Condensation, superfluidity and lasing of coupled light-matter systems.

Jonathan Keeling

Stellenbosch, January 2011

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1. Non-equilibrium model
   - Limiting cases, condensation and lasing
2. Stability of normal state — coherence while in strong coupling
3. Condensed spectrum and superfluidity
   - Current-current response function
   - Power law decay of coherence

Acknowledgements

People:



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EPSRC
Engineering and Physical Sciences
Research Council

Polariton system model

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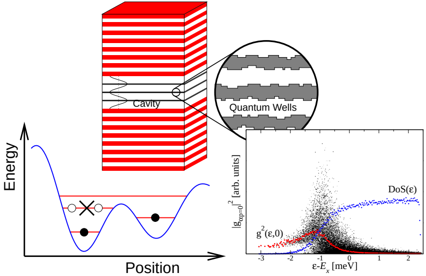
- Disorder-localised excitons
- Treat disorder sites as two-level (exciton/no-exciton)
- Propagating (2D) photons
- Exciton–photon coupling g.



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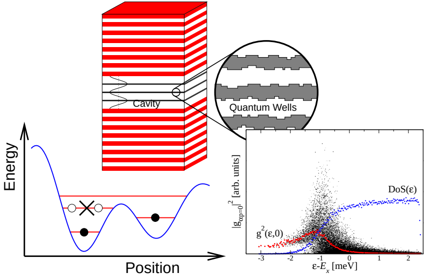

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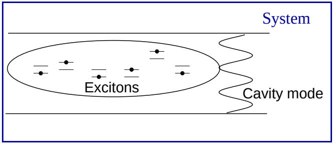

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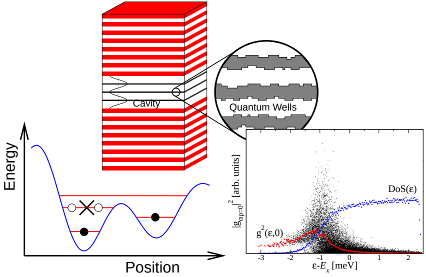

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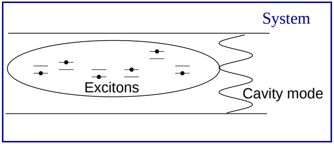

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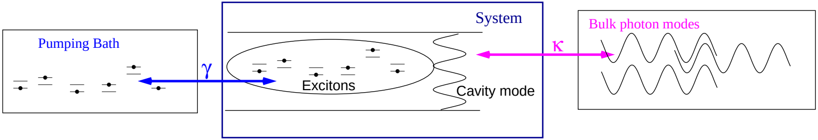

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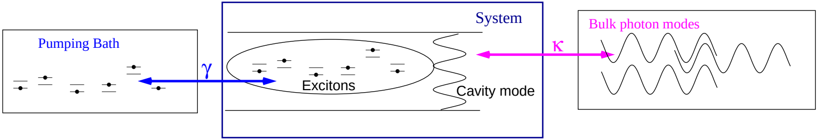

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$$H_{\rm sys,bath} \simeq \sum_{\mathbf{p},\mathbf{k}} \sqrt{\kappa}\psi_{\mathbf{k}}\Psi^\dagger_{\mathbf{p}} + \sum_{\alpha,\beta}\sqrt{\gamma}\left(a^\dagger_\alpha A_\beta + b^\dagger_\alpha B_\beta\right) + \text{H.c.}$$

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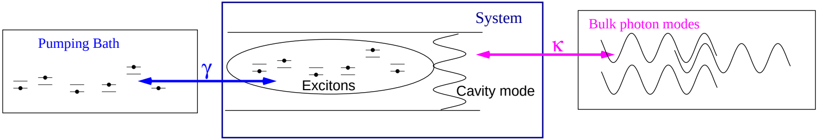

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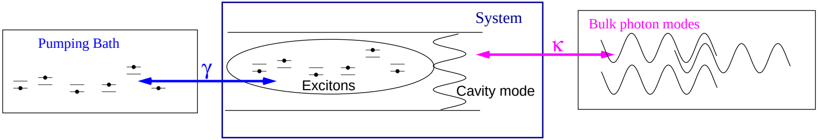

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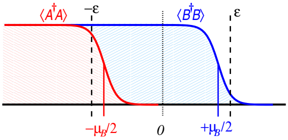
Ψ bath is empty. Pumping bath thermal, μ_B, T:



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$$\chi(\psi_0, \mu_s) = -g^2\gamma \sum_{\text{excitons}} \int \frac{d\nu}{2\pi} \frac{(F_a + F_b)\nu + (F_b - F_a)(i\gamma + \epsilon - \frac{1}{2}\mu_S)}{[(\nu - E)^2 + \gamma^2][(\nu + E)^2 + \gamma^2]}$$

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- Equilibrium limit: finite T set by pumping, need $\kappa \ll \gamma$.

Fluctuations spectrum and stability

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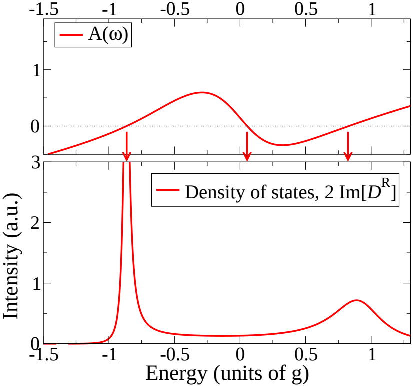

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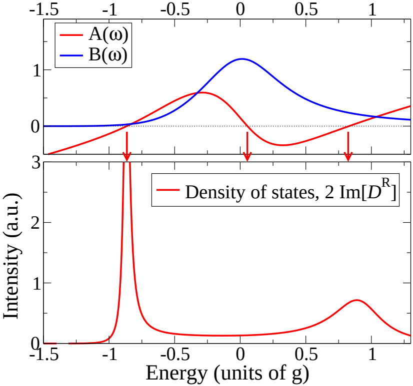

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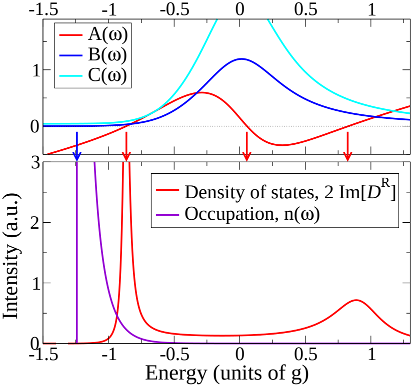

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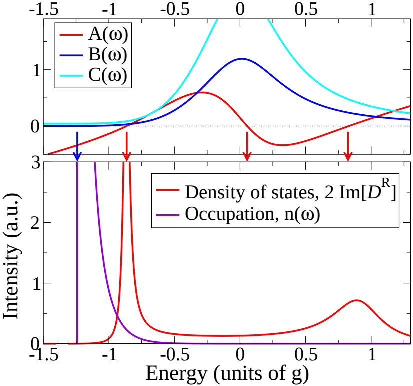

Linewidth, inverse Green's function and gap equation

$$\left[D^R(\omega)\right]^{-1} = 0 \quad \text{at} \quad \omega = \frac{\omega^* + \alpha^2\mu_{\text{eff}} + i\alpha(\mu_{\text{eff}} - \omega^*)}{1+\alpha^2}$$

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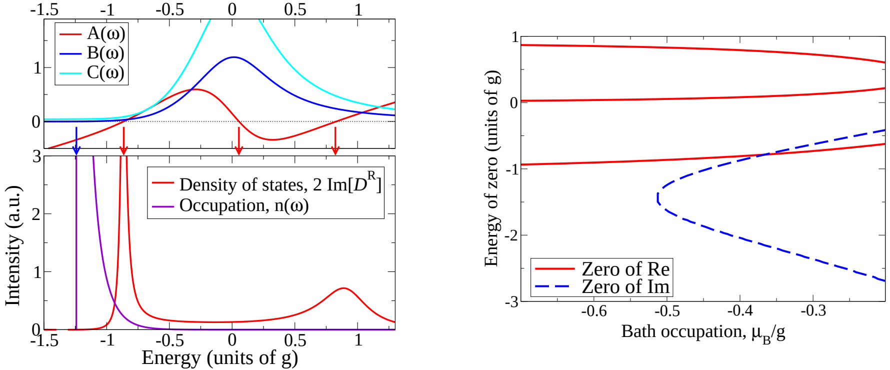
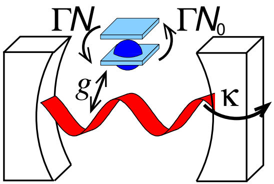


$[D^R]^{-1}$ for a laser



Maxwell-Bloch equations:

$$\partial_t \psi = -i\omega_k \psi - \kappa\psi + gP$$
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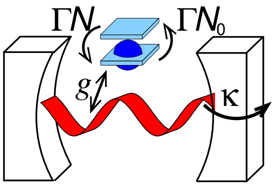

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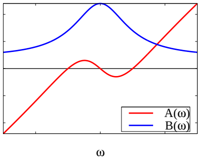

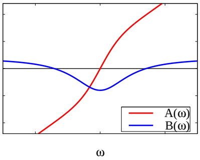

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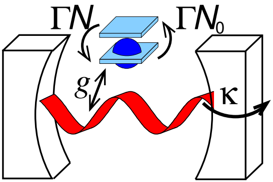

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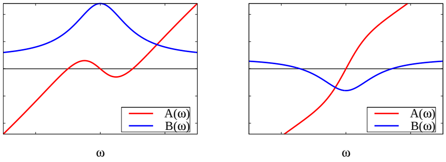

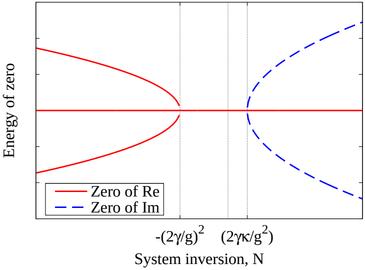

Strong coupling and lasing — low temperature phenomenon

- Laser result: $B(\omega) = \operatorname{Im}\left\{\left[D^R(\omega)\right]^{-1}\right\} = \kappa - 2\gamma\frac{g^2 N_0}{(\omega - 2\epsilon)^2 + 4\gamma^2}$
  Uniformly invert TLS
- Non-equilibrium model

$$B(\omega) = \kappa + \int \frac{d\nu}{2\pi}\sum_\alpha \frac{\gamma^2 g^2 (F_b(\nu+\omega) - F_a(\omega))}{\left[(\nu+\omega-\epsilon)^2+\gamma^2\right]\left[(\nu+\epsilon)^2+\gamma^2\right]}$$

  Inverts low energy part of homogeneous spectrum
- Non-equilibrium model, $T \gg \gamma$

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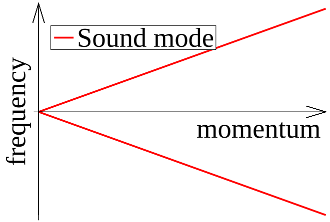
Fluctuations above transition

When condensed

$$\mathrm{Det}\left[D^R(\omega, k)\right]^{-1} = \omega^2 - \xi_k^2$$

With $\xi_k \simeq ck$
Poles:

$$\omega^* = \xi_k$$



- Generic structure of Green's function:

$$[D^R]^{-1} = \begin{pmatrix} \omega + i\gamma_{\rm net} - \epsilon_k - \mu & i\gamma_{\rm net} - \mu \\ -i\gamma_{\rm net} - \mu & -\omega - i\gamma_{\rm net} - \epsilon_k - \mu \end{pmatrix}$$

Fluctuations above transition

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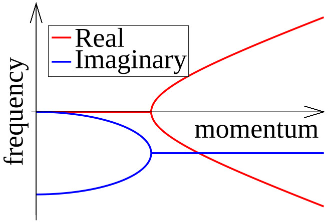
$$\mathrm{Det}\left[D^R(\omega, k)\right]^{-1} = (\omega + i\gamma_{\text{net}})^2 + \gamma_{\text{net}}^2 - \xi_k^2$$

With $\xi_k \simeq ck$
Poles:

$$\omega^* = -i\gamma_{\text{net}} \pm \sqrt{\xi_k^2 - \gamma_{\text{net}}^2}$$



- Generic structure of Green's function:

$$[D^R]^{-1} = \begin{pmatrix} \omega + i\gamma_{\text{net}} - \epsilon_k - \mu & i\gamma_{\text{net}} - \mu \\ -i\gamma_{\text{net}} - \mu & -\omega - i\gamma_{\text{net}} - \epsilon_k - \mu \end{pmatrix}$$

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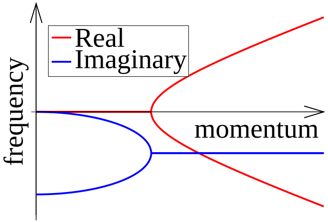

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Non-equilibrium superfluidity checklist

Table 1 | Superfluidity checklist

	Quantized vortices	Landau critical velocity	Metastable persistent flow	Two-fluid hydro-dynamics	Local thermal equilibrium	Solitary waves
Superfluid ^{4}He/cold atom Bose–Einstein condensate	✓	✓	✓	✓	✓	✓
Non-interacting Bose–Einstein condensate	✓	X	X	X	✓	X
Classical irrotational fluid	X	✓	X	✓	✓	✓
Incoherently pumped polariton condensates	✓	X	?	?	X	?
Parametrically pumped polariton condensates	✓	✓	?	?	X	✓

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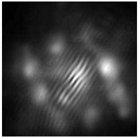

Lagoudakis *et al* Nature Phys. 4, 706 (2008).

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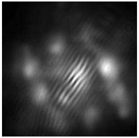

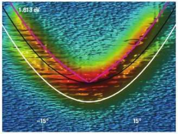

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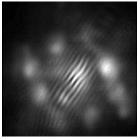

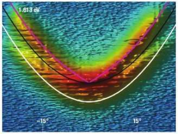

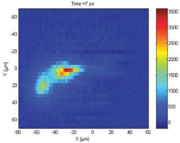

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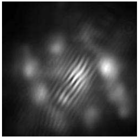
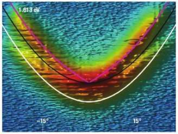

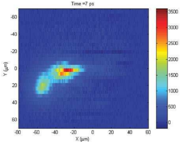

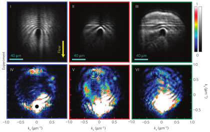

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Asking about non-equilibrium superfluidity

Currrent:

$$\mathbf{J} = \rho\mathbf{v} = \Psi^\dagger i\nabla\Psi = |\Psi|^2\nabla\phi$$

- Response function:

$$\chi_{ij}(\omega = 0, \mathbf{q} \to 0) = \langle [J_i(\mathbf{q}), J_j(-\mathbf{q})] \rangle = \chi_S \frac{q_i q_j}{q^2} + \chi_N \delta_{ij}$$

- In equilibrium, current conservation $\rightarrow \chi_S + \chi_N = \rho_{\text{total}}/m$.
- $\chi_S = \rho_S/m$.
- Given D and $J_i = \psi^\dagger(k+q)\dfrac{2k_i + q_i}{2m}\psi_k$
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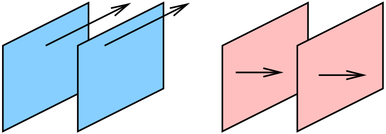

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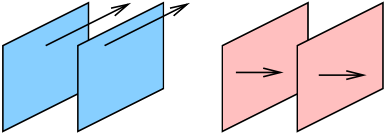

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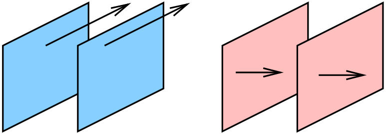

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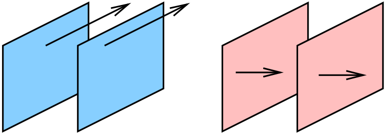

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Calculating superfluid response function

- Using Keldysh generating functional

$$\chi_{ij}(q) = -\frac{i}{2}\frac{d^2\mathcal{Z}[f,\theta]}{df_i(q)d\theta_j(-q)}, \qquad \mathcal{Z}[f,\theta] = \int \mathcal{D}\psi \exp(iS[f,\theta])$$

- f, θ couple as force/response current.

$$S[f,\theta] = S + \sum_{k,q} \begin{pmatrix} \bar{\psi}_{cl} & \bar{\psi}_q \end{pmatrix}_{k+q} \begin{pmatrix} \theta_i & f_i + \theta_i \\ f_i - \theta_i & -\theta_i \end{pmatrix}_q \frac{2k_i + q_i}{2m} \begin{pmatrix} \psi_{cl} \\ \psi_q \end{pmatrix}_k$$

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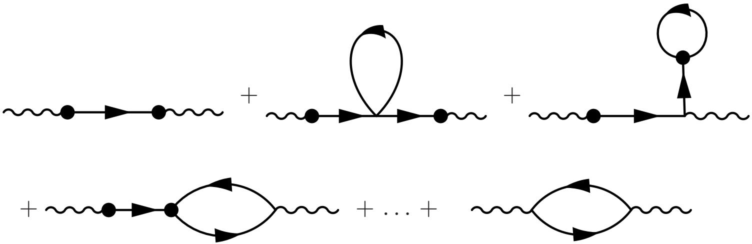

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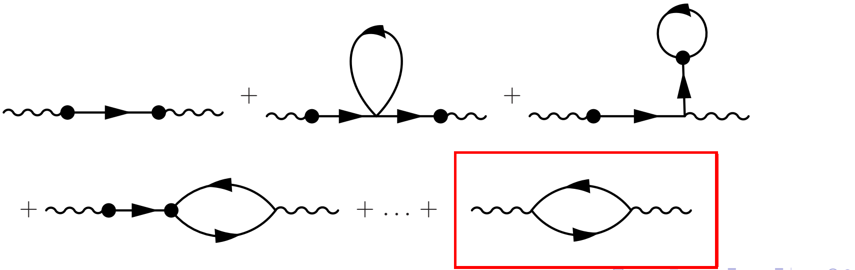

Non-equilibrium superfluid response

- Superfluid response exists because:


$$= \frac{i\psi_0 q_i}{2m}\,(1,-1)\,D^R(q,\omega=0)\begin{pmatrix}1\\-1\end{pmatrix}\frac{i\psi_0 q_j}{2m}$$

- $D^R(\omega=0) \propto 1/\epsilon_q$ despite pumping/decay — superfluid response exists.
- Normal density:

$$\rho_N = \int d^d k\, \epsilon_k \int \frac{d\omega}{2\pi}\mathrm{Tr}\left[\sigma_z D^K \sigma_z (D^R + D^A)\right]$$

- Is affected by pump/decay:
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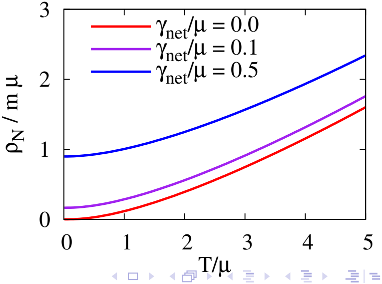

1. Non-equilibrium model
   - Limiting cases, condensation and lasing

2. Stability of normal state — coherence while in strong coupling

3. Condensed spectrum and superfluidity
   - Current-current response function
   - Power law decay of coherence

Correlations in a 2D Gas

Correlations (in 2D):

$$g_1(\mathbf{r}, \mathbf{r}') = \left\langle \psi^\dagger(\mathbf{r}, t)\psi(0, r') \right\rangle \simeq |\psi_0|^2 \exp\left[-D^{<}_{\phi\phi}(t, r, r') \right]$$

- $D^{<} = D^K - D^R + D^A$
- Generally, get: $\left\langle \psi^\dagger(r, t)\psi(0, 0) \right\rangle \simeq$
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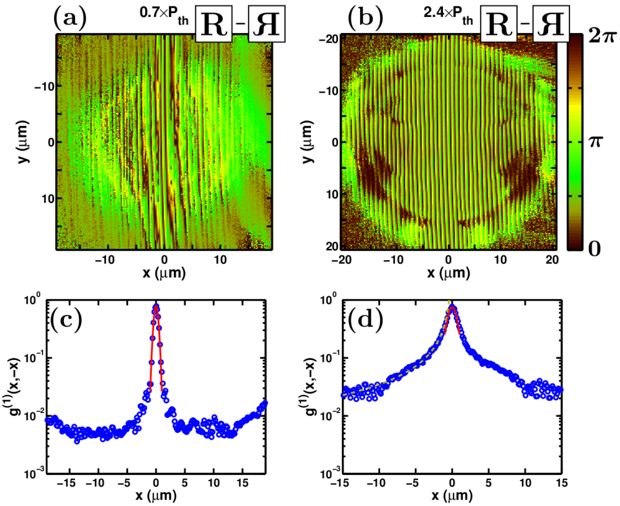
Power law experiment



G. Rompos, Y. Yamamoto et al., submitted

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- Experimentally, $a_P \simeq 1.1$
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- In equilibrium $a_p = mk_B T/2\pi\hbar^2 n_s < 1/4$ (BKT transition)
- Non-equilibrium theory depends on thermalisation.

$$\left[D^{-1}\right]^K(\omega) = \begin{pmatrix} 2if(\mu+\omega) & 0 \\ 0 & 2if(\mu-\omega) \end{pmatrix}$$

A In Eqbm, $f(x) = \gamma_{\text{net}} \coth(\beta(x-\mu)/2)$.
Singularity at $k=0$ pole: recover equilibrium a_p

B Generally, no singularity at $x=\mu$, $f(x) \simeq \zeta$.
$a_p \propto \zeta$.

G. Rompos, Y. Yamamoto et al., submitted

Power law experiment — non-equilibrium theory

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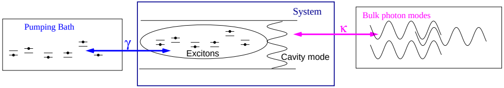
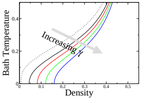
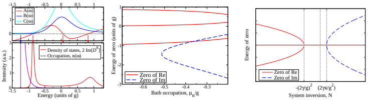
Summary

- Non-equilibrium mean field theory of polaritons



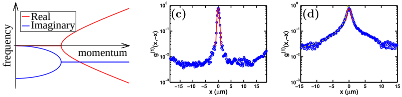


- Strong-coupling & condensation vs lasing.



- Survival of superfluid response

- Change to condensate spectrum and consequences



Extra slides

Limits of gap equation

Gap equation:

$$(\mu_s - \omega_0 + i\kappa)\,\psi = \chi(\psi, \mu_s)\psi$$

- Local density limit:

Limits of gap equation

Gap equation:

$$(\mu_s - \omega_0 + i\kappa)\,\psi = \chi(\psi, \mu_s)\psi$$

- Local density limit: Gross-Pitaevskii equation

$$\left(i\partial_t + i\kappa - \left[V(r) - \frac{\nabla^2}{2m}\right]\right)\psi(r) = \chi(\psi(r,t))\psi(r,t)$$

Nonlinear, complex susceptibility $\chi(\psi(r,t))$

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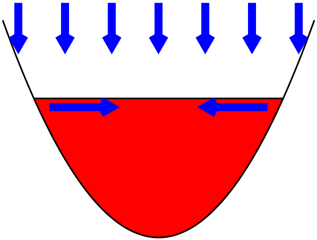
Gross-Pitaevskii equation: Harmonic trap

$$i\partial_t\psi = \left[-\frac{\nabla^2}{2m} + \frac{m\omega^2}{2}r^2 + U|\psi|^2 + i\left(\gamma_{\text{eff}} - \kappa - \Gamma|\psi|^2\right)\right]\psi$$



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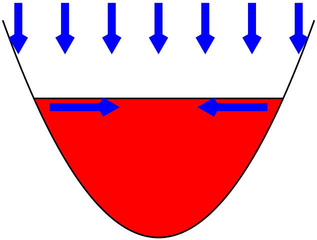
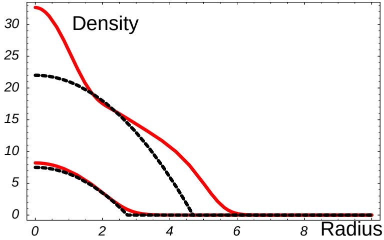
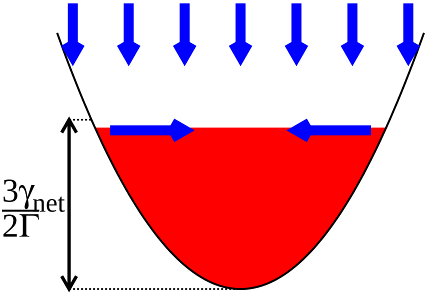




Stability of Thomas-Fermi solution

$$\frac{1}{2}\partial_t\rho+\nabla\cdot(\rho\mathbf{v}) = (\gamma_{\text{net}} - \Gamma\rho)\rho$$



Stability of Thomas-Fermi solution

High m modes: $\delta\rho_{n,m} \simeq e^{im\theta} r^m \ldots$

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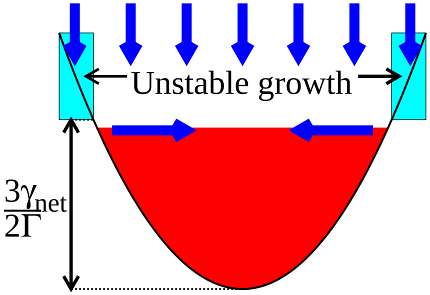

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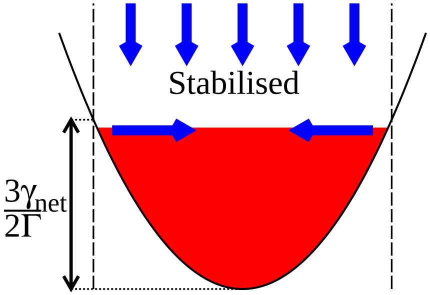

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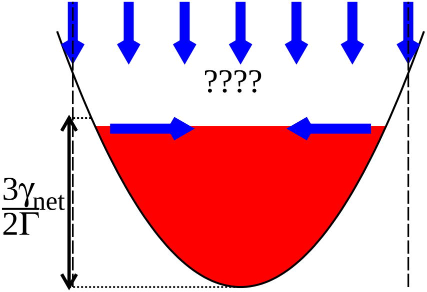
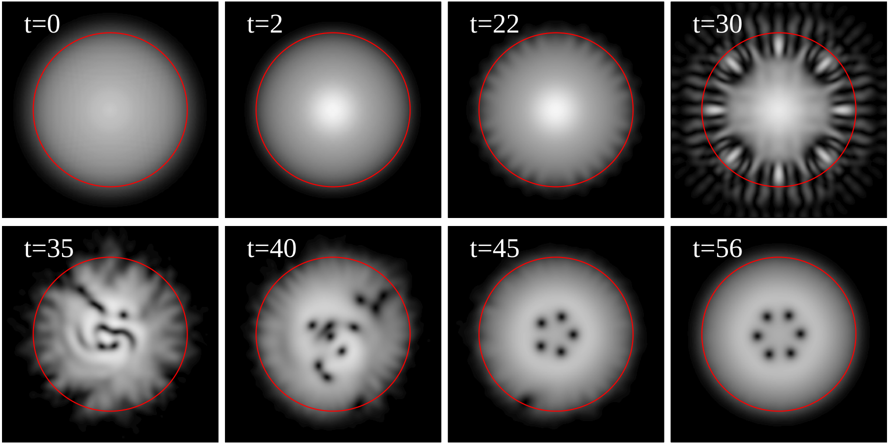
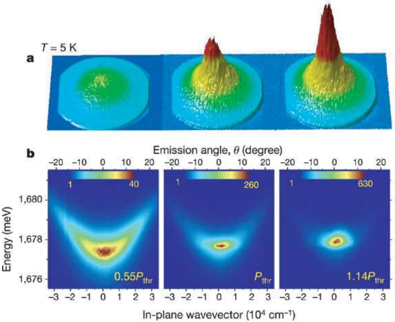
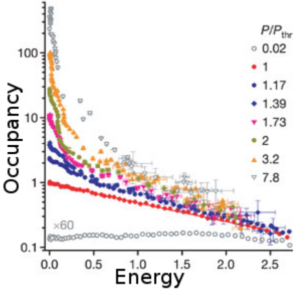


Time evolution:



Polariton experiments: Momentum/Energy distribution





[Kasprzak, et al., Nature, 2006]

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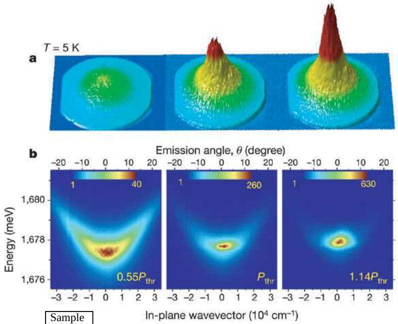
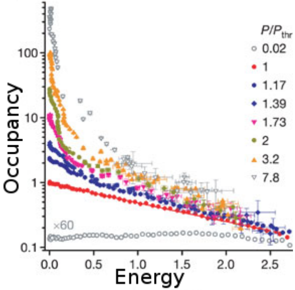
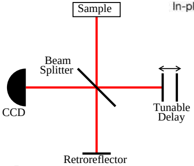
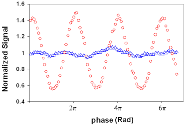
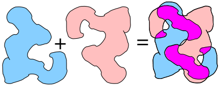


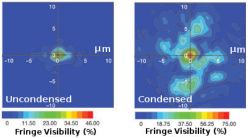






Coherence map:

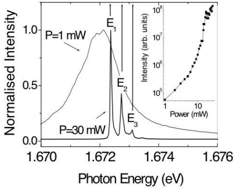


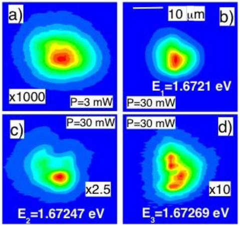


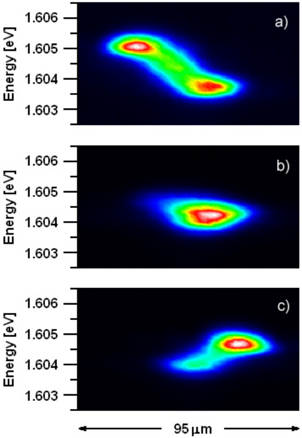
[Kasprzak, et al., Nature, 2006]

Other polariton condensation experiments

- Stress traps for polaritons
  [Balili *et al* Science 316 1007 (2007)]
- Temporal coherence and line narrowing
  [Love *et al* Phys. Rev. Lett. 101 067404 (2008)]

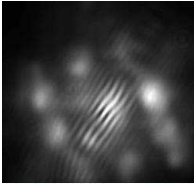




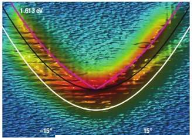


Other polariton condensation experiments

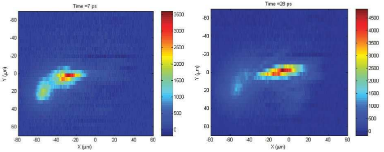
- Quantised vortices in disorder potential
  [Lagoudakis *et al* Nature Phys. 4, 706 (2008)]



- Changes to excitation spectrum
  [Utsunomiya *et al* Nature Phys. 4 700 (2008)]

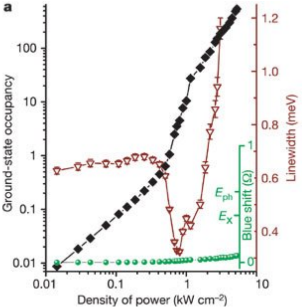


- Soliton propagation
  [Amo *et al* Nature 457 291 (2009)]
- Driven superfluidity
  [Amo *et al* Nature Phys. (2009)

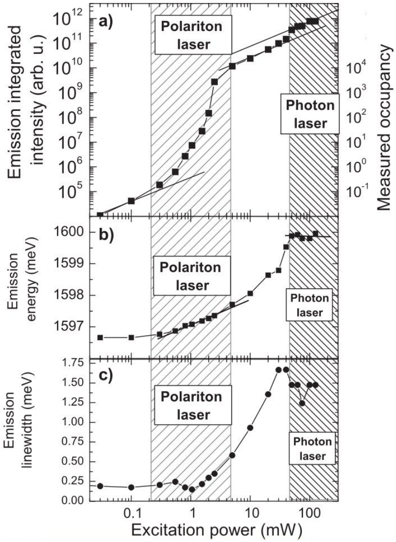


Polariton experiments: Strong coupling

[Bajoni *et al* PRL 2008]



[Kasprzak, et al., Nature, 2006]



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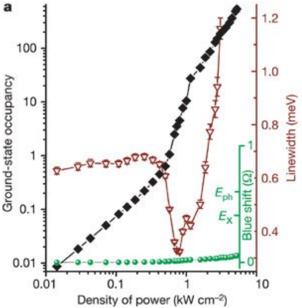

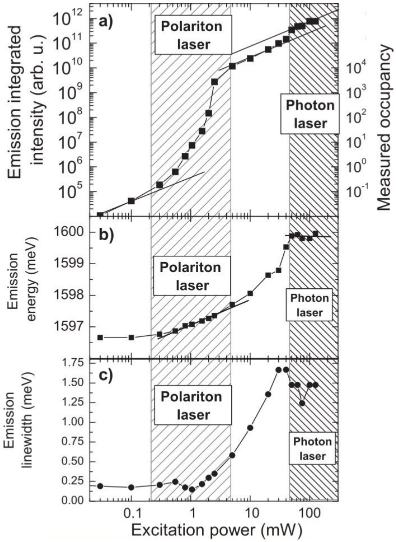

[Kasprzak, et al., Nature, 2006]

Strong coupling via:

- Small blueshift compared to Ω_R
- Polaritonic dispersion, $m > m_{\mathrm{phot}}$
- Separate photon threshold

Equilibrium: Mean-field theory

$$H_{\mathrm{sys}} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \psi^\dagger_{\mathbf{k}} \psi_{\mathbf{k}} + \sum_{\alpha} \left[\epsilon_\alpha \left(b^\dagger_\alpha b_\alpha - a^\dagger_\alpha a_\alpha \right) + \frac{g_{\alpha,\mathbf{k}}}{\sqrt{\mathrm{A}}} \psi_{\mathbf{k}} b^\dagger_\alpha a_\alpha + \mathrm{H.c.} \right]$$

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Self-consistent polarisation and field

$$\left[-i\partial_t - \omega_0 + \frac{\nabla^2}{2m} \right] \psi = -\frac{1}{\sqrt{A}} \sum_{\alpha} g_{\alpha} P_{\alpha}$$

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$$\rho = |\psi|^2 + \frac{1}{A} \sum_\alpha \left[\frac{1}{2} - \frac{\epsilon_\alpha - \mu}{4E_\alpha} \tanh(\beta E_\alpha) \right]$$

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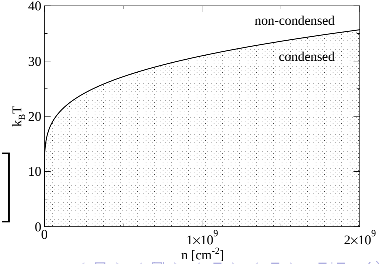
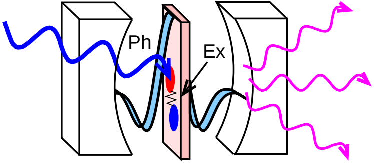
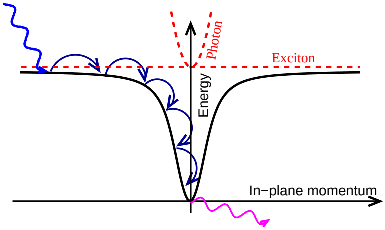


Non-equilibrium: Timescales





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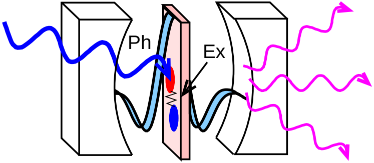
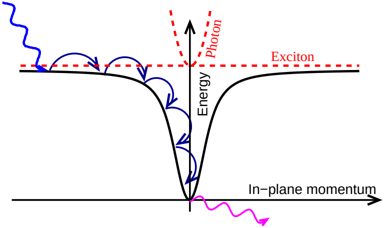
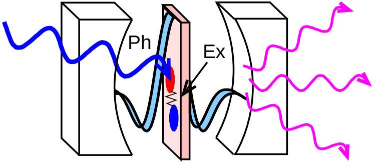


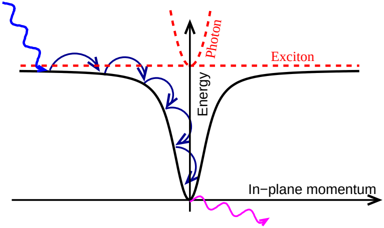


	Lifetime	Thermalisation
Atoms	10s	10ms
Excitons[a]	50ns	0.2ns
Polaritons	5ps	0.5ps
Magnons[b]	1μs(??)	100ns(?)

[a] Coupled quantum wells. [Hammack et al PRB 76 193308 (2007)]
[b] Yttrium Iron Garnett. [Demokritov et al, Nature 443 430 (2007)]

Non-equilibrium: Timescales





	Lifetime	Thermalisation	Linewidth	Temperature	
Atoms	10s	10ms	2.5×10^{-13}meV	10^{-8}K	10^{-9}meV
Excitons[a]	50ns	0.2ns	5×10^{-5}meV	1K	0.1meV
Polaritons	5ps	0.5ps	0.5meV	20K	2meV
Magnons[b]	1μs(??)	100ns(?)	2.5×10^{-6}meV	300K	30meV

[a] Coupled quantum wells. [Hammack et al PRB 76 193308 (2007)]
[b] Yttrium Iron Garnett. [Demokritov et al, Nature 443 430 (2007)]

Spin in terms of ~~two~~four-level systems

To include spin, replace 2 level system with 4 levels: $|0\rangle, |L\rangle, |R|\rangle, |LR\rangle$

- Bi-exciton binding $E_{XX} \leftrightarrow U_1$
- Mean-field: find polarisation given ψ_L, ψ_R.
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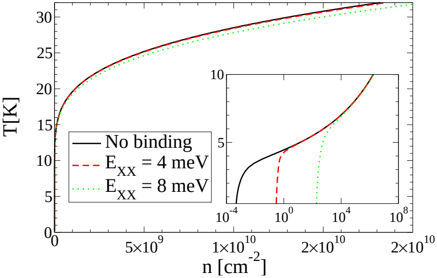

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Polariton spin degree of freedom

- Left- and Right-circular polarised states.
- Spinor Gross-Pitaevskii equation:

$$i\partial_t\psi_L = \left[-\frac{\nabla^2}{2m} + V(r) + U_0|\psi_L|^2 \right.$$
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- Two-mode case (neglect spatial variation): [Wouters PRB '08]
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Non-equilibrium spinor system: two-mode model

Write:

$$\psi_L = \sqrt{R+z}e^{i\phi+i\theta/2}, \quad \psi_R = \sqrt{R-z}e^{i\phi-i\theta/2}$$

Josephson regime: $J_1 \ll U_1 R$, $z \ll R$,

$$\dot{\theta} = -\Delta - 4U_1 z,$$

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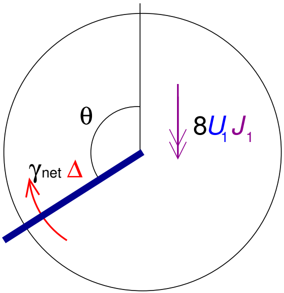

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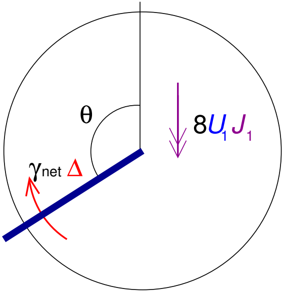
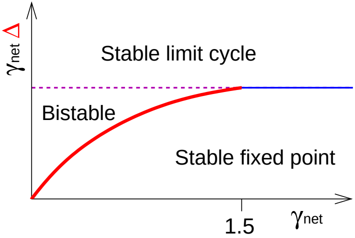
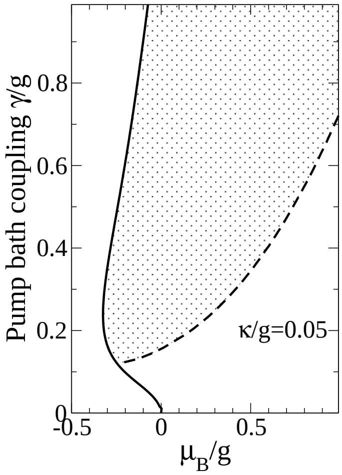




Zero temperature phase diagram

$$\tilde{\omega}_0 - i\kappa = g^2\gamma \sum_\alpha \int \frac{d\nu}{2\pi} \frac{(F_a + F_b)\nu + (F_b - F_a)(\tilde{\epsilon}_\alpha + i\gamma)}{[(\nu - E_\alpha)^2 + \gamma^2][(\nu + E_\alpha)^2 + \gamma^2]}.$$



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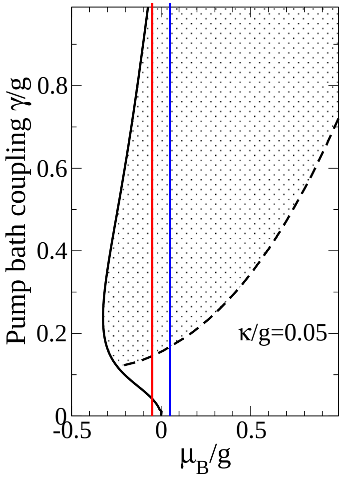

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