Denotation, Paradox and Multiple Meanings*

Stephen Read
Arché Research Centre
University of St Andrews, Scotland
slr@st-andrews.ac.uk

Abstract

In line with the Principle of Uniform Solution, Graham Priest has challenged advocates like myself of the "multiple-meanings" solution to the paradoxes of truth and knowledge, due to the medieval logician Thomas Bradwardine, to extend this account to a similar solution to the paradoxes of denotation, such as Berry's, König's and Richard's. I here rise to this challenge by showing how to adapt Bradwardine's principles of truth and signification for propositions to corresponding principles of denotation and signification for descriptive phrases, applying them to give a "multiple-meanings" solution to the denotational paradoxes.

1 Paradox

Arthur Prior [1958] observed that Epimenides' claim that all Cretans are liars cannot have been the only claim made by a Cretan. Suppose it were and that it was true: then it would be a lie, and so false; but if false, then not a lie and if the only Cretan utterance, true. Prior inferred that Epimenides, or another Cretan, must have said something else, and indeed, something true. Thomas Bradwardine, writing 600 years before Prior, appreciated the point but drew a more radical and more plausible conclusion. For it is in itself paradoxical to suppose that pure reflection on Epimenides' utterance could

*Graham Priest and I became firm friends over forty years ago, when he came to St Andrews on a fixed-term Lectureship. We bounced ideas off one another from the start, and from time to time have agreed sufficiently on some topic to write joint papers. One thing he has never convinced me of, however (or at least, not long enough to survive his leaving the room), is dialetheism. I remain firmly committed to the law of Contravalence (that nothing can be both true and false) as much as to the laws of Non-Contradiction and Excluded Middle. But my ideas about logic, logical paradoxes and logical consequence, such as they are, would not have developed so fruitfully without his constructive criticisms.

reveal to us that there was another Cretan utterance, let alone a true one. Rather, Bradwardine inferred, it is impossible for any utterance to mean only that that very utterance is a lie.[1] Similarly, no Cretan utterance can signify only that all Cretans are liars, if they all were liars. More generally, suppose that s signifies only that s is false, and suppose that s is false. It follows that something it signifies must fail to obtain, that is, it will not be false but true. So s must also signify that s is true, for Bradwardine claimed that signification is closed under consequence. Hence s cannot signify only that it is false.

The argument appeals to a principle that Bradwardine [2010, ¶6.3] cites as his second postulate:

> "Every utterance signifies or means as a matter of fact or absolutely everything which follows from it as a matter of fact or absolutely."[2]

We can unpack this as claiming that the meaning of every utterance encompasses everything that follows from it either of necessity or even contingently, depending on how things are.[3] Thus Bradwardine allows both for inherently (absolutely) paradoxical utterances, and for (merely) contingent paradox, depending on contingent matters of fact, e.g., on who uttered it, or on the existence or non-existence of other utterances and whether they are true or false.

We can formalize Bradwardine's postulate as a closure principle:[4]

$$(\forall p, q)((p \Rightarrow q) \to (\mathbf{Sig}(s, p) \to \mathbf{Sig}(s, q))) \tag{P2}$$

He also invokes his account of truth and falsity:

$$\mathbf{Tr}(s) =_{df} (\exists p)\mathbf{Sig}(s, p) \wedge (\forall p)(\mathbf{Sig}(s, p) \to p) \tag{Tr}$$

that is, an utterance is true just when it is significative (there is something it signifies) and everything it signifies obtains, and

$$\mathbf{Fa}(s) =_{df} (\exists p)(\mathbf{Sig}(s, p) \wedge \neg p) \tag{Fa}$$

---

[1] Bradwardine [2010, ¶ad A.4.3]. See also Read [2009, §2]. For the purposes of this paper, I will assume that the additional chapter contained in Appendix A to Bradwardine [2010] is indeed by Bradwardine. If not, it is certainly by an adherent and advocate of his views on the matter.

[2] Bradwardine [2010, ¶6.3]: *Quelibet propositio significat sive denotat ut nunc vel simpliciter omne quod sequitur ad istam ut nunc vel simpliciter.*

[3] The externalist implications of this account of meaning or signification are explored in Cameron [2012]. One might worry that the principle is too strong, just as many have objected to the unconstrained closure of knowledge under consequence. I explored ways to restrict Bradwardine's principle in Read [2015a].

[4] I've argued in a number of places, e.g., Read [2015b, pp. 399-400], that Bradwardine's second postulate should be interpreted as a closure principle, although Bradwardine's formal statement does not have that form. But he repeatedly applies it in that way.

that is, an utterance is false just when something it signifies fails to obtain.[5] (Tr) and (Fa) entail Bivalence, that every significative utterance is either true or false, and Contravalence, that none is both.[6]

From these principles, Bradwardine was able to show, as above, that every utterance which signifies its own falsity also signifies its own truth. Take Epimenides' claim that all Cretans are liars, for example. Since Epimenides was himself a Cretan, this claim entails that he is a liar, and hence that his own claim is a lie, so by (P2), his claim signifies that it is itself a lie. Now take any utterance, call it s, which signifies its own falsehood, and, in contrast to the earlier proof, suppose it signifies other things as well (as Epimenides' utterance does—it not only signifies that all Cretans are liars, but also that it itself is a lie, that Aenesidemus, also a Cretan, is a liar, that there are liars, and so on). If s is false, something it signifies must fail to obtain (by Fa), so if it is not something else it signifies that fails, it must be that it is false that fails, that is, if it's false and whatever else it signifies obtains, it follows that it is true. But clearly it signifies that it is false and whatever else it signifies, so by (P2) it signifies that it is true.[7] So any utterance which signifies that it is false also signifies that it is true. In particular, Epimenides' claim that all Cretans are liars also signifies, not only that it is itself false, but also that it is true (if all Cretans were liars). But it cannot be both true and false, so things cannot be wholly as it signifies, so by (Fa) it is false. Moreover, we cannot infer from the fact that it is false that it is true, for (Tr) requires for its truth that everything it signifies obtain, and that is impossible. Nothing can be both true and false.

This is Bradwardine's "multiple-meanings" solution to Epimenides' paradox. It can be extended to deal with Eubulides' Liar ('What I am saying is false'), the postcard (or 'yes'-'no') paradox, the 'no'-'no' paradox, Curry's paradox, the validity paradox and many others.[8] What, however, of the para-

[5]Restall [2008, p. 229-30] pointed out that Bradwardine's theory of signification collapses to triviality if '$\rightarrow$' is taken to be material implication. In Read [2008, pp. 206-7] I observed that Bradwardine's argument works and his theory is non-trivial when '$\rightarrow$' is taken as relevant implication.

[6]Whereas the variable 's' in (P2), (Tr) and (Fa) is a normal first-order variable ranging over utterances, the variables 'p' and 'q' are second-order propositional variables, that is, they should not be instantiated by *names* of propositions but by sentences expressing those propositions. For defence of the coherence of propositional quantification, see, e.g., Read [2006, 2007, 2008] or Rumfitt [2014].

[7] I noted in Read [2011, p. 231] that, strictly speaking, Bradwardine appeals here to a stronger principle than (P2):

$$(\forall p, q, r)((p \wedge q \Rightarrow r) \rightarrow (\mathbf{Sig}(s, p) \wedge \mathbf{Sig}(s, q) \rightarrow \mathbf{Sig}(s, r)))$$

in order to infer that if s signifies both that p and that q then s signifies that both p and q.

[8]See, e.g., Read [2006, 2010].

doxes of denotation?

2 Denotational Paradox

Graham Priest [2006b] observed that the denotational paradoxes are somewhat different from the usual semantic paradoxes, and the object of insufficient attention. Nonetheless, he thinks they share enough features with other paradoxes that they should yield to the same solution—"The Principle of Uniform Solution: same kind of paradox, same kind of solution."[9] Hence, any putative solution to the semantic paradoxes that cannot be adapted to deal with the denotational paradoxes is *ipso facto* inadequate. Of course, the slogan, "same paradox, same solution", is equivalent to "different solution, different paradox", threatening to undermine his point completely. The converse of Priest's principle is much more plausible: same solution, same kind of paradox. If they do yield to the same solution, so much the better for that solution, and the search for another solution can be called off; while if they do not, the possibility of a separate solution bringing out their different character is still open. The same point applies to the set-theoretic paradoxes. If they yield to the same solution, good, but if not, that in itself suggests they are different in kind.[10]

As a matter of fact, the Principle of Uniform Solution was invoked by Aristotle in ch. 24 of *De Sophisticis Elenchis*. He there argues against solving the Hooded Man paradox by reference to the fallacy of the relative and the absolute on the ground that the Hooded Man is the same kind of puzzle as 'Is this dog your father?', so they should have the same solution. The fallacious example is: 'This dog is a father, this dog is yours, so this dog is your father'. This puzzle does not yield to the fallacy of the relative and the absolute, so that cannot be right for the Hooded Man either—rather, they are both to be solved by reference to the fallacy of accident: just because the same thing is F and is G, it doesn't follow that it is an FG. Just because Coriscus is in a hood and is known to you, it doesn't follow that he is known to you in a hood; just because this dog is yours and a father, it doesn't follow that it is your father.

So the real question is whether Bradwardine's solution can be adapted to the denotational paradoxes, or whether a different solution is needed. If the latter, that in itself will suggest that the paradoxes are sufficiently different; while if Bradwardine's solution can be suitably adapted, that shows they are sufficiently similar and adds weight to the solution in further exploiting its explanatory character. So let us turn to that issue.

The simplest of the paradoxes of denotation is Berry's. Consider the de-

[9] Priest [2002, §11.5]; see also Priest [2006b, p. 140].

[10] As Priest [2002, §17.2, p. 287 n. 39] himself notes.

scription, 'the least integer not denoted in fewer than 19 syllables'.[11] Assuming that definite descriptions denote at most one thing, and given that there are only finitely many descriptions with fewer than 19 syllables, there must be a least integer not so denoted. But the above description denotes it in 18 syllables. Contradiction.

The paradox lends itself to many variations. König's paradox focuses on the description 'the least indefinable ordinal', assuming that definability requires a unique description.[12] Given that there are uncountably many ordinals but only countably many descriptions,[13] there must be a least ordinal which is not definable (since the ordinals are well-ordered), which has just been defined by that very description. Richard's paradox considers definable real numbers between 0 and 1, of which again, there must be only countably many, hence they are listable.[14] Nonetheless, diagonalization defines the real number whose ith term differs (in some determinate way) from the ith term of the ith number on the list, which is not on the list. Contradiction.

Priest [2002, §4.9] identifies in Berkeley's Master Argument for idealism a paradox which he dubs "Berkeley's paradox". This paradox uses an indefinite description, 'something I will never think about'. It denotes something indefinitely, and enables us to think about an arbitrary one of the things we will never think about, which we just have. This transforms into a paradox of denotation: the expression 'something not denoted' denotes something not denoted. Contradiction.

In his discussion of Bradwardine's solution to the liar paradox, Priest [2012] challenged its adherents to extend the "multiple-meanings" solution to solve these paradoxes of denotation, offering reasons why he thought this might be a challenge too far. I here rise to that challenge.

3 Denotation

As Priest [2012, p. 158] says, to deal with these paradoxes, a theory of descriptions and an account of denotation are needed. Each of the paradoxes in §2 uses a descriptive phrase of the form '$\nu x \phi x$', where 'ν' is a variable-binding term operator (usually abbreviated to 'vbto'), variously the definite description operator, ι (the ϕ), an indefinite description operator, ϵ (a/some ϕ), or the least number (or ordinal) operator, μ (the least ϕ).[15] There are various ways of

[11] Russell [1908, p. 223], Priest [2002, §9.3], Priest [2006a, p. 16].

[12] König [1967] (in van Heijenoort [1967]), Priest [2002, §9.3].

[13] Assuming a finite vocabulary for composing descriptions of finite length.

[14] Richard [1967], Priest [2002, p. 132], Priest [2006b, p. 139].

[15] Here, and throughout the paper, 'ϕ' ranges over properties, denoted by λ-terms, and ϕx represents both a formula ϕ of any complexity containing zero or more occurrences of the variable 'x' free (and possibly other variables) and its β-transform $[(\lambda y)\phi y]x$. Here ϕy (in

dealing with these expressions, either as incomplete expressions, defining them away (as did Russell, for example), or as singular terms, whether taking them always to denote (so when there is no ϕ, taking them to denote some one and the same arbitrary object, such as 0, or perhaps different objects), or allowing them to be empty (so when there is no ϕ, taking them to denote nothing). The Russellian account is not appropriate for present purposes, since it denies that descriptive terms exist at all, defining them away existentially, whereas our task is to give an account of denoting which avoids the paradoxes of denotation. To be sure, denying they denote at all is one (path to a) solution, but a less radical path is to show how these phrases can denote non-paradoxically.

An example of the second, essentially Fregean, way of dealing with definite descriptions standardly takes the two axioms:[16]

$$\exists! x\phi x \to \phi(\iota x\phi x) \tag{ι-$F1$}$$

$$\neg\exists! x\phi x \to \iota x\phi x = \iota x\bot \tag{ι-$F2$}$$

where $\exists! x\phi x$ abbreviates $\exists x\forall y(\phi y \leftrightarrow x = y)$. So if there is a unique ϕ, $\iota x\phi x$ (that is, the ϕ) is indeed ϕ; if not, '$\iota x\phi x$' denotes some constant thing, the same for all empty or non-unique descriptions.[17] The consequence is that all terms are taken to denote, even empty terms such as 'the greatest natural number', or incomplete singular terms like 'the table'. Of course, 'the greatest natural number' doesn't denote the greatest natural number, since there isn't one. Nonetheless, it must denote something. '$\iota x\phi x$' always denotes on the Fregean account.

Priest's theory of denotation invokes both a Gödel-numbering operator, forming a Gödel-term $\langle\tau\rangle$ from each term τ, and a binary relation D of denotation, with the *Denotation Principle*:[18]

$$\forall x(D(\langle\tau\rangle, x) \leftrightarrow x = \tau) \tag{Den}$$

Note that denotation is a function: a singular term 'τ' denotes at most one object. What is missing in this account, however, is any mention of the content of the term 'τ'. Expressions like 'the negative square root of 2' or 'the

general, $\phi\tau$) results from ϕx by replacing all free occurrences of 'x' by 'y', respectively, 'τ', ensuring in the usual way that no variables are accidentally bound.

[16]See, e.g., da Costa [1980, p. 138], Read [1993, §5]. In addition, for all vbtos ν, we have alpha-conversion: $\nu x\phi x = \nu y\phi y$, and extensionality: $\forall x(\phi x \leftrightarrow \psi x) \to \nu x\phi x = \nu x\psi x$. My justification for calling this a "Fregean" account is Frege's discussion in [2013, Part III 1 (a) 1, esp. §§63-64]: "[A] more precise stipulation needs to be made here, so that for every object it is determined which object the half of it is; otherwise it is not permissible to use the expression, 'the half of x', with the definite article ... [P]roper names are inadmissible that do not actually designate an object."

[17]This is a simplifying assumption, for the purposes of this paper. In line with footnote 5, not all contradictions are equivalent, and so different empty descriptions should really be allowed to denote different objects.

[18]See Priest [2006a, p. 25].

smallest positive integer' denote what they do in virtue of their content. That is reflected in (ι-$F1$) and (ι-$F2$). If there is exactly one ϕ, then the ϕ is not only ϕ, but 'the ϕ' denotes it in virtue of its being ϕ (and the only ϕ). If not, then 'the ϕ' denotes $\iota x\bot$.

In particular, consider once again the term, 'the greatest natural number'. This cannot denote a natural number greater than all others, for there is no natural number greater than all others. But to realise that, we again need to examine the content of the term. It is because of the descriptive content of the term, and all it implies, that it fails to denote a natural number greater than all others, for nothing can have that property, which is inherently contradictory. Indeed, as we will see, we can sustain a theory on which it does denote—but not a natural number greater than all others.[19] We also need to realise that an incomplete singular term, like 'the table', say, also fails to denote what one might expect, since there is no unique thing which satisfies the description 'table'—unless context adds further information to the description to fix a unique table.

Accordingly, what we need is a notion of the signification of a term. Where 'τ' is a (meta-)variable over terms, let $\mathbf{Sig}(\langle\tau\rangle,\phi)$ express the fact that a term 'τ' signifies (various) properties, ϕ. Then we require that the singular term 'τ' denotes anything that has all the properties which it signifies, if there is one:

$$\exists! x(\forall\phi)(\mathbf{Sig}(\langle\tau\rangle,\phi)\to\phi x)\to\forall x(\forall\phi(\mathbf{Sig}(\langle\tau\rangle,\phi)\to\phi x)\to D(\langle\tau\rangle,x)) \qquad (\textsc{Den}\text{-}B1)$$

A second principle covers the case where nothing satisfies everything that 'τ' signifies:[20]

$$\neg\exists x(\forall\phi)(\mathbf{Sig}(\langle\tau\rangle,\phi)\to\phi x)\to D(\langle\tau\rangle,\iota x\bot) \qquad (\textsc{Den}\text{-}B2)$$

But what if more than one thing has all the properties that 'τ' signifies? 'τ' is a singular term, so it should denote just one thing, if at all. The answer is to add a further principle regarding the signification of a term 'τ'. The same issue arises with Bradwardine's account of the signification of sentences. For example, Bradwardine clearly thinks, quite naturally, that 'All Cretans are liars' signifies that all Cretans are liars, and that 'What Socrates says is false' signifies that what Socrates says is false. Although not explicitly stated, the general principle he accepts is that $(\forall p)\mathbf{Sig}(\langle p\rangle,p)$.[21] In the same way, a term like 'the Hooded Man' signifies being the Hooded Man, and 'what Socrates

[19] *Pace* Priest [1997b], who shows that, with sufficient violence to logic, we can even construct a model containing a greatest natural number, which the description does denote.

[20] We will see in §6 that the paradoxes can be strengthened to rule out the possibility of avoiding contradiction by supposing that the terms do not denote at all.

[21] See, e.g., Read [2015b, p. 400].

says' signifies being what Socrates says. In general:[22]

$$\mathbf{Sig}(\langle\tau\rangle, (\lambda x)x = \tau). \tag{But}$$

(But) guarantees that if anything has all the properties that 'τ' signifies, then a unique thing does, namely, τ. For suppose $\forall\phi(\mathbf{Sig}(\langle\tau\rangle, \phi) \to \phi x)$ and $\forall\phi(\mathbf{Sig}(\langle\tau\rangle, \phi) \to \phi y)$. Then $\mathbf{Sig}(\langle\tau\rangle, (\lambda x)x = \tau) \to x = \tau$ and $\mathbf{Sig}(\langle\tau\rangle, (\lambda x)x = \tau) \to y = \tau$. By (But), $\mathbf{Sig}(\langle\tau\rangle, (\lambda x)x = \tau)$. So $x = \tau = y$. Hence, if anything has all the properties 'τ' signifies, then only one thing does.

In line with Bradwardine's observation that a proposition may signify more than may appear, and that signification is closed under consequence, we should also require that the signification of terms be closed under consequence:[23]

$$(\forall\phi, \psi)((\forall x)(\phi x \Rightarrow \psi x) \to (\mathbf{Sig}(\langle\tau\rangle, \phi) \to \mathbf{Sig}(\langle\tau\rangle, \psi)) \tag{Clo}$$

Similarly, Bradwardinian versions of (ι-$F1$) and (ι-$F2$) must be predicated on the assumption that something uniquely satisfies everything that '$\iota x\phi x$' signifies:

$$\exists! x\forall\psi(\mathbf{Sig}(\langle\iota x\phi x\rangle, \psi) \to \psi x) \to \phi(\iota x\phi x) \tag{ι-$B1$}$$

and

$$\neg\exists! x\forall\psi(\mathbf{Sig}(\langle\iota x\phi x\rangle, \psi) \to \psi x) \to \iota x\phi x = \iota x\bot \tag{ι-$B2$}$$

We can augment (Den-$B2$) with an exclusion principle, ensuring that 'τ' denotes $\iota x\bot$ only when nothing satisfies everything that 'τ' signifies:

$$\tau = \iota x\bot \leftrightarrow \neg(\exists! x)(\forall\phi)(\mathbf{Sig}(\langle\tau\rangle, \phi) \to \phi x) \tag{Exc}$$

The justification for adding (Exc) is two-fold: first, by the thought that, despite Frege's practice, it is good to keep the denotation of terms that something satisfies distinct from those that nothing satisfies; secondly, that it allows us to preserve the *Denotation Principle* (Den) and the requirement that all terms denote. However, (Exc) does lay down a stiff and puzzling requirement on the denotation of the contradictory term '$\iota x\bot$', namely, that nothing satisfies everything it signifies—that it cannot be exactly characterized. That requirement can be defended: after all, what could satisfy the characterization $(\lambda x)\bot$? However, it does suggest that whatever it is that contradictory terms denote, it does not, and could not, exist. In other words, it suggests that the theory is not so much Fregean, as Meinongian, or rather, noneist. Although Priest [2005, p. ix] remarks that "noneism is naturally committed to the idea that every term denotes something," there is more to noneism than that: it claims that everything is something, and some things don't exist (and may even qualify the claim that every term denotes something, to claim, e.g., that every term

[22] As Butler [1765, Preface, p. 37] wrote, "everything is what it is and not another thing".

[23] If necessary, we can generalize (Clo) in the same way (P2) was generalized in foonote 7.

denotes some thing or things, as in Priest [2005, ch. 8]). Indeed, the present view is not Meinong's: Meinong was committed to the *Independence Principle*, of the independence of *Sosein* from *Sein*, and the *Characterization Principle*: that an object has the properties it is characterized as having, regardless of whether it exists.[24] That's false: in particular, $\iota x\bot$ does not have the property $(\lambda x)\bot$, nor many of the other properties signified by terms that denote it, as we will see.[25]

Given (Exc), we can show that 'τ' always denotes τ. For suppose

$$\exists x(\forall\psi)(\mathbf{Sig}(\langle\tau\rangle,\psi)\to\psi x)$$

Then for some y,

$$(\forall\psi)(\mathbf{Sig}(\langle\tau\rangle,\psi)\to\psi y)$$

in particular,

$$\mathbf{Sig}(\langle\tau\rangle,(\lambda x)x=\tau)\to y=\tau$$

But $\mathbf{Sig}(\langle\tau\rangle,(\lambda x)x=\tau)$, by (But). So $y=\tau$. Hence

$$(\forall\psi)(\mathbf{Sig}(\langle\tau\rangle,\psi)\to\psi\tau)$$

Moreover, suppose $(\forall\psi)(\mathbf{Sig}(\langle\tau\rangle,\psi)\to\psi z)$. Then by the same reasoning, $z=\tau$, so $y=z$, whence $\exists!x(\forall\psi)(\mathbf{Sig}(\langle\tau\rangle,\psi)\to\psi x)$, and $(\forall\psi)(\mathbf{Sig}(\langle\tau\rangle,\psi)\to\psi\tau)$, so by (Den-$B$1), $D(\langle\tau\rangle,\tau)$. On the other hand, if

$$\neg\exists x(\forall\psi)(\mathbf{Sig}(\langle\tau\rangle,\psi)\to\psi x)$$

then by (Den-B2), $D(\langle\tau\rangle,\iota x\bot)$ and by (Exc) $\tau=\iota x\bot$, so $D(\langle\tau\rangle,\tau)$. Either way, $D(\langle\tau\rangle,\tau)$, and so $(\exists x)D(\langle\tau\rangle,x)$, that is, all terms denote.

It follows that D is functional. For suppose that $D(\langle\tau\rangle,x)$ and $D(\langle\tau\rangle,y)$. Then either $\exists x(\forall\phi)(\mathbf{Sig}(\langle\tau\rangle,\phi)\to\phi x)$ or not. If the former, then as we showed above, $\exists!x(\forall\phi)(\mathbf{Sig}(\langle\tau\rangle,\phi)\to\phi x)$, whence $x=y$. But if $\neg\exists x(\forall\phi)(\mathbf{Sig}(\langle\tau\rangle,\phi)\to\phi x)$, then $D(\langle\tau\rangle,\iota x\bot)$, so by (Exc) $x=\iota x\bot=y$.

Finally, (Den) immediately follows, given that D is functional: for if $x=\tau$, then since $D(\langle\tau\rangle,\tau)$, we have $D(\langle\tau\rangle,x)$; conversely, if $D(\langle\tau\rangle,x)$, then since $D(\langle\tau\rangle,\tau)$ and D is functional, $x=\tau$. So $\forall x(D(\langle\tau\rangle,x)\leftrightarrow x=\tau)$.

4 Berry's Paradox

Take the description, 'the least natural number not denoted in English in fewer than 75 characters'. Let 'Bx' abbreviate $\neg\exists y(Nx\wedge(\ell y<75)\wedge Dyx)$, where

[24] See, e.g., [Priest, 2005, p. vii].

[25] Priest [2005, p. 84] proposes that such descriptions characterize what they denote only at other, non-actual worlds. But, like me, he also believes that these other worlds don't exist. See Priest [2005, §7.3] and Read [2005].

ℓy gives the number of characters in the term y. Call the members of the set $\{x : Bx\}$ the Berry numbers, that is, all the numbers not denoted in fewer than 75 characters. Since there are finitely many (alphanumeric) characters in English, say m, there are at most m^{75} names containing fewer than 75 characters, and so at most m^{75} natural numbers are denoted by such terms, a finite number. Hence there are a countable infinity of Berry numbers. The Berry numbers are natural numbers and thus well-ordered, and so $\{x : Bx\}$ has a least member, $\mu y By$, that is, $\iota y(By \wedge \forall x(Bx \to y \leq x))$. As a matter of fact, $\ell\langle \mu y By\rangle = 74$. Let π abbreviate $\mu y By$. Then we may be tempted to argue:

$$N\pi \wedge (\ell\langle\pi\rangle = 74) \wedge D(\langle\pi\rangle, \pi)$$

$$\text{so } \exists y(N\pi \wedge (\ell y = 74) \wedge D(y, \pi))$$

$$\text{whence } \exists y(N\pi \wedge (\ell y < 75) \wedge D(y, \pi))$$

But $B\pi$, so $\neg\exists y(N\pi \wedge (\ell y < 75) \wedge D(y, \pi))$. Contradiction.

The error lies in the claim that 'π', that is, 'the least natural number not denoted in English in fewer than 75 characters', denotes a Berry number smaller than all the others. What the paradox really shows is that the Berry term 'π' is inherently contradictory. We can show that, although $\mathbf{Sig}(\langle\pi\rangle, \lambda x(Bx \wedge \forall y(By \to x \leq y)))$, '$\pi$' signifies more than just $\lambda x(Bx \wedge \forall y(By \to x \leq y))$, that is, being a Berry number smaller than all the others. First, suppose being such a Berry number were all 'π' signified. Then by (Den-$B1$), if n were that Berry number, 'π' would denote n. But 'π' has 74 characters, so n would be denoted by a description with fewer than 75 characters. Since 'π' signifies being a Berry number smaller than all the others, it would follow from (Clo) that 'π' signified being denoted by a description with fewer than 75 characters. So being a Berry number smaller than all the others would not be the only thing 'π' signified. Consequently, being the least Berry number is not all that 'π' signifies.

Now suppose 'π' signifies being a Berry number smaller than all the others and ψ, where ψ encapsulates everything else that 'π' signifies. Once again, if n were a Berry number smaller than all the others and ψ obtained, 'π' would denote n, by (Den-$B1$). But 'π' has 74 characters, so n would be denoted by a description with 74 characters. Since 'π' signifies being a Berry number smaller than all the others and being ψ, it follows from (Clo) that 'π' signifies being denoted in fewer than 75 characters. So 'π' signifies both being a Berry number smaller than all the others, that is, not being denoted with fewer than 75 characters, and being so denoted. So 'π' is implicitly contradictory, and there is nothing which has all and only the properties signified by 'π', and consequently 'π' denotes the contradictory object: $\pi = \iota x\bot$.

What we have shown is that 'π' does not denote something not denoted in fewer than 75 characters, for the description is implicitly contradictory, purporting to denote something which both is and is not denoted in fewer than 75 characters. The description denotes something which is denoted in fewer than 75 characters, for it is denoted by 'the least number not denoted in fewer than 75 characters', that is, 'π'. Of course, whatever that object is, it is not a number not denoted in fewer than 75 characters smaller than all the others, for it is denoted by 'π'. Meinong's Characterization Principle must be denied.

One might be tempted to express this result as denying that 'π' denotes the least number not denoted in fewer than 75 characters, i.e., π—that is, as showing that $\neg D(\langle\pi\rangle, \pi)$. Priest [2012, p. 157] rightly says that such a result would be "something of a *reductio* of the Bradwardine line." Actually, we have seen that the Bradwardinian theory is compatible with the universal truth of $D(\langle\tau\rangle, \tau)$. Nonetheless, the description, 'the least natural number not denoted in English in fewer than 75 characters', does not denote a number not denoted in fewer than 75 characters. And in fact this paradoxical observation answers closely to the Principle of Uniform Solution. Recall that Bradwardine responds to the Liar by saying it is false. As Field [2006, p. 715] notes, this has a similarly puzzling and paradoxical air: the Liar says of itself that it is false, so Maudlin (on whom Field is commenting) and Bradwardine both say that the Liar is false and (since the Liar says that the Liar is false) that it is false that the Liar is false. But the Liar is false (according to Bradwardine) not because it isn't false, but because it isn't true—that is, something else it signifies fails to obtain. The Liar sentence signifies not only that the Liar sentence is false (which is the case) but also that it is true (which is not so)—that is why it is false.

Priest [2012, p. 156] shows that Bradwardine is committed to a similar result about the heterologicality paradox. An object satisfies a predicate if it has all the properties that the predicate signifies, in symbols:

$$(\forall x)(x \boldsymbol{\$} \langle\psi\rangle \leftrightarrow (\forall\phi)(\mathbf{Sig}(\langle\psi\rangle, \phi) \to \phi x))$$

Consider the predicate '$\neg x \boldsymbol{\$} x$', and suppose $\neg\langle\neg x \boldsymbol{\$} x\rangle \boldsymbol{\$} \langle\neg x \boldsymbol{\$} x\rangle$. Then for some ϕ,

$$\mathbf{Sig}(\langle\neg x \boldsymbol{\$} x\rangle, \phi) \wedge \neg\phi(\langle\neg x \boldsymbol{\$} x\rangle).$$

We may assume in line with (BUT) that $\mathbf{Sig}(\langle\neg x \boldsymbol{\$} x\rangle, (\lambda x)\neg x \boldsymbol{\$} x)$, and let ψ conjoin everything else that $\langle\neg x \boldsymbol{\$} x\rangle$ signifies, so

$$\mathbf{Sig}(\langle\neg x \boldsymbol{\$} x\rangle, (\lambda x)(\neg x \boldsymbol{\$} x \wedge \psi x)).$$

Then, since something '$\neg x \boldsymbol{\$} x$' signifies is not satisfied by '$\neg x \boldsymbol{\$} x$', it follows that either $\langle\neg x \boldsymbol{\$} x\rangle \boldsymbol{\$} \langle\neg x \boldsymbol{\$} x\rangle$ or $\neg\psi(\langle\neg x \boldsymbol{\$} x\rangle)$. So if $\psi(\langle\neg x \boldsymbol{\$} x\rangle)$ and $\neg\langle\neg x \boldsymbol{\$} x\rangle \boldsymbol{\$} \langle\neg x \boldsymbol{\$} x\rangle$, then $\langle\neg x \boldsymbol{\$} x\rangle \boldsymbol{\$} \langle\neg x \boldsymbol{\$} x\rangle$. But $\mathbf{Sig}(\langle\neg x \boldsymbol{\$} x\rangle, (\lambda x)(\neg x \boldsymbol{\$} x \wedge \psi x))$, so

$$\mathbf{Sig}(\langle\neg x \boldsymbol{\$} x\rangle, (\lambda x)x \boldsymbol{\$} x).$$

That is, '$\neg x\$x$' signifies not only that '$\neg x\$x$' does not signify itself, but also that it does, exactly parallel with the conclusion that any proposition signifying that it is itself not true also signifies that it itself is true. Consequently, as before, it follows that '$\neg x\$x$' cannot satisfy itself, since it signifies contradictory properties. And again as before, this does not suffice to infer that it does satisfy itself, since nothing can satisfy contradictory properties.

Thus not only does Bradwardine's theory deal with the heterological and Berry paradoxes, it does so in an entirely similar way to the other semantic paradoxes, such as the Liar, employing the closure principle to show that truth, satisfaction and denotation make contradictory demands of such paradoxical terms. $D(\langle\pi\rangle, \pi)$ and $\pi = \iota x\bot$, so $D(\langle\pi\rangle, \iota x\bot)$. Nonetheless, $\exists\phi(\mathbf{Sig}(\langle\pi\rangle, \phi) \wedge \neg\phi\pi)$, in particular, $\neg B\pi$, even though $\mathbf{Sig}(\langle\pi\rangle, \lambda x B x)$.

5 Berkeley's Paradox

König's and Richard's paradoxes can be dealt with in much the same way as Berry's but require more technical apparatus from set theory. We should turn, therefore, to Berkeley's paradox. To recall, this is the paradox prompted by thinking about something not thought about, namely, what is denoted by the expression 'something not denoted'.

Priest extracts 'Berkeley's paradox' from what Gallois [1974] dubbed Berkeley's master argument for idealism. Berkeley challenges his realist opponent to conceive of things that are not conceived:

> "That you conceive them unconceived or unthought of ... is a manifest repugnancy ... The mind ... is deluded to think it can and doth conceive bodies unthought of or without the mind, though at the same time they are apprehended by or exist in it self."[26]

Consequently, Berkeley claims, the idea of mind-independent objects, existing unperceived and unthought of, is incoherent.

Whatever the merits of this argument, Priest [2002, pp. 69-70] distils from it the paradox set out above, that the phrase 'something not denoted' denotes something not denoted, and hence something that is denoted. Note that 'something not denoted' is an indefinite description, whose logical behaviour is given by variants of (ι-$B1$) and (ι-$B2$):[27]

$$\exists x \forall \psi(\mathbf{Sig}(\langle\epsilon x\phi x\rangle, \psi) \to \psi x) \to \phi(\epsilon x \phi x) \qquad (\epsilon\text{-}B1)$$

[26] *Principles of Human Knowledge* §23, in Berkeley [1837, p. 12]. See also his *Three Dialogues between Hylas and Philonous*, The First Dialogue [1837, p. 53].

[27] (ϵ-$B1$) and (ϵ-$B2$) adapt the usual Hilbertian axiom for indefinite descriptions, $(\exists x)\phi x \to \phi(\epsilon x \phi x)$ in the same way that (ι-$B1$) and (ι-$B2$) adapt the Fregean axioms for definite descriptions. See, e.g., Leisenring [1969, p. 40] and Priest [2002, §4.6]. da Costa [1980, p. 139] points out that (ϵ-$B2$) follows from the extensionality axiom (see footnote 16).

$$\neg\exists x\forall\psi(\mathbf{Sig}(\langle\epsilon x\phi x\rangle,\psi)\rightarrow\psi x)\rightarrow\epsilon x\phi x=\iota x\bot \qquad (\epsilon\text{-}B2)$$

Note that '$\epsilon x\phi x$' is a singular term, denoting a single object, either one of the ϕs or $\iota x\bot$. Although more than one thing may satisfy ϕ (e.g., being a table), '$\epsilon x\phi x$' must signify more than that in order to pick out $\epsilon x\phi x$ from the ϕs uniquely—e.g., by (But), '$\epsilon x\phi x$' signifies $(\lambda x)(x=\epsilon x\phi x)$. As this shows, the signification of '$\epsilon x\phi x$' will not be purely descriptive. In particular, it will signify a choice function.[28]

We can formalize the indefinite description 'something not denoted' with the existing notation as $(\epsilon x)\neg(\exists y)Dyx$—let us abbreviate this as ρ. Then it seems that $D(\langle\rho\rangle,\rho)$, while by definition, $\neg(\exists y)D(y,\rho)$, a contradiction.

The mistake, as before, is to think that nothing denotes ρ, just because $\mathbf{Sig}(\langle\rho\rangle,(\lambda x)\neg(\exists y)Dyx)$ and $D(\langle\rho\rangle,\rho)$. So we must ask, what properties, besides $(\lambda x)\neg(\exists y)Dyx$, does '$\rho$' signify? If that were all 'ρ' signified, and if 'ρ' denoted some object e (and not $\iota x\bot$), then by (ϵ-$B1$), $\neg(\exists y)D(y,e)$, but at the same time, $D(\langle\rho\rangle,e)$, a contradiction. So 'ρ' must signify more than that—call it ψ. Then by (Den-$B1$), if e were not denoted and $\psi(e)$, 'ρ' would denote e. Since 'ρ' signifies not being denoted and being ψ, it follows by (Clo) that 'ρ' signifies being denoted by 'ρ', and so being denoted by something. So 'ρ' signifies contradictory properties—both being denoted by 'ρ' and not being denoted at all, so by (Den-$B2$), $D(\langle\rho\rangle,\iota x\bot)$.

6 Hilbert-Bernays' Paradox

We have developed our theory of descriptions on the Fregean basis that all descriptions denote—if not something satisfying the description, then some arbitrary object which serves as the denotation of all unsatisfiable descriptions. Priest [2005, §8.3] reminds us of a paradox due to Hilbert and Bernays which seems to undermine this assumption.[29] Consider the definite description, 'the successor of the denotation of this description'. Given that no number is its own successor, it seems that this description cannot possibly denote. For if it denoted some number n, then it would also denote $n+1$. Given that the denotation of definite descriptions is unique (if it exists), it follows that $n=n+1$. Contradiction. The only possible solutions seem to be that the description does not denote, or denotes more than one thing.

Priest [2005, §§8.5-8.6] explores the latter possibility, that the description 'the successor of the denotation of this description' denotes more than one thing, namely, both what it denotes and its successor. This not only runs counter to the natural presumption in the theory of singular terms that their

[28] See, e.g., Hilbert [1967, p. 466] and Corcoran and Herring [1971, p. 649].

[29] See also Priest [1997a], Priest [2006b, §6].

denotation is unique, it also leads swiftly to contradiction, contradictions of the laws of identity which Priest is willing to countenance.[30]

But the other option, that the description does not denote, is not open to us either, as Priest shows.[31] As before, let

$$(\mu x)\phi x = \iota x(\phi x \wedge \forall y(\phi y \to x \leq y))$$

and let

$$(\eta x)\phi x = \iota y(((\exists x)\phi x \to y = (\mu x)\phi x) \vee (\neg(\exists x)\phi x \to y = \iota x\bot))$$

Then if there is a ϕ, $(\eta x)\phi x$ will be the least ϕ, while if there isn't, $(\eta x)\phi x$ will be $(\iota x)\bot$. But that just identifies $(\eta x)\phi x$ with $(\mu x)\phi x$, for by definition, $(\mu x)\phi x$ is the least ϕ if there is a least ϕ and $(\iota x)\bot$ if there isn't. What this shows us, however, is that taking '$(\iota x)\phi x$' to denote even when there is no unique ϕ is not an unmotivated decision, but can be forced on us by careful choice of ϕ.

Priest [2006b, p. 147] also observes that mention of successor in the above paradox is only one special case.[32] One can develop the paradox for any number-theoretic function, $f : \mathbb{N} \to \mathbb{N}$, and show formally that any such function has a fixed point—though many functions, such as successor, clearly do not. Let δ be a denotation function from arithmetic terms to their denotations, that is, such that for all τ, $D(\langle\tau\rangle, \delta\langle\tau\rangle)$. So $\delta\langle\tau\rangle = \tau$. Then if δ is representable, the usual diagonalization lemma can be reworked to show that, for any $f : \mathbb{N} \to \mathbb{N}$, there is a term '$\chi$' for which

$$\chi = f(\delta\langle\chi\rangle) = f(\chi),$$

so every number-theoretic function has a fixed point. Contradiction.

Hilbert and Bernays' response is to conclude that the denotation function is not arithmetic, and so cannot be represented in arithmetic (if arithmetic is consistent), any more than arithmetic truth can, as recorded in Tarski's Theorem.[33] Priest [1997a, p. 47] observes, however, that, sound as this conclusion may be for formal arithmetic, it still leaves open the paradox in natural language, just as Tarski's Theorem cuts no ice with the Liar paradox. For natural language does appear to have a denotation function, instanced here by the description 'the successor of the denotation of this description'. Priest's own solutions to this paradox (that in [Priest, 2006b] involving both truth-value

[30] Priest [2005, §8.7] avoids the consequence that some number is its own successor, from which it would follow that $0 = 1$, by denying the substitutivity of identicals.

[31] Priest [1997a, §7], Priest [2005, §8.4] and Priest [2006b, §6].

[32] See also Priest [1997a, p. 46] and Priest [2005, p. 158].

[33] Hilbert and Bernays [1939, pp. 268-9]. On Tarski's Theorem, see, e.g., Boolos and Jeffrey [1980, p. 176].

gluts and truth-value gaps, and that in [Priest, 2005] necessitating violations of traditional laws of identity) differ from his dialetheic solution to the other paradoxes. The paradox does not seem readily to fit his common Inclosure schema.[34] If this is right, by the Principle of Uniform Solution the Hilbert-Bernays paradox is different in kind from the other paradoxes, of denotation and of truth. But we can already see that the description at the heart of the paradox is inherently contradictory, and so likely to submit to the standard Bradwardinian solution.

Let σ be short for the definite description 'the successor of the denotation of "σ"', and suppose

$$\mathbf{Sig}(\langle\sigma\rangle, (\lambda x)(x = s(\sigma) \wedge \psi x))$$

gives the whole signification of 'σ', where $s(x)$ is the successor function. Suppose $D(\langle\sigma\rangle, n)$. Then $\sigma = n$, by (Den). But if $n = s(\sigma)$ and ψn, $n = s(\sigma) = s(n)$. So by (Clo), the signification of 'σ' is contradictory. Consequently, since, as we saw, every description denotes, 'σ' must denote something, namely, $\iota x\bot$.

7 Conclusion

Graham Priest challenged supporters of Thomas Bradwardine's "multiple-meanings" diagnosis of the logical paradoxes to show how Bradwardine's idea can be adapted to solve the paradoxes of denotation. He suggested [Priest, 2012, p. 158] that Bradwardine's only option was to deny that the expressions in question denoted at all, since the idea that some expression 'τ' did not denote τ seemed too far-fetched even for him. But properly understood, it is not at all far-fetched and entirely in keeping with Bradwardine's approach. The paradox is only apparent. Take 'the greatest natural number'. It can't denote a natural number greater than all the others, since there is no natural number greater than all the others. Instead, it denotes the contradictory object, $\iota x\bot$. So, in a sense, $\iota x\bot$ is the greatest natural number, but of course, $\iota x\bot$ does not satisfy that description, for nothing does. The Characterization Principle is false. Just as Bradwardine says that the Liar sentence is false while also saying that 'The Liar sentence is false' is false, so too his approach leads naturally to the conclusion that expressions like 'something not denoted' denote something denoted, not something not denoted. For such paradoxical descriptive phrases cannot denote something possessing all the properties they signify, since they are implicitly contradictory and nothing has all the properties in question. In particular, $\iota x\bot$ does not have the "property" $(\lambda x)\bot$, on pain of contradiction.

[34] On the Inclosure schema as a uniform diagnosis of the paradoxes, see, e.g., Priest [2002, §9.4].

The analysis above shows that a coherent account can be given of the apparently paradoxical descriptive phrases in Berry's, Berkeley's and Hilbert-Bernays' paradoxes (and others) in keeping with Bradwardine's principles, resolving the paradoxes, and at the same time, just as Bradwardine maintained such standard logical principles as the laws of Bivalence and Contravalence, preserving the Fregean demand that all such phrases have denotation.

References

George Berkeley. *Works*. Charles Daly, London, 1837.

G. Boolos and R. Jeffrey. *Computability and Logic*. Cambridge UP, Cambridge, 2nd edition, 1980.

Thomas Bradwardine. *Insolubilia*. Peeters, Leuven, 2010. S. Read, Ed. and Eng. Tr.

J. Butler. *Fifteen Sermons*. Robert Horsfield, London, 5th edition, 1765.

Margaret Cameron. Meaning: foundational and semantic theories. In John Marenbon, editor, *The Oxford Handbook of Medieval Philosophy*, pages 342–62. Oxford UP, Oxford, 2012.

J. Corcoran and J. Herring. Notes on a semantic analysis of variable binding term operators. *Logique et Analyse*, 55:644–57, 1971.

Newton da Costa. A model-theoretical approach to variable binding term operators. In A. Arruda et al., editors, *Mathematical Logic in Latin America*, pages 133–62. North-Holland, Amsterdam, 1980.

H. Field. Maudlin's *Truth and Paradox*. *Philosophy and Phenomenological Research*, 73:713–20, 2006.

G. Frege. *Basic Laws of Arithmetic: Derived using Concept-Script*. Oxford University Press, Oxford, 2013. Translated and edited by Philip A. Ebert and Marcus Rossberg, with Crispin Wright.

A. Gallois. Berkeley's master argument. *The Philosophical Review*, 83:55–69, 1974.

D. Hilbert. The foundations of mathematics. In van Heijenoort [1967], pages 464–79. Translation of 'Die Grundlagen der Mathematik', *Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität* 6 (1928): 65-85.

D. Hilbert and P. Bernays. *Grundlagen der Mathematik*. Springer, Berlin, 1939.

J. König. On the foundations of set theory and the continuum problem. In van Heijenoort [1967], pages 145–9. Translation of 'Über die Grundlagen der Mengenlehre und das Kontinuumproblem', *Mathematische Annalen* 61 (1905): 156-60.

A.C. Leisenring. *Mathematical Logic and Hilbert's ϵ-Symbol*. Macdonald, London, 1969.

G. Priest. On a paradox of Hilbert and Bernays. *Journal of Philosophical Logic*, 26:45–56, 1997a.

G. Priest. Inconsistent models of arithmetic, (i): the finite case. *Journal of Philosophical Logic*, 26:223–45, 1997b.

G. Priest. *Beyond the Laws of Thought*. Clarendon Press, Oxford, 2nd edition, 2002.

G. Priest. *Towards Non-Being*. Clarendon Press, Oxford, 2005.

G. Priest. *In Contradiction*. Clarendon Press, Oxford, 2nd edition, 2006a.

G. Priest. The paradoxes of denotation. In K. Bolander, V.F. Hendrix, and S.A. Pedersen, editors, *Self-Reference*, pages 137–50. CSLI, Stanford, CA, 2006b.

G. Priest. Read on Bradwardine on the Liar. In Catarina Dutilh Novaes and Ole Thomassen Hjortland, editors, *Insolubles and Consequences: Essays in Honour of Stephen Read*, pages 155–61. College Publications, London, 2012.

A.N. Prior. Epimenides the Cretan. *Journal of Symbolic Logic*, 23:261–66, 1958.

S. Read. The slingshot argument. *Logique et Analyse*, 143-4:195–218, 1993.

S. Read. The unity of the fact. *Philosophy*, 80:317–42, 2005.

S. Read. Symmetry and paradox. *History and Philosophy of Logic*, 27:307–18, 2006.

S. Read. Bradwardine's revenge. In J.C. Beall, editor, *Revenge of the Liar*, pages 250–261. Oxford UP, Oxford, 2007.

S. Read. Further thoughts on Tarski's T-scheme and the Liar. In J.C. Rahman, Tero Tulenheimo, and Emmanuel Genot, editors, *Unity, Truth and the Liar: the Modern Relevance of Medieval Solutions to the Liar Paradox*, pages 205–225. Springer, Berlin, 2008.

S. Read. Plural signification and the Liar paradox. *Philosophical Studies*, 145: 363–75, 2009.

S. Read. The validity paradox. In M. Peliš, editor, *The Logica Yearbook 2009*, pages 209–21. College Publications, London, 2010.

S. Read. Miller, Bradwardine and the truth. *Discusiones Filosóficas*, 12(18): 229–35, 2011.

S. Read. Paradox, closure and indirect speech reports. *Logica Universalis*, 9: 237–51, 2015a.

S. Read. Truth, signification and paradox. In T. Achourioti, H. Galinon, J. Martínez Fernández, and K. Fujimoto, editors, *Unifying the Philosophy of Truth*, pages 393–408. Springer, Berlin, 2015b.

G. Restall. Modal models for Bradwardine's theory of truth. *Review of Symbolic Logic*, 1:225–40, 2008.

J. Richard. The principles of mathematics and the problem of sets. In van Heijenoort [1967], pages 142–4. Translation of 'Les principes des mathématiques et le problème des ensembles', *Revue générale des sciences pures et appliquées* 16 (1905): 541.

I. Rumfitt. Truth and meaning. *Aristotelian Society Supplementary Volume*, 88:21–55, 2014.

B. Russell. Mathematical logic as based on the theory of types. *American Journal of Mathematics*, 30:222–62, 1908.

J. van Heijenoort, editor. *From Frege to Gödel: a source book in mathematical logic*. Harvard University Press, Cambridge, Mass., 1967.