisited

We can now understand why the rules for 'tonk' do not introduce a legitimate connective. tonk-I and tonk-E do not lie in the appropriate harmonious relation. There is no bar to introducing a connective with tonk-I. That gives unexceptionable assertion conditions for 'A tonk B'.

$X : A$ tonk B	is an immediate consequence of	$X : A$	(tonk-I)

But tonk-E is not then justified as a way of using 'A tonk B', so introduced, to make further inferences. For the ground for asserting 'A tonk B' does not warrant assertion of the conclusion of tonk-E, viz B. Note that $A = \mathsf{cf}(\text{bunch}(A \text{ tonk } B))$. Hence, avoiding setting up the machinery for bunch(A tonk B) in the usual way, the appropriate elimination rule corresponding to tonk-I reduces to:

$Y(X) : C$ is an immediate consequence of $X : A$ tonk B and $Y(A) : C$

In order to use this rule to infer that $X \vdash B$ from $X \vdash A$ tonk B we would need to show that $A \vdash B$. But that is exactly what we cannot in general show, and what the mistaken rule for tonk-E allows.

There is a connective 'tonk'. Its introduction rule says that a proof of A suffices to assert 'A tonk B'. If this is the sole ground for asserting 'A tonk B', that is, the condition is also necessary, then 'A tonk B' says no more and no less than A. But then its elimination rule is as given above, or, even more simply,

$X : A$ is an immediate consequence of $X : A$ tonk B (tonk-E)

The use of an expression includes both correct and incorrect uses. The task of logic is to establish a theory demarcating the correct from the incorrect uses of logical expressions, that is, the valid from the invalid arguments. Unless we mark the correct uses from the mistakes, no content will be given to the expression. The meaning of the logical expressions is constituted by those correct uses. We must examine those arguments which are accounted valid, those invalid, and it is they which determine the sense of the connectives. Gentzen's insight, taken up by Prawitz, was that the sense of a connective is given by its introduction rule. A theory of meaning for logic should be formulated in terms of rules of proof. Such a theory requires a harmony between rules for asserting a formula (introduction rules) and rules for drawing conclusions from it (elimination rules). Prawitz claimed that such a conception reveals the classical account of truth as it comes out in the theory of negation to be incorrect. We have seen that this does not follow.

Theory of meaning looks to a holistic explanation, recognising the dependency of meaning on use. Applied to logic, it means that the sense of the logical particles depends on the prior existence of a structure of argumentation. It is not any homomorphic map onto truth-values, propositions or whatever, but the content of proof, which reveals the meanings of the logical connectives.

Chapter 10

Conclusion

It is a plausible thought that an argument is valid if and only if it is impossible for the premises to be true and the conclusion false. For if it is valid, then the truth of the premises guarantees that of the conclusion, so that if the premises are true, so is the conclusion. Conversely, if it is impossible for the premises to be true and the conclusion false, then the truth of the premises rules out the falsity of the conclusion, and so guarantees its truth. Hence the argument is valid.

We saw that this condition for the validity of arguments appears to throw up some unfortunate and curious consequences. First, if the premises of an argument cannot be true *simpliciter*, then *a fortiori* the premises cannot be true and the conclusion false regardless of what the conclusion is. Hence a contradiction entails any proposition whatever. Secondly, a conjunction cannot be true and either of its conjuncts false. So a conjunction entails each of its conjuncts.

A second plausible thought was that a conditional follows from some other propositions if and only if its consequent follows from its antecedent in conjunction with those other propositions. This idea is often known by the epithet, 'The Deduction Equivalence', or indeed, when shown true of some formal system, 'The Deduction Theorem'. Applying it to the two entailment claims of the previous paragraph, we found first, that any conditional with a false antecedent is true, and secondly, that so is any conditional with a true consequent. But from the necessity of the condition given in the last paragraph for the validity of an argument (the first plausible thought), it followed that no conditional with true antecedent and false consequent is true. In other words, the truth-value of 'if P then Q' is completely determined by the truth-values of P and Q. So we seemed to be committed to the thesis that 'if' is material, the usual truth-functional implication.

'If' cannot be material. That has been shown repeatedly over the last eighty or more years.[1] We looked at one case in particular, where the truth-valud assignment given by the theory that 'if' is material is clearly wrong, and would lead one to count as valid arguments which are invalid. So there must be some error in the two plausible thoughts. That error lies in supposing the conjunction used in each thought to be the normal extensional conjunction which cannot be true if either of its conjuncts is false. We can retain the Standard Account of Validity (that it be impossible for the premises to be true and the conclusion false) and the connection between validity and the truth of the conditional (that a conditional is true contingently on some other propositions if and only if its consequent follows from its antecedent and those other propositions) provided 'and' used in each criterion is interpreted as an intensional conjunction which means only that the truth of either conjunct does not preclude that of the other. Such a conjunction is undoubtedly true if each conjunct is true; but it may also be true when one or the other is false, provided only that the truth of the one does not rule out the truth of the other.

What we did, therefore, was distinguish the Standard Account of Validity from the Classical Account of Validity. In the latter, the conjunction is interpreted extensionally. Similarly, we distinguished the Standard Deduction Equivalence, connecting validity and the conditional, from the Classical Deduction Equivalence. Once again, in the latter, both the conjunction and the conditional are interpreted extensionally. There is something right in the standard accounts. But the Classical Accounts are wrong, for they have false consequences.

The fully-worked out formal theory of the intensional conjunction needed to give the correct account of validity and the Deduction Equivalence incorporates a connective familiarly known as 'fusion'. The important fact about fusion is its admission of exportation and importation relative to the relevant conditional:

$$A \times B \to C \dashv\vdash A \to (B \to C)$$

Note that '$A \times B$' is not equivalent to any combination $\partial(\gamma(A, B))$, where ∂ is an alethic modal operator and γ is a binary truth-function. We saw that the neatest formal treatment of fusion is to distinguish two operators on premises, comma and semicolon, by structural rules (so comma is essentially set union, while semicolon is, in **R**, an associative and commutative

[1] Starting with H. MacColl, *Symbolic Logic and its Applications* (London, 1906), ch. 10, esp. pp. 75, 78.

operation satisfying absorption: $X; X \leq X$), and to extend this distinction to the connectives '&' and '$\times$' through their introduction rules:

$$\begin{array}{rllll} A\&B & \text{follows from} & X, Y & \text{iff} & A \text{ follows from } X \text{ and } B \text{ from } Y \\ \text{while } A \times B & \text{follows from} & X; Y & \text{iff} & A \text{ follows from } X \text{ and } B \text{ from } Y \end{array}$$

The elimination rules can then be given as consequences of the introduction rules in the usual way. For the introduction rule for a connective gives the conditions under which it may be asserted—it makes the meaning of the proposition containing that connective manifest.

In the elimination rule we are not concerned with the grounds for the proposition's assertion, but with what consequences we can draw from the assertion, now made. Given the doctrine of publicity of meaning, that there is no more to meaning than what is manifested in assertion, the elimination rule can legitimately draw from an assertion no more than the grounds put in. The rules pair off in a way which places the formula itself simply as a staging-post, a record of its assertion-conditions.

This is even true of negation, classical negation, when one realises that negation is a formal relation between two propositions, not itself a connective. On the realist view adopted here, propositions come in pairs of an assertion and a denial, and double negation is simply a complication introduced by unnecessary notation.

Fusion has played a considerable and valuable role in the ternary (possible-worlds) semantics for **R**, **E** and related logics. In some ways, therefore, it may appear surprising that the semantics has retained the Classical Account of Validity. The reason is clear on consideration, however. The semantics was designed in an attempt to explain to a sceptical classical logician how relevant logic works, not why it should be adopted. Fusion was the key to giving an appropriate truth-condition for the relevant conditional. '$A \to B$' is true at a if whenever A is true at b, B is true at a world 'fusing' a and b, $a; b$. The ternary relation is then introduced by the definition: $Rabc$ holds iff $a; b \leq c$.[2] This so-called 'operational semantics' was one of the main influences on the development of the proof-theory which we considered in chapter 4. Different logics place different constraints on the ternary relation R, and accordingly treat the fusion of premises in inference differently. The structural rules, identifying the structure of these premise combinations, become the primary distinguishers of the different logical systems.

Fusion lies at the heart of the analysis in this book. That it is distinct

[2]See, e.g., K. Fine, 'Models for Entailment', *Journal of Philosophical Logic*, 3 (1974), pp. 347-72.

from extensional conjunction is crucial at every step. It expresses a logical connection between two propositions, that the truth of one does not preclude that of the other—in other words, that they can (logically) be true together. It gives accounts of consequence and of the conditional which preserve the underlying intuitions, while avoiding the unacceptable consequences, of the two original paradoxes, that any argument whose premises cannot be true must be sound regardless of its conclusion, and that any necessarily true proposition follows from any proposition whatever. Fusion is also linked in **R** by the familiar De Morgan transformations to an intensional disjunction requiring similar discernment from its truth-functional look-alike. Probing further, we found that fusion yields a definition of relevance with the correct logical features—reflexivity, symmetry and non-transitivity—while being tied closely to validity and the conditional. Tthe need for fusion in the Relevant Account of Validity explains by its absence how, when (in Meyer's biblical words) 'preaching to the Gentiles in their own tongue', classical semantics for relevant logic requires a non-homophonic semantics. In contrast, when object- and metalanguage are on a par, the Relevant Account of Validity leads smoothly to completeness and soundness proofs for a formal logic incorporating the intuitions that rejected the earlier paradoxical conclusions.

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