Non-equilibrium coherence in light-matter systems: condensation, lasing and the superradiance transition.

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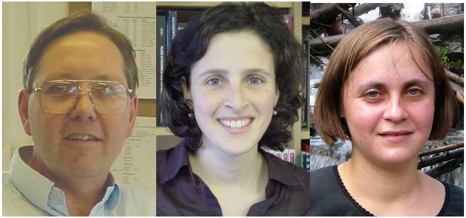
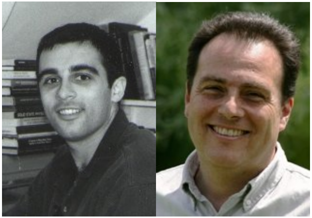
M. J. Bhaseen, B. D. Simons.

ICTP, June 2010

Acknowledgements

People:





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EPSRC
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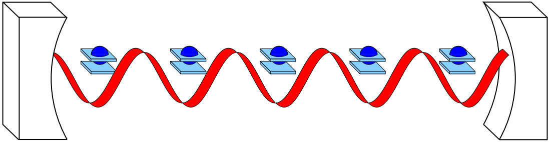
Dicke model & Superradiance phase transition



$$H = \omega_0 S^z + \omega \psi^\dagger \psi + g \left(\psi^\dagger S^- + \psi S^+ \right)$$

[Hepp, Lieb, Ann. Phys. 1973]

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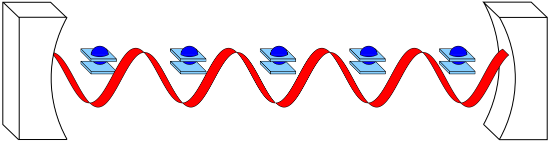

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Mean-field: $|\Psi\rangle = |\Omega\rangle \rightarrow e^{\lambda \psi^\dagger + \eta S^+} |\Omega\rangle$

Spontaneous polarisation if: $Ng^2 > \omega \omega_0$

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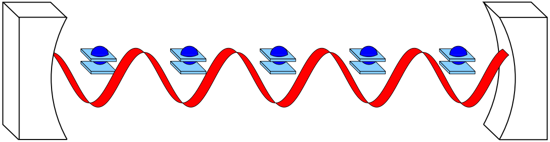

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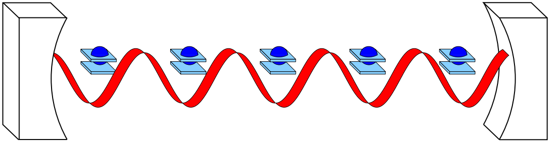

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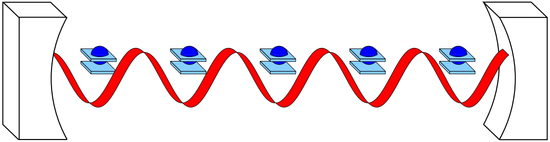

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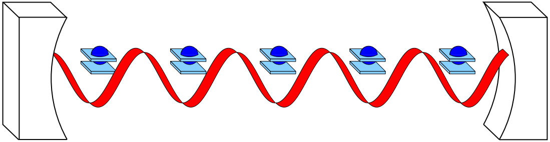

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For large N, $\omega \to \omega + 4N\zeta$. Need $Ng^2 > \omega_0(\omega + 4N\zeta)$.
But $g^2/\omega_0 < 4\zeta$. **No transition** [Rzazewski *et al* Phys. Rev. Lett 1975]

Dicke phase transition: ways out

Problem: $g^2/\omega_0 < 4\zeta$ for intrinsic parameters.

- Introduce chemical potential:

- Dissociate g, ω_0, e.g. Raman scheme: $\omega_0 \ll \omega$.
  [Baumann *et al* Nature 2010]

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Problem: $g^2/\omega_0 < 4\zeta$ for intrinsic parameters. **Solutions**:

- Introduce chemical potential:
  - $H \to H - \mu(S^z + \psi^\dagger\psi)$, need: $g^2 N = (\omega - \mu)(\omega_0 - \mu)$
  - Initially inverted state — dynamical coherence [Bonifacio & Preparata PRA 1970]
  - Pumped system — polariton condensation/lasing
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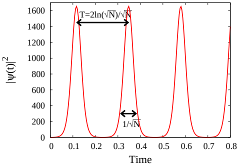

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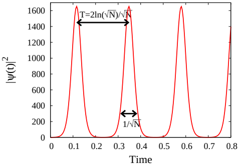

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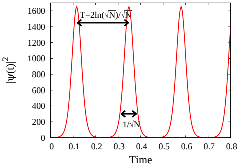

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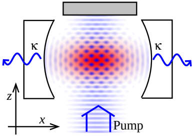
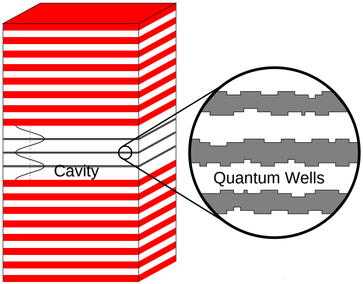


1. Introduction
    - Dicke model and superradiance
2. Microcavity Polariton condensation
    - Polariton experiments
    - Model Hamiltonian & non-equilibrium approach
    - Condensation vs lasing
3. Superradiance in atom-cavity system
4. Conclusions

Microcavity polaritons



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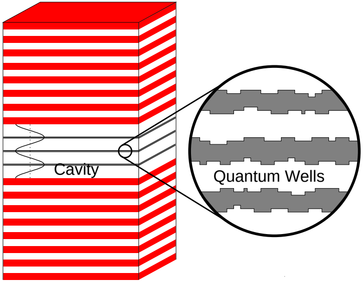
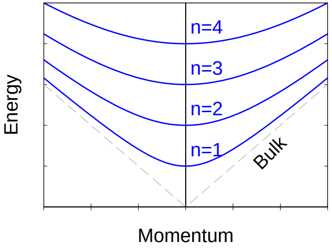


Cavity photons:

$$\omega_k = \sqrt{\omega_0^2 + c^2 k^2}$$
$$\simeq \omega_0 + k^2/2m^*$$
$$m^* \sim 10^{-4} m_e$$



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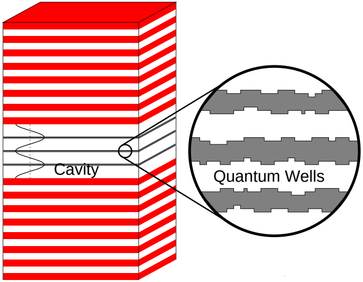

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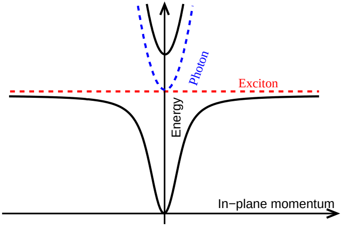

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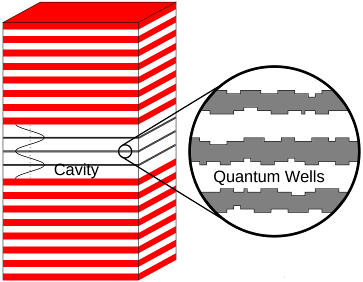

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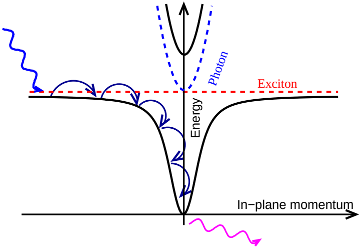
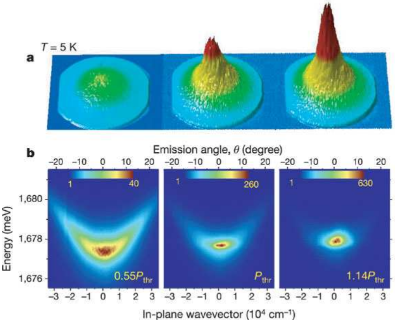
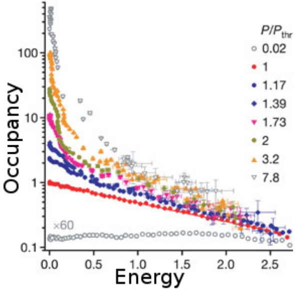


Polariton experiments: Momentum/Energy distribution





[Kasprzak, et al., Nature, 2006]

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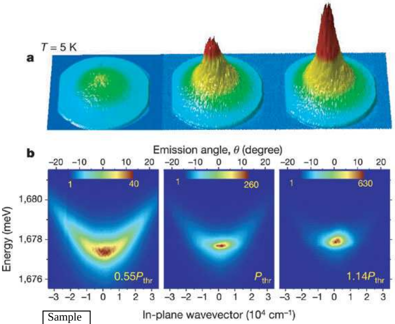

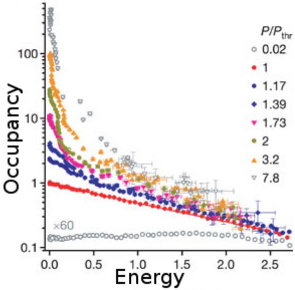

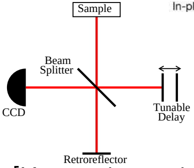

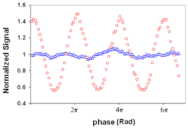

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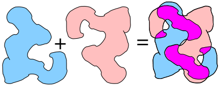
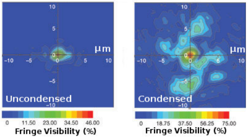
Coherence map:

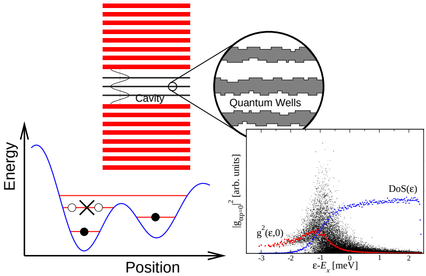




Polariton system model

Polariton model

- Disorder-localised excitons
- Treat disorder sites as two-level (exciton/no-exciton)
- Propagating (2D) photons
- Exciton–photon coupling g.



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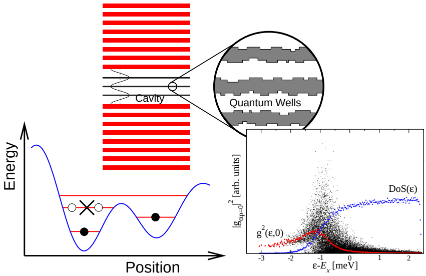
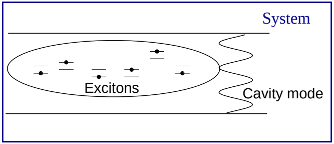


$$H_{\text{sys}} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \psi^\dagger_{\mathbf{k}} \psi_{\mathbf{k}} + \sum_{\alpha} \left[\epsilon_\alpha S^z_\alpha + \frac{1}{\sqrt{\mathrm{A}}} g_{\alpha,\mathbf{k}} \psi_{\mathbf{k}} S^+_\alpha + \mathrm{H.c.} \right]$$



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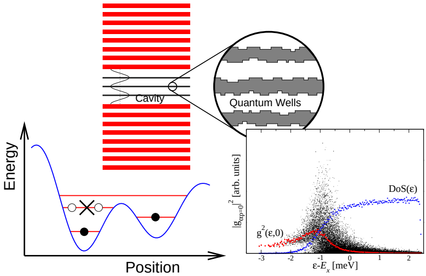

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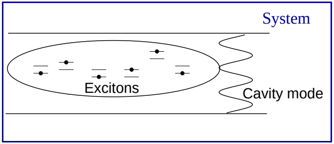
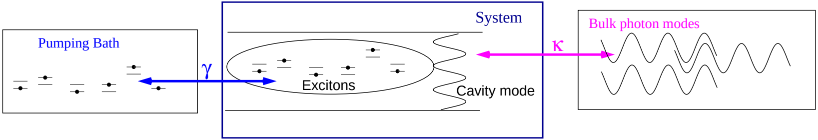


Non-equilibrium model: baths



$$H = H_{\text{sys}} + H_{\text{sys,bath}} + H_{\text{bath}}$$

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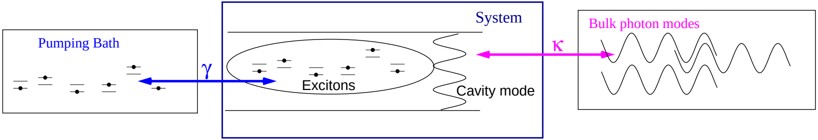

$$H = H_{\text{sys}} + {\color{blue}H_{\text{sys,bath}}} + H_{\text{bath}}$$

Schematically: pump ${\color{red}\gamma}$, decay ${\color{red}\kappa}$

$${\color{blue}H_{\text{sys,bath}}} \simeq \sum_{\mathbf{p},\mathbf{k}} \sqrt{{\color{red}\kappa}}\,\psi_{\mathbf{k}} \Psi^{\dagger}_{\mathbf{p}} + \sum_{\alpha,\beta} \sqrt{{\color{red}\gamma}} \left(a^{\dagger}_{\alpha} A_{\beta} + b^{\dagger}_{\alpha} B_{\beta} \right) + \text{H.c.}$$

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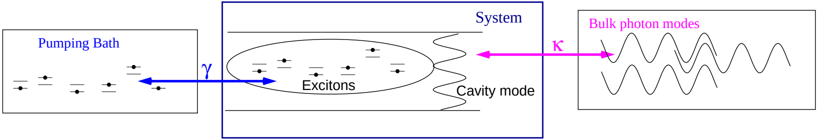

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Bath correlations, $\langle \Psi^\dagger \Psi \rangle$, $\langle A^\dagger A \rangle$, $\langle B^\dagger B \rangle$ fixed:

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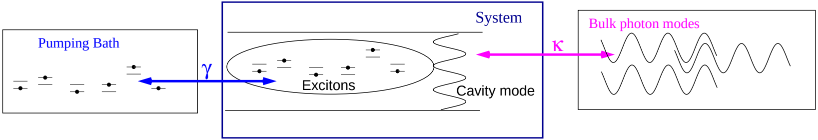

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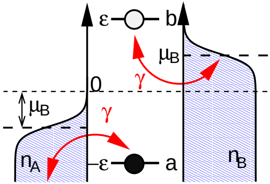
Ψ bath is empty. Pumping bath thermal, μ_B, T:



Non-equilibrium theory; mean-field

Look for mean-field solution, $\psi(\mathbf{r}, t) = \psi_0 e^{-i\mu_S t}$.

- Laser limit: $F_{a,b}(\nu) \to F_{a,b}$ ($T \to \infty$).
- Equilibrium limit: $\kappa \to 0, \gamma \to 0$.

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$$(i\partial_t - \omega_0 + i\kappa)\,\psi_0 = \sum_\alpha g_\alpha \langle S^-_\alpha \rangle$$

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$$(\mu_s - \omega_0 + i\kappa)\,\psi_0 = \chi(\psi_0, \mu_s)\psi_0$$

Susceptibility:

$$\chi(\psi_0, \mu_s) = -g^2\gamma \sum_{\text{excitons}} \int \frac{d\nu}{2\pi} \frac{(F_a + F_b)\nu + (F_b - F_a)(i\gamma + \epsilon_\alpha - \frac{1}{2}\mu_S)}{[(\nu - E_\alpha)^2 + \gamma^2][(\nu + E_\alpha)^2 + \gamma^2]}$$

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Susceptibility: $E_\alpha^2 = \epsilon_\alpha^2 + g^2|\psi_0|^2$

$$\chi(\psi_0, \mu_s) = -g^2\gamma \sum_{\text{excitons}} \int \frac{d\nu}{2\pi} \frac{(F_a + F_b)\nu + (F_b - F_a)(i\gamma + \epsilon_\alpha - \frac{1}{2}\mu_S)}{[(\nu - E_\alpha)^2 + \gamma^2][(\nu + E_\alpha)^2 + \gamma^2]}$$

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$$\chi(\psi_0, \mu_s) = -g^2\gamma \sum_{\text{excitons}} \int \frac{d\nu}{2\pi} \frac{(F_a + F_b)\nu + (F_b - F_a)(i\gamma + \epsilon_\alpha - \frac{1}{2}\mu_S)}{[(\nu - E_\alpha)^2 + \gamma^2][(\nu + E_\alpha)^2 + \gamma^2]}$$

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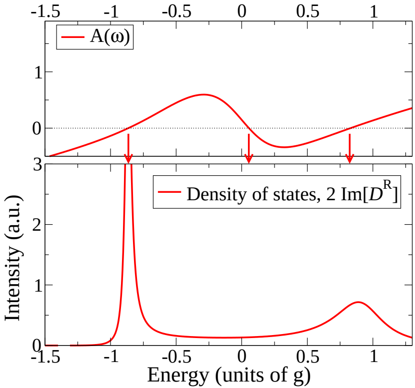

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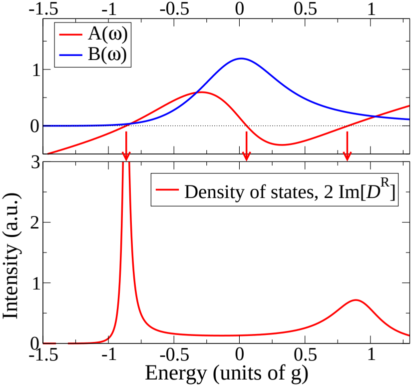

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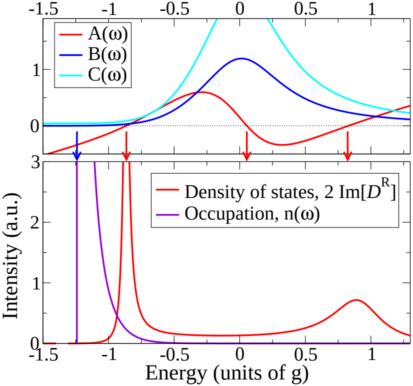

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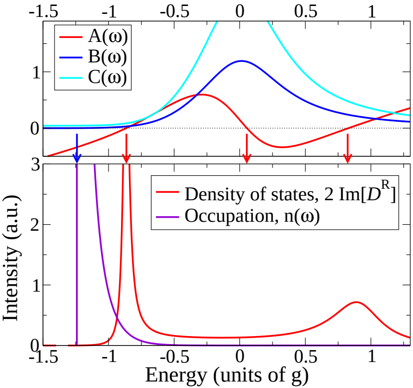
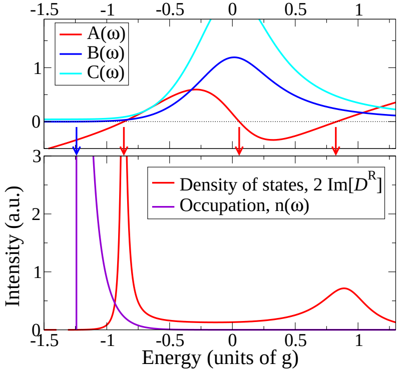
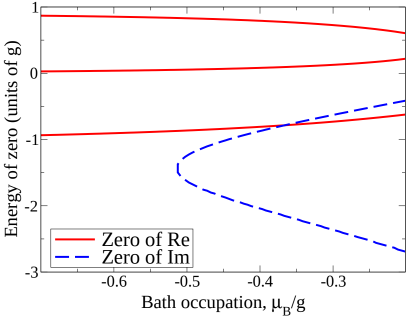
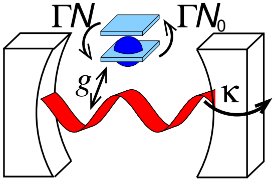



A(ω)
B(ω)
C(ω)
Density of states, 2 Im[D^R]
Occupation, n(ω)
Intensity (a.u.)
Energy (units of g)


Zero of Re
Zero of Im
Energy of zero (units of g)
Bath occupation, μ_B/g

$[\mathcal{D}^R]^{-1}$ for a laser



Maxwell-Bloch equations:

$$\partial_t \psi = -i\omega_k \psi - \kappa\psi + gP$$
$$\partial_t P = -2i\epsilon P - 2\gamma P + g\psi N$$
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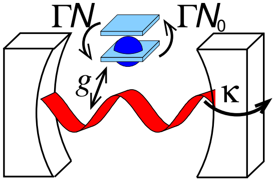

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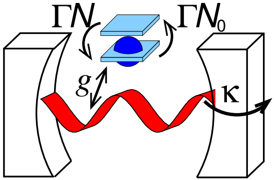

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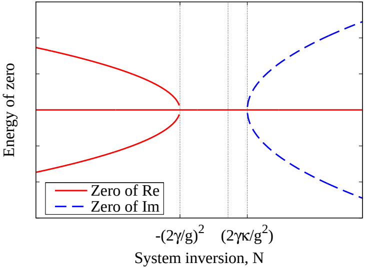
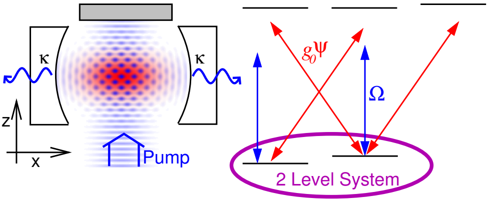


Extended Dicke model



2 Level system, $|\Downarrow\rangle, |\Uparrow\rangle$:

$\Downarrow$: $|k_x, k_z\rangle = |0, 0\rangle$,

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$$H = \omega\psi^\dagger\psi + \omega_0 S^z$$

N atoms: $|\mathbf{S}| = N/2$

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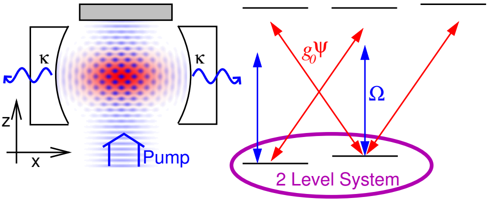

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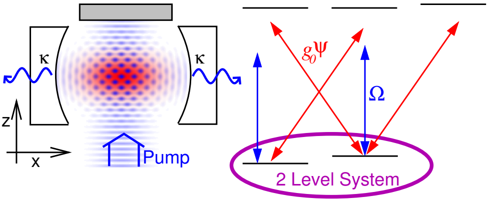

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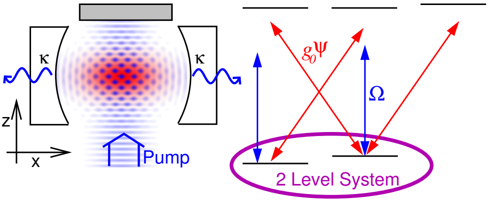

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Add decay:

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$$\partial_t S^z = +ig(\psi + \psi^\dagger)(S^- - S^+)$$

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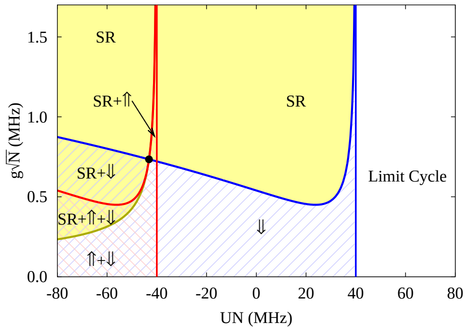
Phase diagram



- $|UN| < \omega/2$, Regular SR, $S^+ = S^-$
- $UN < -\omega/2$, 2nd SR soln $\psi = -\psi^\dagger$
- $UN > \omega/2$ **No SR Fixed point**

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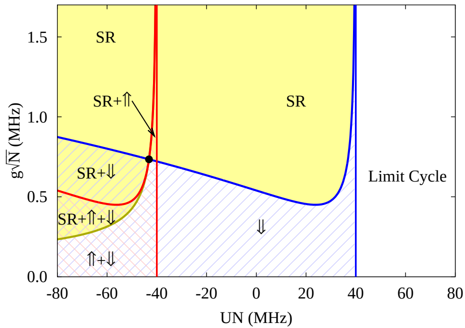

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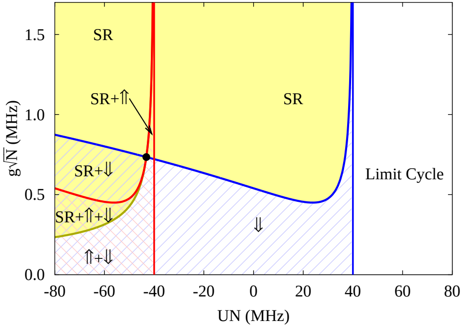

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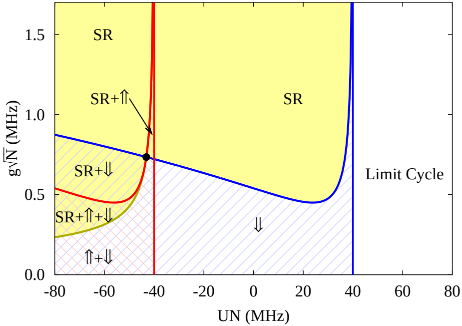

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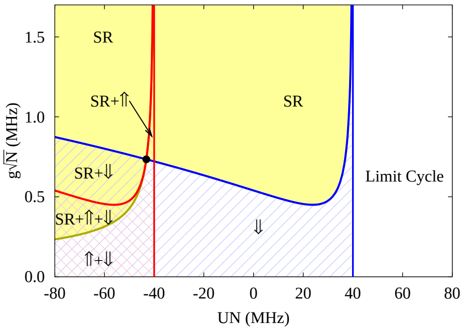

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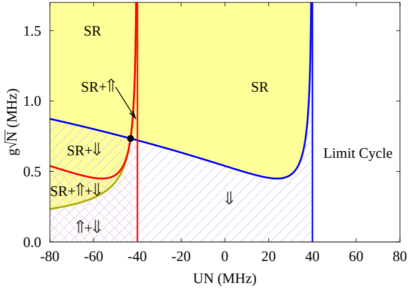
Large U and persistent oscillations



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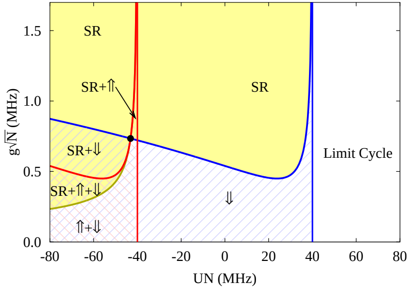

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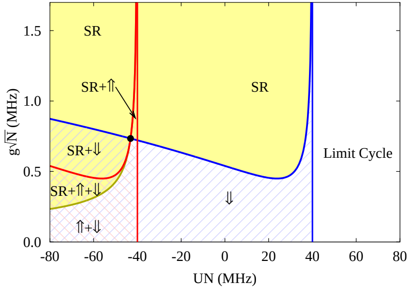

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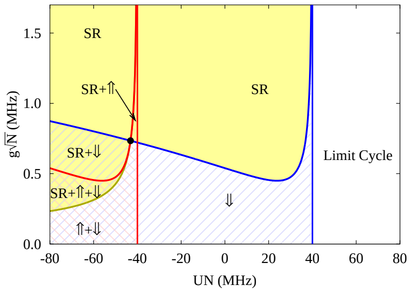

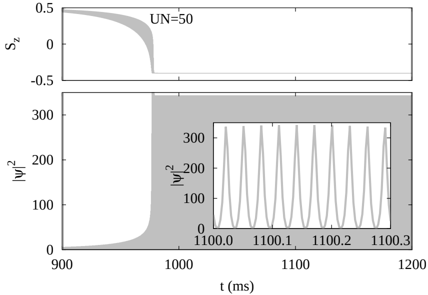

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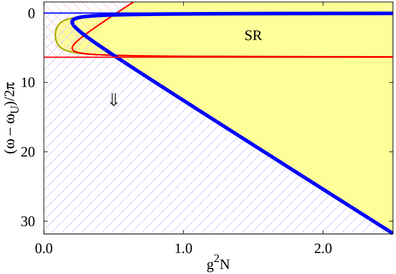
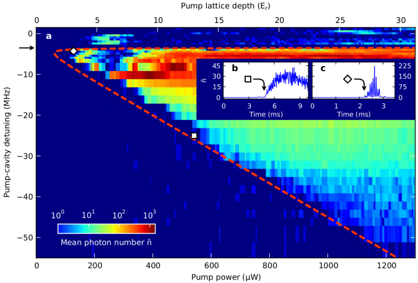
Comparison to experiment $UN = -40\text{MHz}$



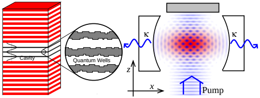
[**JK** *et al* `arXiv:1002.3108`]



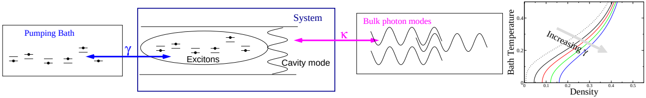
[Baumann *et al* Nature 2010]

Summary

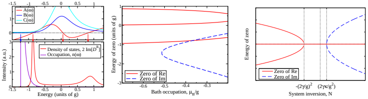
- Non-equilibrium Dicke relevant to increasing number of systems



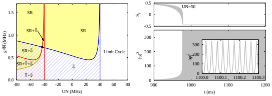
- Effects of pumping on mean-field theory



- Strong-coupling & condensation vs lasing.



- Atomic realisation: many phases & non-trivial dynamics



Extra slides

5 Introduction
- Other types of superradiance

6 Polaritons
- Other polariton experiments
- Equilibrium results
- Non-equilibrium polariton timescales
- Limits of gap equation
- T=0 Keldysh results

7 Cold atom Dicke
- Zero U boundaries
- Fixed points vs U.
- Comparison to expt

Dicke effect and superradiance without a cavity

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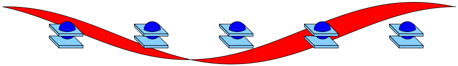

[Dicke, Phys. Rev. 1954]

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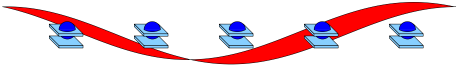

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[Dicke, Phys. Rev. 1954]

Dicke effect and superradiance without a cavity

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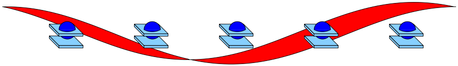

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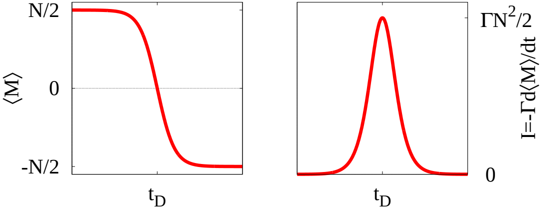

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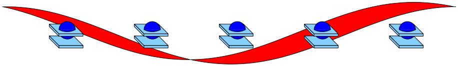

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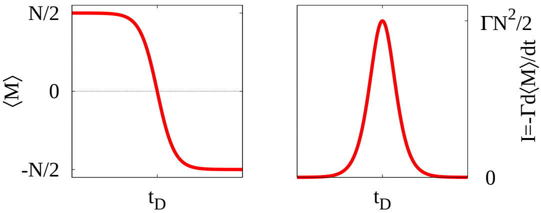
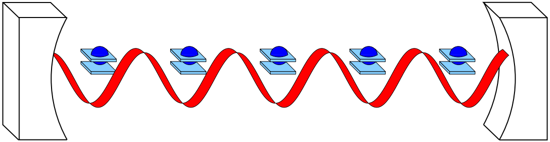


[Dicke, Phys. Rev. 1954]

Problem: dipole-dipole interactions dephase.

Collective radiation with a cavity: Dynamics



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Single cavity mode: oscillations

[Bonifacio and Preparata PRA 1970; **JK** PRA 2009]

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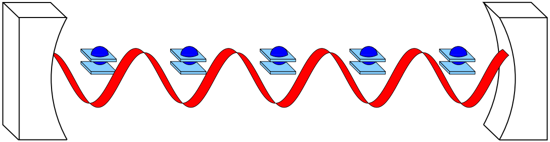

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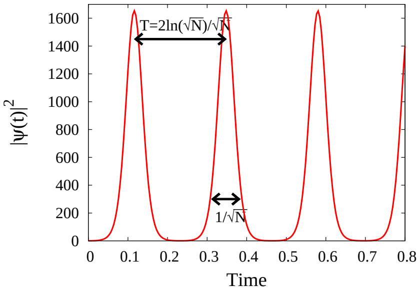

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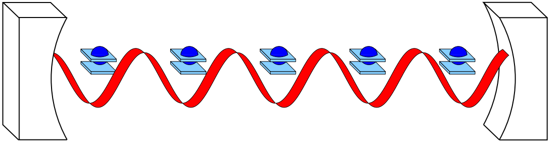

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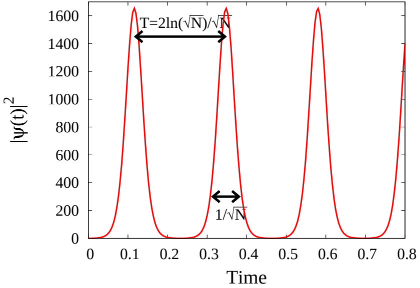
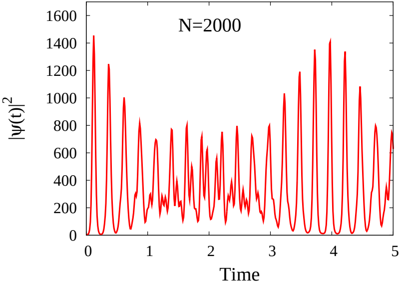
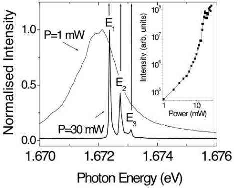


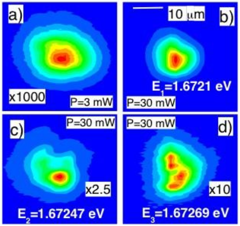


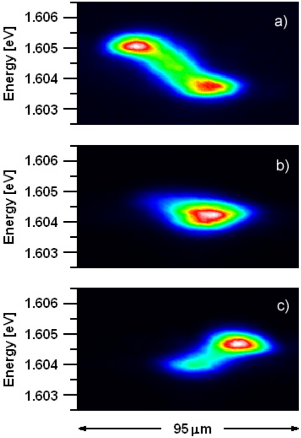
[Bonifacio and Preparata PRA 1970; **JK** PRA 2009]

Other polariton condensation experiments

- Stress traps for polaritons
  [Balili *et al* Science 316 1007 (2007)]
- Temporal coherence and line narrowing
  [Love *et al* Phys. Rev. Lett. 101 067404 (2008)]

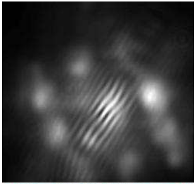




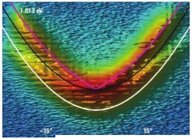


Other polariton condensation experiments

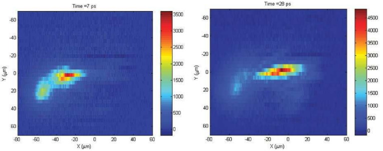
- Quantised vortices in disorder potential
  [Lagoudakis *et al* Nature Phys. 4, 706 (2008)]



- Changes to excitation spectrum
  [Utsunomiya *et al* Nature Phys. 4 700 (2008)]

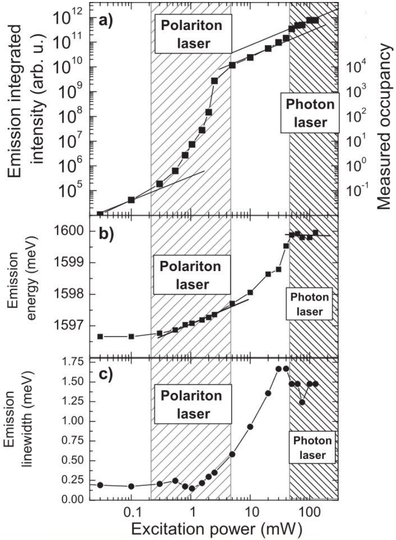


- Soliton propagation
  [Amo *et al* Nature 457 291 (2009)]
- Driven superfluidity
  [Amo *et al* Nature Phys. (2009)]



Polariton experiments: Strong coupling

[Bajoni *et al* PRL 2008]



Equilibrium: Mean-field theory

$$H_{\mathrm{sys}} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \psi^\dagger_{\mathbf{k}} \psi_{\mathbf{k}} + \sum_{\alpha} \left[\epsilon_\alpha \left(b^\dagger_\alpha b_\alpha - a^\dagger_\alpha a_\alpha \right) + \frac{g_{\alpha,\mathbf{k}}}{\sqrt{\mathrm{A}}} \psi_{\mathbf{k}} b^\dagger_\alpha a_\alpha + \mathrm{H.c.} \right]$$

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Self-consistent polarisation and field

$$\left[- i\partial_t - \omega_0 + \frac{\nabla^2}{2m} \right] \psi = -\frac{1}{\sqrt{A}} \sum_{\alpha} g_{\alpha} P_{\alpha}$$

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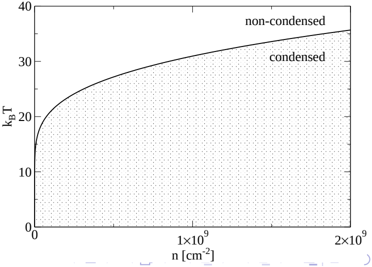
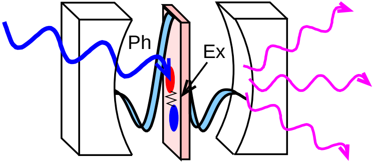
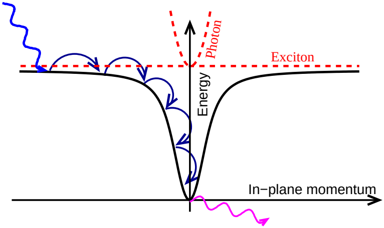


Non-equilibrium: Timescales





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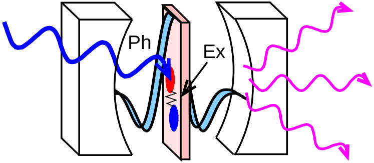
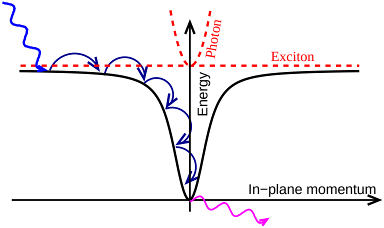
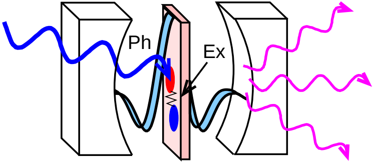
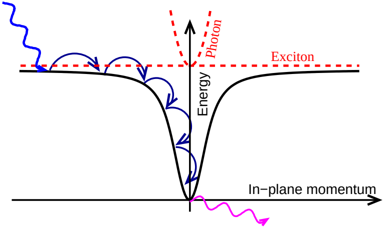




	Lifetime	Thermalisation
Atoms	10s	10ms
Excitons[a]	50ns	0.2ns
Polaritons	5ps	0.5ps
Magnons[b]	1μs(??)	100ns(?)

[a] Coupled quantum wells. [Hammack et al PRB 76 193308 (2007)]
[b] Yttrium Iron Garnett. [Demokritov et al, Nature 443 430 (2007)]





	Lifetime	Thermalisation	Linewidth	Temperature	
Atoms	10s	10ms	2.5×10^{-13}meV	10^{-8}K	10^{-9}meV
Excitons[a]	50ns	0.2ns	5×10^{-5}meV	1K	0.1meV
Polaritons	5ps	0.5ps	0.5meV	20K	2meV
Magnons[b]	1μs(??)	100ns(?)	2.5×10^{-6}meV	300K	30meV

[a] Coupled quantum wells. [Hammack et al PRB 76 193308 (2007)]
[b] Yttrium Iron Garnett. [Demokritov et al, Nature 443 430 (2007)]

Limits of gap equation

$$\mu_s - \omega_0 + i\kappa = -g^2\gamma \sum_{\text{excitons}} \int \frac{d\nu}{2\pi} \frac{(F_a + F_b)\nu + (F_b - F_a)(i\gamma + \epsilon_\alpha - \frac{1}{2}\mu_S)}{[(\nu - E_\alpha)^2 + \gamma^2][(\nu + E_\alpha)^2 + \gamma^2]}$$

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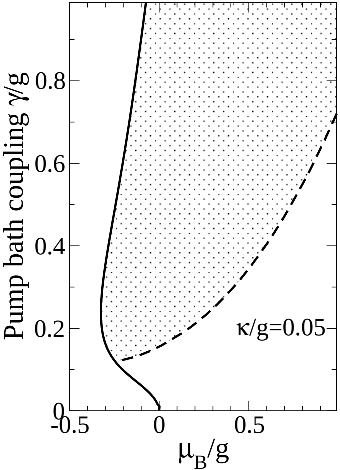
Zero temperature phase diagram

$$\tilde{\omega}_0 - i\kappa = g^2\gamma \sum_\alpha \int \frac{d\nu}{2\pi} \frac{(F_a + F_b)\nu + (F_b - F_a)(\tilde{\epsilon}_\alpha + i\gamma)}{[(\nu - E_\alpha)^2 + \gamma^2][(\nu + E_\alpha)^2 + \gamma^2]}.$$



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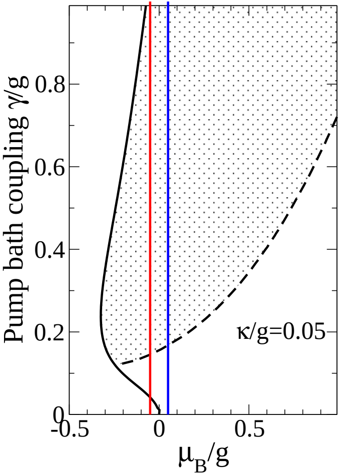

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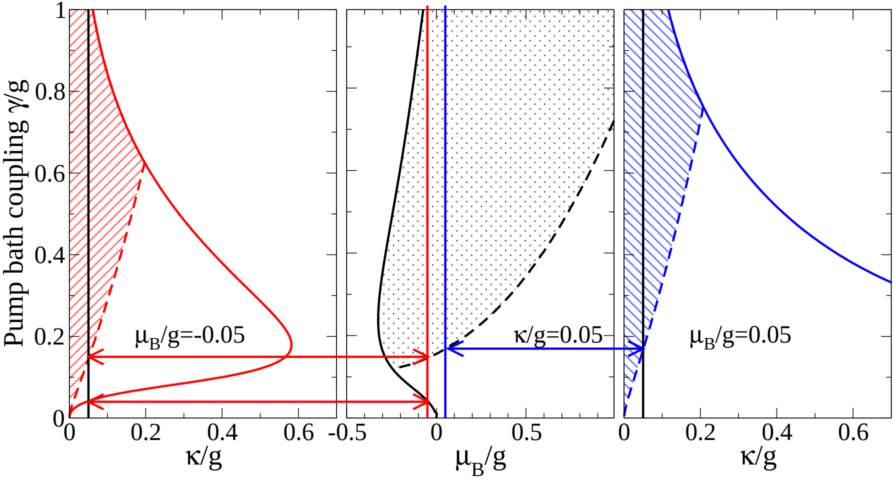
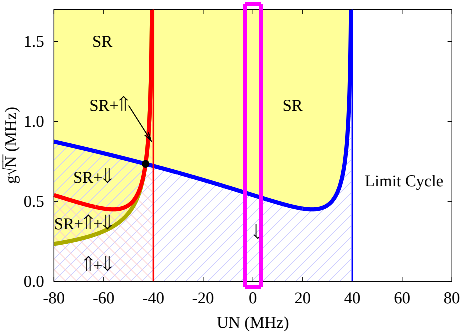
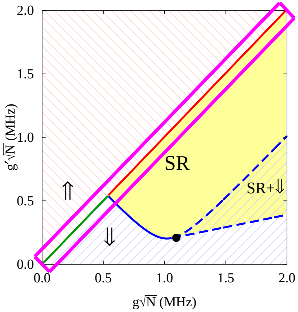


$U = 0$, different g, g'





$$H = \omega\psi^\dagger\psi + \omega_0 S^z + g(\psi^\dagger S^- + \psi S^+) + g'(\psi^\dagger S^+ + \psi S^-) + U S_z \psi^\dagger \psi.$$

Fixed points at $U = 0$

Fixed points $\dot{\mathbf{S}}, \dot{\psi} = 0$.

- $\Uparrow, \Downarrow$: $S^z = \pm N/2, \psi = 0$ always
- SR: $\psi \neq 0$ if g, g' large.

Fixed points at $U = 0$

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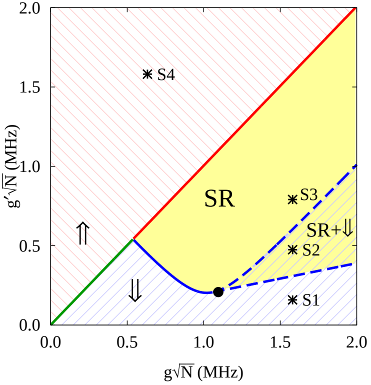

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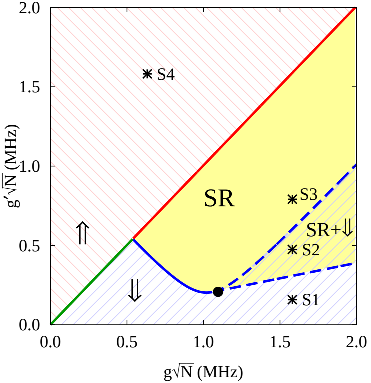

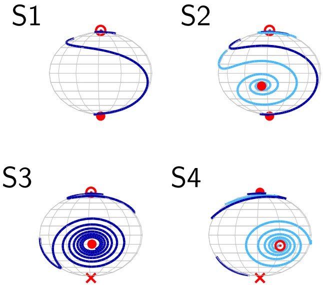

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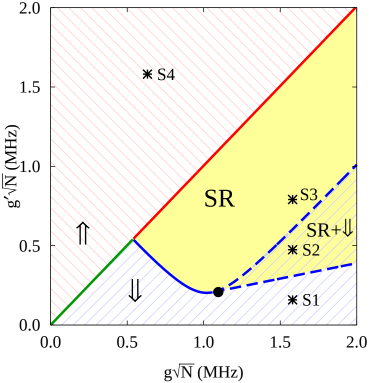

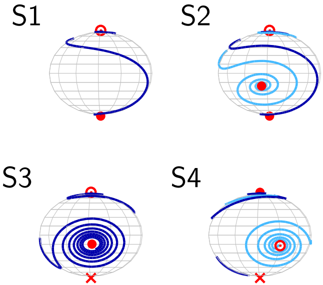

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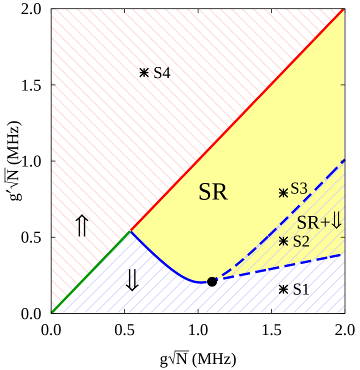
Boundaries $U = 0$

$\kappa \neq 0$



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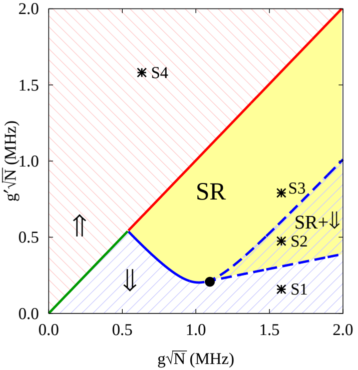

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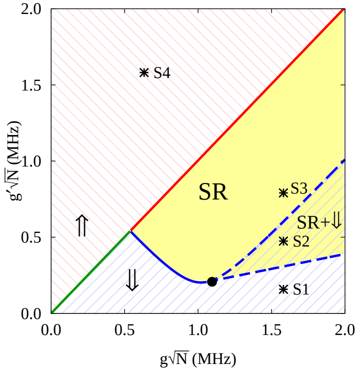
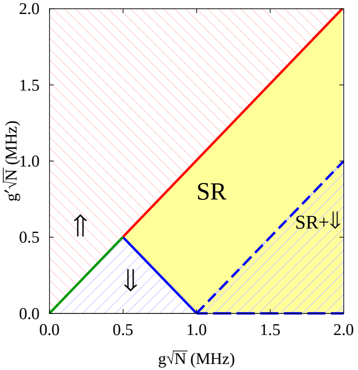
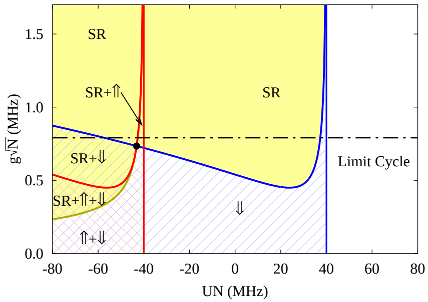


$\kappa = 0$:

—$N(g+g')^2 = \omega\omega_0$



Numerical confirmation of fixed points



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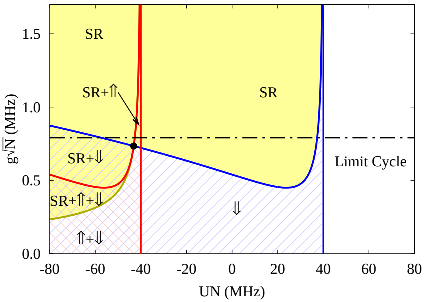

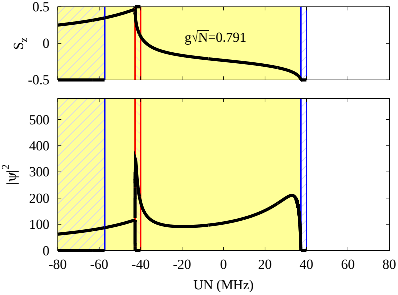

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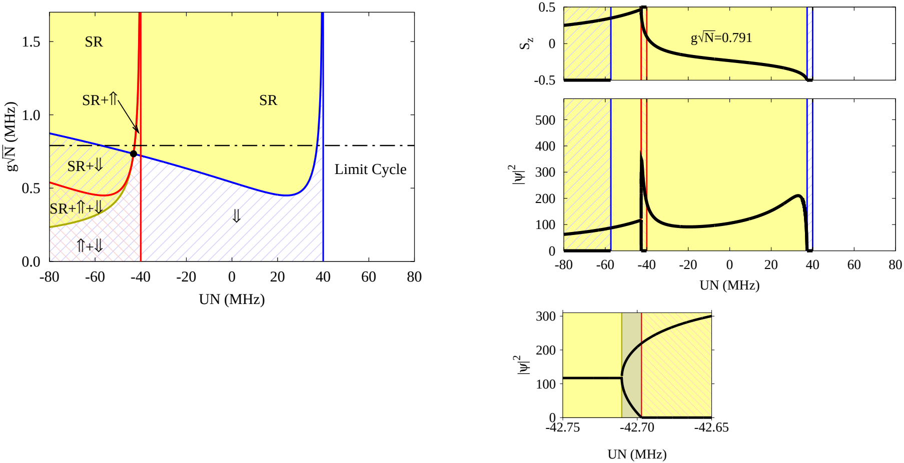

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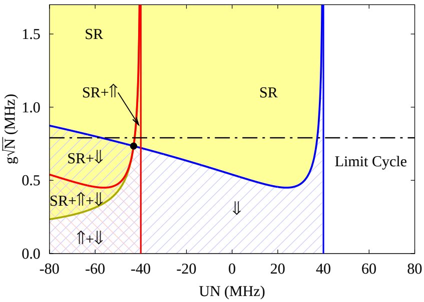
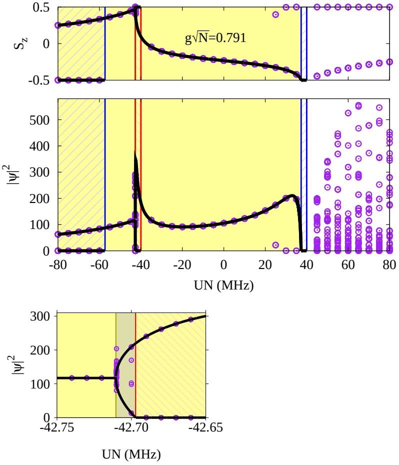
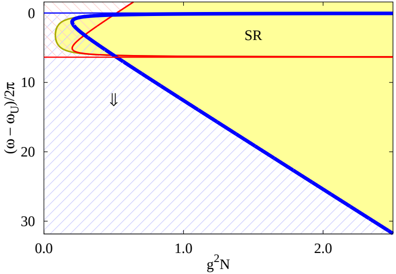


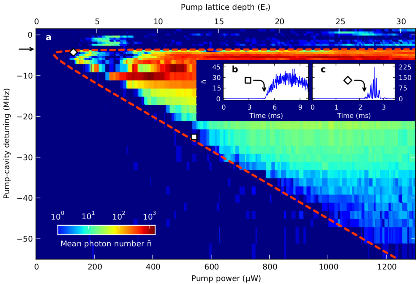
$T = 360\text{ms}$



Comparison to experiment $UN = -40\text{MHz}$



[**JK** *et al* `arXiv:1002.3108`]



[Baumann *et al* Nature 2010]