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General-Elimination Stability

Abstract. General-elimination harmony articulates Gentzen's idea that the elimination-rules are justified if they infer from an assertion no more than can already be inferred from the grounds for making it. Dummett described the rules as not only harmonious but stable if the E-rules allow one to infer no more and no less than the I-rules justify. Pfenning and Davies call the rules locally complete if the E-rules are strong enough to allow one to infer the original judgement. A method is given of generating harmonious general-elimination rules from a collection of I-rules. We show that the general-elimination rules satisfy Pfenning and Davies' test for local completeness, but question whether that is enough to show that they are stable. Alternative conditions for stability are considered, including equivalence between the introduction- and elimination-meanings of a connective, and recovery of the grounds for assertion, finally generalizing the notion of local completeness to capture Dummett's notion of stability satisfactorily. We show that the general-elimination rules meet the last of these conditions, and so are indeed not only harmonious but also stable.

Keywords: harmony, general-elimination rules, stability, local completeness, grounds, Gentzen.

1. Introduction

The concept of general-elimination rules is inspired by Gerhard Gentzen's famous remark in [11] (translated in [40, p. 80]):

> "The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions."

He was alluding to the introduction- (I-) and elimination- (E-) rules of his natural deduction calculi. Gentzen [40, p. 81] continued:

> "It should be possible to display the E-inferences as unique functions of their corresponding I-inferences."

The purpose behind general-elimination rules is to exhibit this functional dependency by casting all the E-rules on the model of the familiar $\vee$E-rule for disjunction, *aka* proof by cases. What $\vee$E encapsulates is the idea that whatever follows both from α and from β must follow from $\alpha \vee \beta$. More generally, a GE-rule states that what follows from a formula is whatever follows from the grounds for assertion of that formula.[1]

[1]For the notion of general-elimination rule, see [22, esp. §1.2] and [42].

Consider $\vee$I: it has two cases:

$$\frac{\alpha}{\alpha \vee \beta}\ \vee\mathrm{I}_1 \qquad \frac{\beta}{\alpha \vee \beta}\ \vee\mathrm{I}_2$$

The meaning conferred on $\alpha \vee \beta$ is that it is assertible just when either α or β are assertible. Moreover, since assertions of α and β exhaust the grounds for assertion of $\alpha \vee \beta$, what follows from an assertion of $\alpha \vee \beta$ must either follow from α or from β. We thus obtain the single elimination-rule:

$$\frac{\alpha \vee \beta \qquad \begin{matrix}[\alpha]^1 \\ \gamma\end{matrix} \qquad \begin{matrix}[\beta]^2 \\ \gamma\end{matrix}}{\gamma}\ \vee\mathrm{E}(1,2)$$

We can read this as saying that γ follows from $\alpha \vee \beta$ iff γ follows from α and γ follows from β.

In general, rules may discharge assumptions.[2] Even though $\vee$I discharges no assumptions, $\vee$E discharges the assumptions of α and β.

Similar considerations apply to $\to$I:

$$\frac{\begin{matrix}[\alpha]^1 \\ \beta\end{matrix}}{\alpha \to \beta}\ \to\mathrm{I}(1)$$

The meaning conferred on $\alpha \to \beta$ is that it is assertible just in case β is assertible on the basis of α, with the assumption of α being discharged by the rule.

By the same reasoning that justified $\vee$E, what follows from the grounds for $\alpha \to \beta$ must follow from an assertion of $\alpha \to \beta$:

$$\frac{\alpha \to \beta \qquad \begin{matrix}\begin{bmatrix}\alpha \\ \vdots \\ \beta\end{bmatrix}^1 \\ \gamma\end{matrix}}{\gamma}\ \to\mathrm{E}(1)$$

We can read this as saying that γ follows from the assumption of β dependent on α iff γ follows from $\alpha \to \beta$. The $\to$E-rule allows us to discharge the assumption of β dependent on α. That is, the rule discharges the sequent

[2] In these cases, we enclose each discharged assumption in square brackets, and cross-reference it with the rule that discharges it.

$\alpha \Rightarrow \beta$. The use of ellipses marks the fact that the assumption is that of a sequent.[3] We will see how assumptions of sequents, marked by ellipses, can be replaced by derivations in the discussion of harmony in §3 below. Note that, even though the sequent $\alpha \Rightarrow \beta$ is discharged by $\rightarrow$E, the rule does not discharge the assumption α in the minor premise.[4]

The plan for the rest of the paper is as follows. In §2, we set out the formal analysis of derivations and rules presupposed in the rest of the paper. In §3 we describe the procedure by which GE-rules for a connective are generated by a set of I-rules for it. The notion of harmony (of E-rules with I-rules) is described and we show that the GE-rules are in harmony with the I-rules from which they are generated. §4 introduces the notion of stability between the I- and E-rules, which requires not only that the E-rules be in harmony with the I-rules, but also that the I-rules be in harmony with the E-rules. The remaining sections are dedicated to: i) distinguishing formal properties of I- and E-rules which are candidates for capturing the harmony of I-rules with E-rules; and ii) showing that I- and GE-rules for a connective satisfy all of these properties. Consequently, the I-rules for a connective are in harmony with its GE-rules. Since the GE-rules are harmonious with the I-rules from which they are generated, it follows that the I- and GE-rules of any connective are stable.

2. Formal Analysis

The aim of this section is to present the notion of derivation in systems S of the sort that will be presupposed in the paper.

Definition 1. *(Sequent)*

- *Any wff is a sequent of level 1.*
- *If Γ is a finite set of wffs, and α is a wff, $\Gamma \Rightarrow \alpha$ is a sequent of level 2.*

[3] The notion of sequent and the precise use of ellipses employed here are explained formally in §2.

[4] $\rightarrow$E is what Peter Schroeder-Heister calls a "higher-level" rule (thinking of the sequent $\alpha \Rightarrow \beta$ as itself a rule). We will consider for the purposes of this paper only introduction-rules allowing at most for the discharge of the assumption of formulae. Thus, only elimination-rules generated by the GE-procedure allowing at most for the discharge of sequents whose assumptions are formulae are considered. In general, higher-level assumptions can consist in sequents whose assumptions are sequents nested to any depth, in a manner close to the account of higher-level rules offered in [35]. For a more recent and approachable discussion, see [37].

- *If Γ is a finite set of wffs or sequents of level 2, and α is a wff, $\Gamma \Rightarrow \alpha$ is a sequent of level 3.*

Notation: $\alpha, \beta, \gamma, \ldots$ are metavariables over wffs. π, π_i, π_{ik} $(i \in I, k \in K)$ are metavariables over sequents, of all levels unless specified otherwise. Γ, Δ, Θ are metavariables over sets of sequents. A sequent $\Gamma \Rightarrow \alpha$ where $\Gamma = \{\pi_1, \ldots, \pi_n\}$ is written $\pi_1, \ldots, \pi_n \Rightarrow \alpha$.[5]

DEFINITION 2. *The antecedent, $a(\pi)$, and the succedent, $s(\pi)$, of a sequent π are defined as follows:*

- $a(\pi) = \begin{cases} \varnothing & \text{if } \pi \text{ is of level 1} \\ \pi_1, \ldots, \pi_n & \text{if } \pi = \pi_1, \ldots, \pi_n \Rightarrow \gamma \end{cases}$
- $s(\pi) = \begin{cases} \pi & \text{if } \pi \text{ is of level 1} \\ \gamma & \text{if } \pi = \pi_1, \ldots, \pi_n \Rightarrow \gamma \end{cases}$

DEFINITION 3. *(Rules of Inference)*

- *The form of an instance of a rule of inference, $\mathscr{J}$, is an $n+1$-tuple of sequents of level 2 or 3, $\pi_i = \Gamma_i \Rightarrow \alpha_i$, $1 \leq i \leq n+1$, asserting that π_{n+1} is a consequence of $\pi_1, \ldots, \pi_n$, written*

$$\frac{\pi_1 \quad \ldots \quad \pi_n}{\pi_{n+1}}\ \mathscr{J}$$

- *In general, $\Gamma_{n+1} = \bigcup_{i=1}^{n}(\Gamma_i - \Theta_i) \cup \Theta_{n+1}$. If Θ_i is non-empty, we say that $\mathscr{J}$ discharges the sequents in Θ_i (wffs, or sequents of level 2).*
- – *If $n = 0$, $\mathscr{J}$ is a rule of level 0.*
  – *Otherwise, if $\mathscr{J}$ discharges no assumptions, $\mathscr{J}$ is of level 1, and*
  – *if the instances of $\mathscr{J}$ discharge assumptions of at most level n, then $\mathscr{J}$ is of level $n+1$.*[6]

Notation: Where π_i is $\Gamma_i \Rightarrow \alpha_i$, $1 \leq i \leq n+1$, the form of an instance of a rule $\mathscr{J}$

$$\frac{\pi_1 \quad \ldots \quad \pi_n}{\pi_{n+1}}\ \mathscr{J}$$

[5] Although the constituents of derivations will be sequents, following Gentzen's presentation in [12, §5], the systems considered, like his, are natural deduction systems, with I- and E-rules.

[6] Note that in a sequent $\Gamma \Rightarrow \alpha$, Γ consists of wffs or sequents of at most level 2, so assumptions discharged can only be wffs or sequents of level 2, as in footnote 4.

may also be written as

$$\frac{\pi_1 \quad \dots \quad \pi_n}{\alpha_{n+1}} \mathscr{I}$$

or as

$$\frac{\alpha_1 \quad \dots \quad \alpha_n}{\alpha_{n+1}} \mathscr{I}$$

If Θ_i is the set of discharged sequents in Γ_i, then the form may also be written as

$$\frac{\begin{matrix}[\Theta_1] & & [\Theta_n] \\ \alpha_1 & \dots & \alpha_n\end{matrix}}{\alpha_{n+1}} \mathscr{I}$$

When $\pi \in \Theta_j$, $1 \leq j \leq n$ is a sequent of level 2 of the form $\gamma_1, \dots, \gamma_k \Rightarrow \beta$, π may be written as

$$\begin{matrix}\gamma_1, \dots, \gamma_k \\ \vdots \\ \beta\end{matrix}$$

DEFINITION 4. *(Rules of S)*

Operational Rules *The operational rules of S, consisting of I-rules giving the meaning of the connectives and E-rules generated by the GE-procedure (to be described in §3), will vary depending on the system, of, e.g., minimal, intuitionistic or classical logic, considered. The rules* $\vee I_1$, $\vee I_2$, $\vee E$, $\to I$ *and* $\to E$ *of §1 are written in sequent (*aka *logistic*[7]*) notation as follows:*

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta} \vee I_1 \quad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta} \vee I_2 \quad \frac{\Gamma \Rightarrow \alpha \vee \beta \quad \Delta, \alpha \Rightarrow \gamma \quad \Theta, \beta \Rightarrow \gamma}{\Gamma, \Delta, \Theta \Rightarrow \gamma} \vee E$$

$$\frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta} \to I \qquad \frac{\Gamma \Rightarrow \alpha \to \beta \quad \Delta, \alpha \Rightarrow \beta \Rightarrow \gamma}{\Gamma, \Delta \Rightarrow \gamma} \to E$$

Structural Rules

$$\frac{}{\alpha \Rightarrow \alpha} Ass_1 \qquad \frac{\Gamma, \Theta \Rightarrow \alpha \quad \Theta \Rightarrow \alpha, \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \beta} Slice$$

$$\frac{\Gamma \Rightarrow \alpha}{\pi, \Gamma \Rightarrow \alpha} Thin \qquad \frac{\Gamma_1 \Rightarrow \alpha_1 \quad \dots \quad \Gamma_n \Rightarrow \alpha_n}{(\bigcup_1^n \alpha_i) \Rightarrow \beta, \bigcup_1^n \Gamma_i \Rightarrow \beta} Ass_2$$

[7]As termed by Francez and Dyckhoff [9, p. 615]. See also [10, p. 17]. Vacuous discharge is handled by use of *Thin*.

Ass_1 is a rule of level 0, *Thin*, Ass_2, $\wedge$I, $\vee$I$_1$ and $\vee$I$_2$ are rules of level 1, $\wedge$E$_1$, $\wedge$E$_2$, $\vee$E and $\to$I are rules of level 2, and *Slice* and $\to$E are rules of level 3.

The structural rule Ass_2 formalizes the introduction of a higher-level assumption in the course of a derivation, as described in [37, p. 1191]. Similarly, *Slice* can be thought of as a higher-level 'Cut for sequents': in light of Definition 12 in §8 below, we could express *Slice* as saying that if $\Theta \Rightarrow \alpha$ is derivable from Γ and $\Delta \Rightarrow \beta$ is derivable from $\Theta \Rightarrow \alpha$, then $\Delta \Rightarrow \beta$ is derivable from Γ.[8]

Definition 5. *(Derivation) A derivation in S is a finite tree of sequents in which every non-initial sequent (that is, any sequent which is not a leaf) is a consequence by a rule of inference of S of the sequents immediately above it, and every leaf is the consequence of a rule of S of level 0.*

Notation: Let $\mathfrak{A}_i, \mathfrak{B}_i$ $(i \in I)$ be metavariables over derivations. We now show how to rewrite derivations given in sequent (or logistic) notation in the perhaps more familiar notation using only wffs.

An application of the rule Ass_1 may be written simply as

$$\alpha$$

If $\mathfrak{A}$ is a derivation of the sequent $\alpha \Rightarrow \beta, \Gamma \Rightarrow \beta$ resulting from the application of the rule Ass_2 to the terminal node $\Gamma \Rightarrow \alpha$ of derivation $\mathfrak{B}$, rewritten as $\mathfrak{B}'$, then $\mathfrak{A}$ may be written as

$$\begin{array}{c}\mathfrak{B}' \\ \vdots \\ \beta\end{array}$$

If $\mathfrak{A}$ is a derivation of the sequent $\Gamma_{n+1} \Rightarrow \alpha_{n+1}$ resulting from the application of a rule $\mathscr{J}$ to the terminal nodes $\Gamma_i \Rightarrow \alpha_i$ of derivations $\mathfrak{B}_i$, rewritten as $\mathfrak{B}'_i$, $1 \leq i \leq n$, then $\mathfrak{A}$ may be written as

$$\frac{\mathfrak{B}'_1 \quad \ldots \quad \mathfrak{B}'_n}{\alpha_{n+1}}\ \mathscr{J}$$

If the application of $\mathscr{J}$ discharges the sequents $\pi_{i1}, \ldots \pi_{in_i}$ in Γ_i, then $\mathfrak{A}$ may be written as

$$\frac{\mathfrak{B}''_1 \quad \ldots \quad \mathfrak{B}''_n}{\alpha_{n+1}}\ \mathscr{J}$$

[8] *Slice* generalizes Gentzen's Cut rule to higher-level sequents in a manner not unlike that in which Cut was a specialization of Hertz's more general rule Syllogism: see Gentzen's discussion in [14, pp. 31-2]. Proposition 1 below shows that *Cut* (for formulae) is derivable from *Slice*.

where each $\mathfrak{B}_i''$ results from $\mathfrak{B}_i'$ by replacing α by $[\alpha]$ if π_{ij}, $1 \leq j \leq n_i$, is a formula α, and replacing

$$\begin{array}{c} \alpha_{i1}, \ldots, \alpha_{ik} \\ \vdots \\ \alpha \end{array}$$

by

$$\left[\begin{array}{c} \alpha_{i1}, \ldots, \alpha_{ik} \\ \vdots \\ \alpha \end{array} \right]$$

if π_{ij} is a sequent of the form $\alpha_{i1}, \ldots, \alpha_{ik} \Rightarrow \alpha$.

DEFINITION 6. *(Derivability) We say that α is derivable from Γ in S, written $\Gamma \vdash_S \alpha$, if there is a derivation in S whose root sequent is $\Gamma' \Rightarrow \alpha$, where $\Gamma' \subseteq \Gamma$.*

PROPOSITION 1. *The following structural rules are derivable:*

1. $\dfrac{\alpha \Rightarrow \alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \beta}\ \mathit{Refl}$
2. $\dfrac{\Gamma \Rightarrow \alpha \quad \alpha, \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \beta}\ \mathit{Cut}$
3. $\dfrac{\Gamma \Rightarrow \alpha \quad \beta, \Delta \Rightarrow \gamma}{\alpha \Rightarrow \beta, \Gamma, \Delta \Rightarrow \gamma}\ \mathit{Trans}$

PROOF. 1.

$$\dfrac{\dfrac{}{\alpha \Rightarrow \alpha}\ \mathit{Ass}_1 \qquad \alpha \Rightarrow \alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \beta}\ \mathit{Slice}$$

2.

$$\dfrac{\Delta, \alpha \Rightarrow \beta \qquad \dfrac{\Gamma \Rightarrow \alpha}{\Gamma, \alpha \Rightarrow \beta \Rightarrow \beta}\ \mathit{Ass}_2}{\Gamma, \Delta \Rightarrow \beta}\ \mathit{Slice}$$

3.

$$\dfrac{\dfrac{\Gamma \Rightarrow \alpha}{\alpha \Rightarrow \beta, \Gamma \Rightarrow \beta}\ \mathit{Ass}_2 \qquad \beta, \Delta \Rightarrow \gamma}{\alpha \Rightarrow \beta, \Gamma, \Delta \Rightarrow \gamma}\ \mathit{Cut}$$

■

3. General-Elimination Harmony

Now take an arbitrary connective $*$, forming a wff $*(\alpha_1, \ldots, \alpha_n)$, for short $*\vec{\alpha}$, having m I-rules with $0 \leq j \leq n_i$ premises, each I-rule $*\mathrm{I}_i$ $(0 \leq i \leq m)$ of the form (in logistic notation):[9]

$$\frac{\Gamma_{i,1}, \Theta_{i,1} \Rightarrow \alpha_{i,1} \quad \ldots \quad \Gamma_{i,n_i}, \Theta_{i,n_i} \Rightarrow \alpha_{i,n_i}}{\bigcup_{j=0}^{n_i} \Gamma_{i,j} \Rightarrow *\vec{\alpha}} *\mathrm{I}_i$$

The set $\bigcup_{j=1}^{n_i} \Theta_{i,j}$ $(1 \leq j \leq n_i)$ is the set of formulae (i.e., sequents of level 1) discharged by the rule. We let $\Gamma_{i,0} = \varnothing$ unless $n_i = 0$.

For each $0 \leq l \leq n_i$, we let

$$g_l(*\mathrm{I}_i) = \begin{cases} \varnothing & \text{if } l = 0 \\ \alpha_{i,l} & \text{if } \Theta_{i,l} = \varnothing \\ \Theta_{i,l} \Rightarrow \alpha_{i,l} & \text{otherwise} \end{cases}$$

The elements in $g(*\mathrm{I}_i) = \bigcup_{l=1}^{n_i} g_l(*\mathrm{I}_i)$ are the *grounds* on which the ith I-rule permits assertion of $*\vec{\alpha}$. For example, α is the ground on which $\vee\mathrm{I}_1$ permits assertion of $\alpha \vee \beta$; $\alpha \Rightarrow \beta$ is the ground on which $\rightarrow$I permits assertion of $\alpha \rightarrow \beta$.

The GE-procedure developed in [32, p. 563] assigns $\prod_{i=1}^{m} n_i$ E-rules to the I-rules for $*$. For each connective, $*$, with m I-rules, let $j \in \bigtimes_{i=1}^{m} n_i$, where $\bigtimes_{i=1}^{m} n_i$ is the set of all functions j with domain $\{1, \ldots, m\}$, where $1 \leq j(i) \leq n_i$, for all $i \in \{1, \ldots, m\}$. Then the jth E-rule has the general-elimination form:[10]

$$\frac{\Gamma \Rightarrow *\vec{\alpha} \qquad \Delta_1, g_{j(1)}(*\mathrm{I}_1) \Rightarrow \gamma \quad \ldots \quad \Delta_m, g_{j(m)}(*\mathrm{I}_m) \Rightarrow \gamma}{\Gamma, \bigcup_{k=1}^{m} \Delta_k \Rightarrow \gamma} *\mathrm{E}_j$$

which we can also write as

$$\frac{*\vec{\alpha} \qquad \begin{matrix} [g_{j(1)}(*\mathrm{I}_1)]^1 \\ \gamma \end{matrix} \quad \ldots \quad \begin{matrix} [g_{j(m)}(*\mathrm{I}_m)]^m \\ \gamma \end{matrix}}{\gamma} *\mathrm{E}_j(1, \ldots, m)$$

[9]See Definition 3. The format here, requiring $*\vec{\alpha}$ to occur as "terminal symbol" [13, p. 80] of the I-rule, excludes "general introduction rules" (see [22] and [20]), where $*\vec{\alpha}$ is also allowed to occur only as a discharged assumption.

[10]If $*$ has no I-rules, then g is undefined, so $*$E has no minor premises, whence $*$ is equivalent to $\bot$, whose only E-rule is: $\dfrac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow \gamma} \bot\mathrm{E}$. Note that if $*$ has any I-rule with no premises, then $\prod_{i=1}^{m} n_i = 0$, and accordingly, $*$ has no E-rule. In this case, $*$ is equivalent to $\top$, whose I-rule is $\dfrac{}{\Gamma \Rightarrow \top} \top\mathrm{I}$. See [33, p. 296 n. 5].

Note that each E-rule discharges, in each minor premise, exactly one of the grounds of the corresponding I-rule if it has any grounds, and discharges nothing otherwise.

Dummett's idea in [4, pp. 396-7] in introducing the notion of "proof-theoretical harmony", was to articulate Gentzen's suggestion that the meaning of a connective encapsulated in its I-rules should justify the inferences permitted by the E-rules. Harmony requires that the inferences drawn (by the elimination-rules) from an assertion be "consonant" (*loc.cit.*) with the grounds for that assertion (given in the introduction-rules). That each GE-rule $\ast\mathrm{E}_j$ is justified by the collection of I-rules is shown by a kind of redundancy in the role of $\ast\vec{\alpha}$, namely, that whatever can be inferred from $\ast\vec{\alpha}$ can already be inferred from one of the grounds for asserting it. The scheme

$$\dfrac{\dfrac{\begin{matrix}[\Theta_{i,1}]^1\\ \mathfrak{A}_1\\ \alpha_{i,1}\end{matrix} \quad \dots \quad \begin{matrix}[\Theta_{i,n_i}]^{n_i}\\ \mathfrak{A}_{n_i}\\ \alpha_{i,n_i}\end{matrix}}{\ast\vec{\alpha}}\,\ast\mathrm{I}_i(1,\dots,n_i) \qquad \dots \qquad \begin{matrix}[g_{j(i)}(\ast\mathrm{I}_i)]^i\\ \mathfrak{B}_i\\ \gamma\end{matrix} \qquad \dots}{\gamma}\,\ast\mathrm{E}_j(1,\dots,m)$$

simplifies to

$$\begin{matrix}\Theta_{i,j(i)}\\ \mathfrak{A}_i\\ \alpha_{i,j(i)}\\ \mathfrak{B}_i\\ \gamma\end{matrix}$$

However, this glosses over an important distinction as to whether $\Theta_{i,j(i)}$ is empty or not. Spelling it out in the more explicit logistic notation, to reveal the dependencies more clearly, we observe that we can simplify the scheme

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{A}_1\\ \Gamma_{i,1},\Theta_{i,1}\Rightarrow\alpha_{i,1}\end{matrix} \quad \dots \quad \begin{matrix}\mathfrak{A}_{n_i}\\ \Gamma_{i,n_i},\Theta_{i,n_i}\Rightarrow\alpha_{i,n_i}\end{matrix}}{\bigcup_{k=1}^{n_i}\Gamma_{i,k}\Rightarrow\ast\vec{\alpha}}\,\ast\mathrm{I}_i \qquad \begin{matrix}\mathfrak{B}_1\\ \Delta_1,g_{j(1)}(\ast\mathrm{I}_1)\Rightarrow\gamma\end{matrix} \quad \dots \quad \begin{matrix}\mathfrak{B}_m\\ \Delta_m,g_{j(m)}(\ast\mathrm{I}_m)\Rightarrow\gamma\end{matrix}}{\bigcup_{k=1}^{n_i}\Gamma_{i,k},\bigcup_{l=1}^{m}\Delta_l\Rightarrow\gamma}\,\ast\mathrm{E}_j$$

by taking the ith minor premise of the application of $\ast\mathrm{E}_j$ together with the $j(i)$th premise of $\ast\mathrm{I}_i$ and applying either *Cut* or *Slice*:

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{A}_{j(1)}\\ \Gamma_{i,j(i)},\Theta_{i,j(i)}\Rightarrow\alpha_{i,j(i)}\end{matrix} \quad \begin{matrix}\mathfrak{B}_i\\ \Delta_i,g_{j(i)}(\ast\mathrm{I}_i)\Rightarrow\gamma\end{matrix}}{\Gamma_{i,j(i)},\Delta_i\Rightarrow\gamma}\,\textit{Cut/Slice}}{\bigcup_{k=1}^{n_i}\Gamma_{i,k},\bigcup_{l=1}^{m}\Delta_l\Rightarrow\gamma}\,\textit{Thin}$$

since either

- $\Theta_{i,j(i)} = \varnothing$, in which case $g_{j(i)}(*\mathrm{I}_i) = \alpha_{i,j(i)}$ and we apply *Cut*, or
- $\Theta_{i,j(i)} \neq \varnothing$, in which case $g_{j(i)}(*\mathrm{I}_i) = \Theta_{i,j(i)} \Rightarrow \alpha_{i,j(i)}$ and we apply *Slice*.[11]

The existence of such a reduction shows that whatever follows from $*\vec{\alpha}$ already follows from what justified its assertion in the first place. The E-rules draw no conclusion not already warranted by the grounds for asserting $*\vec{\alpha}$. Accordingly, the GE-rules generated by the GE-procedure lie in harmony with the I-rules—they exhibit what in [9] is called "general-elimination harmony".[12]

To take an example, for the harmony of $\vee$I and $\vee$E we show that we can close the assumptions of α and of β by the grounds for their assertion, instead of discharging them by the application of $\vee$E:

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{A}\\ \alpha\end{matrix}}{\alpha \vee \beta}\,\vee\mathrm{I} \qquad \begin{matrix}[\alpha]^1\\ \mathfrak{B}_1\\ \gamma\end{matrix} \qquad \begin{matrix}[\beta]^2\\ \mathfrak{B}_2\\ \gamma\end{matrix}}{\gamma}\,\vee\mathrm{E}(1,2) \qquad \text{reduces to} \qquad \begin{matrix}\mathfrak{A}\\ \alpha\\ \mathfrak{B}_1\\ \gamma\end{matrix}$$

Similarly, we show the harmony of $\rightarrow$I and $\rightarrow$E by closing the assumption of $\alpha \Rightarrow \beta$ by replacing it by a derivation of β dependent on α which justifies the assertion of $\alpha \rightarrow \beta$, rather than by discharging the assumption of $\alpha \Rightarrow \beta$ through $\rightarrow$E:

$$\dfrac{\dfrac{\begin{matrix}[\alpha]^1\\ \mathfrak{A}_1\\ \beta\end{matrix}}{\alpha \rightarrow \beta}\rightarrow\mathrm{I}(1) \qquad \begin{matrix}\left[\begin{matrix}\alpha\\ \vdots\\ \beta\end{matrix}\right]^2\\ \mathfrak{A}_2\\ \gamma\end{matrix}}{\gamma}\rightarrow\mathrm{E}(2) \qquad \text{reduces to} \qquad \begin{matrix}\alpha\\ \mathfrak{A}_1\\ \beta\\ \mathfrak{A}_2\\ \gamma\end{matrix}$$

[11] In general, in the presence of other connectives, we may need to perform so-called permutative reductions to permute the application of $*\mathrm{E}_j$ with other E-rules to bring $*\mathrm{I}_i$ and $*\mathrm{E}_j$ into contact. See, e.g., [5, p. 112], [6, p. 250], [42, §2] and [34, §2]. It will, of course, be necessary to show, in any particular logic, that the result of simplification is still a correct derivation, which will depend *inter alia* on the conditions on discharge of assumptions and on the satisfaction of side conditions. Note that the GE-procedure is defined only for I-rules whose parametric assumptions have no side conditions. See §§8 and 9 below for a discussion of I- and E-rules with side conditions.

[12] For further details, see [28, 29] and [32]. The general-elimination procedure proposed in [9] is significantly different from that given here.

In the more explicit logistic notation, it reads:

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{A}_1\\ \Gamma, \alpha \Rightarrow \beta\end{matrix}}{\Gamma \Rightarrow \alpha \to \beta}\to\mathrm{I} \quad \begin{matrix}\mathfrak{A}_2\\ \Delta, \alpha \Rightarrow \beta \Rightarrow \gamma\end{matrix}}{\Gamma, \Delta \Rightarrow \gamma}\to\mathrm{E}$$

which reduces to

$$\dfrac{\begin{matrix}\mathfrak{A}_1\\ \Gamma, \alpha \Rightarrow \beta\end{matrix} \quad \begin{matrix}\mathfrak{A}_2\\ \Delta, \alpha \Rightarrow \beta \Rightarrow \gamma\end{matrix}}{\Gamma, \Delta \Rightarrow \gamma}\,\mathit{Slice}$$

Thus an assumption can be closed in two different ways: by discharging it by the application of a rule, or by establishing it by means of a derivation.

In fact, →E is not as unfamiliar as it may seem. Letting $\gamma = \beta$, we obtain as a special case a derivation of β from $\alpha \to \beta$ and α:

$$\dfrac{\alpha \to \beta \quad \begin{bmatrix}\alpha\\ \vdots\\ \beta\end{bmatrix}^1}{\beta}\to\mathrm{E}(1)$$

or more explicitly in the logistic notation:

$$\dfrac{\dfrac{}{\alpha \to \beta \Rightarrow \alpha \to \beta}\,Ass_1 \quad \dfrac{\dfrac{}{\alpha \Rightarrow \alpha}\,Ass_1}{\alpha, \alpha \Rightarrow \beta \Rightarrow \beta}\,Ass_2}{\alpha, \alpha \to \beta \Rightarrow \beta}\to\mathrm{E}$$

confirming that the undischarged assumptions are $\alpha \to \beta$ and α.[13] This simplifies to the familiar Modus (Ponendo) Ponens:

$$\dfrac{\alpha \to \beta \quad \alpha}{\beta}\,\mathrm{MPP} \qquad \text{that is,} \qquad \dfrac{\Gamma \Rightarrow \alpha \to \beta \quad \Delta \Rightarrow \alpha}{\Gamma, \Delta \Rightarrow \beta}$$

by two applications of *Cut*:

$$\dfrac{\Delta \Rightarrow \alpha \quad \dfrac{\Gamma \Rightarrow \alpha \to \beta \quad \dfrac{\cdots}{\alpha, \alpha \to \beta \Rightarrow \beta}\,\text{(as above)}}{\Gamma, \alpha \Rightarrow \beta}\,\mathit{Cut}}{\Gamma, \Delta \Rightarrow \beta}\,\mathit{Cut}$$

[13] This derivation of β from $\alpha \to \beta$ and α can already be found, with slightly different notation, in [35, p. 1295]. See also [31, §2.4].

Thus MPP is a special case of $\to$E. $\to$E tells us that since β follows from α and $\alpha \Rightarrow \beta$, then it follows from α and $\alpha \to \beta$.

Conversely, MPP justifies $\to$E as follows: given the premises of $\to$E, namely,

$$\alpha \to \beta \qquad \text{and} \qquad \begin{array}{c} \alpha \\ \vdots \\ \beta \\ \mathfrak{A} \\ \gamma \end{array}$$

or in the explicit notation

$$\Gamma \Rightarrow \alpha \to \beta \qquad \text{and} \qquad \Delta, \alpha \Rightarrow \beta \overset{\mathfrak{A}}{\Rightarrow} \gamma$$

we can replace the assumption of $\alpha \Rightarrow \beta$ by an actual derivation of β dependent on α and $\alpha \to \beta$ by MPP, and proceed to γ:

$$\begin{array}{c} \dfrac{\alpha \to \beta \qquad \alpha}{\beta}\ \text{MPP} \\ \mathfrak{A} \\ \gamma \end{array}$$

that is,

$$\dfrac{\dfrac{\Gamma \Rightarrow \alpha \to \beta \qquad \dfrac{}{\alpha \Rightarrow \alpha}\ Ass_1}{\Gamma, \alpha \Rightarrow \beta}\ \text{MPP} \qquad \Delta, \alpha \Rightarrow \beta \overset{\mathfrak{A}}{\Rightarrow} \gamma}{\Gamma, \Delta \Rightarrow \gamma}\ Slice$$

4. General-Elimination Stability

Dummett [6, p. 287] remarks that:

> "Harmony is an excessively modest demand ... It does not show that ... we are accustomed to draw all those consequences we should be entitled to draw."

He proposed that not only should the E-rules be justified by the I-rules, but conversely, the I-rules should be seen as justified by the E-rules. The E-rules are in harmony with the I-rules when we cannot infer more by the E-rules than is warranted by the I-rules; conversely, the I-rules are in harmony with the E-rules when they allow one to assert no more than is warranted by the inferences drawn by the E-rules. Dummett described the rules as "stable" when the rules form a perfect circle, so that starting with the I-rules, forming the E-rules that they warrant, and from there forming the I-rules that these E-rules warrant in turn, one returns to the original I-rules; and similarly, if starting with the E-rules, and performing both justification procedures, one arrives back at the original E-rules.[14] Consequently, stability ensures that not only do the E-rules allow one to infer no more than is justified by the meaning conferred by the I-rules, they permit one to infer no less than is so justified. Stability ensures the consonance behind the original conception of harmony.[15] We will say that the E-rules for a connective are "inversely harmonious" (relative to a set of I-rules) when the I-rules for the connective are harmonious with respect to those E-rules, and so when the I-rules allow one to infer nothing more than is warranted by the E-rules. A collection of I- and E-rules is stable just when the E-rules are both harmonious and inversely harmonious.

Establishing stability lay behind Lorenzen's inversion principle:

> "A general formulation of an 'inversion principle' would be, e.g.: given a system of rules such that for the derivation of an expression p_0 only the rules
>
> $$p_1 \to p_0; \dots ; p_n \to p_0$$
>
> (possibly containing bound variables) are needed, then for every expression p, in which certain variables do not occur free, the meta-rule
>
> $$[p_1 \to p; \dots ; p_n \to p] \to (p_0 \to p)$$
>
> is valid."[16]

[14]Dyckhoff and Francez [9, p. 614] (see also [10, p. 80]) and Schroeder-Heister [37, p. 1205] interpret Dummett's remark [6, p. 287], "the demand that such a condition be met goes beyond the requirement of harmony: we may call it 'stability'," to refer to the converse of harmony, whereby the I-rules are justified by the meaning conferred by the E-rules. But Dummett applies the term 'harmony' to both justification procedures. It is when the procedures bring one back full circle that, for Dummett, we have stability. He writes (*loc.cit.*): "to verify that stability obtains, we have to appeal to *both* justification procedures."

[15]See also [44, p. 506], where it is proposed that the E-rules "*stabilize or delimit* the meaning of the logical constant concerned, by saying, in effect, of the given I-rules: 'These are the only ways in which this constant can be introduced'."

[16]"Eine allgemeine Formulierung eines 'Inversionsprinzips' wäre etwa: Ist ein System von Regeln vorgegeben, so daß zur Ableitung einer Aussage p_0 nur die Regeln $p_1 \to p_0; \dots ; p_n \to p_0$ (evtl. mit gebundenen Variablen) benutzt werden können, so gilt für jede Aussage p, in der gewisse Variable nicht frei vorkommen, die Metaregel $p_1 \to p; \dots ; p_n \to p \dot{\to} p_0 \to p$." [18, p. 176] In [19, §1.4], Lorenzen attempted to spell out the necessary conditions on the bound variables, later corrected by Hermes in [15]. See also [36, §4].

That is, whatever follows from the direct grounds for the assertion of a proposition alone must follow from that proposition. This is a statement of inverse harmony.

Prawitz also formulated an inversion principle. He wrote:

> "Observe that an elimination rule is, in a sense, the inverse of the corresponding introduction rule: by an application of an elimination rule one essentially only restores what had already been established if the major premiss of the application was inferred by an application of an introduction rule." [26, p. 33]

Thus, according to Prawitz's inversion principle, whatever follows from an assertion must follow from the direct grounds for that assertion. This is a statement of harmony.

Negri and von Plato claim to follow Gentzen and Prawitz in formulating their "general inversion principle":

> "Whatever follows from the direct grounds for deriving a proposition must follow from that proposition." ([22, p. 6]; cf. [23, p. 243])

Moriconi and Tesconi [21, p. 111] comment that Negri and von Plato's principle is the very opposite of Prawitz's inversion principle, and remark that, in point of fact, Negri and von Plato's "general inversion principle is very akin to Lorenzen's meta-rule".

What primarily concerns Lorenzen is the "completeness" of the E-rules. As quoted above, he wrote: "[if] for the derivation of an expression p_0 *only* [such and such] rules ... are needed, then for *every* expression p, the meta-rule ... is valid." (Our emphases.) Putting Prawitz's and Lorenzen's principles together, we have the demand for stability. As Moriconi and Tesconi put it:

> "Moreover, considered together with the ('sufficiency'-)condition stated by Lorenzen's meta-rule ["the task of generating (the strongest) elimination rule"—p. 108], the above ('necessity'-condition) [that no more follows from A than from its grounds] not only justifies but also determines a set of elimination rules. In fact, the two conditions are the same as saying that from a proposition A exactly the same thing follows as follows its immediate grounds." [21, p. 110]

Lorenzen's inversion principle, designed to establish "completeness", in other words, inverse harmony, thus completes the circle that Dummett used to define stability.

We have already seen that the E-rules generated by the general-elimination procedure are harmonious relative to the I-rules from which they are generated. Our aim is to show what we will call 'general-elimination stability', that the GE-procedure in fact generates a stable set of E-rules from an arbitrary collection of I-rules. The E-rules generated by the procedure allow one to infer everything warranted by the I-rules from which they are generated, and nothing more.

Prawitz [27, p. 246] conjectured that his rules for minimal logic were stable:

> "Conjecture: *The set of inference rules derivable in* **M** *is the maximum set of inference rules which is valid with respect to some justification* $\mathscr{J}$."

We will show that the GE-rules are inversely harmonious (relative to the I-rules from which they are generated) and so, since they are harmonious, they are stable, thus establishing Prawitz's conjecture for **M**, indeed, for all the systems S of §2.

We consider four formal properties of E-rules with the aim of variously capturing the notion of inverse harmony in a formal manner, and show that GE-rules satisfy all the properties considered. The first property, considered in §5, is called "local completeness" in [25]. Roughly, the E-rules for a connective $*$ are locally complete just in case there is a proof of each formula $*\vec{\alpha}$ from itself only by applications of all the I-rules and non-vacuous applications of all the E-rules for $*$. We consider a different property of E-rules in §6, based on Schroeder-Heister and Olkhovikov's idea ([38], [39]) of the introduction-meaning and the elimination-meaning of a connective. The I- and E-meanings of a connective are metalinguistic formulae expressing the meaning conferred on a connective by the set of I- and E-rules, respectively. The formal property proposed in §6 to capture inverse harmony is the property that the E-rules for a connective have when its E-meaning implies its I-meaning.

But in §7 we cast doubt on whether the properties proposed in §§5-6 really do capture inverse harmony. For the E-rules for quantum disjunction are locally complete with respect to the I-rules for disjunction and quantum disjunction's I- and E-meanings are equivalent. Given the weakness of quantum disjunction, these results cast doubt on whether the properties of local completeness and implication of I-meanings by E-meanings do succeed in formally capturing inverse harmony.

In §8 we consider the property of the E-rules for a connective $*$ whereby one can, by application of the E-rules for $*$, derive the grounds for asserting

$*\vec{\alpha}$ from $*\vec{\alpha}$ itself, but give reasons to think that this property also fails to capture the notion of inverse harmony successfully. Finally, in §9 we introduce the idea of generalized local completeness. As the name indicates, generalized local completeness is a generalization of local completeness. Showing that the GE-rules satisfy generalized local completeness, we propose that this property successfully captures the informal notion of inverse harmony. We conclude that the GE-procedure does indeed generate stable rules, thus fulfilling Gentzen's aim of displaying the E-rules as unique functions of the I-rules, and confirming Prawitz's Conjecture for all the systems S of §2.

5. Local Completeness

The requirement that "we can apply the E-rules to a judgment to recover enough knowledge to permit reconstruction of the original judgment" is called "local completeness" by Pfenning and Davies [25, p. 513].[17] That way, they say, the E-rules will be "sufficiently strong".

If taken literally, it is hard to see how this "expansion" can be available in general. The formula $\alpha \vee \beta$ is an obvious counterexample: one cannot in general recover $\alpha \vee \beta$ after applying $\vee$E to $\alpha \vee \beta$. However, Pfenning and Davies' subsequent discussion of $\Box\alpha$ and their argument for the local completeness of their rules for '$\Box$' [25, p. 518] (see also [3, pp. 561-2]) show that they would accept the following expansion as showing the local completeness of the rules for '$\vee$':[18]

$$\dfrac{\alpha \vee \beta \quad \dfrac{[\alpha]^1}{\alpha \vee \beta}\,\vee\mathrm{I} \quad \dfrac{[\beta]^2}{\alpha \vee \beta}\,\vee\mathrm{I}}{\alpha \vee \beta}\,\vee\mathrm{E}(1,2)$$

Dyckhoff and Francez [9, p. 623 n.] write:

> "Conjunction expansion and disjunction expansion ... exhibit different relative ordering of I-rules and E-rules application."

In fact, a conjunctive expansion can be carried out in either order, I- before E- or E- before I-. But disjunctive expansion, and in general the expansion with respect to any formula constructed with a connective with more than one I-rule, can only be achieved by applying the I-rules before, or subordinately to, the E-rules.

Let us make this precise:

[17] Local completeness is also discussed in [3].

[18] The disjunctive expansion is given explicitly in [9, p. 624] and in [10, p. 108].

Definition 7. *(Local completeness) We take the E-rules for a connective $*$ to be* locally complete *just in case, for any $\vec{\alpha}$, there is a derivation of $*\vec{\alpha}$ depending solely on $*\vec{\alpha}$ and obtained only by applications of the rule of assumptions (i.e., of the structural rules* Ass_1 *and* Ass_2 *in §2), applications of all the I-rules for $*$, and applications of all the E-rules for $*$, and having $*\vec{\alpha}$ as its only undischarged assumption.*[19]

Local completeness is our first attempt to capture the property of inverse harmony formally. As we now show, any connective with I- and E-rules framed according to the general-elimination procedure admits an expansion of the form previously described.

Theorem 1 (Local Completeness). *Let $*$ be an n-ary connective, for any $n \in \mathbb{N}$, governed by a set of introduction- and general-elimination-rules as above. Then there is a derivation of $*\vec{\alpha}$ (in any system S of the sort defined in §§2-3 containing the I- and GE-rules for $*$), for any $\vec{\alpha}$, depending solely on $*\vec{\alpha}$ and obtained only by applications of the rules Ass_1, Ass_2, all the I-rules for $*$ and all the general-elimination rules for $*$.*

The general strategy is to provide a recipe for constructing a derivation of $*\vec{\alpha}$ from undischarged assumptions of $*\vec{\alpha}$.[20] The recipe appeals to a function $Pf()$ from a class of m-tuples (understood as functions with domain $\{1, 2, \ldots, m\}$), technically representing the E-rules for $*$, to the set of derivations. For each such m-tuple j, $Pf(j)$ outputs a derivation of $*\vec{\alpha}$ from $*\vec{\alpha}$, and possibly other open assumptions, in which every I-rule for $*$ is applied, as well as every E-rule for $*$ lexicographically less than or equal to the rule corresponding to j. When j corresponds to the final E-rule for $*$, $Pf(j)$ is a derivation of $*\vec{\alpha}$ from $*\vec{\alpha}$ without any other open assumptions.

Proof. There are three cases to consider, namely, i) when there are no I-rules; ii) when there is an I-rule with no premises; and iii) when all the I-rules have at least one premise. If there are no I-rules, then (as noted in footnote 10 above) $*$ is '$\bot$' and we have the following derivation:

[19] Of course, there is vacuously an expansion consisting of $*\vec{\alpha}$ itself, with no applications of $*$I or $*$E.

[20] The strategy is rather different from that suggested by Dyckhoff and Francez in the proof-sketch in [9, p. 625] for their different version of the E-rules. However, their construction, applying E-rules before I-rules, does not seem to cover connectives like '$\vee$'. Nor can their sketch be filled out to establish local completeness for certain of their GE-rules, as shown in [33] and [39]. See [10, p. 103] and §6 below.

$$\dfrac{\overline{\bot \Rightarrow \bot}\ Ass_1}{\bot \Rightarrow \bot}\ \bot\mathrm{E}$$

If there is one I-rule with no assumptions, then (again as noted in footnote 10 above), $*$ is '$\top$', it has only one I-rule and no GE-rules, and we have the following derivation:

$$\overline{\top \Rightarrow \top}\ \top\mathrm{I}$$

If all the I-rules have at least one premise, assume without loss of generality that $*$ has m I-rules, where each I-rule $*\mathrm{I}_i$ has n_i premises, $1 \leq i \leq m$, $n_i \geq 1$.

Definition 8. *Let*

- $end(\mathfrak{A})$ *be the end sequent of derivation* $\mathfrak{A}$
- $\Omega = \times_{i=1}^{m} n_i$
- *For each* $j \in \Omega$, $1 \leq i \leq m$: $\xi_{ij}(k) = \begin{cases} j(k) & \textit{if } k \neq i, \\ 0 & \textit{if } k = i \end{cases}$
- $\Xi = \bigcup_{j\in\Omega} \bigcup_{i=1}^{m} \{\xi_{ij}\}$
- $\Omega^+ = \Omega \cup \Xi$
- *For each* i *such that* $1 \leq i \leq m$, $\delta_i : \Omega \to \Omega^+$ *such that*

$$\delta_i(j)(k) = \begin{cases} j(k) & \textit{if } k \neq i \\ j(i) - 1 & \textit{if } k = i \end{cases}$$

- $\Omega_m = \langle n_1, \ldots, n_m \rangle$.

Each function $j \in \Omega$ represents the corresponding rule $*\mathrm{E}_j$. Moreover, each function $\xi_{ij} \in \Xi$ represents the rule $*\mathrm{I}_i$, as determined as a function of j. The function $Pf()$ previously mentioned associates with each rule $*\mathrm{E}_j$ a derivation $Pf(j)$ whose last step is an application of $*\mathrm{E}_j$. The ith minor premise of this application of $*\mathrm{E}_j$ is a derivation whose last step is the rule represented by $\delta_i(j)$. Thus, the role of δ is to determine the rule that is the last step of the ith premise of the application of $*\mathrm{E}_j$ in $Pf(j)$. This will become clearer when we introduce the function $Pf()$ in Definition 10 below. If $i - 1 \neq 0$, then $\delta_i(j)$ represents the rule $*\mathrm{E}_{\delta_i(j)}$ that discharges all the same assumptions as $*\mathrm{E}_j$ except the ith, discharging the $(i-1)$th assumption instead. If $i - 1 = 0$, then $\delta_i(j) = \xi_{ij}$, and so $\delta_i(j)$ represents the rule $*\mathrm{I}_i$.

Note that, like j itself, $\delta_i(j)$ is also a function with domain $\{1, \ldots, m\}$, so on occasion we will write $\delta_i(j)$ as δ_{ij}. The m-tuple Ω_m is the last function of Ω according to Ω's lexicographical ordering, and so Ω_m represents the last of $*$'s E-rules given the ordering of $*$'s E-rules based on the lexicographical ordering of Ω.

Definition 9. *Let η be a function mapping sequents π of level 1 or 2 to derivations in any system S such that if π is a formula α, then $\eta(\pi)$ is the derivation*

$$\dfrac{}{\alpha \Rightarrow \alpha}\ Ass_1$$

and if π is a sequent $\beta_1, \ldots, \beta_n \Rightarrow \alpha$, then $\eta(\pi)$ is the derivation

$$\dfrac{\dfrac{}{\beta_1 \Rightarrow \beta_1}\ Ass_1 \quad \ldots \quad \dfrac{}{\beta_n \Rightarrow \beta_n}\ Ass_1}{\beta_1, \ldots \beta_n \Rightarrow \alpha, \beta_1, \ldots, \beta_n \Rightarrow \alpha}\ Ass_2$$

The function $\eta()$ will be used to generate derivations of sequents suitable for the application of the I-rules of $*$ from the grounds for introducing $*$. If a ground $g_l(*\mathrm{I}_i)$ is simply a formula α, then $\eta(g_l(*\mathrm{I}_i))$ is a derivation of α from itself. And if $g_l(*\mathrm{I}_i)$ is the sequent $\beta_1, \ldots, \beta_n \Rightarrow \alpha$, then $\eta(g_l(*\mathrm{I}_i))$ is a derivation of the sequent with succedent α, the succedent of $g_l(*\mathrm{I}_i)$, and antecedents $\beta_1, \ldots, \beta_n \Rightarrow \alpha$, the ground $g_l(*\mathrm{I}_i)$ itself, as well as $\beta_1, \ldots, \beta_n$, the ground's antecedents.

Definition 10. *For each $j \in \Omega^+$, and where $l \in \Omega$, let:*

$$Pf(j) = \begin{cases} \dfrac{\eta(g_1(*I_i)) \quad \ldots \quad \eta(g_{n_i}(*I_i))}{\bigcup_{k=1}^{n_i} g_k(*I_i) \Rightarrow *\vec{\alpha}}\ *I_i & \text{if } j = \xi_{il} \in \Xi \\ \dfrac{\dfrac{}{*\vec{\alpha} \Rightarrow *\vec{\alpha}}\ Ass_1 \quad Pf(\delta_1(j)) \quad \ldots \quad Pf(\delta_m(j))}{\bigcup_{i=1}^{m} \bigcup_{k=j(i)+1}^{n_i} g_k(*I_i), *\vec{\alpha} \Rightarrow *\vec{\alpha}}\ *E_j & \text{if } j \in \Omega \end{cases}$$

The function $Pf()$ is used to generate the required derivation of $*\vec{\alpha} \Rightarrow *\vec{\alpha}$. The function's role may be understood by considering how to construct a derivation of $*\vec{\alpha} \Rightarrow *\vec{\alpha}$ witnessing the claim made in Theorem 1. The bottom node, $*\vec{\alpha} \Rightarrow *\vec{\alpha}$, is obtained by an application of $*\mathrm{E}_{\Omega_m}$, the last of the E-rules according to the lexicographical ordering on Ω. The ith minor premise of the application of $*\mathrm{E}_{\Omega_m}$ is a derivation of a sequent $\pi_{\delta_{i\Omega_m}}$ whose succedent is $*\vec{\alpha}$ and whose antecedents are $*\vec{\alpha}$ and the ground $g_{\Omega_m(i)}(*\mathrm{I}_i)$, which is discharged by $*\mathrm{E}_{\Omega_m}$.

The sequent $\pi_{\delta_{i\Omega_m}}$ is obtained by an application of the E-rule $*\mathrm{E}_{\delta_{i\Omega_m}}$ 'immediately' before $*\mathrm{E}_{\Omega_m}$ with respect to i, in the sense that $*\mathrm{E}_{\delta_{i\Omega_m}}$ discharges all of the assumptions discharged by $*\mathrm{E}_{\Omega_m}$, except that it does not discharge the ground $g_{\Omega_m(i)}(*\mathrm{I}_i)$. Rather, it discharges the ground $g_{\Omega_m(i)-1}(*\mathrm{I}_i)$. We construct the derivation tree in this way, that is, by inserting an application of the E-rule $*\mathrm{E}_{\delta_{ij}}$ in the ith minor premise of the application of the E-rule $*\mathrm{E}_j$. The end sequent $\pi_{\delta_{ij}}$ of the application of $*\mathrm{E}_{\delta_{ij}}$ has $*\vec{\alpha}$ as its succedent and the antecedents of $\pi_{\delta_{ij}}$ are the antecedents of π_j (i.e., the antecedents of the end sequent of the application of $*\mathrm{E}_j$), plus the ground $g_{j(i)}(*\mathrm{I}_i)$, which gets discharged by the application of $*\mathrm{E}_j$ immediately below in the derivation.

Constructing the derivation tree in this bottom-up way, inserting an application of the E-rule $*\mathrm{E}_{\delta_{ij}}$ in the ith minor premise of the application of the E-rule $*\mathrm{E}_j$, eventually we reach the case when $j(i)-1=0$. In this case, $\delta_{ij} \notin \Omega$, and so one cannot place an application of an E-rule for $*$ as the ith minor premise of $*\mathrm{E}_j$, as there is no such rule $*\mathrm{E}_{\delta_{ij}}$. Rather, $\delta_{ij} = \xi_{ij}$, and so what gets placed as the ith minor premise of the application of $*\mathrm{E}_j$ is an application of $*\mathrm{I}_i$.

The premises of this application of $*\mathrm{I}_i$ are $\eta(g_k(*\mathrm{I}_i))$, $1 \leq k \leq n_i$. The succedent of the end sequent of the application of $*\mathrm{I}_i$ is $*\vec{\alpha}$ and the antecedents of the end sequent of the application of $*\mathrm{I}_i$ are the grounds $g_k(*\mathrm{I}_i)$, for all k such that $1 \leq k \leq n_i$. These were the assumptions that were getting discharged by the applications of the E-rules for $*$ as we went further up in the construction of the derivation tree, following the ith minor premise of each application of the E-rule below.

To see why the antecedents of the end sequent of the application of $*\mathrm{I}_i$ are exactly the grounds $g_k(*\mathrm{I}_i)$, for all k such that $1 \leq k \leq n_i$, note that:

(i) if $g_k(*\mathrm{I}_i)$ is a formula, then this formula is the only antecedent of the end sequent of $\eta(g_k(*\mathrm{I}_i))$, and so it is the only sequent that is carried from the antecedent of the end sequent of $\eta(g_k(*\mathrm{I}_i))$ to the antecedent of the end sequent of the application of $*\mathrm{I}_i$; and

(ii) if $g_k(*\mathrm{I}_i)$ is $\beta_1,\ldots,\beta_n \Rightarrow \alpha$, then the antecedents of the end sequent of $\eta(g_k(*\mathrm{I}_i))$ are $g_k(*\mathrm{I}_i)$ and $\beta_1,\ldots,\beta_n$. Since an application of $*\mathrm{I}_i$ discharges $\beta_1,\ldots,\beta_n$, the only sequent in the antecedent of the end sequent of $\eta(g_k(*\mathrm{I}_i))$ that is carried to the antecedents of the end sequent of the application of $*\mathrm{I}_i$ is $g_k(*\mathrm{I}_i)$.

So, the antecedents of the end sequent of the application of $*\mathrm{I}_i$ are exactly the grounds $g_k(*\mathrm{I}_i)$, for all k such that $1 \leq k \leq n_i$. Thus, starting with an application of $*\mathrm{E}_{\Omega_m}$ one can construct a derivation of $*\vec{\alpha} \Rightarrow *\vec{\alpha}$, by following each minor premise i of the application $*\mathrm{E}_j$ in the way just described. This procedure is captured by the function $Pf()$. One starts bottom up with $Pf(\Omega_m)$, and then applies $Pf()$ to the elements in Ω less than or equal to Ω_m.

We establish two general results about the function $Pf()$.

LEMMA 1. *$Pf(j)$ is a derivation in S, for each $j \in \Omega^+$ and system S of the kind defined in §2 containing the I- and GE-rules for $*$.*

PROOF. (Lemma 1) The proof is by well-founded induction on Ω^+. So, consider first the case where $j \in \Xi = \Omega^+ - \Omega$. Let $\mathfrak{A}$ be the derivation obtained by applying $*\mathrm{I}_i$ to $end(\eta(g_1(*\mathrm{I}_i)), \ldots, end(\eta(g_{n_i}(*\mathrm{I}_i))$. It will be shown that $a(end(\mathfrak{A})) = a(end(Pf(j)))$ and $s(end(\mathfrak{A})) = s(end(Pf(j)))$, and so that $Pf(j) = \mathfrak{A}$.

Since, for each $1 \leq k \leq n_i$, $s(end(\eta(g_k(*\mathrm{I}_i)))) = s(g_k(*\mathrm{I}_i))$, it follows that $s(end(\mathfrak{A})) = *\vec{\alpha} = s(end(Pf(j)))$. Moreover, for each $1 \leq k \leq n_i$: $a(end(\eta(g_k(*\mathrm{I}_i)))) = g_k(*\mathrm{I}_i)$ if $g_k(*\mathrm{I}_i)$ is a sequent of level 1, and

$$a(end(\eta(g_k(*\mathrm{I}_i)))) = g_k(*\mathrm{I}_i), a(g_k(*\mathrm{I}_i))$$

if $g_k(*\mathrm{I}_i)$ is a sequent of level 2. Note that, for each $1 \leq k \leq n_i$, $a(g_k(*\mathrm{I}_i))$ is discharged by $*\mathrm{I}_i$. So, $a(end(\mathfrak{A})) = \bigcup_{k=1}^{n_i} g_k(*\mathrm{I}_i) = a(end(Pf(j)))$. Therefore, if $j \in \Xi$, then $Pf(j)$ is a derivation in S.

For the case of $j \in \Omega$, assume that $Pf(\delta_i(j))$ are derivations, for all $1 \leq i \leq m$, and let $\mathfrak{A}$ be the derivation obtained by applying $*\mathrm{E}_j$ to

$$end(Pf(\delta_1(j))), \ldots, end(Pf(\delta_m(j)))$$

as minor premises. Clearly, $s(end(Pf(\delta_i(j)))) = *\vec{\alpha}$, by the definition of $Pf()$. So $s(end(\mathfrak{A})) = *\vec{\alpha} = s(end(Pf(j)))$ since the last step of $\mathfrak{A}$ consists in an application of $*\mathrm{E}_j$.

If $\delta_{ij} \in \Xi$, then $j(i) = 1$, $\delta_{ij} = \xi_{ij}$ and so $a(end(Pf(\delta_{ij}))) = \bigcup_{k=1}^{n_i} g_k(*\mathrm{I}_i)$ by the definition of $Pf()$. Thus,

$$a(end(Pf(\delta_i(j)))) - g_{j(i)}(*\mathrm{I}_i) = \bigcup_{k=j(i)+1}^{n_i} g_k(*\mathrm{I}_i)$$

If $\delta_{ij} \in \Omega$, then

$$\delta_{ij} = \begin{cases} j(l) & \text{if } l \neq i \\ j(i) - 1 & \text{if } l = i \end{cases}$$

and so, by the definition of $Pf()$,

$$a(end(Pf(\delta_{ij}))) = \bigcup_{l \neq i} \bigcup_{k=j(l)+1}^{n_l} g_k(*\mathrm{I}_l) \cup \bigcup_{k=j(i)}^{n_i} g_k(*\mathrm{I}_i), *\vec{\alpha}$$

Thus, $a(end(Pf(\delta_{ij}))) - g_{j(i)}(*\mathrm{I}_i) = \bigcup_{i=1}^{m} \bigcup_{k=j(i)+1}^{n_i} g_k(*\mathrm{I}_i)$. Since the last step of $\mathfrak{A}$ consists in an application of $*\mathrm{E}_j$ we have that:

$$
\begin{aligned}
a(end(\mathfrak{A})) &= \bigcup_{i=1}^{m}(a(end(Pf(\delta_{ij}))) - g_{j(i)}(*\mathrm{I}_i)), *\vec{\alpha} \\
&= \bigcup_{i=1}^{m} \bigcup_{k=j(i)+1}^{n_i} g_k(*\mathrm{I}_i), *\vec{\alpha} \\
&= a(end(Pf(j)))
\end{aligned}
$$

Hence, if $j \in \Omega$, then $Pf(j)$ is a derivation in S. Therefore, for all $j \in \Omega^+$, $Pf(j)$ is a derivation in S. ■

LEMMA 2. *$\forall j, j' \in \Omega$ s.t. $j' \leq j$: $*E_{j'}$ is used in $Pf(j)$.*

PROOF. (Lemma 2) We prove the result by well-founded induction on Ω. Suppose that $\forall j' \in \Omega$ s.t. $j' < j$: $\forall f \in \Omega$ s.t. $f \leq j'$: $*\mathrm{E}_f$ is used in $Pf(j')$. We show that $\forall j' \in \Omega$ s.t. $j' \leq j$: $*\mathrm{E}_{j'}$ is used in $Pf(j)$.

Take an arbitrary $j' \in \Omega$ s.t. $j' \leq j$. If $j' = j$, then it is clear that $*\mathrm{E}_{j'}$ is used in $Pf(j)$. Suppose instead that $j' \neq j$. Then, $\exists i$ s.t. $1 \leq i \leq m$ and $j' \leq \delta_{ij}$. It follows from the I.H. that $*\mathrm{E}_{j'}$ is used in $Pf(\delta_i(j))$. Hence, $*\mathrm{E}_{j'}$ is used in $Pf(j)$. ■

Now, note that $\Omega_m(i) = n_i$, and so

$$\bigcup_{k=\Omega_m(i)+1}^{n_i} g_k(*\mathrm{I}_i) = \bigcup_{k=n_i+1}^{n_i} g_k(*\mathrm{I}_i) = \varnothing$$

which means that

$$\bigcup_{i=1}^{m} \bigcup_{k=\Omega_m(i)+1}^{n_i} g_k(*\mathrm{I}_i) = \varnothing.$$

So

$$Pf(\Omega_m) = \dfrac{\dfrac{}{*\vec{\alpha} \Rightarrow *\vec{\alpha}}\, Ass_1 \qquad Pf(\delta_1(\Omega_m)) \quad \ldots \quad Pf(\delta_m(\Omega_m))}{*\vec{\alpha} \Rightarrow *\vec{\alpha}}\, *\mathrm{E}_{\Omega_m}$$

From Lemma 1 it follows that $Pf(\Omega_m)$ is indeed a derivation. Note that all the I-rules for $*$ are used in $Pf(\pi_1)$, where, for each $1 \leq i \leq m$, $\pi_1(i) = 0$, if $*\mathrm{I}_i$ has no premises, and otherwise $\pi_1(i) = 1$. Note also that, for every $j', j \in \Omega$, if $*\mathrm{E}_{j'}$ is used in $Pf(j)$ then $Pf(j)$ has $Pf(j')$ as a sub-proof, by construction of $Pf()$. Thus, by Lemma 2, $Pf(\pi_1)$ is a sub-proof of $Pf(\Omega_m)$, since $\forall j \in \Omega$: $\pi_1 \leq j$, which means that all the I-rules for $*$ are used in

$Pf(\Omega_m)$. From Lemma 2 it follows that every elimination rule for $*$ is used in $Pf(\Omega_m)$. Therefore, $Pf(\Omega_m)$ is a proof of $*\vec{\alpha}$ from $*\vec{\alpha}$ obtained only by applications of the rules Ass_1, Ass_2, and applications of all the I-rules for $*$ and of all the general-elimination rules for $*$. This completes the proof of Theorem 1. ∎

As an example (given in the more familiar notation using only wffs), consider the following 6-place connective '∇', with introduction rules:

$$\dfrac{\begin{matrix}[\epsilon]^1 & \\ \alpha & \qquad \beta\end{matrix}}{\nabla(\epsilon,\alpha,\beta,\gamma,\xi,\zeta)}\,\nabla\mathrm{I}_1 \qquad \text{and} \qquad \dfrac{\gamma \qquad \xi \qquad \zeta}{\nabla(\epsilon,\alpha,\beta,\gamma,\xi,\zeta)}\,\nabla\mathrm{I}_2$$

The I-rules for '∇' generate the following six GE-rules. Recall the formula from §3: $m = 2, n_1 = 2$ and $n_2 = 3$, so $\prod_{i=1}^{m} n_i = 6$, thus ensuring that the assumption of one premise from each I-rule is discharged by each E-rule (where '$\overline{\nabla}$' abbreviates '$\nabla(\epsilon,\alpha,\beta,\gamma,\xi,\zeta)$'):

$$\dfrac{\overline{\nabla} \quad \begin{matrix}\begin{bmatrix}\epsilon\\ \vdots\\ \alpha\end{bmatrix}^1\\ \psi\end{matrix} \quad \begin{matrix}[\gamma]^2\\ \psi\end{matrix}}{\psi}\,\nabla\mathrm{E}_1(1,2) \qquad \dfrac{\overline{\nabla} \quad \begin{matrix}\begin{bmatrix}\epsilon\\ \vdots\\ \alpha\end{bmatrix}^1\\ \psi\end{matrix} \quad \begin{matrix}[\xi]^2\\ \psi\end{matrix}}{\psi}\,\nabla\mathrm{E}_2(1,2)$$

$$\dfrac{\overline{\nabla} \quad \begin{matrix}\begin{bmatrix}\epsilon\\ \vdots\\ \alpha\end{bmatrix}^1\\ \psi\end{matrix} \quad \begin{matrix}[\zeta]^2\\ \psi\end{matrix}}{\psi}\,\nabla\mathrm{E}_3(1,2) \qquad \dfrac{\overline{\nabla} \quad \begin{matrix}[\beta]^1\\ \psi\end{matrix} \quad \begin{matrix}[\gamma]^2\\ \psi\end{matrix}}{\psi}\,\nabla\mathrm{E}_4(1,2)$$

$$\dfrac{\overline{\nabla} \quad \begin{matrix}[\beta]^1\\ \psi\end{matrix} \quad \begin{matrix}[\xi]^2\\ \psi\end{matrix}}{\psi}\,\nabla\mathrm{E}_5(1,2) \qquad \dfrac{\overline{\nabla} \quad \begin{matrix}[\beta]^1\\ \psi\end{matrix} \quad \begin{matrix}[\zeta]^2\\ \psi\end{matrix}}{\psi}\,\nabla\mathrm{E}_6(1,2)$$

To elucidate how the function $Pf()$ used in the proof of Theorem 1 works, we provide the derivation $Pf(\langle 2,3\rangle)$ in Figure 1, with subproofs $Pf(\langle 1,3\rangle)$ and $Pf(\langle 2,2\rangle)$ in Figures 2 and 3. Note that each sub-proof ending in the application of an E-rule discharging premises (l_1, r_2) corresponds to the derivation $Pf(\langle l,r\rangle)$.

$$\dfrac{\overline{\nabla} \qquad Pf(\langle 1,3\rangle) \qquad Pf(\langle 2,2\rangle)}{\overline{\overline{\nabla}}}\,\nabla\mathrm{E}_6(2_1,3_2)$$

Figure 1. $Pf(\langle 2,3\rangle)$

$$\dfrac{\overline{\nabla} \quad \dfrac{\left[\begin{array}{c}[\epsilon]^1\\ \vdots\\ \alpha\end{array}\right]^{1_1} \quad \beta}{\overline{\nabla}}\,\nabla\mathrm{I}_1(1) \quad \dfrac{\overline{\nabla} \quad \dfrac{\left[\begin{array}{c}[\epsilon]^1\\ \vdots\\ \alpha\end{array}\right]^{1_1} \quad \beta}{\overline{\nabla}}\,\nabla\mathrm{I}_1(1) \quad \dfrac{\overline{\nabla} \quad \dfrac{\left[\begin{array}{c}[\epsilon]^1\\ \vdots\\ \alpha\end{array}\right]^{1_1} \quad \beta}{\overline{\nabla}}\,\nabla\mathrm{I}_1(1) \quad \dfrac{[\gamma]^{1_2} \quad [\xi]^{2_2} \quad [\zeta]^{3_2}}{\overline{\nabla}}\,\nabla\mathrm{I}_2}{\overline{\nabla}}\,\nabla\mathrm{E}_1(1_1,1_2)}{\overline{\nabla}}\,\nabla\mathrm{E}_2(1_1,2_2)}{\overline{\nabla}}\,\nabla\mathrm{E}_3(1_1,3_2)$$

Figure 2. $Pf(\langle 1,3\rangle)$

$$\dfrac{\overline{\nabla} \quad \dfrac{\overline{\nabla} \quad \dfrac{\left[\begin{array}{c}[\epsilon]^1\\ \vdots\\ \alpha\end{array}\right]^{1_1} \quad [\beta]^{2_1}}{\overline{\nabla}}\,\nabla\mathrm{I}_1(1) \quad \dfrac{\overline{\nabla} \quad \dfrac{\left[\begin{array}{c}[\epsilon]^1\\ \vdots\\ \alpha\end{array}\right]^{1_1} \quad [\beta]^{2_1}}{\overline{\nabla}}\,\nabla\mathrm{I}_1(1) \quad \dfrac{[\gamma]^{1_2} \quad [\xi]^{2_2} \quad \zeta}{\overline{\nabla}}\,\nabla\mathrm{I}_2}{\overline{\nabla}}\,\nabla\mathrm{E}_1(1_1,1_2)}{\overline{\nabla}}\,\nabla\mathrm{E}_2(1_1,2_2) \quad \dfrac{\overline{\nabla} \quad \dfrac{\overline{\nabla} \quad \dfrac{\left[\begin{array}{c}[\epsilon]^1\\ \vdots\\ \alpha\end{array}\right]^{1_1} \quad [\beta]^{2_1}}{\overline{\nabla}}\,\nabla\mathrm{I}_1(1) \quad \dfrac{[\gamma]^{1_2} \quad [\xi]^{2_2} \quad \zeta}{\overline{\nabla}}\,\nabla\mathrm{I}_2}{\overline{\nabla}}\,\nabla\mathrm{E}_1(1_1,1_2) \quad \dfrac{[\gamma]^{1_2} \quad [\xi]^{2_2} \quad \zeta}{\overline{\nabla}}\,\nabla\mathrm{I}_2}{\overline{\nabla}}\,\nabla\mathrm{E}_4(2_1,1_2)}{\overline{\nabla}}\,\nabla\mathrm{E}_5(2_1,2_2)$$

Figure 3. $Pf(\langle 2,2\rangle)$

Given this result, if local completeness does successfully capture inverse harmony, then the GE-rules for any connective are inversely harmonious, and a fortiori stable, with respect to the I-rules from which they are generated. But we will question this assumption in §7.

6. Introduction- and Elimination-Meaning

Schroeder-Heister [38] (see also [39]) introduces the idea of the introduction-meaning and the elimination-meaning of a connective, that is, of formulae which express the meaning conferred on a connective by a set of, respectively, I- and E-rules. He proposes an account of stability (which he calls "harmony": [38, p. 339]), according to which a connective has stable I- and E-rules if and only if its I- and E-meanings are equivalent.

Schroeder-Heister describes this approach as "reductive", in contrast to the "foundational" approach taken in [37]. The reductive approach is so called because it involves, in some sense, a reduction of the meaning of a connective to the meaning of the expressions used in the formulation of its I- and E-meanings.

Our project in the present paper is to determine whether the I- and GE-rules for a connective are stable, independently of whether this is achieved by appealing to the meanings of expressions of some language or other. What makes a real distinction between our projects is (as noted in [38, p. 330]) that we are interested in the stability of I- and E-rules where the latter are generated from the former via the GE-procedure, whereas Schroeder-Heister's interest is in the relationship between I- and E-rules in general. Schroeder-Heister's specific formulation of I- and E-meanings concerns (second-order) intuitionistic logic. However, it readily generalizes to other logics and their higher-order counterparts.

These (meta-)formulae belong to a basic vocabulary expressing the central idea of alternative, jointly sufficient and possibly conditional grounds. In the case of '$\wedge$', '$\vee$', '$\to$' and '$\bot$', for example, the I-meaning is simply expressed by the (metalinguistic) formulae $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \to \beta$ and $\bot$, respectively.

Now consider the connective '$\circ$' discussed in [44, p. 506] and [38, p. 345], with I-rules:

$$\frac{\begin{array}{c}[\alpha]^1\\ \beta\end{array}}{\circ(\alpha,\beta,\gamma)}\ \circ\mathrm{I}_1(1) \qquad \text{and} \qquad \frac{\gamma}{\circ(\alpha,\beta,\gamma)}\ \circ\mathrm{I}_2$$

Then $\circ(\alpha,\beta,\gamma)$ can be asserted either if α implies β or if γ obtains, so the I-meaning of $\circ(\alpha,\beta,\gamma)$ is given by the meta-formula $(\alpha \to \beta) \vee \gamma$. In general, the I-meaning of $*$ given by the schematic I-rules in §3 is

$$\bigvee_{i=1}^{m}\bigwedge_{l=0}^{n_i} c(g_l(*\mathrm{I}_i))$$

where

$$c(g_l(*\mathrm{I}_i)) = \begin{cases} g_l(*\mathrm{I}_i) & \text{if } g_l(*\mathrm{I}_i) \text{ is a level 1 sequent and} \\ \bigwedge a(g_l(*\mathrm{I}_i)) \to s(g_l(*\mathrm{I}_i)) & \text{if } g_l(*\mathrm{I}_i) \text{ is a level 2 sequent.}^{21} \end{cases}$$

What of the E-meaning of $*$? First, recall the application of the general-elimination procedure to the case of implication, '$\to$', given in §§1 and 3 above:

$$\dfrac{\alpha \to \beta \qquad \begin{matrix} \left[\begin{matrix} \alpha \\ \vdots \\ \beta \end{matrix}\right]^1 \\ \gamma \end{matrix}}{\gamma} \to\mathrm{E}(1)$$

The E-meaning is given not by the grounds for assertion of $\alpha \to \beta$ (that is, $\alpha \Rightarrow \beta$), but in terms of what can be inferred from those grounds, which is anything which follows from the sequent $\alpha \Rightarrow \beta$. Thus the E-meaning of $\alpha \to \beta$ is represented by the second-order formula:

$$(\forall\zeta)(((\alpha \to \beta) \to \zeta) \to \zeta),$$

that is, if α implies β only if ζ, then ζ, for any ζ.

Now consider $\circ(\alpha, \beta, \gamma)$ again. Since in its I-rule, $m = 2$ and $n_1 = n_2 = 1$, it follows that $\prod_{i=1}^{m} n_i = 1$, and so the rules $\circ\mathrm{I}_1$ and $\circ\mathrm{I}_2$ generate one GE-rule:

$$\dfrac{\circ(\alpha, \beta, \gamma) \qquad \begin{matrix} \left[\begin{matrix} \alpha \\ \vdots \\ \beta \end{matrix}\right]^1 \\ \zeta \end{matrix} \qquad \begin{matrix} [\gamma]^2 \\ \zeta \end{matrix}}{\zeta} \circ\mathrm{E}(1,2)$$

Once again, the E-meaning is given in terms of what can be inferred from the grounds for asserting $\circ(\alpha, \beta, \gamma)$, that is, whatever follows both from the

[21] This account of the I-meaning can already be found in [44, p. 505]. For the case $m = 0$, see footnote 22. If $l = 0$, then $g_l(*\mathrm{I}_i) = \varnothing$ in which case we let $c(\varnothing) = \top$ $(:= \bot \to \bot)$.

sequent $\alpha \Rightarrow \beta$ and from γ. Thus the E-meaning of $\circ(\alpha,\beta,\gamma)$ is represented by the second-order formula:

$$(\forall\zeta)((((\alpha \to \beta) \to \zeta) \wedge (\gamma \to \zeta)) \to \zeta),$$

that is, if α implies β only if ζ, and moreover γ holds only if ζ, then ζ, for any ζ. More generally, the E-meaning of $*$ as given by the $*$E-rules of §3 is:

$$\bigwedge_{j\in\Omega}(\forall\zeta)(\bigwedge_{i=1}^{m}(c(g_{j(i)}(*\mathrm{I}_i)) \to \zeta) \to \zeta)$$

where, as in §5, $\Omega = \times_{i=1}^{m} n_i$, that is, the set of m-tuples j whose ith member $j(i) \leq n_i$.[22]

The property that the E-rules for a connective $*$ have when the E-meaning for $*$ implies the I-meaning for $*$ provides an alternative way of formally capturing the notion of inverse harmony. For example, $(\forall\zeta)(((\alpha \to \zeta) \wedge (\beta \to \zeta)) \to \zeta)$, the E-meaning of '$\vee$', implies $\alpha \vee \beta$, the I-meaning of '$\vee$', and the E-meaning of '$\circ$', $(\forall\zeta)((((\alpha \to \beta) \to \zeta) \wedge (\gamma \to \zeta)) \to \zeta)$, implies $(\alpha \to \beta) \vee \gamma$, the I-meaning of '$\circ$'.

In fact, the I- and E-meanings of both '$\vee$' and '$\circ$' are equivalent. In general, we can show that the E-meanings given by the GE-rules are equivalent to the I-meanings of the set of I-rules from which they are generated:

THEOREM 2. *(Equivalence of I- and E-meanings) The E-meanings given by the GE-rules are equivalent to the I-meanings of the sets of I-rules which generate them.*

PROOF. we show that

$$\bigvee_{i=1}^{m}\bigwedge_{l=0}^{n_i} c(g_l(*\mathrm{I}_i)) \leftrightarrow \bigwedge_{j\in\Omega}(\forall\zeta)(\bigwedge_{i=1}^{m}(c(g_{j(i)}(*\mathrm{I}_i)) \to \zeta) \to \zeta)$$

Left-to-right: Clearly,

$$c(g_l(*\mathrm{I}_i)) \to (\bigwedge_{i=1}^{m}(c(g_{j(i)}(*\mathrm{I}_i)) \to \zeta) \to \zeta)$$

for any ζ, where $j(i) = l \leq n_i$, and so

$$\bigwedge_{l=0}^{n_i} c(g_l(*\mathrm{I}_i)) \to (\forall\zeta)(\bigwedge_{i=1}^{m}(c(g_{j(i)}(*\mathrm{I}_i)) \to \zeta) \to \zeta)$$

[22]When there is no I-rule (i.e., $m = 0$), $\prod_{i=1}^{0} n_i = 1$ and $*\vec{\alpha} = \bot$ (see footnote 10). Then $\bigwedge_{i=1}^{m}(c(g_1(*\mathrm{I}_i)) \to \zeta)$ is empty. Since, by convention, the empty conjunction is $\top$, the E-meaning of '$\bot$' is $(\forall\zeta)(\top \to \zeta)$, that is, $(\forall\zeta)\zeta$. In the case where $n_i = 0$, Ω is empty, and so by the same convention, the E-meaning of $*$ is $\top$.

for every $j \in \Omega$. Hence

$$\bigwedge_{l=0}^{n_i} c(g_l(*\mathrm{I}_i)) \to \bigwedge_{j\in\Omega}(\forall\zeta)(\bigwedge_{i=1}^{m}(c(g_{j(i)}(*\mathrm{I}_i)) \to \zeta) \to \zeta)$$

for every i. The result follows by $\vee$E.

Right-to-left: Suppose $\bigwedge_{j\in\Omega}(\forall\zeta)(\bigwedge_{i=1}^{m}(c(g_{j(i)}(*\mathrm{I}_i)) \to \zeta) \to \zeta)$. Then, by $\wedge$E and $\forall$E,

$$\bigwedge_{i=1}^{m}(c(g_{j(i)}(*\mathrm{I}_i)) \to \bigvee_{i=1}^{m} c(g_{j(i)}(*\mathrm{I}_i))) \to \bigvee_{i=1}^{m} c(g_{j(i)}(*\mathrm{I}_i))$$

for every $j \in \Omega$. Clearly, $c(g_{j(i)}(*\mathrm{I}_i)) \to \bigvee_{i=1}^{m} c(g_{j(i)}(*\mathrm{I}_i))$ for every i, by $\vee$I. Hence, by $\to$E, $\bigvee_{i=1}^{m} c(g_{j(i)}(*\mathrm{I}_i))$ for every $j \in \Omega$. Thus,

$$\bigwedge_{j\in\Omega}\bigvee_{i=1}^{m} c(g_{j(i)}(*\mathrm{I}_i))$$

by $\wedge$I, and so

$$\bigvee_{i=1}^{m}\bigwedge_{j\in\Omega} c(g_{j(i)}(*\mathrm{I}_i))$$

by distributivity, which is equivalent to

$$\bigvee_{i=1}^{m}\bigwedge_{l=0}^{n_i} c(g_l(*\mathrm{I}_i))$$

■

Corollary 1. *(Local Completeness—aliter proof) $*\vec{\alpha}$ follows from the E-meaning of $*$.*

Proof. Clearly, $\bigvee_{i=1}^{m}\bigwedge_{l=0}^{n_i} c(g_l(*\mathrm{I}_i)) \to *\vec{\alpha}$, by $\wedge$E, $*\mathrm{I}_i$ and $\vee$E. So by Theorem 2 we have that

$$\bigwedge_{j\in\Omega}(\forall\zeta)(\bigwedge_{i=1}^{m}(c(g_{j(i)}(*\mathrm{I}_i)) \to \zeta) \to \zeta) \to *\vec{\alpha}$$

■

Equivalence of I- and E-meanings generalizes from the propositional case above to cases such as that of '$\exists$', to be discussed in §8. The E-meaning expressed by $\exists$E is $(\forall\zeta)((\forall u)(\alpha(u/x) \to \zeta) \to \zeta)$ for 'u' not free in ζ, which is intuitionistically and classically equivalent to $(\exists x)\alpha$.

Higher-level rules were simplified in [7] to a form also found in [31, p. 135], [42, p. 545], [22, p. 8] and [9, p. 619] in which only formulae are discharged.[23] In [33], this simplification is called "flattening" of the higher-level rules. The flattened E-rule for '$\rightarrow$' reads:

$$\dfrac{\alpha \rightarrow \beta \qquad \alpha \qquad \begin{matrix}[\beta]^1 \\ \gamma\end{matrix}}{\gamma} \rightarrow\!\mathrm{E}'(1)$$

It was shown in [33] that only in the context of classical *reductio* can we be sure that the flattened rules will be equivalent to the higher-level rules. Intuitionistically, the flattened GE-rules, such as those given in [9], are often weaker than the I-rule warrants, and so are not stable.

Schroeder-Heister and Olkhovikov [39] demonstrate the weakness of flattened E-rules for certain intuitionistic connectives, such as '$\circ$'. Suppose we flatten the GE-rule for '$\circ$' so that only formulae are discharged in the minor premises:

$$\dfrac{\circ(\alpha,\beta,\gamma) \qquad \alpha \qquad \begin{matrix}[\beta]^1 \\ \zeta\end{matrix} \qquad \begin{matrix}[\gamma]^2 \\ \zeta\end{matrix}}{\zeta} \circ\mathrm{E}'(1,2)$$

The E-meaning of '$\circ$' given by $\circ\mathrm{E}'$ is $(\forall\zeta)(\alpha \wedge (\beta \rightarrow \zeta) \wedge (\gamma \rightarrow \zeta)) \rightarrow \zeta$. It is easy to see that this is classically equivalent to $(\forall\zeta)(((\alpha \rightarrow \beta) \rightarrow \zeta) \wedge (\gamma \rightarrow \zeta)) \rightarrow \zeta)$. But it is not intuitionistically equivalent to it. The latter implies the former, but not vice versa. Thus $\circ\mathrm{E}'$ is in harmony with $\circ\mathrm{I}_1$ and $\circ\mathrm{I}_2$, but the rules are not (intuitionistically) stable. $\circ\mathrm{E}'$ is too weak and does not allow one to infer everything that the I-meaning of '$\circ$' warrants.[24]

7. Problems with Quantum Disjunction

We have shown in §§5-6 that the GE-rules of §3 are locally complete and that their E-meanings entail (and indeed, are equivalent to) their I-meanings. However, there is reason to doubt that local completeness and implication of the I-meaning of a connective by its E-meaning really succeed in formally capturing inverse harmony.

[23]However, the simplification was rejected by Dyckhoff for reasons given in [8].

[24]It had already been noted in [44, p. 506] that there "does not seem to be a suitable set of E-rules" (that is, flattened rules) for '$\circ$'.

Consider the elimination-rule for '∨' in quantum logic, which does not allow parametric assumptions in the minor premises:[25]

$$\dfrac{\alpha \vee \beta \qquad \begin{array}{c}[\alpha]^1\\ \gamma\end{array} \qquad \begin{array}{c}[\beta]^2\\ \gamma\end{array}}{\gamma}\ \vee\mathrm{E_Q}(1,2)$$

This rule appears to be locally complete by Pfenning and Davies' test, since one can recover $\alpha \vee \beta$ despite the restriction:[26]

$$\dfrac{\alpha \vee \beta \qquad \dfrac{[\alpha]^1}{\alpha \vee \beta}\ \vee\mathrm{I}_1 \qquad \dfrac{[\beta]^2}{\alpha \vee \beta}\ \vee\mathrm{I}_2}{\alpha \vee \beta}\ \vee\mathrm{E_Q}(1,2)$$

But though $\vee\mathrm{E_Q}$ is in harmony with ∨I (it does not allow one to infer more than ∨I warrants), it is clearly incomplete, since it is weaker than ∨E (which is also in harmony with ∨I) in that it does not permit one to establish the distribution of '∧' over '∨', as ∨E in conjunction with the ∧-rules and ∨I does.

Furthermore, the I- and E-meanings of '$\vee_Q$' are equivalent. The I-meaning of '$\vee_Q$' is $\alpha \vee \beta$, and its E-meaning is $(\forall\zeta)(((\alpha \to \zeta) \wedge (\beta \to \zeta)) \to \zeta)$, and these formulae are equivalent in classical, intuitionistic and quantum logic. Thus, local completeness and implication of I-meanings by E-meanings do not seem to capture the notion of inverse harmony successfully.

The example involving quantum disjunction is contentious. It is unclear whether the ban on parametric assumptions is acceptable in a schematic rule (as opposed, say, to requiring all parametric assumptions to have a certain form, as in the normal ∀I-rule). For example, Humberstone [16, p. 587] criticizes $\vee\mathrm{E_Q}$ for not respecting his demand of *Generality in respect of side formulas* (p. 521). But Humberstone confesses that he has given no argument for his Generality demands, other than to exclude the counterexamples. Francez [10, p. 115] rehearses an argument for the constraint from [17, p. 720], but admits that he finds the argument unconvincing. Indeed, the constraint seems to be violated by standard rules for the quantifiers and for modality. In the same vein, Prawitz [28, p. 35] noted that his modal rules do not satisfy his requirement that the rules be fully schematic.[27] Absent a

[25]See, e.g., [6, pp. 288-9; cf. p. 205].

[26]See [10, p. 95].

[27]Francez [10, p. 115] also notes the case of the modal rules. A way of dealing with modality fully schematically is given in [1].

clear and persuasive formulation of the constraint, the example of quantum disjunction invites further reflection on means to establish inverse harmony and, a fortiori, stability, to which we now turn.

8. Grounds for Assertion

Perhaps it was a mistake to respond to the objection that one cannot use $\vee$I to recover $\alpha \vee \beta$ from the application of $\vee$E to it by allowing the I-rules to precede the minor premises of the application of the E-rule. Rather, what we should require is that one derive the *grounds for assertion of* $\alpha \vee \beta$, or in general, derive the grounds for assertion of $*\vec{\alpha}$ from $*\vec{\alpha}$ after application of $*$E.

For example, we can do this with '$\to$'. Suppose we apply $\to$E to derive β from $\alpha \to \beta$ and α, as in §3:

$$\dfrac{\alpha \to \beta \qquad \left[\begin{array}{c}\alpha\\ \vdots\\ \beta\end{array}\right]^1}{\beta}\,{\to}\mathrm{E}(1)$$

or in the logistic notation

$$\dfrac{\Gamma \Rightarrow \alpha \to \beta \qquad \dfrac{\dfrac{}{\alpha \Rightarrow \alpha}\,Ass_1}{\alpha, \alpha \Rightarrow \beta \Rightarrow \beta}\,Ass_2}{\Gamma, \alpha \Rightarrow \beta}\,\to \mathrm{E}$$

Note that, as we remarked in §3, while the sequent $\alpha \Rightarrow \beta$ is discharged by $\to$E, the assumption of α remains so far undischarged.

We have now obtained, through an application of $\to$E, the grounds for an assertion of $\alpha \to \beta$, namely, an assertion of β on the basis of α: $g_1(\to\mathrm{I}) = \alpha \Rightarrow \beta$. From here, we can proceed to apply $\to$I to discharge the assumption of α and infer $\alpha \to \beta$:

$$\dfrac{\dfrac{\alpha \to \beta \qquad \left[\begin{array}{c}[\alpha]^2\\ \vdots\\ \beta\end{array}\right]^1}{\beta}\,{\to}\mathrm{E}(1)}{\alpha \to \beta}\,{\to}\mathrm{I}(2)$$

that is,

$$\dfrac{\dfrac{\Gamma \Rightarrow \alpha \to \beta \qquad \dfrac{\dfrac{}{\alpha \Rightarrow \alpha}\,Ass_1}{\alpha, \alpha \Rightarrow \beta \Rightarrow \beta}\,Ass_2}{\Gamma, \alpha \Rightarrow \beta}\,\to \mathrm{E}}{\Gamma \Rightarrow \alpha \to \beta}\,\to \mathrm{I}$$

This may seem little different from the derivation of $\alpha \vee \beta$ from $\alpha \vee \beta$ in §5. But that derivation did not derive either of the grounds for assertion of $\alpha \vee \beta$ from $\alpha \vee \beta$. As remarked in §5, that is clearly impossible, since those grounds are an assertion either of α or of β, and neither α nor β follows from $\alpha \vee \beta$.

However, in a multiple-conclusion environment, such as is available in classical logic, one can infer both α and β (disjunctively) from $\alpha \vee \beta$; and one can mimic a multiple-conclusion environment in single-conclusion natural deduction by use of the negation rules. First, note that in a single-conclusion calculus, we can show that if one of the grounds for assertion of $\alpha \vee \beta$ is not available, the other must be, that is, given $\neg\beta$, one can infer α from $\alpha \vee \beta$; and given $\neg\alpha$, one can infer β from it. First, we need *ex falso quodlibet* as $\neg$E:

$$\frac{\neg\alpha \qquad \alpha}{\beta}\;\neg\text{E} \qquad \text{that is,} \qquad \frac{\Gamma \Rightarrow \neg\alpha \quad \Delta \Rightarrow \alpha}{\Gamma, \Delta \Rightarrow \beta}\;\neg\text{E}$$

Then we can argue:

$$\dfrac{\alpha \vee \beta \quad [\alpha]^1 \quad \dfrac{[\beta]^2 \quad \neg\beta}{\alpha}\;\neg\text{E}}{\alpha}\;\vee\text{E}(1,2)$$

that is,

$$\dfrac{\dfrac{}{\alpha \vee \beta \Rightarrow \alpha \vee \beta}\;Ass_1 \quad \alpha \Rightarrow \alpha \quad \dfrac{\dfrac{}{\beta \Rightarrow \beta}\;Ass_1 \quad \dfrac{}{\neg\beta \Rightarrow \neg\beta}\;Ass_1}{\beta, \neg\beta \Rightarrow \alpha}\;\neg\text{E}}{\alpha \vee \beta, \neg\beta \Rightarrow \alpha}\;\vee\text{E}$$

The prohibition on parametric assumptions in the minor premises of $\vee\text{E}_\text{Q}$ prevents construction of the corresponding derivation.

Thus, in order to establish the derivability of grounds in a single-conclusion setting, we must aim to show that we can derive each of the grounds of one of the $*$I-rules from $*\vec{\alpha}$, given the falsehood of at least one of the grounds of each of the other $*$I-rules. First, we need to clarify what is meant by the falsity of a ground and by the derivation of a ground, when a ground is a sequent of level 2.

Definition 11. $\neg\pi = \{\beta_1, \ldots, \beta_l, \neg\gamma\}$ *if* π *is* $\beta_1, \ldots, \beta_l \Rightarrow \gamma$.

Definition 12. *A sequent* $\pi = \beta_1, \ldots, \beta_l \Rightarrow \gamma$ *is derivable from* Γ *if and only if the sequent* $\Gamma, \beta_1, \ldots, \beta_l \Rightarrow \gamma$ *is derivable.*

Also, recall the function $\eta()$ defined in §5. For each $1 \leq i \leq m$, $1 \leq l \leq n_i$, let $\neg\text{E}(g_l(*\text{I}_i), \beta)$ be the following derivation:

$$\dfrac{\eta(g_l(*\text{I}_i)) \quad \dfrac{}{\neg s(g_l(*\text{I}_i)) \Rightarrow \neg s(g_l(*\text{I}_i))}\;Ass_1}{g_l(*\text{I}_i), \neg g_l(*\text{I}_i) \Rightarrow \beta}\;\neg\text{E}$$

that is,

$$\frac{a(g_l(*\mathrm{I}_i)), g_l(*\mathrm{I}_i)) \Rightarrow s(g_l(*\mathrm{I}_i)) \quad \neg s(g_l(*\mathrm{I}_i)) \Rightarrow \neg s(g_l(*\mathrm{I}_i))}{g_l(*\mathrm{I}_i), a(g_l(*\mathrm{I}_i)), \neg s(g_l(*\mathrm{I}_i)) \Rightarrow \beta}\ \neg\mathrm{E}$$

since

$$a(end(\eta(g_l(*\mathrm{I}_i)))) = g_l(*\mathrm{I}_i), a(g_l(*\mathrm{I}_i))$$

$$s(end(\eta(g_l(*\mathrm{I}_i)))) = s(g_l(*\mathrm{I}_i)),$$

and

$$\neg g_l(*\mathrm{I}_i) = a(g_l(*\mathrm{I}_i)), \neg s(g_l(*\mathrm{I}_i)).$$

Theorem 3 (Derivability of Grounds). *For every $1 \leq k \leq m$, and every $j \in \Omega$, $g_{j(k)}(*I_k)$ is derivable from $*\vec{\alpha}$ and $\neg g_{j(i)}(*I_i)$ for every $i \neq k$.*

Proof.

$$\frac{\dfrac{}{*\vec{\alpha} \Rightarrow *\vec{\alpha}}\ Ass_1 \quad \ldots \quad \eta(g_{j(k)}(*\mathrm{I}_k)) \quad \ldots \quad \neg\mathrm{E}(g_{j(i)}(*\mathrm{I}_i), s(g_{j(k)}(*\mathrm{I}_k))) \quad \ldots}{*\vec{\alpha}, \bigcup_{i \neq k} \neg g_{j(i)}(*\mathrm{I}_i), a(g_{j(k)}(*\mathrm{I}_k)) \Rightarrow s(g_{j(k)}(*\mathrm{I}_k))}\ *\mathrm{E}_j$$

since

$$a(end(\eta(g_{j(k)}(*\mathrm{I}_k)))) = g_{j(k)}(*\mathrm{I}_i), a(g_{j(k)}(*\mathrm{I}_i)),$$

$$s(end(\eta(g_{j(k)}(*\mathrm{I}_k)))) = s(g_{j(k)}(*\mathrm{I}_k))$$

and $\neg\mathrm{E}(g_{j(i)}(*\mathrm{I}_i), s(g_{j(k)}(*\mathrm{I}_k)))$ is

$$\frac{\eta(g_{j(i)}(*\mathrm{I}_i)) \quad \neg s(g_{j(i)}(*\mathrm{I}_i)) \Rightarrow \neg s(g_{j(i)}(*\mathrm{I}_i))}{g_{j(i)}(*\mathrm{I}_i), \neg g_{j(i)}(*\mathrm{I}_i) \Rightarrow g_{j(k)}(*\mathrm{I}_k)}\ \neg\mathrm{E}$$

■

Thus, arguably, the single-conclusion setting has available resources enabling us to capture the idea of recovery of grounds and to show that the E-rule for quantum disjunction is inadequate to recover the grounds for assertion. In particular, the derivation of α from $\alpha \vee_{\mathrm{Q}} \beta$ and $\neg\beta$ is not sanctioned by $\vee\mathrm{E}_{\mathrm{Q}}$, since $\neg\beta$ is a parametric assumption in the derivation of α from β.

However, note that in Theorem 3, we use *ex falso quodlibet* (EFQ—in the form of $\neg$E). This need to appeal to EFQ in the single-conclusion setting may be seen as unsatisfactory, since EFQ is only an admissible rule in minimal logic, not a derived rule. Thus, we do not have a proof that the grounds for assertion for $*\vec{\alpha}$ are derivable from $*\vec{\alpha}$ in the case of **M**, as sought in Prawitz's Conjecture.

The single-conclusion approach is also faced with another difficulty. The idea of deriving the grounds for assertion of $*\vec{\alpha}$ from $*\vec{\alpha}$ does not easily generalize to cases where there are side conditions on $*$I or $*$E. Consider, $\exists$I and $\exists$E, for example:

$$\frac{\alpha(t/x)}{(\exists x)\alpha}\,\exists\mathrm{I} \qquad \text{and} \qquad \frac{(\exists x)\alpha \qquad \begin{array}{c}[\alpha(u/x)]^1\\ \zeta\end{array}}{\zeta}\,\exists\mathrm{E}(1)$$

provided that in $\exists$E, 'u' does not occur free in ζ or in any parametric assumptions (that is, 'u' is fresh). As observed in [31, §2.6], $\exists$I can be thought of heuristically as an infinite collection of rules, one for each term 't', so by the GE-procedure, $\exists$E should have infinitely many minor premises, each inferring the same conclusion from $\alpha(t/x)$ for each term 't'. The usual $\exists$E-rule with just one minor premise captures the effect by inferring the uniform conclusion ζ from an arbitrary instance $\alpha(u/x)$, so that the minor premise can go proxy for any of the derivations of the common 't'-free conclusion ζ from $\alpha(t/x)$ by substituting 't' for 'u' throughout the derivation, necessary for the simplification establishing harmony. However, none of these instances of $\alpha(t/x)$ follows from $(\exists x)\alpha$, nor does $\alpha(u/x)$ even.[28] If ζ does follow from $\alpha(u/x)$, then ζ follows from $(\exists x)\alpha$, as $\exists$E states. But there is no way back from $(\exists x)\alpha$ to its grounds, $\alpha(t/x)$.

Moreover, consider the E-rule $\vee\mathrm{E_{Q1}}$, a rule similar to $\vee\mathrm{E_Q}$, but allowing for singleton sets of formulae as parametric assumptions. As previously shown, on the single-conclusion approach the grounds for $\alpha\vee\beta$ are not derivable from $\alpha\vee\beta$ via $\vee\mathrm{E_Q}$, since, e.g., in deriving α from $\alpha\vee\beta$, the derivation relies on the additional premise $\neg\beta$. But on the single-conclusion approach the grounds for $\alpha\vee\beta$ may be derived from $\alpha\vee\beta$ via an application of $\vee\mathrm{E_{Q1}}$. So, if the single-conclusion approach to the derivability of grounds successfully captured the notion of inverse harmony, then the rule $\vee\mathrm{E_{Q1}}$ would be inversely harmonious with $\vee$I. But $\vee\mathrm{E_{Q1}}$ is not inversely harmonious with $\vee$I, since it is weaker than $\vee$E. Whereas $((\gamma\wedge\zeta)\wedge\alpha)\vee((\gamma\wedge\zeta)\wedge\beta)$ is derivable from γ, ζ and $\alpha\vee\beta$ by $\vee$E and $\wedge$I, no such derivation is possible by appealing to $\vee\mathrm{E_{Q1}}$ and $\wedge$I. Since $\vee\mathrm{E_{Q1}}$ is not inversely harmonious with $\vee$I, the single conclusion approach to the derivability of grounds does not successfully capture the notion of inverse harmony.

[28]Some systems of natural deduction, following [30, p. 164], contain a rule of existential instantiation (EI), allowing one to infer $\alpha(u/x)$ from $(\exists x)\alpha$, provided 'u' is fresh. But, since application of (EI) concludes in what Quine calls an "unfinished derivation," it follows, as Pelletier [24, p. 13] notes, that "under most interpretations [the inference from $(\exists x)\alpha$ to $\alpha(u/x)$] is not semantically valid."

Now consider the multiple-conclusion treatment of '$\vee$'. Nodes will consist of multisets of formulae. The rules are:

$$\frac{\Gamma, \alpha}{\Gamma, \alpha \vee \beta}\,\vee\mathrm{I}_1^{\mathrm{M}} \qquad \frac{\Gamma, \beta}{\Gamma, \alpha \vee \beta}\,\vee\mathrm{I}_2^{\mathrm{M}} \qquad \frac{\Gamma, \alpha \vee \beta \quad \begin{array}{c}[\alpha]^1 \\ \Delta, \gamma\end{array} \quad \begin{array}{c}[\beta]^2 \\ \Theta, \gamma\end{array}}{\Gamma, \Delta, \Theta, \gamma}\,\vee\mathrm{E}^{\mathrm{M}}(1,2)$$

$\vee\mathrm{E}^{\mathrm{M}}$ simplifies classically to:

$$\frac{\Gamma, \alpha \vee \beta}{\Gamma, \alpha, \beta}\,\vee\mathrm{E}'^{\mathrm{M}}$$

as shown in [31, §3.2], which can also be written:

$$\frac{\alpha \vee \beta}{\alpha \qquad\quad \beta}\,\vee\mathrm{E}'^{\mathrm{M}}$$

Then the local completeness of the rules for '$\vee$' is immediate: we have derived the grounds for $\alpha \vee \beta$ from $\alpha \vee \beta$ itself:

$$\frac{\alpha \vee \beta}{\dfrac{\alpha}{\alpha \vee \beta}\,\vee\mathrm{I}_1^{\mathrm{M}} \qquad\qquad \dfrac{\beta}{\alpha \vee \beta}\,\vee\mathrm{I}_2^{\mathrm{M}}}\,\vee\mathrm{E}'^{\mathrm{M}}$$

aliter,

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha \vee \beta}{\alpha, \beta}\,\vee\mathrm{E}'^{\mathrm{M}}}{\alpha \vee \beta, \beta}\,\vee\mathrm{I}_1^{\mathrm{M}}}{\alpha \vee \beta, \alpha \vee \beta}\,\vee\mathrm{I}_2^{\mathrm{M}}}{\alpha \vee \beta}\,\textit{Contraction}$$

Moreover, in the multiple-conclusion case we can prove the grounds for assertion, $g_l(*\mathrm{I}_k)$, directly, in disjunction with a choice of other grounds for assertion of $*\vec{\alpha}$:

Theorem 4 (Derivability of Grounds—Multiple-Conclusion Case). *We can derive the multiple conclusion $g_{j(1)}(*I_1), \ldots, g_{j(m)}(*I_m)$ from $*\vec{\alpha}$ for each selection function j.*

Proof. First, we need to formulate the multiple-conclusion rules for $*\vec{\alpha}$:

$$\frac{\Gamma_{i,1}, \Theta_{i,1} \Rightarrow \Delta_{i,1}, \alpha_{i,1} \quad \ldots \quad \Gamma_{i,n_i}, \Theta_{i,n_i} \Rightarrow \Delta_{i,n_i}, \alpha_{i,n_i}}{\bigcup_{j=0}^{n_i} \Gamma_{i,j} \Rightarrow \bigcup_{j=1}^{n_i} \Delta_{i,j}, *\vec{\alpha}}\,*\mathrm{I}_i^{\mathrm{M}}$$

$$\dfrac{\Gamma_0 \Rightarrow \Delta_0, *\vec{\alpha} \quad \Gamma_1, g_{j(1)}(*\mathrm{I}_1^{\mathrm{M}}) \Rightarrow \Delta_1, \gamma \quad \ldots \quad \Gamma_m, g_{j(m)}(*\mathrm{I}_m^{\mathrm{M}}) \Rightarrow \Delta_m, \gamma}{\bigcup_{k=0}^{n_i} \Gamma_k \Rightarrow \bigcup_{k=0}^{n_i} \Delta_k, \gamma} *\mathrm{E}_j^{\mathrm{M}}$$

and we can write the latter as

$$\dfrac{\Delta_0, *\vec{\alpha} \qquad \begin{array}{c}[g_{j(1)}(*\mathrm{I}_1^{\mathrm{M}})]^1 \\ \Delta_1, \gamma\end{array} \qquad \ldots \qquad \begin{array}{c}[g_{j(m)}(*\mathrm{I}_m^{\mathrm{M}})]^m \\ \Delta_m, \gamma\end{array}}{\Delta_0, \Delta_1, \ldots, \Delta_m, \gamma} *\mathrm{E}_j^{\mathrm{M}}(1, \ldots, m)$$

Then:

$$\dfrac{*\vec{\alpha} \Rightarrow *\vec{\alpha} \quad \ldots \quad \dfrac{\eta(g_{j(i)}(*\mathrm{I}_i^{\mathrm{M}}))}{\bigcup_{k \neq i} a(g_{j(k)}(*\mathrm{I}_k^{\mathrm{M}})), g_{j(i)}(*\mathrm{I}_i^{\mathrm{M}}), a(g_{j(i)}(*\mathrm{I}_i^{\mathrm{M}})) \Rightarrow s(g_{j(i)}(*\mathrm{I}_i^{\mathrm{M}})), \bigcup_{k \neq i} s(g_{j(k)}(*\mathrm{I}_k^{\mathrm{M}}))} \mathit{Thin} \quad \ldots}{*\vec{\alpha}, \bigcup_{i=1}^{m} a(g_{j(i)}(*\mathrm{I}_i^{\mathrm{M}})) \Rightarrow \bigcup_{i=1}^{m} s(g_{j(i)}(*\mathrm{I}_i^{\mathrm{M}}))} *\mathrm{E}_j^{\mathrm{M}}$$

since

$$a(end(\eta(g_{j(i)}(*\mathrm{I}_i^{\mathrm{M}})))) = a(g_{j(i)}(*\mathrm{I}_i^{\mathrm{M}})), g_{j(i)}(*\mathrm{I}_i^{\mathrm{M}})$$

and

$$s(end(\eta(g_{j(i)}(*\mathrm{I}_i^{\mathrm{M}})))) = s(g_{j(i)}(*\mathrm{I}_i^{\mathrm{M}})).$$

■

In fact, Theorem 4 also applies to intuitionistic logic, since intuitionistic logic can be given a multiple-conclusion formulation by restricting the premise of →I (and of ¬I and ∀I, if present) to a single formula.[29] If we admit infinitary multiple conclusions, we can derive the infinitary set of all wffs $\alpha(t/x)$ from $(\exists x)\alpha$.

However, even in the multiple-conclusion case derivability of grounds arguably does not capture the informal notion of inverse harmony. Converting the multiple-conclusion sequent calculus *GO†* for quantum logic of [2, p. 226] to natural deduction yields the following E-rule for disjunction:[30]

$$\dfrac{\Gamma, \alpha \vee \beta \qquad \begin{array}{c}[\alpha]^1 \\ \Delta\end{array} \qquad \begin{array}{c}[\beta]^2 \\ \Delta\end{array}}{\Gamma, \Delta} \vee\mathrm{E}_{\mathrm{Q}}^{\mathrm{M}}(1, 2)$$

As in the single-conclusion case, parametric assumptions are not allowed in the minor premises. In the presence of the elimination rule $\vee\mathrm{E}_{\mathrm{Q}}^{\mathrm{M}}$, the grounds

[29] See, e.g., [41, §3.5.10].

[30] In these systems the cut rule is restricted, since otherwise the distribution of '∧' over '∨' would be restored. See [2, p. 247] and [16, p. 301].

for an assertion of $\alpha \vee_{\mathrm{Q}} \beta$ are derivable from $\alpha \vee_{\mathrm{Q}} \beta$ (in a multiple-conclusion setting, the I-rules for quantum disjunction are just $\vee\mathrm{I}_1^{\mathrm{M}}$ and $\vee\mathrm{I}_2^{\mathrm{M}}$):

$$\dfrac{\alpha \vee_{\mathrm{Q}} \beta \qquad \dfrac{[\alpha]^1}{\alpha, \beta}\,\mathit{Thin} \qquad \dfrac{[\beta]^2}{\alpha, \beta}\,\mathit{Thin}}{\alpha, \beta}\,\vee\mathrm{E}_{\mathrm{Q}}^{\mathrm{M}}(1,2)$$

This result appears to show that the multiple-conclusion treatment of the derivability of the grounds of assertion does not successfully capture the informal notion of inverse harmony.

We turn, therefore, to a final way of formally capturing the notion of inverse harmony by generalizing the property of local completeness.

9. Generalized Local Completeness

Recall Lorenzen's inversion principle: it required that the E-rules for a connective $*$ be strong enough not only to derive the judgment $*\vec{\alpha}$, but anything that judgment implies. Pfenning and Davies' proposal (considered in §5 above) required only that the E-rules be strong enough to infer everything necessary to derive the original assertion to which those rules were applied. We now see that Lorenzen was right to require that they be strong enough to infer anything that follows from that assertion.

What is missing in Lorenzen's account, however, and was absent in Davies and Pfenning's formulation of local completeness, is inclusion of the parametric assumptions that may be in play. Accordingly, we need to take account of both issues in generalizing the notion of local completeness to the requirement of generalized local completeness as follows:

Definition 13. *(Generalized Local Completeness)*

- *The E-rules for a connective $*$ are* generally locally complete relative to a nonempty set of I-rules *if and only if for any system S of the sort defined in §2, for all $1 \leq i \leq m$, Γ_i and ζ, if ζ is derivable in S from Γ_i and $g_j(*I_i)$, for all $0 \leq j \leq n_i$, then ζ is derivable in S from $\bigcup_{i=1}^{m} \Gamma_i$ and $*\vec{\alpha}$ by appealing only to derivations of ζ from Γ_i, $g_j(*I_i)$, for all $0 \leq j \leq n_i$, the E-rules for $*$ and structural rules.*
- *The E-rules for a connective $*$ are* generally locally complete relative to an empty set of I-rules *if and only if for any system S of the sort defined in §2 then ζ is derivable in S from $*\vec{\alpha}$ by appealing only to the E-rules for $*$ and structural rules.*

Local completeness is a special case of generalized local completeness in two respects. First, in local completeness, ζ is $*\vec{\alpha}$ itself. Secondly, ζ (i.e., $*\vec{\alpha}$) is derived from $*\vec{\alpha}$ without considering any assumptions besides its grounds. Thus, generalized local completeness imposes a requirement at least as strong as local completeness. As should already be apparent, it imposes a requirement that is strictly stronger. The E-rules for $\vee E_Q$ are locally complete even though they are not generally locally complete: $(\gamma \wedge \alpha) \vee_Q (\gamma \wedge \beta)$ is derivable both from γ, α and from γ, β, even though $(\gamma \wedge \alpha) \vee_Q (\gamma \wedge \beta)$ is not derivable from γ and $\alpha \vee_Q \beta$. The weakness of the E-rules for $\vee_Q$ displayed in the single-conclusion calculus discussed in §8 is another instance of the same phenomenon: even though α is derivable both from α and from β and $\neg\beta$, it is not derivable from $\alpha \vee_Q \beta$ and $\neg\beta$, given the prohibition on parametric assumptions in the minor premises of $\vee E_Q$. Similarly, the $\vee E_{Q1}$-rule is not generally locally complete: $((\gamma \wedge \zeta) \wedge \alpha) \vee_{Q1} ((\gamma \wedge \zeta) \wedge \beta)$ is derivable both from γ, ζ and α, and from γ, ζ and β, even though $((\gamma \wedge \zeta) \wedge \alpha) \vee_{Q1} ((\gamma \wedge \zeta) \wedge \beta)$ is not derivable from γ, ζ and $\alpha \vee_{Q1} \beta$.

We now observe that the E-rules generated by the GE-procedure from the I-rules for any connective $*$ satisfy the requirement of generalized local completeness:

Theorem 5 (Generalized Local Completeness). *Let $*$ be an n-ary connective, for any $n \in \mathbb{N}$, governed by a set of introduction-rules. Then the GE-rules for $*$ are generally locally complete.*

The proof of Theorem 5 is essentially the same as the proof of Theorem 1, and thus we omit it.[31] The proof requires no principles not available in minimal logic, and so avoids the objections to the use of EFQ in the attempt to recover the grounds for assertion mooted in §8. In particular, it finally

[31] The one crucial difference between the two proofs lies in the definition of $Pf()$, with the function now being defined as follows:

$$Pf(j) = \begin{cases} \begin{array}{c} \mathfrak{A}_i \\ \Gamma_i, g_1(*\mathrm{I}_i), \ldots, g_{n_i}(*\mathrm{I}_i) \Rightarrow \zeta \end{array} & \text{if } j = \xi_{il} \in \Xi, \\[2ex] \dfrac{\dfrac{}{*\vec{\alpha} \Rightarrow *\vec{\alpha}}\,Ass_1 \quad Pf(\delta_1(j)) \quad \ldots \quad Pf(\delta_m(j))}{\bigcup_{i=1}^m \bigcup_{k=j(i)+1}^{n_i} g_k(*\mathrm{I}_i), \bigcup_{i=1}^m \Gamma_i, *\vec{\alpha} \Rightarrow \zeta}\,*\mathrm{E}_j & \text{if } j \in \Omega, \end{cases}$$

Note that i) when the set of I-rules is empty, $*$ is '$\bot$' and we obtain the derivation: $\dfrac{\dfrac{}{\bot \Rightarrow \bot}\,Ass_1}{\bot \Rightarrow \zeta}\,\bot\mathrm{E}$; and ii) when one of the I-rules has no premises, $*$ is '$\top$' and we obtain the derivation: $\dfrac{\begin{array}{c}\mathfrak{A} \\ \Gamma \Rightarrow \zeta\end{array}}{\top, \Gamma \Rightarrow \zeta}\,Thin$.

gives a positive answer to Prawitz's Conjecture: the set of inference rules warranted by the meaning conferred on the logical constants by the set of I-rules for each logic S are derivable from the addition to those rules of the GE-rules generated by them.

We have seen that the E-rules for quantum disjunction are not generally locally complete. If generalized local completeness indeed captures the informal notion of inverse harmony, then this is as it should be. Moreover, in the previous section we also noted that derivability of grounds was not easily generalizable to cases where there are side conditions on $*$I or $*$E, another reason for seeking a different formal account of inverse harmony. How does generalized local completeness fare in this respect?

Consider the standard I-rule for the universal quantifier,

$$\frac{\alpha(u/x)}{(\forall x)\alpha}\ \forall\mathrm{I}$$

provided that 'u' does not occur free in any parametric assumptions. In [31] it was shown how the rule $\forall$I can be thought of heuristically as a restriction to a finitary setting of the infinitary rule $\forall\mathrm{I}_\infty$:

$$\frac{\alpha(t_1/x) \qquad \alpha(t_2/x) \qquad \ldots}{(\forall x)\alpha}\ \forall\mathrm{I}_\infty$$

with a premise $\alpha(t_i/x)$ for every term t_i in the language. There, the GE-rules for the universal quantifier,

$$\frac{(\forall x)\alpha \qquad \begin{matrix}[\alpha(t_i/x)]^1 \\ \zeta\end{matrix}}{\zeta}\ \forall\mathrm{E}_i(1)$$

one for each term t_i, were shown to be harmonious, since they are in harmony with $\forall\mathrm{I}_\infty$, and equivalent to the standard E-rule for $\forall$:

$$\frac{(\forall x)\alpha}{\alpha(t/x)}\ \forall\mathrm{E}$$

Moreover, the GE-rules $\forall\mathrm{E}_i$ are generally locally complete with respect to the rule $\forall\mathrm{I}_\infty$. For suppose that we have a derivation of ζ from the (infinitely many) premises of $\forall\mathrm{I}_\infty$ in the context Γ:[32]

$$\begin{matrix}\Gamma, \alpha(t_1/x), \alpha(t_2/x), \ldots \\ \mathfrak{B} \\ \zeta\end{matrix}$$

[32]To avoid misunderstanding, note that $\Gamma, \alpha(t_1/x), \alpha(t_2/x), \ldots$ here denotes the undischarged assumptions on which ζ depends.

Then, in an infinitary setting we can construct the following derivation:[33]

$$\dfrac{(\forall x)\alpha \qquad \dfrac{(\forall x)\alpha \qquad \begin{matrix}\Gamma, [\alpha(t_1/x)]^1, [\alpha(t_2/x)]^2, \dots \\ \mathfrak{B} \\ \zeta\end{matrix}}{\zeta}\ \forall\mathrm{E}_1(1)}{\begin{matrix}\zeta \\ \vdots \\ \vdots\end{matrix}}\ \forall\mathrm{E}_2(2)$$

The proof is infinitely long, and has no last step. Nonetheless, it is a derivation of ζ depending only on Γ and $(\forall x)\alpha$ as open assumptions. By way of analogy, take the set of natural numbers, and successively remove each positive integer, starting with 1. At each finite step, we are left with an infinite set $\{0, n, n+1, \dots\}$ for some n. Nonetheless, once the whole process is completed (of which there is no last step), we are left with the singleton set $\{0\}$. Similarly here: each finite initial segment of the derivation is a derivation of ζ from the infinite set of open assumptions $\Gamma, (\forall x)\alpha, \alpha(t_n/x), \alpha(t_{n+1}/x), \dots$. But once the proof is completed, we have a derivation of ζ from Γ and $(\forall x)\alpha$ alone.

These considerations are, of course, purely heuristic. We do not have an infinite system of proof with infinitely long proofs from infinitely many assumptions. Instead, we simplify the infinitary rule $\forall\mathrm{I}_\infty$ to the standard rule $\forall\mathrm{I}$, which we can now see to be in inverse harmony with the set of E-rules $\forall\mathrm{E}_i$. For suppose that we have the following derivation:

$$\begin{matrix}\Gamma, \alpha(u/x) \\ \mathfrak{A} \\ \zeta\end{matrix}$$

for some term $u = t_i$. Then we can apply $\forall\mathrm{E}_i$ to derive ζ from $\Gamma, (\forall x)\alpha$:

$$\dfrac{(\forall x)\alpha \qquad \begin{matrix}\Gamma, [\alpha(u/x)]^1 \\ \mathfrak{A} \\ \zeta\end{matrix}}{\zeta}\ \forall\mathrm{E}_i(1)$$

Thus, given that generalized local completeness is the appropriate rendering of inverse harmony, this result shows that the GE-rules for the universal quantifier are indeed inversely harmonious.

[33]Here the vertical ellipses represent infinitely long derivations, not assumptions of derivations.

Consider next the infinitary E-rule $\exists\mathrm{E}_\infty$ for the existential quantifier:

$$\dfrac{(\exists x)\alpha \qquad \begin{matrix}[\alpha(t_1/x)]^1\\ \zeta\end{matrix} \qquad \begin{matrix}[\alpha(t_2/x)]^2\\ \zeta\end{matrix} \qquad \cdots}{\zeta}\ \exists\mathrm{E}_\infty(1,2,\ldots)$$

As mentioned in §8, the standard rule $\exists$E can be thought of heuristically as the restriction to the finitary case of the infinitary rule $\exists\mathrm{E}_\infty$.[34] Just as the harmony of the GE-rules for the universal quantifier is established with respect to the infinitary I-rule $\forall\mathrm{I}_\infty$, it is the infinitary rule $\exists\mathrm{E}_\infty$ that should be shown to be generally locally complete. The proof of this result is straightforward. Suppose that, for each i, we have the following derivation:

$$\begin{matrix}\Gamma_i, \alpha(t_i/x)\\ \mathfrak{A}_\mathfrak{i}\\ \zeta\end{matrix}$$

Then we can derive ζ from $\bigcup_i \Gamma_i$ and $(\exists x)\alpha$, showing that the $\exists$-rules are generally locally complete:

$$\dfrac{(\exists x)\alpha \qquad \begin{matrix}\Gamma_1, [\alpha(t_1/x)]^1\\ \mathfrak{A}_1\\ \zeta\end{matrix} \qquad \begin{matrix}\Gamma_2, [\alpha(t_2/x)]^2\\ \mathfrak{A}_2\\ \zeta\end{matrix} \qquad \cdots}{\zeta}\ \exists\mathrm{E}_\infty(1,2,\ldots)$$

Given the robustness of generalized local completeness, we propose that this property does indeed capture the informal notion of inverse harmony. Theorem 5 establishes that the GE-rules are inversely harmonious. This result, in conjunction with their harmony, ensures that the GE-rules are stable with respect to the I-rules from which they are generated.

10. Conclusion

The idea of logical inferentialism is that the rules for logical expressions give them their meaning. General-elimination inferentialism, as one might call it, takes the introduction-rules as prior in conferring meaning, the general-elimination rules being no more than consequences of the meaning so provided. But the E-rules should not merely be justified by the meaning conferred by the I-rules; they should allow one to infer everything that is warranted by that meaning. When they do, the rules are not only harmonious,

[34] See also [20, §8.5].

but stable. GE-harmony is the thesis that the GE-procedure generates GE-rules that are harmonious relative to the I-rules from which they are generated. GE-stability is the claim that the GE-rules are not only harmonious, but also inversely harmonious, that is, they allow one to infer not only no more, but also no less than is warranted by the I-rules from which they are generated.

Four tests for stability have been considered. Local completeness requires that one be able to give a non-vacuous derivation of an arbitrary formula dependent on itself using only the I- and E-rules for its main connective. We showed that the GE-rules satisfy that test, but queried whether the test was adequate, since the clearly unstable rules of quantum disjunction seem to pass the test. Perhaps the $\vee$E-rule of quantum logic should already be excluded as not being fully schematic, but a reason must be given for the exclusion which is not *ad hoc*.

An alternative proposal developed a recent suggestion of capturing the I- and E-meanings in (meta-)formulae expressing, respectively, the grounds for assertion and the inferences justified by such an assertion. It was shown that in the case of the GE-rules, the formulae expressing the I- and E-meanings of a connective are equivalent, both classically and intuitionistically (and indeed, in minimal logic **M**). But quantum disjunction again seems to show that the equivalence is insufficient to demonstrate stability.

A third proposal was that one should be able to derive not just the original assertion itself, but its grounds. In a single-conclusion or multiple-conclusion calculus, for classical or intuitionistic logic, it can be done for propositional logic. Even so, in the case of the single-conclusion calculus, the need to appeal to EFQ entails that the demonstration has not established Prawitz's Conjecture, which was made for minimal logic **M**, in which EFQ is only an admissible rule, and not derivable. Moreover, in the presence of quantifiers (or modality), where there are side conditions on the minor premises, there is no way back from the conclusion to the grounds. Finally, the single-conclusion calculus fails to exclude '$\vee_{\mathrm{Q1}}$'. The multiple-conclusion calculus avoids the first two problems, but still fails to exclude '$\vee_{\mathrm{Q}}$'.

Finally, we returned to the notion of local completeness, generalizing it to require the inclusion of parametric formulae, and the derivation of an arbitrary conclusion, and showed that the rules generated by the GE-procedure satisfy the more stringent test, while excluding the unstable rules for quantum disjunction. Hence, the general-elimination procedure guarantees stability, that is, the E-rules generated by the I-rules according to the general-elimination procedure allow one to infer no more and no less than is justified by the meaning conferred by the I-rules.

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