Collective behaviour and driven-dissipative systems

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Collective behaviour and driven-dissipative systems

1. Nonequilibrium quantum matter

2. Collective behaviour in driven–dissipative systems
   - Transverse field Ising
   - Rabi-Hubbard model

3. Collective dissipation
   - Coupled qubit-cavity systems
   - Bath induced coherence
Nonequilibrium quantum matter

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   - Transverse field Ising
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Driven systems

Open quantum system

\[ \partial_t \rho = -i[\hat{H}, \rho] + \sum_i \kappa_i \mathcal{L}[X_i], \quad \mathcal{L}[X_i] = 2X_i \rho X_i^\dagger - X_i^\dagger X_i \rho - \rho X_i^\dagger X_i \]

Need drive to balance loss

\[ \hat{H} \rightarrow \hat{H} + \hat{V} \cos(\Omega t) \]

External coherent drive:
Driven systems

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Need drive to balance loss

1. External coherent drive:

\[
\hat{H} \rightarrow \hat{H} + \hat{V} \cos(\Omega t)
\]

\[\hat{H} = e^{-i\Omega N} \hat{H} e^{i\Omega N} - \Omega \hat{N}\]

- Neglect fast \(e^{2i\Omega t}\) terms — fast
- Rotating frame — breaks detailed balance with bath.
Driven systems

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\[ \frac{\partial_t \rho}{\partial t} = -i[\hat{H}, \rho] + \sum_i \kappa_i \mathcal{L}[X_i], \quad \mathcal{L}[X_i] = 2X_i \rho X_i^{\dagger} - X_i^{\dagger} X_i \rho - \rho X_i^{\dagger} X_i \]

Need drive to balance loss

1. External coherent drive:

\[ \tilde{H} = \begin{pmatrix}
  h_0 & \nu_{01} \cos(\Omega t) & 0 & \ldots \\
  \nu_{01}^{\dagger} \cos(\Omega t) & h_1 & \nu_{12} \cos(\Omega t) & \ldots \\
  0 & \nu_{12}^{\dagger} \cos(\Omega t) & h_2 & \ldots \\
  \vdots & \vdots & \vdots & \ddots
\end{pmatrix} \]

\[ \tilde{H} = e^{-i\Omega \hat{N}} \hat{H} e^{i\Omega \hat{N}} - \Omega \hat{N} \]

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Need drive to balance loss

1. External coherent drive:

\[ \tilde{\hat{H}} \approx \begin{pmatrix} h_0 & v_{01} & 0 & \cdots \\ v_{01}^\dagger & h_1 - \Omega & v_{12} & \cdots \\ 0 & v_{12}^\dagger & h_2 - 2\Omega & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \]

\[ \tilde{\hat{H}} = e^{-i\Omega \hat{N} t} \hat{H} e^{i\Omega \hat{N} t} - \Omega \hat{N} \]

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\[ \partial_t \rho = -i[H, \rho] + \sum_i \kappa_i L[X_i], \quad L[X_i] = 2X_i \rho X_i^\dagger - X_i^\dagger X_i \rho - \rho X_i^\dagger X_i \]

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Non-equilibrium steady state

External **incoherent** drive:

\[
\partial_t \rho = -i[\hat{H}, \rho] + \sum_i \kappa_i \mathcal{L}[X_i] + \sum_i \gamma_i \mathcal{L}[X_i^\dagger]
\]

- Energy flow through system
- Not thermodynamics — attractors of dynamics
  - Stationary points — extrema of energy?
  - Nontrivial attractors
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Coupled cavity arrays

- Control photon dispersion — lattice

[Hartmann et al. Nat. Phys. ’06; Greentree et al. *ibid* 06; Angelakis et al. PRA ’07]

- X-Hubbard Model, $H = \sum_i H_{X, \text{site}} - J \sum_{\langle ij \rangle} \psi_i^\dagger \psi_j$

[X=Bose, Jaynes-Cummings, Rabi, ... ]
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[Underwood et al. PRA ’12; Nat. Phys ’12]

[Lepert et al. NJP ’11; APL ’13]
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Parametrically pumped BHM

\[ H = -\frac{J}{Z} \sum_{\langle ij \rangle} \psi_i^\dagger \psi_j + \sum_i \left[ \omega_c \psi_i^\dagger \psi_i + U \psi_i^\dagger \psi_i^\dagger \psi_i \psi_i - \Omega \left( \psi_i^\dagger \psi_{i+1} e^{-2i\omega_p t} + \text{H.c.} \right) \right] \]

[Bardyn & Imamoglu, PRL '12]
Parametrically pumped BHM

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\]

Rotating frame, blockade approximation, rescale:

\[
H = -J \sum \left[ \tau_i^+ \tau_{i+1}^- + \tau_{i+1}^+ \tau_i^- + g \tau_i^z + \Delta \left( \tau_i^+ \tau_{i+1}^+ + \tau_{i+1}^- \tau_i^- \right) \right]
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Parametric pumping – equilibrium

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- Equilibrium – transverse field Ising model
  - \( g \) – transverse field, \( g_{\text{crit}} = 1 \).
  - \( \Delta \) – anisotropy.
    \( \Delta = 0: \text{XY}, \ |\Delta| > 0: \text{Ising (X,Y)}. \)

[Bardyn & Imamoglu, PRL ’12]
## Parametric pumping – equilibrium

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\[ \partial_t \rho = -i[H, \rho] + \sum_i \kappa \mathcal{L}[\tau_i^-] \]

\[ \text{Mean-field EOM: } \partial_t \langle \tau_i^\alpha \rangle = F_\alpha(\langle \tau_i^\beta \rangle, \langle \tau_i^\beta \rangle, \langle \tau_i^\beta \rangle) \]

\[ \text{Dynamical attractors, linear stability:} \]
Parametric pumping – open system

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- Dynamical attractors, linear stability:
Why AFM/FM attractors

- Linear stability, fluctuation \( \sim \exp(-i\nu_k t + ikr) \) Linear stability

\[
\nu_k = -i\kappa \pm 2J \sqrt{g^2 + 2g \cos k + (1 - \Delta^2) \cos^2 k}
\]

- \( g \ll -1 \), Dissipation matches ground state
  - Most unstable mode, \( k = 0 \)
- \( g \gg +1 \), Dissipation matches max energy
  - Most unstable mode, \( k = \pi \)

[Joshi, Nissen, Keeling, PRA '13]
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Beyond mean-field

- Matrix-product-operator representation of

\[
\rho = \sum_{\{i_1, i_2, \ldots, i_N\}} \left( \sum_{\{\alpha_j\}} \Gamma_{i_1, \alpha_1}^{[1]} \Lambda_{\alpha_1, \alpha_2}^{[1]} \Gamma_{\alpha_2, \alpha_N-1}^{[2]} \Lambda_{\alpha_N-1, 1}^{[N-1]} \right) \bigotimes_{j=1}^{N} \tau_{i_j}^{i_j}
\]

Vidal, White, Schollwöck, et al. Density matrices: [Zwolak & Vidal, PRL '04]

- No broken symmetry — correlators:

\[\Delta = 1, \kappa = 0.5J;\]
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\]


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\[ \rho = \sum_{\{i_1,i_2,\ldots,i_N\}} \left( \sum_{\{\alpha_j\}} \Gamma_1^{[1]} i_1 \land \Gamma_2^{[2]} i_2 \ldots \Gamma_{N-1}^{[N-1]} i_{N-1} \land \Gamma_N^{[N]} i_N \right) \otimes \tau_j^{i_j} \]


- No broken symmetry — correlators:
  \[ \Delta = 1, \kappa = 0.5J: \]
Correlations

- **AFM vs FM from sign of \( g (\Delta = 1) \)**

\[ h \hat{b}^\dagger \frac{n}{2} \hat{b} \frac{n}{2} + i i = (Z \leftrightarrow)^i_1 \]

\[ h \hat{b}^\dagger \frac{n}{2} \hat{b} \frac{n}{2} + i i = |h \hat{b}^\dagger \frac{n}{2} \hat{b} \frac{n}{2} + i i| \]

- \( \Delta \to 0 \), Analytic spin-wave,

\[ \langle \sigma_{lN}^x \sigma_{lN+1}^x \rangle \propto \exp(-\xi_{cl}) \]
Correlations

- **AFM vs FM from sign of** $g$ ($\Delta = 1$)

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\[
\left| \langle \tau_i^- \tau_{i+1}^\pm \rangle \right| \propto \exp(-\xi_c l)
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Correlations

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  \[ \left| \langle \tau_i^+ \tau_{i+1}^- \rangle \right| \propto \exp(-\xi_c l) \]

\[ \begin{array}{c}
\text{(a)} \\
\xi_c
\end{array} \]
Rabi Hubbard model

\[ H = -J \sum_{\langle ij \rangle} a_i^\dagger a_j + \sum_i h_i^{\text{Rabi}} \]

\[ h_i^{\text{Rabi}} = \omega a_i^\dagger a + \frac{\omega_0}{2} \sigma^z + \left[ a_i^\dagger (g\sigma^- + g'\sigma^+) + \text{H.c.} \right] \]

\[ \omega = \omega_{\text{cavity}} - \omega_{\text{pump}} \]

- \( g, g' \) separately tunable
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\[ \dot{\rho} = -i[H, \rho] + \sum_i \kappa L[a_i] + \gamma L[\sigma_i^-] \]
Rabi Hubbard model – equilibrium

Discrete $\mathbb{Z}_2$ symmetry

Parity Mott lobes

$g = g'$, never degenerate — never superfluid

[Schiró et al. PRL ’12]
Rabi Hubbard model – equilibrium

- Discrete $\mathbb{Z}_2$ symmetry
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$g' / g = 0.5$

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Driven-dissipative system — linear stability

Mean field theory — still large Hilbert space.

- Normal state + fluctuations: \( \rho = \otimes_n (\rho_{ss} + \sum_k \delta \rho_k e^{ik\cdot n - i\nu_k t} + \text{H.c.}) \)
- \( \nu_k \) Eigenvalues of \( M = M_0 - t_k M_1 \), \( t_k = -2J \cos(k) \)
- Unstable if \( \Im[\nu_k] > 0 \)

- Given \( J \), \( |t_k| < 2J \)
- First instability \( k = 0, \pi \)
- \( k \rightarrow \pi/2 \) at large \( J \)
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Follow [Boité et al., PRA 2014]

\[ \text{Max}[\Im(\nu_k)] = \frac{2J}{g = g' = 1} \]

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Rabi-Hubbard model — linear stability

Stability phase diagram:

Rabi-Hubbard model — linear stability

Stability phase diagram:

Steady state correlations:

Rabi-Hubbard model — linear stability

Stability phase diagram:


Steady state correlations:

\[ \langle \sigma^x_{n\sigma_{n+1}} \rangle \]

\[ J=0.4 \]

\[ g=1.5 \]

\[ \text{Separation, } l \]

\[ |i - j| = \uparrow \downarrow \]
Rabi-Hubbard model — linear stability

Stability phase diagram:

\[ \text{Most unstable } k \]

\[ \begin{array}{cccccc}
0 & 0.25 & 0.5 & 0.75 & 1 \\
0 & 0.25 & 0.5 & 0.75 & 1
\end{array} \]

\[ g = g' \]

\[ J \]

\[ \frac{\pi}{2} \]

\[ \pi \]

\[ -2 \]

\[ 0 \]

\[ 2 \]

\[ -2 \]

\[ 0 \]

\[ 2 \]

\[ \langle \sigma_x^n \sigma_x^{n+1} \rangle \]

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\[ J = 0.4 \]

\[ J = 0.6 \]

\[ \text{vs } |i - j| = \updownarrow \]

\[ \langle \sigma_x^n \sigma_x^{n+1} \rangle \]

\[ \text{Separation, } l \]

\[ 0 \]

\[ 4 \]

\[ 8 \]

\[ 12 \]

\[ 16 \]

\[ -0.2 \]

\[ 0 \]

\[ 0.2 \]

\[ 0.4 \]

\[ 0.6 \]

\[ -0.2 \]

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\[ 0.6 \]

\[ -0.2 \]

\[ 0 \]

\[ 0.2 \]

\[ 0.4 \]

\[ 0.6 \]

\[ 0 \]

\[ 4 \]

\[ 8 \]

\[ 12 \]

\[ 16 \]

Rabi-Hubbard model — linear stability

Stability phase diagram:


Steady state correlations:

\[ \langle \sigma_x^n \sigma_x^{n+1} \rangle \]

\[ \langle \sigma^x_n \sigma^x_{n+1} \rangle \]

\[ J=0.4 \]
\[ J=0.6 \]
\[ J=0.8 \]

\[ \text{Separation, } l \]

\[ 0 4 8 12 16 \]

\[ 0 0.2 0.4 0.6 \]

\[ 0 0.2 0.4 0.6 \]

\[ g=1.5 \]
\[ 0 \]
\[ \pi /2 \]
\[ \pi \]

\[ \pi \]

\[ g=g' \]

\[ J \]

\[ \text{Most unstable } k \]

\[ \ldots \text{ vs } |i - j| = \updownarrow \]
Linear stability – limit cycles

If \( \nu_k = \pm \nu'_k + i \nu''_k \) at instability \( \rightarrow \) Limit Cycle

[Lee et al. PRA ’11, Jin et al. PRL ’13, Ludwig & Marquard PRL ’13, Chan et al. arXiV:1501.00979]

Linear stability – limit cycles

- If $\nu_k = \pm \nu'_k + i \nu''_k$ at instability $\rightarrow$ Limit Cycle
  
  [Lee et al. PRA '11, Jin et al. PRL '13, Ludwig & Marquard PRL '13, Chan et al. arXiV:1501.00979]

Phase-boundary Effective model

- Compare phase boundaries

**Ground state:**

\[ g/\omega_0 \]

\[ J/\omega_0 \]

\[ \text{Normal} \]

\[ \text{Ordered} \]

\[ 0 \hspace{1cm} 0.2 \hspace{1cm} 0.4 \hspace{1cm} 0.6 \hspace{1cm} 0.8 \hspace{1cm} 1 \]

\[ J/\omega \]

\[ 0 \]

\[ g/\omega_0 \]

\[ 0 \hspace{1cm} 1 \hspace{1cm} 2 \]

\[ \text{Normal} \]

\[ \text{Ordered} \]

- Ground state, \( J_{\text{crit}} \sim e^{-2g^2/\omega^2} \) at \( g \gg \omega \)

- Dissipation means \( J_{\text{crit}} > J_{\text{min}} \)

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Collective dissipative behaviour
CUNY, April 2015
Phase-boundary Effective model

- Compare phase boundaries

Ground state:

- Driven dissipative:

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- Ground state, $J_{\text{crit}} \sim e^{-2g^2/\omega^2}$ at $g \gg \omega$
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**Driven dissipative:**

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Collective dissipative behaviour

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Consider effective spinor model

\[ H = \sum_i \frac{\Delta}{2} \tau_i^2 - \sum_{\langle ij \rangle} \tilde{J}_x \tau_i^x \tau_i^x + \tilde{J}_y \tau_i^y \tau_i^y, \quad \dot{\rho} = -i[H, \rho] + \ldots \]

Level populations:

If \( \Delta \sim \omega_0 e^{-2g^2/\omega^2} \ll 1 \)

\[ J_{\text{crit}} \approx \frac{\kappa^2 g^2}{\omega^3} + \frac{\omega^3}{16g^2} \]
Phase-boundary Effective model

- Consider effective spinor model

\[ H = \sum_i \frac{\Delta}{2} \tau_i^z - \sum_{\langle ij \rangle} \tilde{J}_x \tau_i^x \tau_j^x + \tilde{J}_y \tau_i^y \tau_j^y, \quad \dot{\rho} = -i[H, \rho] + \ldots \]

- Level populations:

\begin{align*}
\dot{\rho} &= -i[H, \rho] + \ldots \\
\text{if } \Delta &\sim \omega_0 e^{-2g^2/\omega^2} < 1 \\
J_{\text{crit}} &\approx \frac{\kappa^2 g^2}{\omega^3} + \frac{\omega^3}{16g^2}
\end{align*}
Phase-boundary Effective model

- Consider effective spinor model

\[
H = \sum_i \frac{\Delta}{2} \tau_i^z - \sum_{\langle ij \rangle} \tilde{J}_x \tau_i^x \tau_j^x + \tilde{J}_y \tau_i^y \tau_j^y, \quad \dot{\rho} = -i[H, \rho] + \ldots
\]

- Level populations:

![Level populations graph]

- If \( \Delta \sim \omega_0 e^{-2g^2/\omega^2} \ll 1 \)

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J_{\text{crit}} \approx \frac{\kappa^2 g^2}{\omega^3} + \frac{\omega^3}{16g^2}
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$g' \neq g$, Level crossings

- For $g' \neq g$, $\Delta$ can swap sign

If levels/populations in wrong order, FM/AFM switch.
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**Collective dissipative behaviour**

**CUNY, April 2015**
Collective dissipation

1. Nonequilibrium quantum matter

2. Collective behaviour in driven–dissipative systems
   - Transverse field Ising
   - Rabi-Hubbard model

3. Collective dissipation
   - Coupled qubit-cavity systems
   - Bath induced coherence
Collective dephasing

- Real environment is not Markovian
  - [Carmichael & Walls JPA ’73] Requirements for correct equilibrium
  - [Ciuti & Carusotto PRA ’09] Dicke SR and emission

- Cannot assume fixed $\kappa, \gamma$

- Phase transition $\rightarrow$ soft modes

Dicke model linewidth:

$$H = \omega \psi \dagger \psi + N \sum_{i=1}^{\sigma} \epsilon_i \sigma_z^i + g (\sigma_+^i \psi + h. c.) + \sum_{i} \sigma_z^i \sum_{q} \gamma_q (b_{q}^\dagger + b_{q}) + \sum_{q} \beta_q b_{q}^\dagger b_{q}.$$
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\[
H = \omega \psi^\dagger \psi + \sum_{i=1}^{N} \frac{\epsilon_i}{2} \sigma_i^z + g \left( \sigma_i^+ \psi + \text{h.c.} \right) \\
+ \sum_i \sigma_i^z \sum_q \gamma_q \left( b_q^\dagger + b_q \right) + \sum_q \beta_q b_{iq}^\dagger b_q.
\]

[Nissen, Fink et al. PRL ’13]
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[Nissen, Fink et al. PRL ’13]
Collective dephasing & weak lasing


- Toy problem:

\[
\hat{H} = \omega_a a^\dagger a + \omega_b b^\dagger b + (a^\dagger + b^\dagger) \sum_i \xi_i c_i + \text{H.c} + H_{\text{Bath}}
\]

- Standard picture:

\[
\dot{\rho} = -i[H_0, \rho] + \left\{ \gamma_\uparrow \mathcal{L}[a^\dagger + b^\dagger] + \gamma_\downarrow \mathcal{L}[a + b] + \text{degenerate} \right\} + \left\{ \gamma_\uparrow a \mathcal{L}[a^\dagger] + \gamma_\downarrow b \mathcal{L}[b^\dagger] + \text{secularised} \right\}
\]

- Exactly solvable problem — which is correct? Consider \( \langle a^\dagger b \rangle \)
Collective dephasing & weak lasing

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\hat{H} = \omega_a a^\dagger a + \omega_b b^\dagger b + (a^\dagger + b^\dagger) \sum_i \xi_i c_i + H.c + H_{\text{Bath}}
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- Standard picture:

\[
\dot{\rho} = -i[H_0, \rho] + \left\{ \begin{align*}
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\gamma_1 a \mathcal{L}[a^\dagger] + \gamma_1 b \mathcal{L}[b^\dagger] + \text{secularised}
\end{align*} \right.
\]

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Collective dephasing & weak lasing


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\left. \gamma_{\uparrow, a} \mathcal{L}[a^\dagger] + \gamma_{\uparrow, b} \mathcal{L}[b^\dagger] + \ldots \right. \text{ degenerate} \\
\left. \right. \text{ secularised}
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Collective dephasing & weak lasing


- Toy problem:

\[ \hat{H} = \omega_a a^\dagger a + \omega_b b^\dagger b + (a^\dagger + b^\dagger) \sum_i \xi_i c_i + H.c + H_{\text{Bath}} \]

- Standard picture:

\[ \dot{\rho} = -i[H_0, \rho] + \begin{cases} 
\gamma_{\uparrow} \mathcal{L}[a^\dagger + b^\dagger] + \gamma_{\downarrow} \mathcal{L}[a + b] + & \text{degenerate} \\
\gamma_{\uparrow, a} \mathcal{L}[a^\dagger] + \gamma_{\uparrow, b} \mathcal{L}[b^\dagger] + & \text{secularised} 
\end{cases} \]

- Exactly solvable problem – which is correct? Consider \( \langle a^\dagger b \rangle \)
Bath induced coherence

- **Steady state:**
  - If $\omega_a = \omega_b$, then $\langle a^\dagger b \rangle = \langle a^\dagger a \rangle = \langle b^\dagger b \rangle$
  - If $\omega_a \neq \omega_b$ then $\langle a^\dagger b \rangle \ll \langle a^\dagger a \rangle, \langle b^\dagger b \rangle$

- $\mathcal{L}[a + b]$ wrong if $\omega_a \neq \omega_b$

- Residual coherence – non-flat DoS

- Requires non-secular master eqn.

- Approaching steady state:
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- **Approaching steady state:**

![Graph showing coherence变化](image)
Bath induced coherence

Steady state:
- If $\omega_a = \omega_b$, then $\langle a\dagger b \rangle = \langle a\dagger a \rangle = \langle b\dagger b \rangle$
- If $\omega_a \neq \omega_b$ then $\langle a\dagger b \rangle \ll \langle a\dagger a \rangle, \langle b\dagger b \rangle$
- $\mathcal{L}[a + b]$ wrong if $\omega_a \neq \omega_b$
- Residual coherence – non-flat DoS
- Requires non-secular master eqn.

Approaching steady state:

$$\langle a\dagger b \rangle \simeq \exp \left[ -C(\omega_a - \omega_b)^2 t \right] + \langle a\dagger b \rangle \bigg|_{t \to \infty}$$
Summary

- Parametric pumping — non-equilibrium “phases” of transverse field Ising model

Joshi et al. PRA ’13

- Rabi Hubbard model — exotic attractors.

Schiró et al. arXiv:1503.04456

- Collective effects in dephasing

Nissen et al. PRL ’13