

LIGHT-MATTER INTERACTIONS AND QUANTUM OPTICS.

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Contents

Contents	iii
Introduction	v
1 Quantisation of electromagnetism	1
1.1 Revision: Lagrangian for electromagnetism	2
1.2 Eliminating redundant variables	3
1.3 Canonical quantisation; photon modes	5
1.4 Approximations of light-matter coupling	6
1.5 Further reading	7
2 Quantum electrodynamics in other gauges	9
2.1 Freedom of choice of gauge and classical equations	9
2.2 Transformation to the electric dipole gauge	11
2.3 Electric dipole gauge for semiclassical problems	15
2.4 Pitfalls of perturbation	17
2.5 Further reading	18
3 Jaynes Cummings model	19
3.1 Semiclassical limit	19
3.2 Single mode quantum model	20
3.3 Many mode quantum model — irreversible decay	23
3.A Further properties of collapse and revival	27
4 Density matrices for 2 level systems	29
4.1 Density matrix equation for relaxation of two-level system	29
4.2 Dephasing in addition to relaxation	33
4.3 Power broadening of absorption	36
4.4 Further reading	37
5 Resonance Fluorescence	39
5.1 Spectrum of emission into a reservoir	39
5.2 Quantum regression “theorem”	40
5.3 Resonance fluorescence spectrum	42
5.4 Further reading	44
6 Quantum stochastic methods	47
6.1 Quantum jump formalism	47

6.2	Heisenberg-Langevin equations	49
6.3	Fluctuation dissipation theorem	51
6.4	Further reading	52
7	Cavity Quantum Electrodynamics	57
7.1	The Purcell effect in a 1D model cavity	57
7.2	Weak to strong coupling via density matrices	61
7.3	Examples of Cavity QED systems	63
7.4	Further reading	67
8	Superradiance	69
8.1	Simple density matrix equation for collective emission . . .	69
8.2	Beyond the simple model	74
8.3	Further reading	78
9	The Dicke model	79
9.1	Phase transitions, spontaneous superradiance	79
9.2	No-go theorem: no vacuum instability	81
9.3	Radiation in a box; restoring the phase transition	82
9.4	Dynamic superradiance	83
9.5	Further Reading	86
10	Lasers and micromasers	89
10.1	Density matrix equations for a micromaser and a laser . . .	89
10.2	Laser rate equations	92
10.3	Laser Linewidth	93
10.4	Further reading	96
11	More on lasers	99
11.1	Density matrix equation	99
11.2	Spontaneous emission, noise, and β parameter	106
11.3	Single atom lasers	109
11.4	Further reading	112
12	Three levels, and coherent control	113
12.1	Semiclassical introduction	113
12.2	Coherent evolution alone; why does EIT occur	117
12.3	Dark state polaritons	117
12.4	Further reading	120
	Bibliography	123

Introduction

The title *quantum optics* covers a large range of possible courses, and so this introduction intends to explain what this course does and does not aim to provide. Regarding the negatives, there are several things this course deliberately avoids:

- It is not a course on quantum information theory. Some basic notions of coherent states and entanglement will be assumed, but will not be the focus.
- It is not a course on relativistic gauge field theories; the majority of solid state physics does not require covariant descriptions, and so it is generally not worth paying the price in complexity of using a manifestly covariant formulation.
- As far as possible, it is not a course on semiclassical electromagnetism. While at times radiation will be treated classically, this will generally be for comparison to a full quantum treatment, or where such an approximation is valid (for at least part of the radiation).

Regarding the positive aims of this course, they are: to discuss how to model the quantum behaviour of coupled light and matter; to introduce some simple models that can be used to describe such systems; to discuss methods for open quantum systems that arise naturally in the context of coupled light and matter; and to discuss some of the more interesting phenomena which may arise for matter coupled to light. Semiclassical behaviour will be discussed in some sections, both because an understanding of semiclassical behaviour (i.e. classical radiation coupled to quantum mechanical matter) is useful to motivate what phenomena might be expected; and also as comparison to the semiclassical case is important to see what new physics arises from quantised radiation.

The kind of quantum optical systems discussed will generally consist of one or many few-level atoms coupled to one quantised radiation fields. Realisations of such systems need not involve excitations of real atoms, but can instead be artificial atoms, i.e. well defined quantum systems with discrete level spectra which couple to the electromagnetic field. Such concepts therefore apply to a wide variety of systems, and a variety of characteristic energies of electromagnetic radiation. Systems currently studied experimentally include: real atomic transitions coupled to optical cavities[1];

Josephson junctions in microwave cavities (waveguides terminated by reflecting boundaries)[2, 3]; Rydberg atoms (atoms with very high principle quantum numbers, hence small differences of energy levels) in GHz cavities[4]; and solid state excitations, i.e. excitons or trions localised in quantum dots, coupled to a variety of optical frequency cavities, including simple dielectric contrast cavities, photonic band gap materials, and whispering gallery modes in disks[5].

These different systems provide different opportunities for control and measurement; in some cases one can probe the atomic state, in some cases the radiation state. To describe experimental behaviour, one is in general interested in calculating a response function, relating the expected outcome to the applied input. However, to understand the predicted behaviour, it is often clearer to consider the evolution of quantum mechanical state; thus, both response functions and wavefunctions will be discussed. As such, the lectures will switch between Heisenberg and Schrödinger pictures frequently according to which is most appropriate. When considering open quantum systems, a variety of different approaches; density matrix equations, Heisenberg-Langevin equations and their semiclassical approximations, again corresponding to both Schrödinger and Heisenberg pictures.

The main part of this course will start with the simplest case of a single two-level atom, and discuss this in the context of one or many quantised radiation modes. The techniques developed in this will then be applied to the problem of many two-level atoms, leading to collective effects. The techniques of open quantum systems will also be applied to describing lasing, focussing on the “more quantum” examples of micromasers and single atom lasers. The end of the course will consider atoms beyond the two-level approximation, illustrating what new physics may arise. Separate to this main discussion, the first two lectures stand alone in discussing where the simple models of coupled light and matter used in the rest of the course come from, in terms of the quantised theory of electromagnetism.

Lecture 1

Quantisation of electromagnetism in the Coulomb gauge

Our aim is to write a theory of quantised radiation interacting with quantised matter fields. Such a theory, e.g. the Jaynes-Cummings model (see next lecture) has an intuitive form:

$$H_{J.C.} = \sum_k \omega_k a_k^\dagger a_k + \sum_{i,k} [\epsilon_i \sigma_i^z + g_{i,k} \sigma_i^+ a_k + \text{H.c.}] . \quad (1.1)$$

The operator a_k^\dagger creates “a photon” in the mode with wavevector k , and so this Hamiltonian describes a process where a two-level system can change its state with the associated emission or absorption of a photon. The term $\omega_k a_k^\dagger a_k$ then gives the total energy associated with occupation of the mode with energy ω_k . While the rest of the course is dedicated to studying such models of coupled light-matter system, this (and in part the next) lecture will show the relation between such models and the classical electromagnetism of Maxwell’s equations.

To reach this destination, we will follow the route of canonical quantisation; our first aim is therefore to write a Lagrangian in terms of only relevant variables. Relevant variables are those where both the variable and its time derivative appear in the Lagrangian; if the time derivative does not appear, then we cannot define the canonically conjugate momentum, and so cannot enforce canonical commutation relations. The simplest way of writing the Lagrangian for electromagnetism contains irrelevant variables — i.e. the electric scalar potential ϕ and gauge of the vector potential A ; that irrelevant variables exist is due to the gauge invariance of the theory. Since we are not worried about preserving manifest Lorentz covariance, we are free to solve this problem in the simplest way — eliminating the unnecessary variables.

1.1 Revision: Lagrangian for electromagnetism

To describe matter interacting with radiation, we wish to write a Lagrangian whose equations of motion will reproduce Maxwell's and Lorentz's equations:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \dot{\mathbf{E}} \qquad \nabla \cdot \mathbf{E} = \rho / \varepsilon_0 \qquad (1.2)$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{E} = -\dot{\mathbf{B}} \qquad (1.3)$$

$$m_\alpha \ddot{\mathbf{r}}_\alpha = q_\alpha [\mathbf{E}(\mathbf{r}_\alpha) + \dot{\mathbf{r}}_\alpha \times \mathbf{B}(\mathbf{r}_\alpha)]. \qquad (1.4)$$

Equations (1.3) determine the structure of the fields, not their dynamics, and are immediately satisfied by defining $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\nabla\phi - \dot{\mathbf{A}}$. Let us suggest the form of Lagrangian \mathcal{L} that leads to Eq. (1.2) and Eq. (1.4):

$$\mathcal{L} = \sum_\alpha \frac{1}{2} m_\alpha \dot{\mathbf{r}}_\alpha^2 + \frac{\varepsilon_0}{2} \int dV [\mathbf{E}^2 - c^2 \mathbf{B}^2] + \sum_\alpha q_\alpha [\dot{\mathbf{r}}_\alpha \cdot \mathbf{A}(\mathbf{r}_\alpha) - \phi(\mathbf{r}_\alpha)]. \qquad (1.5)$$

Here, the fields \mathbf{E} and \mathbf{B} should be regarded as functionals of ϕ and \mathbf{A} . Note also that in order to be able to extract the Lorentz force acting on individual charges, the currents and charge densities have been written as:

$$\rho(\mathbf{r}) = \sum_\alpha q_\alpha \delta(\mathbf{r} - \mathbf{r}_\alpha), \quad \mathbf{J}(\mathbf{r}) = \sum_\alpha q_\alpha \dot{\mathbf{r}}_\alpha \delta(\mathbf{r} - \mathbf{r}_\alpha). \qquad (1.6)$$

The identification of the Lorentz equation is simple:

$$\begin{aligned} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}_\alpha} &= \frac{d}{dt} [m_\alpha \dot{\mathbf{r}}_\alpha + q_\alpha \mathbf{A}(\mathbf{r}_\alpha)] = m_\alpha \ddot{\mathbf{r}}_\alpha + q_\alpha (\dot{\mathbf{r}}_\alpha \cdot \nabla) \mathbf{A}(\mathbf{r}_\alpha) + q_\alpha \frac{\partial}{\partial t} \mathbf{A}(\mathbf{r}_\alpha) \\ &= \frac{\partial \mathcal{L}}{\partial \mathbf{r}_\alpha} = q_\alpha \nabla [\dot{\mathbf{r}}_\alpha \cdot \mathbf{A}(\mathbf{r}_\alpha) - \phi(\mathbf{r}_\alpha)] \\ &= q_\alpha [(\dot{\mathbf{r}}_\alpha \cdot \nabla) \mathbf{A}(\mathbf{r}_\alpha) + \dot{\mathbf{r}}_\alpha \times (\nabla \times \mathbf{A}(\mathbf{r}_\alpha)) - \nabla \phi(\mathbf{r}_\alpha)], \end{aligned}$$

thus one recovers the Lorentz equation,

$$m_\alpha \ddot{\mathbf{r}}_\alpha = q_\alpha \left[\dot{\mathbf{r}}_\alpha \times [\nabla \times \mathbf{A}(\mathbf{r}_\alpha)] - \nabla \phi(\mathbf{r}_\alpha) - \frac{\partial}{\partial t} \mathbf{A}(\mathbf{r}_\alpha) \right]. \qquad (1.7)$$

Similarly, the equation that results from ϕ can be easily extracted; since $\partial \mathcal{L} / \partial \dot{\phi} = 0$, this becomes

$$\frac{\partial \mathcal{L}}{\partial \phi} = \varepsilon_0 \nabla \cdot \mathbf{E} - \sum_\alpha q_\alpha \delta(\mathbf{r} - \mathbf{r}_\alpha) = 0. \qquad (1.8)$$

Finally, the equations for \mathbf{A} are more complicated, requiring the identity

$$\frac{\partial}{\partial \mathbf{A}} (\nabla \times \mathbf{A})^2 = 2 \nabla \times (\nabla \times \mathbf{A}), \qquad (1.9)$$

which then gives:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{A}}} = -\frac{d}{dt} \varepsilon_0 \mathbf{E} = \frac{\partial \mathcal{L}}{\partial \mathbf{A}} = -\varepsilon_0 c^2 \nabla \times (\nabla \times \mathbf{A}) + \sum_\alpha q_\alpha \dot{\mathbf{r}}_\alpha \delta(\mathbf{r} - \mathbf{r}_\alpha), \qquad (1.10)$$

which recovers the required Maxwell equation

$$-\varepsilon_0 \frac{d}{dt} \mathbf{E} = -\frac{1}{\mu_0} \nabla \times \mathbf{B} + \sum_{\alpha} q_{\alpha} \dot{\mathbf{r}}_{\alpha} \delta(\mathbf{r} - \mathbf{r}_{\alpha}). \quad (1.11)$$

Thus, the Lagrangian in Eq. (1.5), along with the definitions of \mathbf{E} and \mathbf{B} in terms of \mathbf{A} and ϕ produce the required equations.

1.2 Eliminating redundant variables

As mentioned in the introduction, we must remove any variable whose time derivative does not appear in the Lagrangian, as one cannot write the required canonical commutation relations for such a variable. It is clear from Eq. (1.5) that the electric scalar potential ϕ is such a variable. Since $\dot{\phi}$ does not appear, it is also possible to eliminate ϕ directly from the equation $\partial\mathcal{L}/\partial\phi$; using Eq. (1.8) and the definition of \mathbf{E} , this equation gives:

$$-\varepsilon_0 \nabla \cdot \dot{\mathbf{A}} - \varepsilon_0 \nabla^2 \phi - \rho(\mathbf{r}) = 0. \quad (1.12)$$

Rewriting this in Fourier space, one has:

$$\phi(\mathbf{k}) = \frac{1}{k^2} \left(\frac{\rho(\mathbf{k})}{\varepsilon_0} + i\mathbf{k} \cdot \dot{\mathbf{A}}(\mathbf{k}) \right). \quad (1.13)$$

We can now try to insert this definition into the Lagrangian, to eliminate ϕ . To do this, we wish to write \mathbf{E}^2 and \mathbf{B}^2 in terms of ϕ and \mathbf{A} ; it is therefore useful to start by writing

$$-E_j(\mathbf{k}) = ik_j \phi(\mathbf{k}) + \dot{A}_j(\mathbf{k}) = i \frac{k_j}{\varepsilon_0 k^2} \rho(\mathbf{k}) + \left(\delta_{jk} - \frac{k_j k_k}{k^2} \right) \dot{A}_k(\mathbf{k}). \quad (1.14)$$

This means that the electric field depends on the charge density, and on the transverse part of the vector potential, which will be written:

$$A_j^{\perp} \mathbf{k} = \left(\delta_{jk} - \frac{k_j k_k}{k^2} \right) \dot{A}_k(\mathbf{k}). \quad (1.15)$$

The transverse¹ part of the vector potential is by definition orthogonal to the wavevector \mathbf{k} , and so the electric field is the sum of two orthogonal vectors, and so:

$$|\mathbf{E}(\mathbf{k})|^2 = |\dot{\mathbf{A}}^{\perp}(\mathbf{k})|^2 + \frac{1}{\varepsilon_0^2 k^2} |\rho(\mathbf{k})|^2. \quad (1.18)$$

¹The combination:

$$\delta_{jk}^{\perp}(\mathbf{k}) = \left(\delta_{jk} - \frac{k_j k_k}{k^2} \right), \quad (1.16)$$

is the reciprocal space representation of the transverse delta function; with appropriate regularisation[6, Complement A_I], it can be written in real space as:

$$\delta_{jk}^{\perp}(\mathbf{r}) = \frac{2}{3} \delta(\mathbf{r}) \delta_{jk} + \frac{1}{4\pi r^3} \left(3 \frac{r_j r_k}{r^2} - \delta_{jk} \right). \quad (1.17)$$

Similarly, the squared magnetic field in reciprocal space is given by:

$$\begin{aligned} |\mathbf{B}(\mathbf{k})|^2 &= (\mathbf{k} \times \mathbf{A}(\mathbf{k})) \cdot (\mathbf{k} \times \mathbf{A}^*(\mathbf{k})) \\ &= k^2 \left(\delta_{jk} - \frac{k_j k_k}{k^2} \right) A_j(\mathbf{k}) A_k^*(\mathbf{k}) = k^2 |\mathbf{A}^\perp(\mathbf{k})|^2. \end{aligned} \quad (1.19)$$

Thus, the field part of the Lagrangian becomes:

$$\begin{aligned} \frac{\varepsilon_0}{2} \int dV [\mathbf{E}^2 - c^2 \mathbf{B}^2] &= \frac{1}{\varepsilon_0} \int d^3k \frac{1}{k^2} |\rho(\mathbf{k})|^2 \\ &\quad + \varepsilon_0 \int d^3k \left(|\dot{\mathbf{A}}^\perp(\mathbf{k})|^2 - c^2 k^2 |\mathbf{A}^\perp(\mathbf{k})|^2 \right). \end{aligned} \quad (1.20)$$

The notation \int has been introduced to mean integration over reciprocal half-space; since $\mathbf{A}(\mathbf{r})$ is real, $\mathbf{A}^*(\mathbf{k}) = \mathbf{A}(-\mathbf{k})$, thus the two half spaces are equivalent. This rewriting is important to avoid introducing redundant fields in the Lagrangian; the field is either specified by one real variable at all points in real space, or by two real variables at all points in reciprocal half-space. Similar substitution into the coupling between fields and matter, written in momentum space gives:

$$\begin{aligned} \mathcal{L}_{\text{em-matter}} &= 2\Re \int d^3k [\mathbf{J}(\mathbf{k}) \cdot \mathbf{A}^*(\mathbf{k}) - \rho(\mathbf{k}) \phi^*(\mathbf{k})] \\ &= 2\Re \int d^3k \left[\mathbf{J}(\mathbf{k}) \cdot \mathbf{A}^*(\mathbf{k}) - \rho(\mathbf{k}) \left(\frac{\rho^*(\mathbf{k})}{\varepsilon_0 k^2} - \frac{i\mathbf{k}}{k^2} \cdot \dot{\mathbf{A}}^*(\mathbf{k}) \right) \right]. \end{aligned} \quad (1.21)$$

This can be simplified by adding a total time derivative, $\mathcal{L} \rightarrow \mathcal{L} + dF/dt$; such transformations do not affect the equations of motion, since they add only boundary terms to the action. If:

$$F = 2\Re \int d^3k \rho(\mathbf{k}) \left[\frac{-i\mathbf{k} \cdot \mathbf{A}^*(\mathbf{k})}{k^2} \right], \quad (1.22)$$

then

$$\mathcal{L}_{\text{em-matter}} + \frac{dF}{dt} = 2\Re \int d^3k \left[\left(\mathbf{J}(\mathbf{k}) - \dot{\rho}(\mathbf{k}) \frac{i\mathbf{k}}{k^2} \right) \cdot \mathbf{A}^*(\mathbf{k}) - \frac{|\rho(\mathbf{k})|^2}{\varepsilon_0 k^2} \right]. \quad (1.23)$$

Then, using conservation of current, $\dot{\rho}(\mathbf{k}) + i\mathbf{k} \cdot \mathbf{J}(\mathbf{k}) = 0$, one finally has:

$$\mathcal{L}_{\text{em-matter}} + \frac{dF}{dt} = 2\Re \int d^3k \left[\mathbf{J}^\perp(\mathbf{k}) \cdot \mathbf{A}^{\perp*}(\mathbf{k}) - \frac{|\rho(\mathbf{k})|^2}{\varepsilon_0 k^2} \right]. \quad (1.24)$$

Note that this set of manipulations, adding dF/dt has eliminated the longitudinal part of the vector potential from the Lagrangian. The form chosen for F is such that this procedure is equivalent to a gauge transformation; the chosen gauge is the Coulomb gauge. Putting everything together, one has:

$$\begin{aligned} \mathcal{L}_{\text{coulomb}} &= \sum_\alpha \frac{1}{2} m_\alpha \dot{\mathbf{r}}_\alpha^2 - \frac{1}{\varepsilon_0} \int d^3k \frac{1}{k^2} |\rho(\mathbf{k})|^2 \\ &\quad + \varepsilon_0 \int d^3k \left(|\dot{\mathbf{A}}^\perp(\mathbf{k})|^2 - c^2 k^2 |\mathbf{A}^\perp(\mathbf{k})|^2 + 2\Re \left[\mathbf{J}^\perp(\mathbf{k}) \cdot \mathbf{A}^{\perp*}(\mathbf{k}) \right] \right). \end{aligned} \quad (1.25)$$

Thus, the final form has divided the interaction into a part mediated by transverse fields, described by \mathbf{A}^\perp , and a static (and non-retarded) Coulomb interaction. Importantly, there are no irrelevant variables left in Eq. (1.25) The Coulomb term can also be rewritten:

$$V_{\text{coul}} = \frac{1}{\varepsilon_0} \int d^3k \frac{1}{k^2} |\rho(\mathbf{k})|^2 = \sum_{\alpha, \beta} \frac{q_\alpha q_\beta}{8\pi\varepsilon_0 |\mathbf{r}_\alpha - \mathbf{r}_\beta|}. \quad (1.26)$$

Note that since the Coulomb interaction is non-retarded, both the Coulomb and transverse parts of interaction must be included to have retarded interactions between separated charges.

1.3 Canonical quantisation; photon modes

We have in Eq. (1.25) a Lagrangian which can now be treated via canonical quantisation. Since only the transverse part of the field \mathbf{A} appears in Eq. (1.25), we can drop the superscript label in \mathbf{A}^\perp from here on. To proceed, we should first identify the canonical momenta and the Hamiltonian, and then impose canonical commutation relations. Thus,

$$\mathbf{p}_\alpha = \frac{\partial \mathcal{L}_{\text{coulomb}}}{\partial \dot{\mathbf{r}}_\alpha} = m\dot{\mathbf{r}}_\alpha + q_\alpha \mathbf{A}(\mathbf{r}_\alpha) \quad (1.27)$$

$$\mathbf{\Pi}(\mathbf{k}) = \frac{\partial \mathcal{L}_{\text{coulomb}}}{\partial \dot{\mathbf{A}}(\mathbf{k})^*} = \varepsilon_0 \dot{\mathbf{A}}(\mathbf{k}). \quad (1.28)$$

Then, constructing the Hamiltonian by $H = \sum_i P_i \dot{R}_i - \mathcal{L}$, one finds:

$$H = \sum_\alpha \frac{1}{2m_\alpha} [\mathbf{p}_\alpha - q_\alpha \mathbf{A}(\mathbf{r}_\alpha)]^2 + V_{\text{coul}} + \varepsilon_0 \int d^3k \left(\frac{|\mathbf{\Pi}(\mathbf{k})|^2}{\varepsilon_0^2} + c^2 k^2 |\mathbf{A}(\mathbf{k})|^2 \right). \quad (1.29)$$

In order to quantise, it remains only to introduce commutation relations for the canonically conjugate operators. Noting that $\mathbf{A}(\mathbf{r})$ has only two independent components, because it is transverse, it is easiest to write its commutation relations in reciprocal space, introducing directions $\mathbf{e}_{\mathbf{k},n}$ orthogonal to \mathbf{k} with $n = 1, 2$; then:

$$[r_{i,\alpha}, p_{j,\beta}] = i\hbar \delta_{\alpha\beta} \delta_{ij} \quad (1.30)$$

$$[A_{\mathbf{e}_{\mathbf{k},n}}(\mathbf{k}), \Pi_{\mathbf{e}_{\mathbf{k}',n'}}(\mathbf{k}')] = i\hbar \delta(\mathbf{k} - \mathbf{k}') \delta_{nn'}. \quad (1.31)$$

This concludes the quantisation of matter with electromagnetic interactions in the Coulomb gauge. It is however instructive to rewrite the transverse part of the fields in terms of their normal modes. The second line of Eq. (1.29) has a clear similarity to the harmonic oscillator, with a separate oscillator for each polarisation and momentum. Rewriting in normal modes thus means introducing the ladder operators:

$$a_{\mathbf{k},n} = \sqrt{\frac{\varepsilon_0}{2\hbar ck}} \left[ck A_{\mathbf{e}_{\mathbf{k},n}}(\mathbf{k}) + \frac{i}{\varepsilon_0} \Pi_{\mathbf{e}_{\mathbf{k},n}}(\mathbf{k}) \right], \quad (1.32)$$

or, inverted one has:

$$\mathbf{A}(\mathbf{r}) = \sum_{\mathbf{k}} \sum_{n=1,2} \sqrt{\frac{\hbar}{2\varepsilon_0\omega_{\mathbf{k}}V}} \hat{\mathbf{e}}_n \left(a_{\mathbf{k},n} e^{i\mathbf{k}\cdot\mathbf{r}} + a_{\mathbf{k},n}^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}} \right). \quad (1.33)$$

Inserting this into Eq. (1.29) gives the final form:

$$H = \sum_{\alpha} \frac{1}{2m_{\alpha}} [\mathbf{p}_{\alpha} - q_{\alpha}\mathbf{A}(\mathbf{r}_{\alpha})]^2 + \sum_{\mathbf{k}} \sum_{n=1,2} \hbar\omega_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + V_{\text{coul}}. \quad (1.34)$$

1.4 Dipole, two-level, and rotating wave approximations

Equation (1.34) applies to any set of point charges interacting with the electromagnetic field. In many common cases, one is interested in dipoles, with pairs of opposite charges closely spaced, and much larger distances between the dipoles. In this case, there are a number of approximations one can make to simplify calculations. This section will briefly illustrate these approximations: neglect of the A^2 terms, the dipole approximation, projection to two-level systems, and the rotating wave approximation. The study of the approximate models that result will be the subject of the rest of this course.

Neglect of the \mathbf{A}^2 terms in expanding $[\mathbf{p}_{\alpha} - q_{\alpha}\mathbf{A}(\mathbf{r}_{\alpha})]^2$ can be justified in the limit of low density of dipoles; considering only a single radiation mode, the contribution of the \mathbf{A}^2 term can be rewritten using Eq. (1.33) as:

$$\delta H_{A^2} = \frac{N}{V} \frac{q^2 \hbar}{4m\varepsilon_0\omega} \left(a + a^\dagger \right)^2. \quad (1.35)$$

Thus, this term adds a self energy to the photon field, which scales like the density of dipoles. The relative importance of this term can be estimated by comparing it to the other term in the Hamiltonian which is quadratic in the photon operators, $\hbar\omega_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$. Their ratio is given by:

$$\frac{N}{V} \frac{q^2}{4m\varepsilon_0(\omega_{\mathbf{k}})^2} \propto \frac{N}{V} a_B^3 \left(\frac{\mathcal{R}yd}{\hbar\omega_{\mathbf{k}}} \right)^2 \quad (1.36)$$

thus, assuming particles are more dilute than their Bohr radius, neglect of δH_{A^2} is valid for frequencies of the order of the Rydberg for the given bound system of charges.

Turning next to the dipole approximation, consider a system with just two charges: charge $+q$ mass m_1 at $R+r/2$ and $-q$ mass m_2 at $R-r/2$. If $r \ll \lambda$ where λ is a characteristic wavelength of light, then one may assume that $\mathbf{A}(\mathbf{R} + \mathbf{r}/2) \simeq \mathbf{A}(\mathbf{R} - \mathbf{r}/2) \simeq \mathbf{A}(\mathbf{R})$, and so the remaining coupling between radiation and matter is of the form

$$\delta H_{\mathbf{A}\cdot\mathbf{p}} = q \left(\frac{\mathbf{p}_1}{m_1} - \frac{\mathbf{p}_2}{m_2} \right) \cdot \mathbf{A}(\mathbf{R}).$$

Then, in the case that one can write H_0 , a Hamiltonian for the dipole without its coupling to radiation one can use $p/m = \dot{x} = i[H_0, x]/\hbar$, thus giving:

$$H = H_0 - i\frac{q}{\hbar}[H_0, \mathbf{r}] \cdot \mathbf{A}(\mathbf{R}) + \sum_{\mathbf{k}, n=1,2} \hbar\omega_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + V_{\text{coul}}. \quad (1.37)$$

To further simplify, one can now reduce the number of states of the dipoles that are considered; currently, there will be a spectrum of eigenstates of H_0 , and transitions are induced between these states according to $\langle\psi_f|\mathbf{r}|\psi_i\rangle$. Finally, restricting to only the two lowest atomic levels and to a single radiation mode, one has a model of two-level systems (describing matter) coupled to bosonic modes (describing radiation). This model is known as the Jaynes-Cummings model:

$$H = \frac{1}{2} \begin{pmatrix} \epsilon & g(a + a^\dagger) \\ g(a + a^\dagger) & -\epsilon \end{pmatrix} + \hbar\omega_0 a^\dagger a. \quad (1.38)$$

Here we have introduced the energy splitting ϵ between the lowest two atomic levels. In terms of the upper and lower states, $|a\rangle$ and $|b\rangle$, the atom-photon coupling strength g can be written:

$$\frac{g_k}{2} = q \langle b|[H_0, r]|a\rangle \frac{1}{\sqrt{2\varepsilon_0\hbar\omega_k V}} = \frac{\epsilon \mathbf{e}_{k,n} \cdot \mathbf{d}_{ba}}{\sqrt{2\varepsilon_0\hbar\omega_k V}}, \quad (1.39)$$

where $\mathbf{d}_{ba} = q \langle b|\mathbf{r}|a\rangle$ is the dipole matrix element. The matrix notation represents the two levels of the dipole. We assumed H_0 commutes with the parity operator $x \rightarrow -x$, and so the coupling to radiation appears only in the off-diagonal terms in the two-level basis. The final approximation to be discussed here, the rotating wave approximation, is appropriate when $\epsilon \simeq \omega_0$. Considering g as a perturbation, one can identify two terms:

$$\Delta_{\text{co}} = \frac{g}{2} \begin{pmatrix} 0 & a \\ a^\dagger & 0 \end{pmatrix}, \quad \Delta_{\text{cross}} = \frac{g}{2} \begin{pmatrix} 0 & a^\dagger \\ a & 0 \end{pmatrix}, \quad (1.40)$$

where Δ_{co} , the co-rotating terms, ‘‘conserve energy’’; and Δ_{cross} do not. More formally, the effects of Δ_{cross} give second order perturbation terms like $g^2/(\omega_0 + \epsilon)$, while Δ_{co} give the much larger $g^2/(\omega_0 - \epsilon)$.

1.5 Further reading

The discussion of quantisation in the Coulomb gauge in this chapter draws heavily on the book by Cohen-Tannoudji et al. [6].

Questions

Question 1.1: Transverse delta function

Prove that Eq. (1.17) is the Fourier transform of Eq. (1.16). It is helpful to consider a modified version of Eq. (1.17), with a factor $\exp(-mk^2)$, and take $m \rightarrow 0$ only at the end of the calculation.

Question 1.2: Thomas-Reiche-Kuhn sum rule

Prove the equality:

$$[r_\alpha, [H_0, r_\beta]] = \frac{\hbar^2 \delta_{\alpha\beta}}{m}, \quad (1.41)$$

stating the most general form of H_0 for which this is true.

From the expectation of this commutator, show that the dipole matrix elements between any state a and all other states b obeys the relation:

$$\sum_b |d_{ab}|^2 (E_b - E_a) = \sum_\alpha \frac{\hbar^2 q_\alpha^2}{2m} \quad (1.42)$$

Lecture 2

Gauge transformations, and quantum electrodynamics in other gauges

In the previous section, we choose to work in the Coulomb gauge, adding a total time-derivative to the Lagrangian which had the effect of removing all dependence of the Lagrangian on A_{\parallel} . This choice, which is equivalent to choosing to impose $A_{\parallel} \equiv 0$ hugely simplified the subsequent algebra, but is not strictly necessary. This chapter describes the consequences of making other choices for A_{\parallel} ; such choices turn out to correspond to making a unitary transformation in the quantum problem. In the special case of one or a few localised systems of charge — where charges within a system are separated by far less than the wavelength of light — a change of gauge to the *electric dipole gauge* can simplify calculations, and provides further understanding of the relation between the instantaneous Coulomb interaction and the photon-mediated terms in Eq. (1.34).

2.1 Freedom of choice of gauge and classical equations

The classical Lagrangian in the previous section depended on the longitudinal part of the vector potential only in the coupling between matter and electromagnetic fields, Eq. (1.21)

$$\begin{aligned}\mathcal{L}_{\text{em-matter}} &= 2\Re \int d^3k [\mathbf{J}(\mathbf{k}) \cdot \mathbf{A}^*(\mathbf{k}) - \rho(\mathbf{k})\phi^*(\mathbf{k})] \\ &= 2\Re \int d^3k \left[\mathbf{J}(\mathbf{k}) \cdot \mathbf{A}^*(\mathbf{k}) - \rho(\mathbf{k}) \left(\frac{\rho^*(\mathbf{k})}{\varepsilon_0 k^2} - \frac{i\mathbf{k}}{k^2} \cdot \dot{\mathbf{A}}^*(\mathbf{k}) \right) \right].\end{aligned}\tag{1.21}$$

Using the continuity equation $\dot{\rho} + i\mathbf{k} \cdot \mathbf{J} = 0$ to eliminate $J_{\parallel} = i\dot{\rho}/k$ and breaking \mathbf{A} into transverse and longitudinal parts yields

$$\mathcal{L}_{\text{e-m}} = 2\Re \int d^3k \left[\mathbf{J}_{\perp}(\mathbf{k}) \cdot \mathbf{A}_{\perp}^*(\mathbf{k}) + \frac{i}{k} \dot{\rho}(\mathbf{k}) A_{\parallel}^*(\mathbf{k}) - \frac{|\rho(\mathbf{k})|^2}{\varepsilon_0 k^2} + \frac{i}{k} \rho(\mathbf{k}) \dot{A}_{\parallel}^*(\mathbf{k}) \right]. \quad (2.1)$$

It is therefore clear that taking any functional form for $A_{\parallel} = g(\mathbf{A}_{\perp})$ leads to a change to the Lagrangian which may be written as:

$$\delta L_{\text{gauge}} = \frac{d}{dt} \left\{ 2\Re \int d^3k \left[\frac{i}{k} \rho g^*(\mathbf{A}_{\perp}) \right] \right\}. \quad (2.2)$$

The choice of A_{\parallel} is the choice of gauge; by choosing A_{\parallel} as a function of \mathbf{A}_{\perp} , one can ensure, for example, that $\mathbf{n} \cdot \mathbf{A}(\mathbf{r}) = 0$ for some fixed \mathbf{n} . Since such a condition corresponds only to an additional total derivative, the action changes only by boundary terms, and so the equations of motion are not affected. The definition of canonical momentum however does change.

Rewriting

$$\delta L_{\text{gauge}} = \frac{\partial}{\partial t} \left\{ 2\Re \int d^3k \left[\frac{i}{k} \rho g^*(\mathbf{A}_{\perp}) \right] \right\} + 2\Re \int d^3k \left[\frac{i}{k} \rho \frac{\partial g^*(\mathbf{A}_{\perp})}{\partial \mathbf{A}_{\perp}^*} \dot{\mathbf{A}}_{\perp}(\mathbf{k})^* \right], \quad (2.3)$$

the canonical momentum thus becomes:

$$\mathbf{\Pi}(\mathbf{k}) = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{A}}_{\perp}(\mathbf{k})^*} = \mathbf{\Pi}_{\text{Coulomb}}(\mathbf{k}) + \frac{i\rho(\mathbf{k})}{k} \frac{\partial g^*(\mathbf{A}_{\perp})}{\partial \mathbf{A}_{\perp}} \quad (2.4)$$

Quantum formalisms resulting from different gauges

Once we have found the canonical momentum in a given gauge, we can quantise in this gauge by promoting the dynamical variables and their canonical conjugates to operators, and imposing canonical commutation relations. This means $[\mathbf{A}_{\perp n}(\mathbf{k}), \mathbf{\Pi}_{n'}(\mathbf{k}')] = i\hbar \delta(\mathbf{k} - \mathbf{k}') \delta_{nn'}$, with $\mathbf{\Pi}(\mathbf{k})$ being the *new* canonical momentum in the *new* gauge: The canonical momentum corresponds to a different combination of physical fields in different gauges. If we write

$$F(\mathbf{A}_{\perp}, \rho) = 2\Re \int d^3k \left[\frac{i}{k} \rho g^*(\mathbf{A}_{\perp}) \right], \quad (2.5)$$

then we may compare the two formalisms:

$$\mathbf{A}_{\perp, \text{new}}(\mathbf{k}) = \mathbf{A}_{\perp, \text{old}}(\mathbf{k}) \quad (2.6)$$

$$\mathbf{\Pi}_{\text{new}}(\mathbf{k}) = \mathbf{\Pi}_{\text{old}}(\mathbf{k}) + \frac{\partial F}{\partial \mathbf{A}_{\perp, \text{new}}(\mathbf{k})}. \quad (2.7)$$

If we consider a state $|\pi\rangle$ which is an eigenstate of the old momentum, $\mathbf{\Pi}_{\text{old}}(\mathbf{k})|\pi\rangle = \lambda_{\pi}|\pi\rangle$, then in the new formalism, this state obeys:

$$\lambda_{\pi}|\pi\rangle = \left[\mathbf{\Pi}_{\text{new}}(\mathbf{k}) - \frac{\partial F}{\partial \mathbf{A}_{\perp, \text{new}}(\mathbf{k})} \right] |\pi\rangle \quad (2.8)$$

$$= \left[-i\hbar \frac{\partial}{\partial \mathbf{A}_{\perp, \text{new}}(\mathbf{k})} - \frac{\partial F}{\partial \mathbf{A}_{\perp, \text{new}}(\mathbf{k})} \right] |\pi\rangle \quad (2.9)$$

$$= e^{iF(\mathbf{A}_{\perp}, \rho)/\hbar} \left[-i\hbar \frac{\partial}{\partial \mathbf{A}_{\perp, \text{new}}(\mathbf{k})} \right] e^{-iF(\mathbf{A}_{\perp}, \rho)/\hbar} |\pi\rangle. \quad (2.10)$$

Hence, the eigenstates of the momenta of the old and new formalism are related by the unitary transform $\exp[-iF(\mathbf{A}_\perp, \rho)/\hbar]$ — one can immediately see that this relation of eigenstates of operators works for all canonical momenta, as well as working for the position operators which commute with the unitary transforms. It is worth stressing that the statement proved here is that all gauge transforms correspond to unitary transforms of the quantum formalism, and not the other way around. It is also worth highlighting (as was implicit in the calculation), that this occurred because the gauge transforms corresponded classically to canonical transformations of the Lagrangian (addition of a total time derivative), and that it is more generally true that canonical transformations of classical dynamical systems correspond to unitary transformations of the associated quantum problem.

Formal equivalence of gauges

Formally, since a gauge transformation corresponds to a unitary transformation, calculations of physical quantities in different gauges should exactly match, i.e.

$$\langle \psi^{(1)} | O^{(1)} | \psi^{(1)} \rangle = \langle \psi^{(1)} | U^\dagger U O^{(1)} U^\dagger U | \psi^{(1)} \rangle = \langle \psi^{(2)} | O^{(2)} | \psi^{(2)} \rangle. \quad (2.11)$$

Despite this formal equivalence, there have been long and convoluted arguments about gauge invariance in approximate methods. One particularly dangerous feature of such transformations is that the “natural” identification of a bare and perturbation Hamiltonian can be different between different gauges (this is true of the discussion below), and so eigenstates of the bare Hamiltonian are not a gauge invariant quantity. Two general solutions to such problems are

- To use a basis of states that correspond to eigenstates of a physical (i.e. gauge invariant) operator, such as the mechanical momentum, physical fields, and the total energy.
- To use only the formal equivalence, and demand a transformation of states as well as of operators when switching between gauges.

2.2 Transformation to the electric dipole gauge

The preceding working allows us to directly transform between two quantum pictures, rather than needing to change gauges classically and re-quantize. As an illustration of this in a case where it is particularly clear, this section describes the transformation to the *electric dipole gauge* for a problem of two systems of localised charges, where each system is overall neutral. We consider a hierarchy of scales, such that within a given system all charges are within a distance l and $l \ll \lambda, |\mathbf{R}_A - \mathbf{R}_B|$, where $\mathbf{R}_{A,B}$ are the centres of mass of the two systems. This is illustrated in Fig. 2.1.

Since $l \ll \lambda$, we can approximate the electromagnetic fields as being those at the centre of mass of each system. In the Coulomb gauge, this

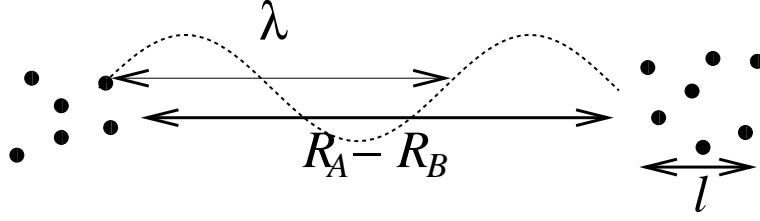


Figure 2.1: Two localised systems, consisting of charges which are closely spaced in each system.

then can be written following the notation in Eq. (1.34) as:

$$H = \sum_{\alpha} \frac{1}{2m_{\alpha}} [\mathbf{p}_{\alpha} - q_{\alpha} \mathbf{A}_{\perp}(\mathbf{R}_A)]^2 + \sum_{\beta} \frac{1}{2m_{\beta}} [\mathbf{p}_{\beta} - q_{\beta} \mathbf{A}_{\perp}(\mathbf{R}_B)]^2 + \sum_{\mathbf{k}, n} \omega_{\mathbf{k}} \hbar a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} + V_{\text{coul}}^{AA} + V_{\text{coul}}^{BB} + V_{\text{coul}}^{AB}, \quad (2.12)$$

in which the Coulomb interaction has been divided into parts limited to each subsystem as well as a cross term. This cross term is instantaneous (as all the Coulomb terms are), while the real physical interaction is retarded. To recover this retardation it is however necessary to include the transverse parts of the fields. From this point onwards we will set $\hbar = 1$

Transformation of Hamiltonian

The unitary transformation that will simplify the results is of the form:

$$U = \exp [i \mathbf{d}_A \cdot \mathbf{A}_{\perp}(\mathbf{R}_A) + i \mathbf{d}_B \cdot \mathbf{A}_{\perp}(\mathbf{R}_B)], \quad \mathbf{d}_A = \sum_{\alpha} q_{\alpha} (\mathbf{r}_{\alpha} - \mathbf{R}_A), \quad (2.13)$$

in which we have introduced the dipole moments $\mathbf{d}_{A,B}$ of the two systems. This transformation clearly commutes with the position operator, and ought to commute with the transverse vector potential (this is shown to be true explicitly below). However, it does not commute with \mathbf{p}_{α} nor $a_{\mathbf{k}}^{\dagger}, a_{\mathbf{k}}$. One instead finds:

$$\begin{aligned} U^{\dagger} \mathbf{p}_{\alpha} U &= \mathbf{p}_{\alpha} - i \frac{\partial}{\partial \mathbf{r}_{\alpha}} \left(i \sum_{\alpha} q_{\alpha} (\mathbf{r}_{\alpha} - \mathbf{R}_A) \cdot \mathbf{A}_{\perp}(\mathbf{R}_A) \right) \\ &= \mathbf{p}_{\alpha} + q_{\alpha} \mathbf{A}_{\perp}(\mathbf{R}_A) - \left(\frac{\partial \mathbf{R}_A}{\partial \mathbf{r}_{\alpha}} \sum_{\alpha'} q_{\alpha'} \right) \cdot \mathbf{A}_{\perp}(\mathbf{R}_A) + \mathcal{O} \left(\frac{l}{\lambda} \right). \end{aligned} \quad (2.14)$$

In this, the term $\sum_{\alpha} q_{\alpha} = 0$ due to neutrality of each subsystem, and the higher order terms in l/λ are ignored due to the long wavelength approximation. Hence all that remains is $U^{\dagger} \mathbf{p}_{\alpha} U \approx \mathbf{p}_{\alpha} + q_{\alpha} \mathbf{A}_{\perp}(\mathbf{R}_A)$.

For the photon annihilation operators no approximations are needed. The unitary transformation can be rewritten as

$$U = \exp \left\{ i \sum_{\mathbf{k}, n} \frac{\hat{\mathbf{e}}_n}{\sqrt{2\varepsilon_0\omega_{\mathbf{k}}V}} \cdot \left[\mathbf{d}_A e^{-i\mathbf{k}\cdot\mathbf{R}_A} + \mathbf{d}_B e^{-i\mathbf{k}\cdot\mathbf{R}_B} \right] a_{\mathbf{k}}^\dagger + \text{H.c.} \right\}, \quad (2.15)$$

which has the form of a shift operator for the annihilation operators:

$$U^\dagger a_{\mathbf{k}} U = a_{\mathbf{k}} + i \frac{\hat{\mathbf{e}}_n}{\sqrt{2\varepsilon_0\omega_{\mathbf{k}}V}} \cdot \left[\mathbf{d}_A e^{-i\mathbf{k}\cdot\mathbf{R}_A} + \mathbf{d}_B e^{-i\mathbf{k}\cdot\mathbf{R}_B} \right] \quad (2.16)$$

From this we can reconfirm that the unitary matrix commutes with the transverse vector potential,

$$\mathbf{A}_\perp(\mathbf{r}) = \sum_{\mathbf{k}, n} \frac{\hat{\mathbf{e}}_n}{\sqrt{2\varepsilon_0\omega_{\mathbf{k}}V}} \left[e^{-i\mathbf{k}\cdot\mathbf{r}} a_{\mathbf{k}}^\dagger + e^{i\mathbf{k}\cdot\mathbf{r}} a_{\mathbf{k}} \right], \quad (2.17)$$

hence,

$$\begin{aligned} U^\dagger \mathbf{A}_\perp(\mathbf{r}) U &= \mathbf{A}_\perp(\mathbf{r}) \\ &+ \sum_{\mathbf{k}, n} \frac{i\hat{\mathbf{e}}_n}{2\varepsilon_0\omega_{\mathbf{k}}V} \left[\hat{\mathbf{e}}_n \cdot \mathbf{d}_A e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{R}_A)} + \hat{\mathbf{e}}_n \cdot \mathbf{d}_B e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{R}_B)} - \text{H.c.} \right]. \end{aligned} \quad (2.18)$$

It is clear that due to the antisymmetry of the sum under $\mathbf{k} \leftrightarrow -\mathbf{k}$ the sum vanishes, so $\mathbf{A}_\perp(\mathbf{r})$ commutes with the transformation. Hence, the transformed form of the Hamiltonian is given by:

$$\begin{aligned} \tilde{H} = U^\dagger H U &= \sum_{\alpha} \frac{\mathbf{p}_{\alpha}^2}{2m_{\alpha}} + \sum_{\beta} \frac{\mathbf{p}_{\beta}^2}{2m_{\beta}} + V_{\text{coul}}^{AA} + V_{\text{coul}}^{BB} + V_{\text{coul}}^{AB} \\ &+ \sum_{\mathbf{k}, n} \left\{ \omega_{\mathbf{k}} \left[a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \sum_{S=A,B} i \frac{\hat{\mathbf{e}}_n \cdot \mathbf{d}_S}{\sqrt{2\varepsilon_0\omega_{\mathbf{k}}V}} \left(e^{-i\mathbf{k}\cdot\mathbf{R}_S} a_{\mathbf{k}}^\dagger - e^{i\mathbf{k}\cdot\mathbf{R}_S} a_{\mathbf{k}} \right) \right] \right. \\ &\left. + \frac{1}{2\varepsilon_0V} \left[(\hat{\mathbf{e}}_n \cdot \mathbf{d}_A)^2 + (\hat{\mathbf{e}}_n \cdot \mathbf{d}_B)^2 + (\hat{\mathbf{e}}_n \cdot \mathbf{d}_A) (\hat{\mathbf{e}}_n \cdot \mathbf{d}_B) 2e^{i\mathbf{k}\cdot(\mathbf{R}_A-\mathbf{R}_B)} \right] \right\}, \end{aligned} \quad (2.19)$$

where the last line has relabelled the sum via $\mathbf{k} \leftrightarrow -\mathbf{k}$.

Transformation of physical fields

To simplify this expression, we need two connected things:

- To identify the combination of photon operators that appears in the second line as the light-matter coupling.
- To simplify the cross terms between the A and B systems.

It is tempting to use the definition of the transverse electric field in the old gauge,

$$\mathbf{E}_\perp(\mathbf{r}) = i \sum_{\mathbf{k}, n} \hat{\mathbf{e}}_n \sqrt{\frac{\omega_k}{2\epsilon_0 V}} \left[e^{-i\mathbf{k}\cdot\mathbf{r}} a_k^\dagger - e^{i\mathbf{k}\cdot\mathbf{r}} a_k \right], \quad (2.20)$$

to identify the combination of operators, however this relation between the physical operators and the annihilation operators is true only in the *Coulomb gauge*, not the *electric dipole gauge*. Transforming this electric field using Eq. (2.16) gives:

$$U^\dagger \mathbf{E}_\perp U = \mathbf{E}_\perp - \sum_{\mathbf{k}, n} \frac{\hat{\mathbf{e}}_n}{2\epsilon_0 V} \left[\hat{\mathbf{e}}_n \cdot \left[\mathbf{d}_A e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{R}_A)} + \mathbf{d}_B e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{R}_B)} \right] + \text{H.c.} \right]. \quad (2.21)$$

This sum is symmetric under $\mathbf{k} \leftrightarrow -\mathbf{k}$ and so does not vanish. Instead it can be identified in terms of the polarisation. Since the dipole matrix element of system A is \mathbf{d}_A , we can write $\mathbf{P}_A(\mathbf{r}) = \mathbf{d}_A \delta(\mathbf{r} - \mathbf{R}_A)$; this places the polarisation at the centre of charges, and is measured w.r.t a reference state in which all the charges are at the centre of mass, i.e. $\forall \mathbf{r}_\alpha = \mathbf{R}_A$. If we then use the transverse delta function to find the transverse part of this expression, we find that:

$$\mathbf{P}_{A,\perp}(\mathbf{r}) = \frac{1}{V} \sum_{\mathbf{k}, n} \hat{\mathbf{e}}_n \left[\hat{\mathbf{e}}_n \cdot \mathbf{d}_A e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{R}_A)} \right] \quad (2.22)$$

and hence:

$$i \sum_{\mathbf{k}, n} \hat{\mathbf{e}}_n \sqrt{\frac{\omega_k}{2\epsilon_0 V}} \left[e^{-i\mathbf{k}\cdot\mathbf{r}} a_k^\dagger - e^{i\mathbf{k}\cdot\mathbf{r}} a_k \right] = \mathbf{E}_\perp(\mathbf{r}) + \frac{\mathbf{P}_\perp(\mathbf{r})}{\epsilon_0} = \frac{\mathbf{D}_\perp(\mathbf{r})}{\epsilon_0} \quad (2.23)$$

This solves our first question; the second is also considerably simplified when we note that the term:

$$\begin{aligned} & \sum_{\mathbf{k}, n} \frac{1}{\epsilon_0 V} (\hat{\mathbf{e}}_n \cdot \mathbf{d}_A) (\hat{\mathbf{e}}_n \cdot \mathbf{d}_B) e^{i\mathbf{k}\cdot(\mathbf{R}_A - \mathbf{R}_B)} = \\ & \int \frac{d^3 r}{\epsilon_0} \left(\frac{1}{V} \sum_{\mathbf{k}, n} \hat{\mathbf{e}}_n \left[\hat{\mathbf{e}}_n \cdot \mathbf{d}_A e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{R}_A)} \right] \right) \cdot \left(\frac{1}{V} \sum_{\mathbf{k}, n} \hat{\mathbf{e}}_n \left[\hat{\mathbf{e}}_n \cdot \mathbf{d}_B e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{R}_B)} \right] \right) \\ & = \int \frac{d^3 r}{\epsilon_0} \mathbf{P}_{A,\perp}(\mathbf{r}) \mathbf{P}_{B,\perp}(\mathbf{r}). \quad (2.24) \end{aligned}$$

Thus, we can write:

$$\begin{aligned} \tilde{H} = & \sum_\alpha \frac{\mathbf{p}_\alpha^2}{2m_\alpha} + \sum_\beta \frac{\mathbf{p}_\beta^2}{2m_\beta} + V_{\text{coul}}^{AA} + V_{\text{coul}}^{BB} + V_{\text{coul}}^{AB} \\ & + \sum_{\mathbf{k}, n} \omega_k a_k^\dagger a_k + \mathbf{D}_\perp(\mathbf{R}_A) \cdot \mathbf{d}_A + \mathbf{D}_\perp(\mathbf{R}_B) \cdot \mathbf{d}_B \\ & + \Sigma_{\text{dipolar}}^A + \Sigma_{\text{dipolar}}^B + \int \frac{d^3 r}{\epsilon_0} \mathbf{P}_{A,\perp}(\mathbf{r}) \mathbf{P}_{B,\perp}(\mathbf{r}). \quad (2.25) \end{aligned}$$

Instantaneous nature of interaction

To complete the simplification, it is necessary to write the cross-term in the Coulomb interaction similarly to Eq. (2.24). The Coulomb interaction arose by writing the longitudinal part of the electric field in terms of the charge density. Inverting this procedure easily gives:

$$V_{\text{coul}}^{AB} = \frac{\varepsilon_0}{2} \int d^3r \left[(\mathbf{E}_{\parallel,A}(\mathbf{r}) + \mathbf{E}_{\parallel,A}(\mathbf{r}))^2 - \mathbf{E}_{\parallel,A}(\mathbf{r})^2 - \mathbf{E}_{\parallel,A}(\mathbf{r})^2 \right] \quad (2.26)$$

$$= \varepsilon_0 \int d^3r \mathbf{E}_{\parallel,A}(\mathbf{r}) \cdot \mathbf{E}_{\parallel,A}(\mathbf{r}) = \frac{1}{\varepsilon_0} \int d^3r \mathbf{P}_{\parallel,A}(\mathbf{r}) \cdot \mathbf{P}_{\parallel,A}(\mathbf{r}). \quad (2.27)$$

The last step made use of the reference distribution of charges; since this distribution has no net charges (each system is neutral, and all charges coincide), then $\nabla \cdot \mathbf{D} = 0$, i.e. $\mathbf{D}_{\parallel} = 0$, so $\mathbf{E}_{\parallel} = \mathbf{P}_{\parallel}/\varepsilon_0$. This allows us to combine:

$$V_{\text{coul}}^{AB} + \int \frac{d^3r}{\varepsilon_0} \mathbf{P}_{A,\perp}(\mathbf{r}) \mathbf{P}_{B,\perp}(\mathbf{r}) = \int \frac{d^3r}{\varepsilon_0} \mathbf{P}_A(\mathbf{r}) \mathbf{P}_B(\mathbf{r}) = 0 \quad (2.28)$$

where the last line follows from the definition of the polarisation, which is zero outside each region of charge.

Hence, the final Hamiltonian is:

$$\begin{aligned} \tilde{H} = & \sum_{\alpha} \frac{\mathbf{p}_{\alpha}^2}{2m_{\alpha}} + \sum_{\beta} \frac{\mathbf{p}_{\beta}^2}{2m_{\beta}} + V_{\text{coul}}^{AA} + V_{\text{coul}}^{BB} + \Sigma_{\text{dipolar}}^A + \Sigma_{\text{dipolar}}^B \\ & + \sum_{\mathbf{k},n} \omega_{\mathbf{k}} a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} + \mathbf{D}_{\perp}(\mathbf{R}_A) \cdot \mathbf{d}_A + \mathbf{D}_{\perp}(\mathbf{R}_B) \cdot \mathbf{d}_B. \end{aligned} \quad (2.29)$$

This transformation has therefore removed the instantaneous interaction between the two systems; all interaction is now described by the quantised \mathbf{D} field. Since $\mathbf{D}_{\parallel} = 0$, the transverse and total fields are equivalent, hence the transverse \mathbf{D} field is physical and retarded.

2.3 Electric dipole gauge for semiclassical problems

The previous section showed the unitary transformation to the *electric dipole gauge* for quantised radiation coupled to quantised matter. For the simpler case of a single localised system, the transformation would be $U = \exp[i\mathbf{d} \cdot \mathbf{A}_{\perp}(\mathbf{0})]$, yielding:

$$\tilde{H} = \sum_{\alpha} \frac{\mathbf{p}_{\alpha}^2}{2m_{\alpha}} + V_{\text{coul}} + \Sigma_{\text{dipolar}} + \sum_{\mathbf{k},n} \omega_{\mathbf{k}} a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} + \mathbf{D}_{\perp}(\mathbf{0}) \cdot \mathbf{d}. \quad (2.30)$$

For comparison, this section describes the even simpler transformation that applies when the radiation is assumed to be classical; i.e. it has no

free dynamics, but rather the operator \mathbf{A}_\perp is replaced by a time dependent field corresponding to some incident radiation. In this case the initial Hamiltonian is:

$$H_{\text{s.c.}} = \sum_{\alpha} \frac{1}{2m_{\alpha}} [\mathbf{p}_{\alpha} - q_{\alpha} \mathbf{A}_{\perp}(0, t)]^2 + V_{\text{coul}}. \quad (2.31)$$

The gauge transformation is formally the same as before, but as now \mathbf{A}_\perp is not an operator but a time-dependent field we must use the result:

$$i\partial_t U|\psi\rangle = U i\partial_t |\psi\rangle + (i\partial_t U)|\psi\rangle = HU|\psi\rangle \quad (2.32)$$

$$i\partial_t |\psi\rangle + (U^\dagger i\partial_t U)|\psi\rangle = U^\dagger HU|\psi\rangle \quad (2.33)$$

and so:

$$\tilde{H} = U^\dagger HU - U^\dagger i\partial_t U. \quad (2.34)$$

The transformation of \mathbf{p}_α is exact as in Eq. (2.14). Since \mathbf{A}_\perp is a c-number, it commutes with the unitary transformation, and so the final result is:

$$\begin{aligned} \tilde{H}_{\text{s.c.}} &= \sum_{\alpha} \frac{\mathbf{p}_{\alpha}^2}{2m_{\alpha}} + V_{\text{coul}} - i\partial_t [i\mathbf{d} \cdot \mathbf{A}_{\perp}(\mathbf{0})] \\ &= \sum_{\alpha} \frac{\mathbf{p}_{\alpha}^2}{2m_{\alpha}} + V_{\text{coul}} - \mathbf{d} \cdot \mathbf{E}_{\perp}(\mathbf{0}) \end{aligned} \quad (2.35)$$

Relating S -matrix elements in different gauges

In the semiclassical case, as well as the formal equivalence between gauges, there are “less formal” relations that can be shown to hold. One example arises if we consider a field $\mathbf{A}_{\perp}(t) = \mathbf{A}_0 \lambda(t) \cos(\omega t)$, where $\lambda(t)$ is a function that goes to 0 at $\pm\infty$, but $\lambda(t) = 1$ for $t_1 < t < t_2$, with smooth interpolation between: Since this means $U(t) = \exp[i\lambda(t)\mathbf{d} \cdot \mathbf{A}_0 \cos(\omega t)]$, it is clear



Figure 2.2: Form of $\lambda(t)$ for S -matrix

that at long times $U(t) = \mathbf{1}$, and hence eigenstates in the two gauges are coincident.

This allows us to consider the S -matrix:

$$S_{ab} = \langle b | e^{iH_0 t_2} U(t_2, t_1) e^{-iH_0 t_1} | a \rangle. \quad (2.36)$$

Since $U = \mathbf{1}$ at long times the bare Hamiltonian H_0 is also the same between the two gauges, so this S -matrix should be gauge invariant. Let us consider its calculation perturbatively in the two gauges

Calculation in $\mathbf{A} \cdot \mathbf{p}$ gauge

In the Coulomb gauge, the perturbation Hamiltonian (to first order in \mathbf{A}_0) is given by:

$$H_{\text{int}}(t) = - \sum_{\alpha} \frac{\mathbf{p}_{\alpha} q_{\alpha}}{m_{\alpha}} \cdot \mathbf{A}_0 \cos(\omega t). \quad (2.37)$$

Using the result of time-dependent perturbation theory:

$$U(t_2, t_1) = \mathbf{1} - i \int_{t_1}^{t_2} d\tau e^{-iH_0(t_2-\tau)} H_{\text{int}}(\tau) e^{-iH_0(\tau-t_1)} + \dots \quad (2.38)$$

we have:

$$\begin{aligned} S_{ab} &= \delta_{ab} - i \int_{t_1}^{t_2} d\tau e^{i\omega_{ba}\tau} \langle b | H_{\text{int}}(\tau) | a \rangle \quad (2.39) \\ &= \delta_{ab} + i \int_{t_1}^{t_2} d\tau e^{i\omega_{ba}\tau} \cos(\omega t) \mathbf{A}_0 \cdot \langle b | \sum_{\alpha} \frac{\mathbf{p}_{\alpha} q_{\alpha}}{m_{\alpha}} | a \rangle \\ &= \delta_{ab} + i2\pi \frac{\delta(\omega_{ba} + \omega) + \delta(\omega_{ba} - \omega)}{2} \mathbf{A}_0 \cdot \langle b | i \sum_{\alpha} [H_0, \mathbf{r}_{\alpha} q_{\alpha}] | a \rangle \\ &= \delta_{ab} - \pi [\delta(\omega_{ba} + \omega) + \delta(\omega_{ba} - \omega)] \omega_{ba} \mathbf{A}_0 \cdot \langle b | \mathbf{d} | a \rangle. \quad (2.40) \end{aligned}$$

In the last line we have used the fact that the bare Hamiltonian depends only on \mathbf{p}_{α} via $\mathbf{p}_{\alpha}^2/2m_{\alpha}$, allowing one to rewrite the matrix element of momentum in terms of the dipole matrix element.

Calculation in $\mathbf{E} \cdot \mathbf{r}$ gauge

In the electric dipole gauge, the perturbation is instead

$$H_{\text{int}}(t) = -\mathbf{d} \cdot \mathbf{E}(t). \quad (2.41)$$

where $\mathbf{E}(t) = -\dot{\mathbf{A}}(t) \approx A_0 \omega \sin(\omega t)$, and so starting from Eq. (2.39) we have:

$$\begin{aligned} S_{ab} &= \delta_{ab} + i \int_{t_1}^{t_2} d\tau e^{i\omega_{ba}\tau} \sin(\omega t) \omega \mathbf{A}_0 \cdot \langle b | \mathbf{d} | a \rangle \\ &= \delta_{ab} - \pi [\delta(\omega_{ba} - \omega) - \delta(\omega_{ba} + \omega)] \omega \mathbf{A}_0 \cdot \langle b | \mathbf{d} | a \rangle, \quad (2.42) \end{aligned}$$

where we have made the same resonant approximation as above. Although functionally different, the two expressions do match, because $\omega_{ba} = \omega$ is ensured by the delta function. For a further example of this kind of equivalence, see question 2.1.

2.4 Pitfalls of perturbation

The idea of writing the free Hamiltonian H_0 is in fact rather subtle to define correctly: Atoms are held together by electromagnetic forces, and as mentioned above, separation into transverse, photon mediated, and Coulomb

terms leads to apparently non-retarded interactions. Further, H_0 differs between different gauges.

The question of most practical relevance is how the parameters in a semi-phenomenological model of a coupled light-matter system should be related to experimentally measured quantities.

2.5 Further reading

The majority of this chapter is also based on discussions in the book by Cohen-Tannoudji et al. [6]. The subtlety mentioned briefly, about how one can define a “free Hamiltonian” for atoms when the binding within an atom is due to the electromagnetic field has been discussed in the context of which gauge should be used in calculating off-resonant transitions. See for example [7–9].

Questions

Question 2.1: Two-photon transitions

If we choose states $|b\rangle, |a\rangle$ such that $\langle b | \mathbf{d} | a \rangle = 0$, then the leading order contribution to the transition comes from two photon processes. Assuming $a \neq b$, the leading order contribution to the S -matrix can be written:

$$S_{ab} = - \int_{t_1}^{t_2} d\tau \int_{t_1}^{\tau} d\tau' e^{i\omega_{bc}\tau} e^{i\omega_{ca}\tau'} \langle b | H_{\text{int}}(\tau) | c \rangle \langle c | H_{\text{int}}(\tau') | a \rangle \quad (2.43)$$

2.1.(a) Making the resonant approximation, show that the expressions for S_{ab} in the two gauges can be written as:

$$S_{ab}^{\mathbf{A}\cdot\mathbf{p}} = 2\pi \frac{\delta(\omega_{ba} - 2\omega)}{4} \sum_c \frac{\omega_{bc}\omega_{ca}}{\omega_{ca} - \omega} \mathbf{A}_0 \cdot \langle b | \mathbf{d} | c \rangle \mathbf{A}_0 \cdot \langle c | \mathbf{d} | a \rangle \quad (2.44)$$

$$S_{ab}^{\mathbf{E}\cdot\mathbf{r}} = 2\pi \frac{\delta(\omega_{ba} - 2\omega)}{4} \sum_c \frac{\omega^2}{\omega_{ca} - \omega} \mathbf{A}_0 \cdot \langle b | \mathbf{d} | c \rangle \mathbf{A}_0 \cdot \langle c | \mathbf{d} | a \rangle \quad (2.45)$$

2.1.(b) Prove the equivalence of these expressions by showing that:

$$D = \sum_c \frac{\omega_{bc}\omega_{ca} - \omega^2}{\omega_{ca} - \omega} \cdot \langle b | d_i | c \rangle \cdot \langle c | d_j | a \rangle = 0 \quad (2.46)$$

if $2\omega = \omega_{ba}$.

[Hint: use the two-photon resonance condition to show that

$$D \propto \sum_c (\omega_{bc} - \omega_{ca}) \cdot \langle b | d_i | c \rangle \cdot \langle c | d_j | a \rangle, \quad (2.47)$$

then rewrite this sum in terms of the commutator $[[H_0, d_i], d_j]$.

Lecture 3

One two-level system coupled to photons: the Jaynes-Cummings model

The aim of this first section is to discuss the interaction between a single two-level system and various different configurations of photon fields. We will begin by discussing the interaction with a semiclassical light field, then consider a single quantised mode of light in an initially coherent state, and finally consider a continuum of quantised modes. In future lectures we will then consider combinations of these cases, to describe realistic cavity QED systems, in which the cavity picks out a particular mode, but coupling to a continuum of background modes also exists. Even in the case of a classical applied field, the behaviour of a two-level system is not entirely trivial, as the two-level system is non-linear, i.e. it does not have an harmonic spectrum.

3.1 Semiclassical limit

On applying a classical light field to a two-level system, we can adapt the Jaynes-Cummings model by neglecting the dynamics of the photon field, and replacing the photon creation and annihilation operators with the amplitude of the time-dependent classical field. This gives the effective Hamiltonian:

$$H = \frac{1}{2} \begin{pmatrix} \epsilon & g\lambda e^{-i\omega_0 t} \\ g\lambda e^{i\omega_0 t} & -\epsilon \end{pmatrix}. \quad (3.1)$$

We have made the rotating wave approximation, and hence only included the positive/negative frequency components of the classical field in the raising/lowering operators for the two-level system. To solve this problem, and find the time-dependent state of the two-level atom, it is convenient to make a transformation to a rotating frame. In general, if

$$U = \begin{pmatrix} e^{if(t)} & 0 \\ 0 & e^{-if(t)} \end{pmatrix}, \quad (3.2)$$

then

$$H \rightarrow \tilde{H} = \begin{pmatrix} H_{11} + \dot{f}(t) & H_{12}e^{-2if(t)} \\ H_{21}e^{2if(t)} & H_{22} - \dot{f}(t) \end{pmatrix}, \quad (3.3)$$

thus, in this case we use $f = -\omega t/2$, giving:

$$H = \frac{1}{2} \begin{pmatrix} \epsilon - \omega & g\lambda \\ g\lambda & -(\epsilon - \omega) \end{pmatrix}. \quad (3.4)$$

Since this transformation affects only the phases of the wavefunction, we can then find the absorption probability by considering the time averaged probability of being in the upper state. The eigenvalues of Eq. (3.2) are $\pm\Omega/2$ where $\Omega = \sqrt{(\epsilon - \omega)^2 + g^2\lambda^2}$. Writing the eigenstates as $(\cos\theta, \sin\theta)^T$, and $(-\sin\theta, \cos\theta)^T$ corresponding respectively to the \pm roots (assuming $\tan(\theta) > 0$). Substituting these forms, one finds the condition: $\tan(2\theta) = g\lambda/(\epsilon - \omega)$. This form is sufficient to find the probability of excitation. If the two-level system begins in its ground state, then the time-dependent state can be written as:

$$\begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} = \sin\theta \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} e^{-i\Omega t/2} + \cos\theta \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} e^{i\Omega t/2} \quad (3.5)$$

$$P_{\text{ex}} = |\psi_\uparrow|^2 = \overline{|2\sin\theta\cos\theta\sin(\Omega t/2)|^2} = \left[\frac{1 - \cos(\Omega t)}{2} \right] \frac{(g\lambda)^2}{(g\lambda)^2 + (\epsilon - \omega)^2}. \quad (3.6)$$

Thus, the probability of excitation oscillates at the Rabi frequency $\Omega = \sqrt{(\epsilon - \omega)^2 + (g\lambda)^2}$, and the amplitude of oscillation depends on detuning. On resonance ($\epsilon = \omega$), one can engineer a state with $|\psi_\uparrow| = |\psi_\downarrow| = 1/\sqrt{2}$, or a state with $|\psi_\uparrow| = 1$ by applying a pulse with a duration $(g\lambda)t = \pi/2$ or π respectively.

3.2 Single mode quantum model

Let us now repeat this analysis in the case of a quantised mode of radiation, starting with the Jaynes-Cummings model in the rotating wave approximation.

$$H = \frac{1}{2} \begin{pmatrix} \epsilon & ga \\ ga^\dagger & -\epsilon \end{pmatrix} + \omega_0 a^\dagger a. \quad (3.7)$$

Rabi oscillations in the Jaynes-Cummings model

In the rotating wave approximation the total number of excitations $N_{\text{ex}} = \sigma^z + a^\dagger a$ is a conserved quantity, thus if we start in a number state of the light field:

$$|n, \downarrow\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0, \downarrow\rangle, \quad (3.8)$$

then the general state at later times can always be written:

$$|\psi(t)\rangle = \alpha(t) |n-1, \uparrow\rangle + \beta(t) |n, \downarrow\rangle. \quad (3.9)$$

Inserting this ansatz in the equation of motion gives

$$i\partial_t \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} = \left[\left(n - \frac{1}{2} \right) \omega \mathbb{1} + \frac{1}{2} \begin{pmatrix} (\epsilon - \omega) & g\sqrt{n} \\ g\sqrt{n} & -(\epsilon - \omega) \end{pmatrix} \right] \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}. \quad (3.10)$$

After removing the common phase variation $\exp[-i(n-1/2)\omega t]$, the general solution can be written in terms of the eigenvectors and values of the matrix in Eq. (3.10), and reusing the results for the semiclassical case, we have the state

$$\begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} = e^{-i(n-1/2)\omega t} \left[\sin \theta e^{-i\Omega t/2} \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} + \cos \theta e^{i\Omega t/2} \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} \right], \quad (3.11)$$

and the excitation probability is thus:

$$P_{\text{ex}} = \left[\frac{1 - \cos(\Omega t)}{2} \right] \frac{g^2 n}{g^2 n + (\epsilon - \omega)^2} \quad (3.12)$$

where we now have $\tan(2\theta) = g\sqrt{n}/(\epsilon - \omega)$ and $\Omega = \sqrt{(\epsilon - \omega)^2 + g^2 n}$.

Collapse and revival of Rabi oscillations

Restricting to the resonant case $\epsilon = \omega$, let us discuss what happens if the initial state does not have a well defined excitation number. Since the oscillation frequency depends on n , the various components of the initial state with different numbers of excitations will oscillate at different frequencies. We can therefore expect interference, washing out the signal. This is indeed seen, but in addition one sees revivals; the signal reappears at much later times, when the phase difference between the contributions of successive number state components is an integer multiple of 2π .

Let us consider explicitly how the probability of being in the excited state evolves for an initially coherent state

$$|\psi\rangle = e^{-|\lambda|^2/2} \exp(\lambda a^\dagger) |0, \downarrow\rangle = e^{-|\lambda|^2/2} \sum_n \frac{\lambda^n}{\sqrt{n!}} |n, \downarrow\rangle. \quad (3.13)$$

If resonant, the general result in Eq. (3.11) simplifies, as $\theta = \pi/4$. Following the previous analysis, the state at any subsequent time is given by:

$$|\psi\rangle = e^{(-|\lambda|^2 + i\omega t)/2} \times \sum_n \frac{(\lambda e^{-i\omega t})^n}{\sqrt{n!}} \left[\cos\left(\frac{g\sqrt{n}t}{2}\right) |n, \downarrow\rangle + \sin\left(\frac{g\sqrt{n}t}{2}\right) |n-1, \uparrow\rangle \right]. \quad (3.14)$$

Thus, the probability of being in the excited atomic state, traced over all possible photon states, is given by

$$\begin{aligned} P_{\text{ex}} &= \sum_n |\langle n, \uparrow | \psi \rangle|^2 = e^{-|\lambda|^2} \sum_n \frac{|\lambda|^{2n}}{n!} \sin^2\left(\frac{g\sqrt{n}t}{2}\right) \\ &= \frac{1}{2} - \frac{1}{2} e^{-|\lambda|^2} \sum_n \frac{|\lambda|^{2n}}{n!} \cos(g\sqrt{n}t). \end{aligned} \quad (3.15)$$

We consider the case $\lambda \gg 1$. In this case, one may see that the coefficients $|\lambda|^{2n}/n!$ are sharply peaked around $n \simeq |\lambda|^2$ by writing:

$$\frac{|\lambda|^{2n}}{n!} = \frac{1}{\sqrt{2\pi n}} \exp[-f(n)], \quad f(n) = n \ln(n) - n - n \ln(|\lambda|^2), \quad (3.16)$$

By differentiating f twice, one finds $f'(n) = \ln(n/|\lambda|^2)$, $f'' = 1/n$, and hence one may expand $f(n)$ near its minimum at $n = |\lambda|^2$ to give:

$$f(n = |\lambda|^2 + x) \simeq -|\lambda|^2 + \frac{x^2}{2|\lambda|^2} \quad (3.17)$$

Using this in Eq. (3.15) gives the approximate result:

$$P_{\text{ex}} \simeq \frac{1}{2} - \frac{1}{2\sqrt{2\pi|\lambda|^2}} \Re \left\{ \sum_m \exp \left[-\frac{m^2}{2|\lambda|^2} + i g t \sqrt{|\lambda|^2 + m} \right] \right\}, \quad (3.18)$$

where it is assumed that $|\lambda|^2$ is large enough that the limits of the sum may be taken to $\pm\infty$.

Three different timescales affect the behaviour of this sum. Concentrating on the peak, near $m = 0$, there is a fast oscillation frequency $g\lambda$, describing Rabi oscillations. Then, considering whether the terms in the sum add in phase or out of phase, there are two time scales: collapse of oscillations occurs when there is a phase difference of 2π across the range of terms $|m| < \sigma_m$ contributing to this sum, i.e. when $gT_{\text{collapse}}(\sqrt{|\lambda|^2 + \sigma_m} - |\lambda|) = \pi$. Taking $\sigma_m = |\lambda|$ from the Gaussian factor, this condition becomes $gT_{\text{collapse}} \simeq 2\pi$. On the other hand, if the phase difference between each successive term in the sum is 2π , then they will rephase, and a revival occurs, with the condition $gT_{\text{revival}}(\sqrt{|\lambda|^2 + 1} - |\lambda|) = 2\pi$, giving $gT_{\text{revival}} = 4\pi|\lambda|$. Thus, the characteristic timescales are given by:

$$T_{\text{oscillation}} \simeq \frac{1}{g|\lambda|}, \quad T_{\text{decay}} \simeq \frac{1}{g}, \quad T_{\text{revival}} \simeq \frac{|\lambda|}{g}. \quad (3.19)$$

and the associated behaviour is illustrated in Fig. 3.1.

Let us now consider how this behaviour may be approximated analytically. The simplest approach might be to replace the sum over number states by integration, however this would inevitably lose the revivals, which are associated with the discreteness of the sum allowing rephasing. Thus, it is necessary to take account of the discreteness of the sum, and hence the difference between a sum and an integral. For this, we make use of Poisson summation which is based on the result:

$$\sum_m \delta(x - m) = \sum_r e^{2\pi i r m} \Rightarrow \sum_m f(m) = \sum_r \int dx e^{2\pi i r x} f(x). \quad (3.20)$$

This allows us to replace the summation by integration, at the cost of having a sum of final results, however in the current context this is very helpful, as the final sum turns out to sum over different revivals. Applying this formula, we may write $P_{\text{ex}} \simeq \frac{1}{2} [1 - \sum_r \Re\{\Lambda(r, |\lambda|, t)\}]$ where

$$\Lambda(r, |\lambda|, t) = \int \frac{dx}{\sqrt{2\pi|\lambda|^2}} \exp \left[2\pi i r x - \frac{x^2}{2|\lambda|^2} + i g t \left(|\lambda| + \frac{x}{2|\lambda|} - \frac{x^2}{8|\lambda|^3} \right) \right] \quad (3.21)$$

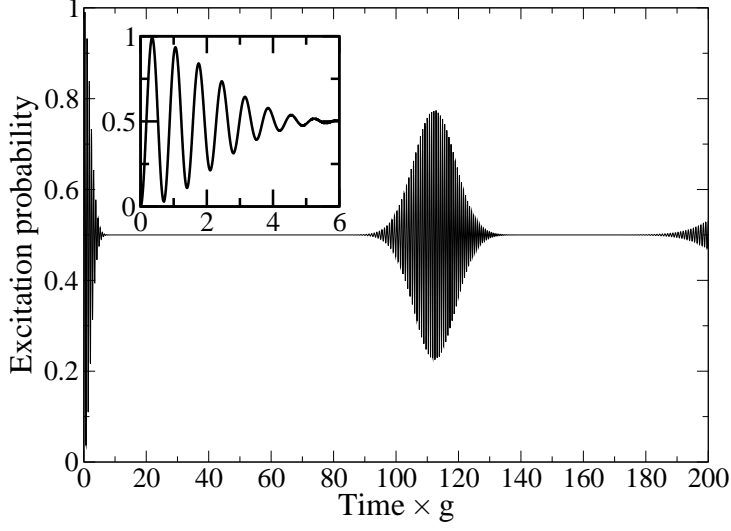


Figure 3.1: Collapse and revival of Rabi oscillations, plotted for $\lambda = \sqrt{200}$.

Completing the square yields:

$$\begin{aligned}
 \Lambda(r, |\lambda|, t) &= \frac{e^{igt|\lambda|}}{\sqrt{2\pi|\lambda|^2}} \int dx \exp \left[-\frac{1}{2|\lambda|^2} \left(1 + \frac{igt}{4|\lambda|} \right) x^2 + i \left(2\pi r + \frac{gt}{2|\lambda|} \right) x \right] \\
 &= \frac{e^{igt|\lambda|}}{\sqrt{2\pi|\lambda|^2}} \exp \left[-\frac{|\lambda|^2 (2\pi r + gt/2|\lambda|)^2}{2 (1 + igt/4|\lambda|)} \right] \\
 &\quad \times \int dx \exp \left[-\frac{1}{2|\lambda|^2} \left(1 + \frac{igt}{4|\lambda|} \right) \left(x - i|\lambda|^2 \frac{2\pi r + gt/2|\lambda|}{1 + igt/4|\lambda|} \right)^2 \right] \\
 &= \frac{e^{igt|\lambda|}}{\sqrt{1 + igt/4|\lambda|}} \exp \left[-\frac{|\lambda|^2 (2\pi r + gt/2|\lambda|)^2}{2 (1 + igt/4|\lambda|)} \right]. \quad (3.22)
 \end{aligned}$$

From this final expression one may directly read off: the oscillation frequency $g|\lambda|$; the characteristic time for the first collapse $2\sqrt{2}/g$ (by taking $r = 0$ and assuming $gt/|\lambda|$ is small); the delay between successive revivals $4\pi|\lambda|/g$; and finally one may also see that the revival at $gt = 4\pi r|\lambda|$ has an increased width and decreased height compared to the initial collapse, given by a factor $\sqrt{1 + \pi^2 r^2}$ as is visible in figure 3.1.

3.3 Many mode quantum model — irreversible decay

So far we have discussed a two level system coupled to a single mode, with light described either classically or quantum mechanically. One important difference between these situations is that when described classically, there was stimulated absorption and stimulated emission (hence the periodic oscillation), but if the classical field had vanished, then there would have been

no transition. For the quantum description, spontaneous emission also existed, in that the state $|0, \uparrow\rangle$ hybridises with the state $|1, \downarrow\rangle$, due to the $+1$ in the $\sqrt{n+1}$ matrix elements. We will now consider the effects of this $+1$ factor when we couple a single two level system to a continuum of radiation modes.

Our Hamiltonian will remain in the rotating wave approximation, but no longer restricted to a single mode of light, and so we have:

$$H = \epsilon S_z + \sum_{n,k} \omega_k a_{\mathbf{k},n}^\dagger a_{\mathbf{k},n} + \sum_k \frac{g_{\mathbf{k},n}}{2} \left[a_{\mathbf{k},n}^\dagger S_- + \text{H.c.} \right] \quad (3.23)$$

where $\omega_k = ck$ and

$$\frac{g_{\mathbf{k},n}}{2} = \frac{\epsilon \mathbf{e}_{\mathbf{k},n} \cdot \mathbf{d}_{ba}}{\sqrt{2\epsilon_0 \omega_k V}}. \quad (3.24)$$

Wigner–Weisskopf approach

To study spontaneous emission, we start from the state $|\mathbf{0}, \uparrow\rangle$, in which there are no photons. The subsequent state can be written as:

$$|\psi\rangle = \alpha(t)|\mathbf{0}, \uparrow\rangle + \sum_k \beta_{\mathbf{k},n}(t)|\mathbf{1}_{\mathbf{k},n}, \downarrow\rangle, \quad (3.25)$$

where $|\mathbf{1}_{\mathbf{k},n}\rangle$ contains a single photon in state k, n . Then, writing the Schrödinger equations as:

$$i\partial_t \alpha = \frac{\epsilon}{2} \alpha + \sum_{\mathbf{k},n} \frac{g_{\mathbf{k},n}}{2} \beta_{\mathbf{k},n}, \quad i\partial_t \beta_{\mathbf{k},n} = \left(-\frac{\epsilon}{2} + \omega_k \right) \beta_{\mathbf{k},n} + \frac{g_{\mathbf{k},n}}{2} \alpha, \quad (3.26)$$

one may solve these equations by first writing $\alpha = \tilde{\alpha} e^{-i\epsilon t/2}$, $\beta_k = \tilde{\beta}_k e^{i(\epsilon/2 - \omega_k)t}$, and then formally solving the equation for $\tilde{\beta}_k$, with the initial condition $\beta_k(0) = 0$, to give:

$$\tilde{\beta}_{\mathbf{k},n} = -i \frac{g_{\mathbf{k},n}}{2} \int_0^t dt' e^{i(\omega_k - \epsilon)t'} \tilde{\alpha}(t'). \quad (3.27)$$

Substituting this into the equation for α gives:

$$\begin{aligned} \partial_t \tilde{\alpha} &= - \int^t dt' \sum_{\mathbf{k},n} \left| \frac{g_{\mathbf{k},n}}{2} \right|^2 e^{i(\omega_k - \epsilon)(t' - t)} \tilde{\alpha}(t') \\ &= - \int^t dt' \int \frac{d\omega}{2\pi} \Gamma(\omega) e^{i(\omega - \epsilon)(t' - t)} \tilde{\alpha}(t') \end{aligned} \quad (3.28)$$

where we have defined:

$$\Gamma(\omega) = 2\pi \sum_{\mathbf{k},n} \left| \frac{g_{\mathbf{k},n}}{2} \right|^2 \delta(\omega - \omega_{\mathbf{k},n}) \quad (3.29)$$

Before evaluating Γ , let us find the resultant behaviour, under the assumption that Γ is a smooth function, meaning that Γ does not significantly vary over the range $\epsilon \pm \Gamma$, i.e. $d\Gamma/d\omega \ll 1$. In this case (which corresponds to

the Markov approximation, meaning a flat effective density of states) the integral over ω in Eq. (3.28) becomes a delta function, so that:

$$\partial_t \tilde{\alpha} = - \int^t dt' \Gamma(\epsilon) \delta(t' - t) \tilde{\alpha}(t') = - \frac{\Gamma(\epsilon)}{2} \tilde{\alpha}(t) \quad (3.30)$$

Thus, the probability of remaining in the excited state decays exponentially due to spontaneous decay, with a decay rate $\Gamma(\epsilon)$ for the probability. The approach used in this section is known as the Wigner–Weisskopf approach.

Effective decay rate

We have not yet calculated the effective decay rate appearing in the previous expression. Let us now use the form of g_k in Eq. (3.24) to evaluate $\Gamma(\epsilon)$. Writing the wavevector in polar coordinates, with respect to the dipole matrix element \mathbf{d}_{ab} pointing along the \mathbf{z} axis (see Fig. 3.2), we then have:

$$\Gamma(\epsilon) = 2\pi \frac{V}{(2\pi)^3} \iiint d\phi d\theta \sin\theta k^2 dk \delta(ck - \epsilon) \frac{\epsilon^2 |d_{ab}|^2}{2\epsilon_0 ck V} \sum_n (\hat{\mathbf{z}} \cdot \hat{\mathbf{e}}_{\mathbf{k}n})^2. \quad (3.31)$$

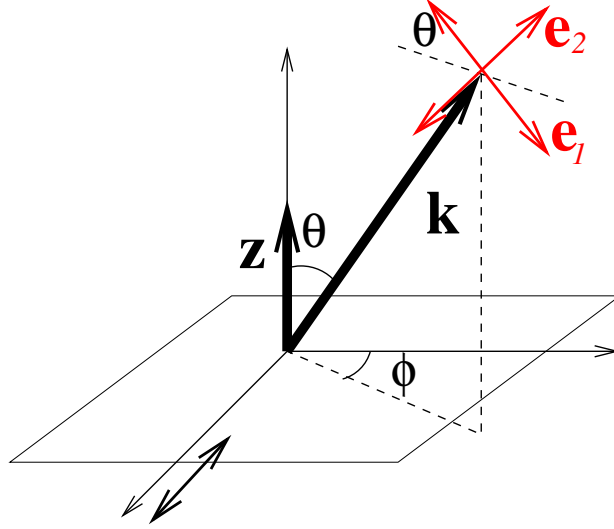


Figure 3.2: Directions of polarisation vectors in polar coordinates

From Fig. 3.2 it is clear that $\hat{\mathbf{z}} \cdot \hat{\mathbf{e}}_{\mathbf{k}2} = 0$ and one may see that $\hat{\mathbf{z}} \cdot \hat{\mathbf{e}}_{\mathbf{k}1} = \sin\theta$. For concreteness, the \mathbf{k} direction and polarisations can be parametrised by the orthogonal set:

$$\hat{\mathbf{k}} = \begin{pmatrix} \sin\theta \cos\phi \\ \sin\theta \sin\phi \\ \cos\theta \end{pmatrix}, \quad \hat{\mathbf{e}}_{\mathbf{k}1} = \begin{pmatrix} \cos\theta \cos\phi \\ \cos\theta \sin\phi \\ -\sin\theta \end{pmatrix}, \quad \hat{\mathbf{e}}_{\mathbf{k}2} = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}. \quad (3.32)$$

Putting all the terms together, one finds:

$$\Gamma(\epsilon) = \frac{\epsilon^2 |d_{ab}|^2}{(2\pi)^2 2\epsilon_0} \int d\phi \int d\theta \sin^3\theta \int \frac{ckcdk}{c^3} \delta(ck - \epsilon) = \frac{1}{4\pi\epsilon_0} \frac{4|d_{ab}|^2 \epsilon^3}{3c^3} \quad (3.33)$$

This formula therefore gives the decay rate of an atom in free space, associated with the strength of its dipole moment and its characteristic energy.

Further reading

The contents of most of this chapter is discussed in most quantum optics books. For example, it is discussed in Scully and Zubairy [10], or Yamamoto and Imamoglu [11].

Questions

Question 3.1: Collapse and revival

Consider the “linear” model:

$$H = \omega a^\dagger a + \epsilon c^\dagger c + g(a^\dagger c + c^\dagger a). \quad (3.34)$$

What is its spectrum? If one starts from a coherent state of photons, $\exp(\lambda a^\dagger)|0, 0\rangle$, what is the expectation of occupation of the “atomic” mode c after time t . Explain why there is no collapse.

Consider now the model [10, Sec. 6.1]:

$$H = \frac{1}{2} \begin{pmatrix} \epsilon & ga\sqrt{a^\dagger a} \\ g\sqrt{a^\dagger a}a^\dagger & -\epsilon \end{pmatrix} + \omega_0 a^\dagger a. \quad (3.35)$$

Once again, find the time evolution of the probability of being in an excited state, and thus find the timescales for oscillation, collapse and revival in this case. [Note that this problem is exactly solvable, no approximations should be required].

Question 3.2: Noisy classical driving

Let us consider the problem in Eq. (3.1) perturbatively; i.e. to leading order in $g\lambda$; for non-degenerate perturbation theory to be valid here, we must for the moment assume we are away from resonance. Show that in first order time-dependent perturbation theory, the probability of being in the excited state initially grows as:

$$P_{\text{ex}}(T) = T \int d\tau X(t + \tau)X^*(t), \quad X(t) = \frac{g\lambda}{2} e^{i(\omega_0 - \epsilon)t}. \quad (3.36)$$

[Hint: Find the standard time-dependent perturbation result for the amplitude in the excited state, then rewrite the amplitude squared to bring out a factor of time delay T .]

Now, consider the case where in addition to the desired driving there is some random field, $e^{i\omega_0 t} \rightarrow e^{i\omega_0 t + \eta(t)}$. If $\eta(t)$ is a Gaussian random variable with $\langle \eta(t) \rangle = 0$, and $\langle \eta(t)\eta(t') \rangle = \Gamma_2 \delta(t - t')$, show that the transition rate now becomes

$$W = \frac{P_{\text{ex}}(T)}{T} = \frac{\Gamma_2}{2} \frac{g^2 |\lambda|^2}{(\epsilon - \omega_0)^2 + \Gamma_2^2} \quad (3.37)$$

3.A Further properties of collapse and revival

When considering collapse and revivals of the Jaynes-Cummings oscillations, we discussed only the probability of finding the atom in an excited state. We can however understand more of where this collapse and revival originates from by considering the entanglement between the two-level system and the photon field. The entanglement is defined in terms of $S = -\text{Tr}(\rho_{TLS} \ln \rho_{TLS})$, where ρ_{TLS} is the reduced density matrix of the two level system:

$$\rho_{TLS} = \sum_n \langle n|\psi\rangle\langle\psi|n\rangle, \quad (3.38)$$

summing over e.g. number states of the photon field. Using our previous results for Rabi oscillations, the full state of the photon field can be written as:

$$|\psi(t)\rangle = \sum_n \alpha_n(t)|n-1, \uparrow\rangle + \sum_n \beta_n(t)|n, \downarrow\rangle. \quad (3.39)$$

with (restricted to the resonant case):

$$\begin{pmatrix} \alpha_n(t) \\ \beta_n(t) \end{pmatrix} = \frac{\lambda^n}{\sqrt{n!}} e^{-|\lambda|^2/2} \begin{pmatrix} -i \sin(g\sqrt{nt}/2) \\ \cos(g\sqrt{nt}/2) \end{pmatrix}. \quad (3.40)$$

In terms of these, the reduced density matrix will have the form:

$$\rho_{TLS} = \sum_n \begin{pmatrix} |\alpha_n|^2 & \alpha_{n+1}^* \beta_n \\ \beta_n^* \alpha_{n+1} & |\beta_n|^2 \end{pmatrix} = \begin{pmatrix} 1/2 + Z & X + iY \\ X - iY & 1/2 - Z \end{pmatrix}. \quad (3.41)$$

For a two dimensional density matrix, since the trace must be unity, by normalisation, the eigenvalues (and hence the entanglement entropy) are entirely determined by the determinant. In terms of the above parametrisation, the determinant is $1/4 - R^2$ where $R^2 = X^2 + Y^2 + Z^2$. If $|R| = 1/2$, the determinant is zero, and so the eigenvalues are 1, 0, i.e. this is a pure state, with no entanglement. If $|R| = 0$, then the determinant is $1/4$, and the eigenvalues are $1/2, 1/2$, with maximum entanglement. Thus, by plotting $1/4 - R^2$ we have a non-linear parametrisation of the entropy. The vector \mathbf{R} is the Bloch vector parametrisation of the density matrix, something we will make much use of in subsequent lectures.

From the form of the diagonal matrix elements, and the sum in Eq. (3.15) we have that $1/2 + Z = P_{\text{ex}}$ and so

$$Z = -\frac{1}{2} \sum_r \Re \left\{ \frac{e^{igt|\lambda|}}{\sqrt{1+igt/4|\lambda|}} \exp \left[-\frac{|\lambda|^2 (2\pi r + gt/2|\lambda|)^2}{2(1+igt/4|\lambda|)} \right] \right\}. \quad (3.42)$$

This is non-zero only near the revivals, so if we consider what happens away from the revivals (which is the majority of possible times), then we have $|R|^2 = X^2 + Y^2$, which can be written as:

$$\begin{aligned} |R|^2 &= \left| \sum_n |\lambda|^{2n} n! e^{-|\lambda|^2} \frac{\lambda}{2\sqrt{n+1}} \right. \\ &\quad \left. \times \left[\sin \left(\frac{gt}{2} (\sqrt{n} + \sqrt{n+1}) \right) + \sin \left(\frac{gt}{2} (\sqrt{n} - \sqrt{n+1}) \right) \right] \right|^2 \end{aligned} \quad (3.43)$$

In this expression, the sum and difference sine terms can be treated separately. The first term, depending approximately on $\sin(gt\sqrt{n})$ will behave very similarly to the excitation probability, with the only difference being this is the imaginary part, rather than the real part, of the complex expression. Thus, once again, away from revivals, this term is small. The last term is however different. Expanding for large n , one has:

$$|R| \simeq \left| \sum_n |\lambda|^{2n} n! e^{-|\lambda|^2} \frac{\lambda}{2\sqrt{n+1}} \sin\left(\frac{gt}{4\sqrt{n}}\right) \right|$$

$$\simeq \frac{1}{2} \left| \Im \left\{ \exp\left(i\frac{gt}{4|\lambda|} - \frac{g^2 t^2}{8|\lambda|^4}\right) \right\} \right|, \quad (3.44)$$

where the last line replace the sum by an integral (not worrying about revivals of this discrete sum). This looks superficially similar to the excitation probability, but the timescales are different — the oscillation timescale here is $t \sim 4|\lambda|/g$; this is the same timescale as the revivals of the excitation probability. Such a result makes sense — collapse of the excitation probability occurred because of dephasing, i.e. decoherence, arising from entanglement between the two level system and the photon field. The revival timescale is thus connected to the period of the entanglement oscillations.

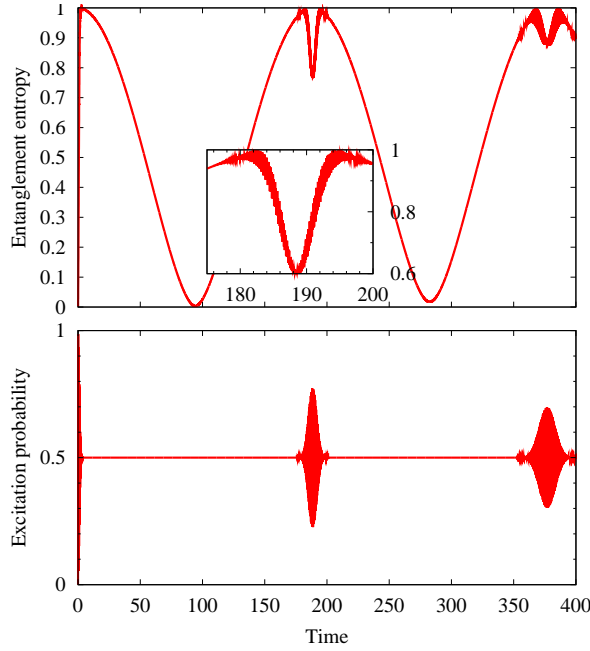


Figure 3.3: Oscillations of the entanglement (top panel) and its connection to collapse and revivals of the excitation probability (bottom panel) in the case of $\lambda = 30$.

Lecture 4

Two level systems coupled to many modes: Density matrix description

The aim of this lecture is to repeat the physics discussed at the end of the last lecture — i.e. the relaxation of a two-level system coupled to a continuum of radiation modes — but to present it in the context of the density matrix of the two-level system. This framework will then allow us to consider other relaxation and decoherence processes, such as those arising due to noise sources, or inhomogeneous broadening in an ensemble of atoms. We will then illustrate the use of this approach by discussing the behaviour of a two-level system illuminated by a coherent state of light, but able to radiate into a continuum of modes. In this lecture we will study the steady state of this system, and in subsequent lectures we will look at the spectrum of emitted light, which will require knowing more about the full time-evolution of the system. The formalism we develop here will be useful also when we later consider the laser, and its more quantum mechanical variants of single atom lasers and micromasers.

4.1 Density matrix equation for relaxation of two-level system

We begin by considering again the problem of a two-level system coupled to a continuum of radiation modes, which in the previous lecture we solved via the Wigner-Weisskopf approach, of eliminating the Schrödinger equations for the state of the radiation field. Let us now apply the same idea in a more formal scheme, by eliminating the the behaviour of the continuum modes in order to write a closed equation for the density matrix of the system. If we define the system density matrix as the result of tracing over all “reservoir” degrees of freedom, we can write an equation of motion for this quantity as:

$$\frac{d}{dt}\rho_S = -i\text{Tr}_R\{[H_{SR}, \rho_{SR}]\}. \quad (4.1)$$

in which S means state of the system, R the reservoir (in this case, the continuum of photon modes), and Tr_R traces over the state of the reservoir. After removing the free time evolution of the reservoir and system fields, the remaining system-reservoir Hamiltonian can be written in the interaction picture as:

$$H_{SR} = \sum_k \frac{g_k}{2} \left(\sigma^- a_k^\dagger e^{i(\omega_k - \epsilon)t} + \sigma^+ a_k e^{i(\epsilon - \omega_k)t} \right). \quad (4.2)$$

In general Eq. (4.1) will be complicated to solve, as interactions between the system and reservoir lead to entanglement, and so the full density matrix develops correlations. This means that the time evolution of the system would depend on such correlations, and thus on the history of system reservoir interactions. However, the same assumptions as in the previous lecture will allow us to make a Markov approximation, meaning that the evolution of the system depends only on its current state — i.e. neglecting memory effects of the interaction.

For a general interaction, there are a number of approximations which are required in order to reduce the above equation to a form that is straightforwardly soluble.

- We assume the interaction is weak, so $\rho_{SR} = \rho_S \otimes \rho_R + \delta\rho_{SR}$, where $\delta\rho_{SR} \simeq \mathcal{O}(g_k)$. Since $\rho_S = \text{Tr}_R(\rho_{SR})$, it is clear that $\text{Tr}_R(\delta\rho_{SR}) = 0$, however the result of $\text{Tr}_R(H_{SR}\delta\rho_{SR})$ may be non-zero. Such correlations will be assumed to be small, but must be non-zero in order for there to be any influence of the bath on the system. In order to take account of these small correlation terms, generated by the coupling, we use the formal solution of the density matrix equation $\rho_{SR}(t) = -i \int^t dt' [H_{SR}(t'), \rho_{SR}(t')]$ to write:

$$\frac{d}{dt}\rho_S(t) = -\text{Tr}_R \left\{ \int_0^t dt' [H_{SR}(t), [H_{SR}(t'), \rho_{SR}(t')]] \right\}. \quad (4.3)$$

We may now make the assumption of small correlations by assuming that $\rho_{SR}(t') = \rho_S(t') \otimes \rho_R(t')$, and that only the correlations generated at linear order need be considered.

- We further assume the reservoir to be large compared to the system, and so unaffected by the evolution, thus $\rho_R(t') = \rho_R(0)$. Along with the previous step, this is the Born approximation.
- Finally, if the spectrum of the bath is dense (i.e. the spacing of energy levels is small), then the trace over bath modes of the factors $e^{i\omega_k t}$ will lead to delta functions in times. This in effect means that we may replace $\rho_S(t') = \rho_S(t)$. This is the Markov approximation.

One thus has the final equation:

$$\frac{d}{dt}\rho_S(t) = -\text{Tr}_R \left\{ \int_0^t dt' [H_{SR}(t), [H_{SR}(t'), \rho_S(t) \otimes \rho_R(0)]] \right\}. \quad (4.4)$$

To evaluate the trace over the reservoir states, we need the trace of the various possible combinations of reservoir operators coming from H_{SR} . These will involve pairs of operators a_k, a_k^\dagger . Using cyclic permutations, these trace terms can be written as thermal average $\langle X \rangle = \text{Tr}_R(\rho_R X)$. For bulk photon modes, the relevant expectations are:

$$\langle a_k a_{k'} \rangle = \langle a_k^\dagger a_{k'}^\dagger \rangle = 0, \quad \langle a_k^\dagger a_{k'} \rangle = \delta_{kk'} n_k, \quad \langle a_k a_{k'}^\dagger \rangle = \delta_{kk'} (n_k + 1). \quad (4.5)$$

In these expressions we have assumed $n_k \neq 0$, i.e. we have allowed for a thermal occupation of the photon modes. The results of the previous lecture assumed an initial vacuum state, and thus $n_k = 0$.

With these expectations, we can then write the equation of motion for the density matrix explicitly, writing $\xi_k = \exp[i(\omega_k - \epsilon)(t - t')]$, to give:

$$\begin{aligned} \frac{d}{dt} \rho_S(t) = & - \sum_k \left| \frac{g_k}{2} \right|^2 \int_0^t dt' [(\sigma^- \sigma^+ n_k \xi_k + \sigma^+ \sigma^- (n_k + 1) \xi_k^*) \rho_S \\ & - (\sigma^- \rho \sigma^+ (n_k + 1) + \sigma^+ \rho \sigma^- n_k) (\xi_k + \xi_k^*) \\ & + \rho_S (\sigma^- \sigma^+ n_k \xi_k^* + \sigma^+ \sigma^- (n_k + 1) \xi_k)]. \end{aligned} \quad (4.6)$$

In this expression, the first term in parentheses comes from the expression $\text{Tr}_R [H_{SR}(t) H_{SR}(t') \rho_{SR}]$, the second comes from the conjugate pair $\text{Tr}_R [H_{SR}(t) \rho_{SR} H_{SR}(t')]$ and $t \leftrightarrow t'$ [hence $(\xi_k + \xi_k^*)$], and the final term comes from $\text{Tr}_R [\rho_{SR} H_{SR}(t') H_{SR}(t)]$.

At this point, we can now simplify the above expression by using the assumed flat density of states of the photon modes, so that just as in the Wigner-Weisskopf approach, we may write:

$$\sum_k \left| \frac{g_k}{2} \right|^2 \xi_k = \Gamma \delta(t - t'), \quad \sum_k \left| \frac{g_k}{2} \right|^2 n_k \xi_k = \Gamma \bar{n} \delta(t - t'). \quad (4.7)$$

Here we have assumed that not only is the density of states flat, but that also n_k is sufficiently flat, that we might write an averaged occupation \bar{n} . Such an approximation is only valid at high enough temperatures — we will discuss this in some detail in a few lectures time.

$$\begin{aligned} \frac{d}{dt} \rho_S(t) = & - \frac{\Gamma}{2} \bar{n} (\sigma^- \sigma^+ \rho_S - 2\sigma^+ \rho_S \sigma^- + \rho_S \sigma^- \sigma^+) \\ & - \frac{\Gamma}{2} (\bar{n} + 1) (\sigma^+ \sigma^- \rho_S - 2\sigma^- \rho_S \sigma^+ + \rho_S \sigma^+ \sigma^-) \end{aligned} \quad (4.8)$$

The factor of $1/2$ here comes from the regularised half integral of the delta function. The two lines in this expression correspond to stimulated absorption (which exists only if $\bar{n} > 0$), and leads to excitation of the two-level system, and to emission (both stimulated and spontaneous). If $\bar{n} = 0$, only emission survives, and the results of the previous lecture are recovered for the relaxation to equilibrium.

One can see the behaviour from this equation most clearly by writing the equations for the diagonal components:

$$\partial_t \rho_{\uparrow\uparrow} = -\frac{\Gamma}{2} \bar{n} (-2\rho_{\downarrow\downarrow}) - \frac{\Gamma}{2} (\bar{n} + 1) (2\rho_{\uparrow\uparrow}) = -\partial_t \rho_{\downarrow\downarrow} \quad (4.9)$$

Then, using $\rho_{\downarrow\downarrow} = 1 - \rho_{\uparrow\uparrow}$, one may find the steady state probability of excitation is given by:

$$0 = \Gamma [\bar{n}(1 - \rho_{\uparrow\uparrow}) - (\bar{n} + 1)\rho_{\uparrow\uparrow}] \quad \rightarrow \quad \rho_{\uparrow\uparrow} = \frac{\bar{n}}{2\bar{n} + 1}. \quad (4.10)$$

This formula is as expected, such that if one uses the Bose-Einstein occupation function for the photons, $\bar{n} = [\exp(\beta\epsilon) - 1]^{-1}$, the excitation probability of the two-level system is the thermal equilibrium expression.

Equations of motion with coherent driving

As a slightly less trivial example of the effect of the bath, let us now turn to the combination of coherent driving and decay. We will assume from hereon that the reservoir photon modes are empty¹. The evolution of the density matrix is then controlled by:

$$\partial_t \rho = -i[H, \rho] - \frac{\Gamma}{2} (\sigma^- \sigma^+ \rho_S - 2\sigma^+ \rho_S \sigma^- + \rho_S \sigma^- \sigma^+) \quad (4.11)$$

where H describes the coherent, Hamiltonian dynamics, which we will take to be given by:

$$H = \epsilon \sigma^z + \frac{g\alpha}{2} e^{-i\omega t} \sigma^+ + \text{H.c.} \quad (4.12)$$

One may then write out the equations of motion for each component explicitly, giving

$$\partial_t \rho_{\uparrow\uparrow} = -i \left[\frac{\epsilon}{2} \rho_{\uparrow\uparrow} + \frac{g\alpha}{2} e^{-i\omega t} \rho_{\downarrow\uparrow} - \rho_{\uparrow\downarrow} \frac{g\alpha}{2} e^{i\omega t} - \rho_{\uparrow\uparrow} \frac{\epsilon}{2} \right] - \Gamma \rho_{\uparrow\uparrow}, \quad (4.13)$$

$$\partial_t \rho_{\uparrow\downarrow} = -i \left[\frac{\epsilon}{2} \rho_{\uparrow\downarrow} + \frac{g\alpha}{2} e^{-i\omega t} \rho_{\downarrow\downarrow} - \rho_{\uparrow\uparrow} \frac{g\alpha}{2} e^{-i\omega t} - \rho_{\uparrow\downarrow} \left(-\frac{\epsilon}{2} \right) \right] - \frac{\Gamma}{2} \rho_{\uparrow\downarrow}. \quad (4.14)$$

These two equations are sufficient, since the unit trace property of a density matrix implies $\rho_{\downarrow\downarrow} = 1 - \rho_{\uparrow\uparrow}$, and the off diagonal elements are complex conjugate of one another by the Hermitian structure of the density matrix. To solve these equations, it is convenient to first go to a rotating frame, so that one writes $\rho_{\uparrow\downarrow} = P e^{-i\omega t}$, and secondly to introduce the inversion, $Z = (\rho_{\uparrow\uparrow} - \rho_{\downarrow\downarrow})/2$. In terms of these variables, one has:

$$\partial_t Z = -i \frac{g\alpha}{2} (P^* - P) - \Gamma \left(Z + \frac{1}{2} \right) \quad (4.15)$$

$$\partial_t P = i(\omega - \epsilon)P + i \frac{g\alpha}{2} 2Z - \frac{\Gamma}{2} P. \quad (4.16)$$

To further simplify, we may separate the real and imaginary parts of $P = X + iY$, and write $\Delta = \omega - \epsilon$ for the detuning of the optical pump, so that one has the three coupled equations:

¹This limit is frequently relevant in cavity quantum Electrodynamics experiments, since the confined photon modes start at energies of the order of $1\text{eV} \sim 10^4\text{K}$, so the thermal population of such photon modes is negligible.

$$\partial_t X = -\Delta Y - \frac{\Gamma}{2} X \quad (4.17)$$

$$\partial_t Y = +\Delta X + g\alpha Z - \frac{\Gamma}{2} Y \quad (4.18)$$

$$\partial_t Z = -g\alpha Y - \Gamma \left(Z + \frac{1}{2} \right). \quad (4.19)$$

These are Bloch equations for the Bloch vector parametrisation of the density matrix (in the rotating frame):

$$\rho = \begin{pmatrix} \frac{1}{2} + Z & X - iY \\ X + iY & \frac{1}{2} - Z \end{pmatrix}. \quad (4.20)$$

The three terms, $\Delta, g\alpha, \Gamma$ correspond to: rotation about the Z axis (i.e. phase evolution); rotation about the X axis (i.e. excitation — nb the observation that the rotation rate is $g\alpha$, so a duration $g\alpha t = \pi$ would lead to the inverted state); and Γ causes the length of the Bloch vector to shrink.

As noted in the appendix to lecture 3, for a two level system, the eigenvalues are directly given by the determinant of the density matrix. Hence, the entanglement entropy, describing the decoherence of the system due to its entanglement with the reservoir, depends monotonically on the length of the Bloch vector, $X^2 + Y^2 + Z^2$.

Before solving Eq. (4.17–4.19), we will first consider other kinds of dephasing that may affect the density matrix evolution, leading to a generalised version of these equations, allowing for both relaxation and pure dephasing.

4.2 Dephasing in addition to relaxation

In addition to relaxation, where excitations are transferred from the system to the modes of the environment, pure dephasing is also possible, in which the populations are unaffected but their coherence is reduced. In considering this kind of dephasing, it is useful to also broaden our view from considering a single two level system, to describing measurements on an ensemble of many two level systems. For the moment we will assume that even if there are many two level systems, they all act independently. As such, the expectation of any measurement performed on this ensemble can be given by:

$$\langle X \rangle = \sum_i Tr(\rho_i X) = Tr(\bar{\rho} X), \quad \text{where } \bar{\rho} = \sum_i \rho_i. \quad (4.21)$$

We thus have two types of decoherence one may consider:

- Broadening within a two-level system, coming from time dependent shifts of energies (and possibly also coupling strengths). A variety of sources of such terms exist; examples include collisional broadening of gaseous atoms, where the rare events in which atoms approach

each other lead to shifts of atomic energies. In solid state contexts, motions of charges nearby the artificial atom can also lead to time dependent noise.

- Inhomogeneous broadening, where the parameters of each two-level system vary, and so off-diagonal matrix elements, involving time dependent coherences, are washed out in $\bar{\rho}$. This is particularly an issue in solid state experiments, where the two-level systems may involve the dimensions of a fabricated or self-assembled artificial atom, and these dimensions may vary between different systems.

To describe both such effects together, we consider adding noise,

$$\epsilon \rightarrow \epsilon + \eta(t). \quad (4.22)$$

This noise will then lead to a decay of the off diagonal correlations, via:

$$P(t) \rightarrow P(t) \left\langle \exp \left(i \int_0^t \eta(t') dt' \right) \right\rangle \quad (4.23)$$

We will take $\langle \eta(t) \rangle = 0$, and two distinct limits for the correlations of η .

Fast noise limit

In order to extract a tractable model of time-dependent noise, the simplest limit to consider is that $\eta(t)$ has a white noise spectrum, i.e. it is Gaussian correlated with:

$$\langle \eta(t)\eta(t') \rangle = 2\gamma_0 \delta(t - t'). \quad (4.24)$$

This assumption is equivalent to assuming that the noise is “fast”, i.e. that any correlation timescale it does possess is sufficiently short compared to the dynamics of the system that it may be neglected.

With this assumption of instantaneous Gaussian correlations, one may then expand the exponential in Eq. (4.23) and write:

$$\begin{aligned} P(t) &\rightarrow P(t) \left\langle 1 + i \int_0^t dt' \eta(t') - \frac{1}{2!} \iint_0^t dt' dt'' \eta(t') \eta(t'') + \dots \right\rangle \\ &= P(t) \left[1 - \frac{2\gamma_0}{2!} \int_0^t dt' + \frac{(2\gamma_0)^2}{4!} 3 \left(\int_0^t dt' \right)^2 + \dots \right] = P(t) e^{-\gamma_0 t} \end{aligned} \quad (4.25)$$

with the combinatoric factors coming from the number of possible pairings in the Wick decomposition of $\langle \eta^n \rangle$.

This noise thus leads to an enhanced decoherence rate for the off diagonal terms, so that $\Gamma/2$ should be replaced by $1/T_2 = (\Gamma/2) + \gamma_0$ in the density matrix equations. One may equivalently define the relaxation time $1/T_1 = \Gamma$. In general $T_2 < T_1$, as dephasing is faster than relaxation, however the factors of two in the above mean that the strict inequality is $T_2 < 2T_1$, with equality holding when there is only relaxation.

The Bloch equations are thus given by:

$$\partial_t X = -\Delta Y - \frac{1}{T_2} X \quad (4.26)$$

$$\partial_t Y = +\Delta X + g\alpha Z - \frac{1}{T_2} Y \quad (4.27)$$

$$\partial_t Z = -g\alpha Y - \frac{1}{T_1} \left(Z + \frac{1}{2} \right). \quad (4.28)$$

Static limit (inhomogeneous broadening)

To describe inhomogeneous broadening, η should represent the system-dependent variation of energies, and should have no time dependence. The same behaviour is also relevant if time-dependent noise terms vary slowly on the timescale of an experiment. (For example, slow noise may arise from a nearby impurity have multiple stable charging states, so that the charge environment can take multiple different values; if the charging and discharging of this impurity is slow compared to the experiment, this energy shift will effectively give inhomogeneous broadening when averaged over multiple experimental shots.)

For this case η is static, but randomly distributed, and Eq. (4.23) becomes:

$$P(t) \rightarrow P(t) \int d\eta p(\eta) e^{i\eta t}. \quad (4.29)$$

If we consider a Lorentzian probability distribution for η with width γ_1 , i.e. $p(\eta) = 2\gamma_1/(\gamma_1^2 + \eta^2)$, then one finds $P(t) \rightarrow P(t) \exp(-\gamma_1|t|)$.

If one has both noise terms as in the previous section, and inhomogeneous broadening, one may then distinguish T_2 as defined above, and T_2^* , where $1/T_2 = (\Gamma/2) + \gamma_0 + \gamma_1$.

Distinguishing T_2 and T_2^*

Since both γ_0 and γ_1 lead to decay of the coherence functions, a simple experiment measuring coherence will see the lifetime T_2^* . However, the dynamics of each individual two level system remains coherent for the longer timescale T_2 . Such extra coherence can easily be seen in any sufficiently non-linear measurement. An example of how the difference of T_2 and T_2^* can be measured is given by photon echo, illustrated in Fig. 4.1.

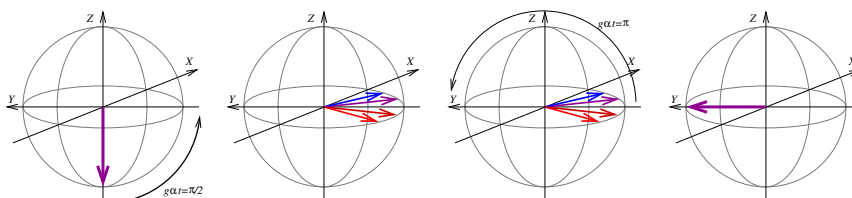


Figure 4.1: Cartoon of photon-echo experiment, which removes dephasing due to inhomogeneous broadening, leaving only true dephasing rate to reduce intensity of revival.

From the Bloch equations, Eq. (4.26–4.28), it is clear that a resonant pulse with duration $g\alpha t = \pi/2$ brings the Bloch vector to the equator. From there, inhomogeneous broadening means that (in a rotating frame at the mean frequency) the Bloch vectors for each individual two-level system start to spread out. However, by applying a second pulse with $g\alpha t = \pi$, a π rotation about the X axis means that whichever Bloch vector spread at the fastest rate is now furthest from the Y axis, and the subsequent evolution sees the vectors re-converge. At the final time, the coherence of the resulting pulse will have been reduced by γ_0 and Γ , but γ_1 will have had no effect.

4.3 Power broadening of absorption

Having developed this formalism, we may now use this to re-examine the question we started the previous lecture with — what is the effect of shining classical light on a two-level atom, but now accounting for decay. Whereas in the previous lecture, the absence of decay meant that the result was Rabi oscillations, with decay a steady state is eventually found. Looking for steady state solutions of Eq. (4.26–4.28) one finds from the equations for X and Z that:

$$Y = \frac{-1}{g\alpha T_1} \left(Z + \frac{1}{2} \right), \quad X = -\Delta T_2 Y = \frac{\Delta T_2}{g\alpha T_1} \left(Z + \frac{1}{2} \right). \quad (4.30)$$

Substituting these into the equation for Y , and recalling $P_{\text{ex}} = Z + 1/2$, gives:

$$0 = P_{\text{ex}} \left[\Delta \frac{\Delta T_2}{g\alpha T_1} + \frac{1}{T_2} \frac{1}{g\alpha T_1} + g\alpha \right] - \frac{g\alpha}{2} \quad (4.31)$$

After re-arranging, one then finds:

$$P_{\text{ex}} = \frac{1}{2} \frac{T_1 T_2 (g\alpha)^2}{(T_2 \Delta)^2 + 1 + T_1 T_2 (g\alpha)^2} = \frac{1}{2} \frac{(T_1/T_2)(g\alpha)^2}{\Delta^2 + [1 + T_1 T_2 (g\alpha)^2]/T_2^2} \quad (4.32)$$

The addition of damping to the equations of motion has thus had a number of important consequences:

- It is responsible for a steady state existing at all (but note that this steady state is in a frame rotating at the pump frequency ω , so the physical coherence is time dependent.)
- It gives a finite width to the resonance, even at weak pump powers — in the Hamiltonian case, as $g\alpha \rightarrow 0$, the width of the resonance peak vanished.
- It modifies the overall amplitude at resonance, by a factor depending on the power, such that at $g\alpha \rightarrow 0$, there is no response.

One feature of this absorption probability formula that can be remarked on is that the linewidth depends on the intensity of radiation. This is a consequence of the nonlinearity implicit in a two-level system. Such

dependence is not so surprising given that in the Hamiltonian dynamics, the resonance width already depended on the field strength. One may note that if one considered an harmonic atomic spectrum, rather than a two-level atom, no such broadening would be seen (see question 4.2).

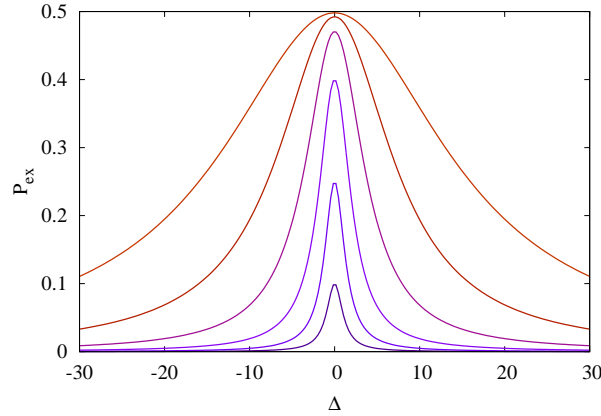


Figure 4.2: Power broadening of absorption — probability of excitation vs atom-photon detuning for increasing field strength.

4.4 Further reading

Once again, the contents of this chapter can be found in most standard quantum optics books, e.g.[10, 12]. The discussion of the resonance fluorescence problem in particular can be found in chapter 4 of Meystre and Sargent III [12].

Questions

Question 4.1: Fast noise via density matrix equations

As another way to consider fast noise terms, consider interaction between the two-level system and a bath of radiation modes given by the interaction picture coupling:

$$H_{SR} = \sum_n \sigma^z \zeta_n \left(B_n e^{-i\omega_n t} + B_n^\dagger e^{i\omega_n t} \right) \quad (4.33)$$

where B_n are the quantised modes of a field whose field strength shifts the energies of the two-level system.

4.1.(a) By making the Born-Markov approximation, show that this leads to a density matrix equation of the form:

$$\partial_t \rho|_{\text{noise}} = -\gamma_0 \left(\frac{\rho}{2} - 2\sigma^z \rho \sigma^z \right), \quad (4.34)$$

and determine how γ_0 depends on the temperature of the reservoir if the reservoir follows a thermal Bose-Einstein distribution.

4.1.(b) Show that this describes pure dephasing, i.e. it has no effect on the diagonal elements of the density matrix, and leads to a decay of the off-diagonal terms $\partial_t \rho_{\uparrow\downarrow} = -\gamma_0 \rho_{\uparrow\downarrow}$.

Question 4.2: Absence of power broadening for an harmonic oscillator

Consider an harmonic oscillator mode $H_{\text{sys}} = \epsilon b^\dagger b$, coupled to a continuum of radiation modes via the interaction picture Hamiltonian:

$$H_{SR} = \sum_k \frac{g_k}{2} \left(b a_k^\dagger e^{i(\omega_k - \epsilon)t} + b^\dagger a_k e^{i(\epsilon - \omega_k)t} \right). \quad (4.35)$$

4.2.(a) Show that (for an empty bath) this leads to the density matrix equation of motion:

$$\partial_t \rho = -\frac{\Gamma}{2} \left(b^\dagger b \rho - 2b \rho b^\dagger + \rho b^\dagger b \right) \quad (4.36)$$

4.2.(b) Now consider adding the Hamiltonian dynamics describing pumping by a coherent field, with $H = (g\alpha/2)(b^\dagger e^{-i\omega t} + \text{H.c.}) + \epsilon b^\dagger b$. Determine the density matrix equation of motion in the Fock basis (i.e. number state basis).

4.2.(c) Show that the ansatz:

$$\rho_{nm} \propto \frac{1}{\sqrt{n!m!}} \left(\frac{-ig\alpha/2}{-i\Delta + \Gamma/2} \right)^n \left(\frac{ig\alpha/2}{i\Delta + \Gamma/2} \right)^m \quad (4.37)$$

satisfies this equation of motion.

4.2.(d) From the normalised density matrix, find the average excitation probability, and show that there is no power broadening.

Lecture 5

Resonance Fluorescence

At the end of the last lecture, we found the steady state excitation probability for a two-level system pumped by a coherent light field. This steady state emerged due to the competition between coherent pumping, and incoherent decay into the continuum of radiation modes. The aim of this lecture is to discuss the spectrum of this fluorescence, when the atom is driven near resonance. This is intended both as an illustration of applying the density matrix equation of motion approach to more complex problems, and also to reveal further aspects of the behaviour of a two-level system.

5.1 Spectrum of emission into a reservoir

Our aim is to calculate the spectrum of the emission into the continuum of photon modes. Formally, the spectrum of radiation is given (via the Wiener-Khintchine theorem, applicable to a stationary process, which ours is) by:

$$I(\nu) = \int_{-\infty}^{\infty} e^{i\nu t} \langle E^+(t) E^-(0) \rangle = 2\Re \left[\int_0^{\infty} e^{i\nu t} \langle E^+(t) E^-(0) \rangle \right] \quad (5.1)$$

where we have divided:

$$E(t) = E^+(t) + E^-(t), \quad E^-(t) = \sum_k \mathcal{E}_k a_{\mathbf{k},n}(t) \mathbf{e}_{n,\mathbf{k}}, \quad (5.2)$$

with $\mathcal{E}_k = \sqrt{\omega_k/2\epsilon_0 V}$. The first challenge is to write the time dependent reservoir operators in terms of system operators. If we work in the Heisenberg picture, then this is straightforward, the interaction Hamiltonian was:

$$H_{SR} = \sum_k \frac{g_k}{2} \left(\sigma^- a_k^\dagger e^{i(\omega_k - \epsilon)t} + \sigma^+ a_k e^{i(\epsilon - \omega_k)t} \right). \quad (5.3)$$

and so the solution to the Heisenberg equation for $a_k(t)$ can be written as:

$$a_k(t) = \frac{g_k}{2} \int_{-\infty}^t dt' e^{i(\omega_k - \epsilon)(t-t')} \sigma^-(t') + a_k(-\infty). \quad (5.4)$$

We will neglect second term $a_k(-\infty)$, since we assume the state at $t = -\infty$ to be a vacuum, i.e. the only photons are those coming from the atom. In this case we may write:

$$E^-(t) = \int^t dt' \sum_k \frac{\mathcal{E}_k g_k}{2} e^{i(\omega_k - \epsilon)(t' - t)} \sigma^-(t') \mathbf{e}_{n, \mathbf{k}}, \quad (5.5)$$

Because (at least near resonance) $\mathcal{E}_k \propto g_k$ the sum in the above expression is Markovian as in the previous lecture, and so we may write $E^-(t) \propto \sigma^-(t)$, hence up to an overall constant, we have:

$$I(\nu) \propto 2\Re \left[\int_0^\infty e^{i\nu t} \langle \sigma^+(t) \sigma^-(0) \rangle \right]. \quad (5.6)$$

This reduces the problem to that of finding a two-time correlation function of the open quantum system. This is however something we have not yet calculated — the density matrix completely describes single time correlation functions, but in general more knowledge is required to find the correlation of operators at two different times. However, for Markovian baths it turns out that it is possible to relate this two-time correlation to the density matrix evolution via the quantum regression theorem.

5.2 Quantum regression “theorem”

The quantum regression theorem can be stated in terms of the time evolution of single-time correlation functions, governed by:

$$\langle B(t + \tau) \rangle = \sum_i \alpha_i(\tau) \langle B_i(t) \rangle, \quad (5.7)$$

where $B_i(t)$ is some complete set of operators, and the functions $\alpha_i(\tau)$ solve the averaged equations of motion for $\langle B_i(t) \rangle$. Then, the theorem states that one can write:

$$\langle A(t) B(t + \tau) C(t) \rangle = \sum_i \alpha_i(\tau) \langle A(t) B_i(t) C(t) \rangle, \quad (5.8)$$

Note that ordering is important, as A, B_i, C are non-commuting operators.

In order to prove the theorem, it is first necessary to state more explore further what is implied by the Markovian approximation.

Significance of the Markovian approximation

Formally stated, a Markovian system is one where the future evolution is governed by the current state, and not by any history of the system. In the current context, current state means the state of the system, not including the baths. Thus, the assumption is that one can write an equation of motion for the system density matrix that depends only on the current value of the system density matrix. Therefore, the baths must have no memory of previous states of the system.

Generically, without the Markovian approximation, the equation of motion of the system density matrix could be written as

$$\frac{d\rho_S(t)}{dt} = -i[H_S, \rho_S(t)] + \int_{-\infty}^t dt' \Gamma(t-t') \rho_S(t'). \quad (5.9)$$

The term $\Gamma(t-t')$ describes how the state of the bath influences the system at time t , and depends on the state of the system at the earlier time t' . The Markovian approximation is that $\Gamma(t-t') = \Gamma\delta(t-t')$. If one describes the bath by a sum over many independent modes, these delta correlated response functions imply a dense spectrum of bath modes; this is what was used in Eq. (4.7).

There is another consequence of the Markovian limit, which was also used earlier in Eq. (4.4). The full evolution really depends on ρ_{SR} ; validity of the Markovian approximation then requires that the state of the system is sufficiently described by $\rho_S = \text{Tr}_R \rho_{SR}$. This means that correlations between the system and the reservoir must be unimportant, and so it is sufficient to write $\rho_{SR}(t) = \rho_S(t) \otimes \rho_R(t)$. It is this statement, as we see next, that implies the quantum regression theorem.

Under what physical conditions is the Markovian approximation a good description of a real system? Firstly, to have truly delta correlated noise requires a flat spectrum of the baths, as we have already assumed. However, a flat spectrum is only required for the bath modes that couple almost resonantly to the system (i.e. $|\omega_j - \omega| \leq \Gamma$). If one considers a reservoir which is not in its vacuum state — i.e. with some thermal occupation — then a Markovian description of the system requires that not only is the density of states flat, but that we may also approximate:

$$\sum_k n_k \left| \frac{g_k}{2} \right|^2 e^{i(\omega_k - \omega)t} = \bar{n} \Gamma \delta(t).$$

i.e. we require that n_k varies slowly for over the range of k for which $|\omega_j - \omega| \leq \Gamma$; e.g. for a thermal distribution, we require $T \gg \Gamma$. This is clearly always an approximation; the consequences of this approximation will be investigated further in the next lecture.

Proof of regression theorem

The Markovian approximation implies that the evolution of single-time expectations can be written as follows, in terms of the unitary evolution $U(\tau)$ of system and reservoir:

$$\begin{aligned} \langle B(t+\tau) \rangle &= \text{Tr}_S \text{Tr}_R \left(U^\dagger(\tau) B(t) U(\tau) \rho_S(t) \otimes \rho_R(t) \right) \\ &= \text{Tr}_S \left[B(t) \text{Tr}_R \left(U(\tau) \rho_S(t) \otimes \rho_R(t) U^\dagger(\tau) \right) \right] \\ &= \text{Tr}_S \left[B(t) \text{Tr}_R \left(\rho_S(t+\tau) \otimes \rho_R(t+\tau) \right) \right]. \end{aligned} \quad (5.10)$$

Here we made use of the cyclic property of the trace, and the relation between evolution of density matrices and that of Heisenberg operators. In

an analogous way, the two-time expectation evolves according to:

$$\begin{aligned}
\langle A(t)B(t+\tau)C(t) \rangle &= \text{Tr}_S \text{Tr}_R \left(A(t)U^\dagger(\tau)B(t)U(\tau)C(t)\rho_S(t) \otimes \rho_R(t) \right) \\
&= \text{Tr}_S \left[B(t)\text{Tr}_R \left(U(\tau) [C(t)\rho_S(t)A(t)] \otimes \rho_R(t)U^\dagger(\tau) \right) \right] \\
&= \text{Tr}_S \left[B(t)\text{Tr}_R \left([C\rho_S A](t+\tau) \otimes \rho_R(t+\tau) \right) \right].
\end{aligned} \tag{5.11}$$

Thus, the only difference between the right hand sides of Eq. (5.10) and Eq. (5.11) is to replace the initial system density matrix ρ_S with the product $C\rho_S A$. However, in the Markovian approximation, we know that Eq. (5.10) can be solved using Eq. (5.7), for any initial density matrix. Thus, one can use this solution to give

$$\begin{aligned}
\langle A(t)B(t+\tau)C(t) \rangle &= \sum_i \alpha_i(\tau) \langle B_i(t) \rangle \Big|_{\rho_S(t) \rightarrow C(t)\rho_S(t)A(t)} \\
&= \sum_i \alpha_i(\tau) \text{Tr}_S [B_i(t)C(t)\rho_S(t)A(t)] \\
&= \sum_i \alpha_i(\tau) \text{Tr}_S [A(t)B_i(t)C(t)\rho_S(t)]
\end{aligned} \tag{5.12}$$

which is the desired result in Eq. (5.12).

Thus, to find a two-time correlation in practise, the scheme is this: find the general time-dependent solution of one of the operators, and how it depends on initial expectation values — find the explicit form of Eq. (5.7), and then insert the second operator(s) in these initial expectations.

5.3 Resonance fluorescence spectrum

From the quantum regression theory, we now have that:

$$I(\nu) \propto 2\Re \left[\int_0^\infty e^{i\nu t} \text{Tr} [\sigma^+ \rho(t)] \Big|_{\rho(t=0)=\sigma^- \rho_0} \right], \tag{5.13}$$

i.e. we should solve the density matrix equation of motion to find $\rho(t)$ given the initial condition $\sigma^- \rho_0$, where ρ_0 is the equilibrium density matrix.

Since the equation of motion for the density matrix is a first order differential equation, $\partial_t \rho = M[\rho]$, where M is a linear super-operator, we can easily solve by Laplace transforming to give:

$$M[\rho(s)] - s\rho(s) = -\sigma^- \rho_0, \tag{5.14}$$

where the right hand side encodes the initial conditions. From the Laplace transform $\rho(s)$, the Fourier transform corresponding to the intensity spectrum is then given by:

$$I(\nu) \propto 2\Re \left\{ \text{Tr} [\sigma^+ \rho(s = -i\nu + 0)] \right\}. \tag{5.15}$$

Let us now specify the various parts of this for the case of resonance fluorescence, i.e. $\Delta = \omega - \epsilon = 0$. We will work in the rotating frame as in the previous lecture. The steady state condition then becomes:

$$\rho_0 = \frac{1}{2[1 + T_1 T_2 (g\alpha)^2]} \begin{pmatrix} T_1 T_2 (g\alpha)^2 & iT_2 (g\alpha) \\ -iT_2 (g\alpha) & 2 + T_1 T_2 (g\alpha)^2 \end{pmatrix} \quad (5.16)$$

(making use of $X = 0$ and Y, Z as calculated in the previous lecture.)

The important part of the spectrum however comes from the super-operator in Eq. (5.14). If we write density matrices as a vector:

$$\rho = \begin{pmatrix} \rho_{\uparrow\uparrow} & \rho_{\downarrow\uparrow} \\ \rho_{\uparrow\downarrow} & \rho_{\downarrow\downarrow} \end{pmatrix} \quad \rightarrow \quad \rho = \begin{pmatrix} \rho_{\uparrow\uparrow} \\ \rho_{\downarrow\uparrow} \\ \rho_{\uparrow\downarrow} \\ \rho_{\downarrow\downarrow} \end{pmatrix}, \quad (5.17)$$

then in this four dimensional space, the super-operator can be written as a 4×4 matrix, following Eq. (4.15)

$$M = \begin{pmatrix} -1/T_1 & -ig\alpha/2 & ig\alpha/2 & 0 \\ -ig\alpha/2 & -1/T_2 & 0 & ig\alpha/2 \\ ig\alpha/2 & 0 & -1/T_2 & -ig\alpha/2 \\ 1/T_1 & ig\alpha/2 & -ig\alpha/2 & 0 \end{pmatrix}. \quad (5.18)$$

One can thus find the full density matrix as:

$$I(\nu) \propto -2\Re \left[(0 \ 0 \ 1 \ 0) [M - (-i\nu + 0^+) \mathbf{1}]^{-1} \begin{pmatrix} 0 \\ 0 \\ T_1 T_2 (g\alpha)^2 \\ +iT_2 (g\alpha) \end{pmatrix} \right]. \quad (5.19)$$

When inverting the 4×4 matrix, there will in general be four poles. However, because the density matrix evolution is trace-preserving, one of these poles will necessarily be at zero, corresponding to the conserved quantity. The corresponding eigenvalue will be the steady state (the trace-preserving property ensures that the steady state is non-trivial). Since we know that in the steady state there is a non-zero polarisation, the weight of this zero pole will not in general vanish. This means there is a delta-function peak in the spectrum. Such a peak describes the elastic scattering of pump photons into the radiation modes.

The remainder of the poles will describe inelastic scattering. To find these poles, we should solve:

$$0 = \text{Det} [M + i\nu] = i\nu \left(i\nu - \frac{1}{T_2} \right) \left[\left(i\nu - \frac{1}{T_2} \right) \left(i\nu - \frac{1}{T_1} \right) + (g\alpha)^2 \right]. \quad (5.20)$$

This has the zero mode as discussed above, and three other poles:

$$\nu = -\frac{i}{T_2}, \quad \nu = -i \left(\frac{1}{2T_1} + \frac{1}{2T_2} \right) \pm \sqrt{(g\alpha)^2 - \left(\frac{1}{2T_1} - \frac{1}{2T_2} \right)^2}. \quad (5.21)$$

The first of these will describe a broadened resonance near elastic scattering (remembering that this is in the rotating frame). The second term describes (at large enough α), a resonance shifted by $\pm\alpha$. When the weights of these poles are calculated, as shown in Fig. 5.1, the central peak has twice the weight of the other two peaks. This characteristic shape is known as the Mollow triplet, and has a simple interpretation.

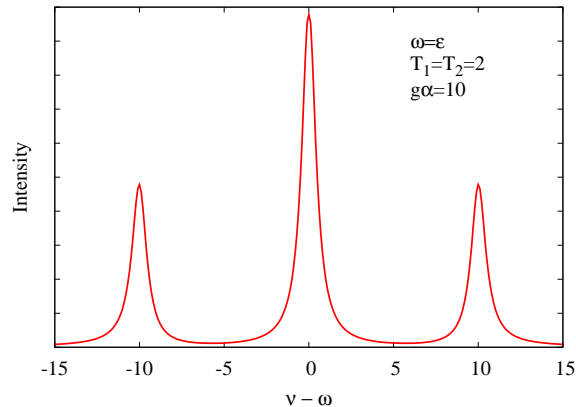


Figure 5.1: Spectrum of resonance fluorescence, from Eq. (5.19).

Interpretation: of the Mollow triplet

The origin of the Mollow triplet is the effective energy levels of the two-level system in the presence of a coherent field. If one considers in general a two-level system coupled to photons, there are Rabi oscillations. In the frequency domain, these oscillations correspond to a splitting of the eigenstates resulting from hybridising the states with n excitations; i.e. the states $|n-1, \uparrow\rangle, |n, \downarrow\rangle$ mix and lead to eigenstates split by $g\sqrt{n}$ (if on resonance).

If the system can now decay, it will emit an excitation into the continuum, and undergo a transition to a state with one fewer excitation in the system. However, starting from one of the hybridised eigenstates with n excitations, it is possible to undergo a transition to either of the eigenstates with $n-1$ excitations — non-zero matrix elements connect them all. Thus, in general there are four transition frequencies.

In the limit of a strong classical field, all four of these matrix elements have the same strength, and moreover, as indicated in Fig. 5.2, the Rabi splitting becomes almost n independent when $n \gg 1$, so two of the frequencies will match. This then leads to the characteristic 1:2:1 ratio seen in Figure 5.1.

5.4 Further reading

The discussion of the resonance fluorescence problem in particular can be found in chapter 16 of Meystre and Sargent III [12]. The discussion of the quantum regression theorem in this lecture is however more closely related

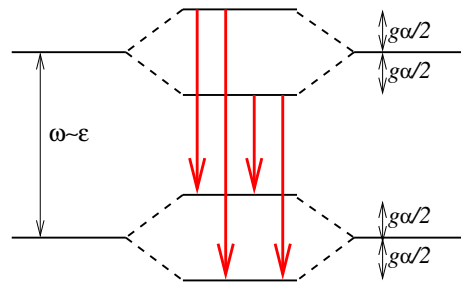


Figure 5.2: Characteristic behaviour at high excitation number, with $n \simeq |\alpha|^2$, demonstrating a triplet of transition frequencies, with the intensity ratio 1:2:1.

to the density matrix based approach given in Scully and Zubairy [10]; The Onsager-Lax (quantum regression) theorem was introduced by Lax [13].

Lecture 6

Quantum stochastic methods and limitations of the Markovian approximation.

The previous two lectures, have introduced the density matrix equations of motion to describe a system coupled to a bath, and have shown how the Markovian approximation implicit in these equations of motion leads one to the quantum regression theorem. This lecture reviews the foundations of these methods for open systems. Two further formalisms are presented, which can be seen as the stochastic equations of motion for which the probability evolution is that of the density matrix. In the latter of these — the Heisenberg-Langevin equations — the consequences of the approximations required for a Markovian equation of motion are more explicit; these are discussed in terms of the fluctuation dissipation theorem.

We will consider just the problem of a system coupled to a single decay bath, which gave us the equation of motion:

$$\begin{aligned} \frac{d}{dt}\rho(t) = & -\frac{\kappa}{2} \left[\bar{n} \left(aa^\dagger \rho - 2a^\dagger \rho a + \rho aa^\dagger \right) \right. \\ & \left. + (\bar{n} + 1) \left(a^\dagger a \rho - 2\rho a^\dagger + \rho a^\dagger a \right) \right]. \end{aligned} \quad (6.1)$$

6.1 Quantum jump formalism

Consider evolution under Eq. (6.1) when $\bar{n} = 0$, i.e. when only decay can occur. Then, after a short time δt , one has:

$$\begin{aligned} \rho_S(t + \delta t) = & \rho_S(t) - \frac{\kappa \delta t}{2} \left(a^\dagger a \rho_S + \rho_S a^\dagger a \right) + \kappa \delta t a \rho_S a^\dagger + \mathcal{O}(\delta t^2) \\ = & \left(1 - \frac{\kappa \delta t}{2} a^\dagger a \right) \rho_S \left(1 - \frac{\kappa \delta t}{2} a^\dagger a \right) + \sqrt{\kappa \delta t} a \rho_S \sqrt{\kappa \delta t} a^\dagger + \mathcal{O}(\delta t^2). \end{aligned} \quad (6.2)$$

These two contributions to the density matrix can be interpreted as the conditional density matrices that arise under “no photon loss” and “one photon loss”; they are added together because of the trace over states of

the bath. The probability of these two final states can be found from the trace of the conditional density matrices, hence:

$$P_{\text{one photon loss}} = \text{Tr} \left(\kappa \delta t a \rho_s a^\dagger \right) = \kappa \delta t \langle a^\dagger a \rangle. \quad (6.3)$$

This is as one intuitively expects; the probability of losing a single photon in $[t, t + \delta t]$ is given by the rate of loss of each photon multiplied by the total number of photons. Thus, the evolution can be written in terms of the probabilities of transition to the (suitable normalised)

$$|\psi\rangle \rightarrow \begin{cases} \frac{a|\psi\rangle}{\sqrt{\langle\psi|a^\dagger a|\psi\rangle}} & P = \kappa \delta t \langle\psi|a^\dagger a|\psi\rangle \\ \frac{(1 - \kappa \delta t a^\dagger a/2)|\psi\rangle}{\sqrt{\langle\psi|1 - \kappa \delta t a^\dagger a|\psi\rangle}} & P = 1 - \kappa \delta t \langle\psi|a^\dagger a|\psi\rangle \end{cases} \quad (6.4)$$

This *quantum jump* formalism can be understood as describing the loss process in terms of the environment repeatedly measuring the number of photons in the cavity, and hence decohering the states with different numbers of photons.

Example: Photon loss from a single-photon state

The above procedure can be illustrated for a particularly simple case: if the initial state contains a single photon then from the density matrix equation:

$$\begin{aligned} \partial_t \langle n+k|\rho|n\rangle = & -\frac{\kappa}{2}(2n+k)\langle n+k|\rho|n\rangle \\ & + \kappa \sqrt{(n+1)(n+k+1)}\langle n+k+1|\rho|n+1\rangle \end{aligned} \quad (6.5)$$

it is clear that the initial condition $\forall n : \langle n+k|\rho|n\rangle = 0$ is preserved except for $k=0$, and so the only equations to solve are

$$\partial_t \langle 1|\rho|1\rangle = -\partial_t \langle 0|\rho|0\rangle = -\kappa \langle 1|\rho|1\rangle. \quad (6.6)$$

so the full solution is:

$$\rho = e^{-\kappa t} |1\rangle\langle 1| + (1 - e^{-\kappa t}) |0\rangle\langle 0|. \quad (6.7)$$

In the *quantum jump* formalism it is easy to reproduce this result; at each time step we have that:

$$|1\rangle \rightarrow \begin{cases} |0\rangle & P = \kappa \delta t \\ |1\rangle & P = 1 - \kappa \delta t \end{cases} \quad (6.8)$$

A more complicated example is given in Question 6.1, for decay of a coherent state. In that case, since a coherent state is unchanged under the loss of a photon, one has the counterintuitive result that it is failure to observe a photon that leads to the density matrix evolving. This is less surprising if one considers that the state has an uncertain number of photons to start with, and failure to observe any photons escape indicates that the wavefunction has been projected by observation onto a state with fewer photons.

6.2 Heisenberg-Langevin equations

Just as one can solve a closed quantum-mechanical problem in either the Schrödinger or Heisenberg pictures, quantum stochastic approaches can be constructed in either picture. The previous sections generally described a stochastic evolution of the wavefunction, this section instead describes stochastic evolution of the system operators. Starting once again from the same coupled bath and system, and working in the rotating frame as normal, one has the Hamiltonian of Eq. (4.2),

$$H_{SR} = \sum_k \zeta_k \left(ab_k^\dagger e^{i(\omega-\omega_k)t} + a^\dagger b_k e^{-i(\omega-\omega_k)t} \right). \quad (4.2)$$

From this Hamiltonian, the Heisenberg operator equations of motion are:

$$i\dot{a} = [a, H_{SR}] = \sum_k \zeta_k b_k e^{-i(\omega-\omega_k)t} \quad (6.9)$$

$$i\dot{b}_k = [b_k, H_{SR}] = \zeta_k a e^{i(\omega-\omega_k)t}. \quad (6.10)$$

To eliminate the bath degrees of freedom, one can integrate Eq. (6.10), and substitute it in Eq. (6.9). Thus,

$$b_k(t) = -i\zeta_k \int_0^t dt' e^{i(\omega-\omega_k)t'} a(t') + b_k(0) \quad (6.11)$$

which, substituted into Eq. (6.9), gives:

$$\dot{a} = - \int_0^t dt' \left[\sum_k \zeta_k^2 e^{i(\omega-\omega_k)(t'-t)} \right] a(t') - i \left[\sum_k \zeta_k b_k(0) e^{-i(\omega-\omega_k)t} \right]. \quad (6.12)$$

Then, defining the second term in brackets as $F_a(t)$, and making the same Markovian approximation:

$$\sum_k \zeta_k^2 F(\omega_k) \rightarrow \int \frac{d\nu}{2\pi} \kappa F(\nu) \quad (6.13)$$

as in Eq. (4.7), so that the first term in brackets becomes $\kappa\delta(t-t')$, this equation becomes:

$$\dot{a} = -\frac{1}{2}\kappa a + F_a(t). \quad (6.14)$$

Here $F_a(t)$ is a stochastic operator; it has quantum mechanical commutation relations related to its definition in terms of b_k ; but b_k is a bath operator, and the state of the bath is random — drawn from a thermal ensemble, so b_k has different expectations for each realisation of the bath.

Preservation of commutation relations

The operator nature of $F_a(t)$ is apparent if one considers the commutation relations:

$$\begin{aligned} [F_a(t), F_a^\dagger(t')] &= \sum_{k,k'} \zeta_k \zeta_{k'} [b_k, b_{k'}^\dagger] e^{-i(\omega-\omega_k)t+i(\omega-\omega_{k'})t'} \\ &= \sum_k \zeta_k^2 e^{-i(\omega-\omega_k)(t-t')} = \kappa\delta(t-t'). \end{aligned} \quad (6.15)$$

This non-commuting nature is essential to ensure the preservation of commutation relations in the time-evolution of the operators $a(t)$. The solution of Eq. (6.14) can be written

$$a(t) = a(0)e^{-\kappa t/2} + \int_0^t dt' F_a(t') e^{-\kappa(t-t')/2}. \quad (6.16)$$

Using this, one can write:

$$\begin{aligned} [a(t), a^\dagger(t)] &= [a(0), a^\dagger(0)] e^{-\kappa t} \\ &+ \int_0^t dt' \int_0^t dt'' [F_a(t'), F_a^\dagger(t'')] e^{-\kappa t + \kappa(t'+t'')/2}, \end{aligned} \quad (6.17)$$

in which we have assumed no correlation between the initial state of the system and that of the bath. Then, using $[a, a^\dagger] = 1$ in the initial state, and the commutator in Eq. (6.15) this becomes:

$$\begin{aligned} [a(t), a^\dagger(t)] &= e^{-\kappa t} + \int_0^t dt' \kappa e^{-\kappa t + \kappa t'} \\ &= e^{-\kappa t} + [1 - e^{-\kappa t}] \end{aligned} \quad (6.18)$$

This shows that the existence of the non-commuting stochastic terms was essential to preserve the operator commutation relations.

Finite bath occupation; other correlations

To reproduce all the results of the density-matrix formalism, it is necessary also to have expressions for the correlation functions of the stochastic operators $F_a(t)$ as well as their commutator. Given the form for the commutator, all correlation functions can be found from this and the anticommutator,

$$\begin{aligned} \langle \{F_a(t), F_a^\dagger(t')\} \rangle &= \sum_{k,k'} \zeta_k \zeta_{k'} \langle \{b_k, b_{k'}^\dagger\} \rangle e^{-i(\omega - \omega_k)t + i(\omega - \omega_{k'})t'} \\ &= \sum_k \zeta_k^2 (2n_k + 1) e^{-i(\omega - \omega_k)(t-t')} \\ &= \kappa \int \frac{d\nu}{2\pi} \coth\left(\frac{\beta\nu}{2}\right) e^{-i(\omega - \nu)(t-t')} \end{aligned} \quad (6.19)$$

$$\approx \kappa \coth\left(\frac{\beta\omega}{2}\right) \delta(t - t'). \quad (6.20)$$

The last line here makes the additional assumption of $T \gg \kappa$, i.e. that the temperature is large compared to the bandwidth, with the bandwidth taken approximately equal to the imaginary width. This assumption is clearly necessary if the entire formalism is to be Markovian (i.e. to depend only on current state, rather than history), and is identical to the approximation made in Eq. (4.7). The validity and limitations of this last approximation are the subject of the next section.

6.3 Fluctuation dissipation theorem and the Markovian approximation

The fluctuation-dissipation theorem relates the correlation function of an operator with its response function. For the photon operators considered here, the correlation function can be defined by:

$$C(\Omega) = \int_{-\infty}^{\infty} d\tau C(\tau) e^{i\Omega\tau}, \quad C(\tau) = \frac{1}{2} \left\langle \left\{ a(t), a^\dagger(t+\tau) \right\} \right\rangle. \quad (6.21)$$

The response function can be found by considering the response to a perturbation:

$$\delta H = \sum_{\Omega} f_{\Omega} e^{i\Omega t} a^\dagger + \text{H.c.} \quad (6.22)$$

The response function $\alpha(\Omega)$ is given by

$$\langle a(t) \rangle = \sum_{\Omega} \alpha(\Omega) f_{\Omega} e^{i\Omega t}. \quad (6.23)$$

The discussion below uses the Heisenberg-Langevin formalism to evaluate the relation between these operators with and without the second Markovian approximation above.

The response function is the same in either case, since it is averaged and so does not depend on the stochastic operator $F_a(t)$. In the following it will be necessary to distinguish between the bare operators $a(t)$ and the gauge transformed versions $\tilde{a}(t) = a(t)e^{-i\omega t}$. Thus,

$$\left(\partial_t + \frac{\kappa}{2} \right) \langle \tilde{a}(t) \rangle = i \sum_{\Omega} f_{\Omega} e^{i(\Omega-\omega)t} \quad (6.24)$$

where we have used the Heisenberg equation following from the gauge transformed perturbation Hamiltonian. This yields

$$\langle a(t) \rangle = e^{i\omega t} \langle \tilde{a}(t) \rangle = e^{i\omega t} \sum_{\Omega} \frac{if_{\Omega} e^{i(\Omega-\omega)t}}{i(\Omega-\omega) + \kappa/2} = \sum_{\Omega} \frac{f_{\Omega} e^{i\Omega t}}{(\Omega-\omega) - i\kappa/2} \quad (6.25)$$

hence $\alpha(\Omega) = 1/[(\Omega-\omega) - i\kappa/2]$.

Next, consider the fluctuation correlation function at long times, so that the initial conditions are not involved. In this case, Eq. (6.16) should become:

$$a(t) = e^{i\omega t} \int_{-\infty}^t dt' F_a(t') e^{-\kappa(t-t')/2} \quad (6.26)$$

and so:

$$C(\tau) = \frac{e^{-i\omega\tau}}{2} \int_{-\infty}^t dt' \int_{-\infty}^{t+\tau} dt'' \left\langle \left\{ F_a(t'), F_a^\dagger(t'') \right\} \right\rangle e^{-\kappa(2t+\tau-t'-t'')/2} \quad (6.27)$$

We now have two choices; we may either use the exact relation in Eq. (6.19), or the Markov approximation in Eq. (6.20).

Without second Markovian approximation

Considering first Eq. (6.19), we have:

$$\begin{aligned}
C(\tau) &= \frac{e^{-i\omega\tau}}{2} \int_{-\infty}^t dt' \int_{-\infty}^{t+\tau} dt'' \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \kappa \coth\left(\frac{\beta\nu}{2}\right) e^{i(\omega-\nu)(t''-t')-\kappa(2t+\tau-t'-t'')/2} \\
&= \frac{e^{-i\omega\tau}}{2} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \kappa \coth\left(\frac{\beta\nu}{2}\right) \frac{e^{i(\omega-\nu)(t+\tau-t)}}{[i(\omega-\nu)+\kappa/2][-i(\omega-\nu)+\kappa/2]} \\
&= \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \kappa \coth\left(\frac{\beta\nu}{2}\right) \frac{e^{-i\nu\tau}}{(\omega-\nu)^2 + (\kappa/2)^2} \tag{6.28}
\end{aligned}$$

hence it is immediately clear that:

$$C(\Omega) = \coth\left(\frac{\beta\Omega}{2}\right) \frac{\kappa/2}{(\omega-\Omega)^2 + (\kappa/2)^2} = \coth\left(\frac{\beta\Omega}{2}\right) \Im[\alpha(\Omega)] \tag{6.29}$$

which is the required fluctuation dissipation theorem.

With second Markovian approximation

In contrast, if we use Eq. (6.20) we have that:

$$\begin{aligned}
C(\tau) &= \frac{e^{-i\omega\tau}}{2} \int_{-\infty}^t dt' \int_{-\infty}^{t+\tau} dt'' \kappa \coth\left(\frac{\beta\omega}{2}\right) \delta(t'-t'') e^{-\kappa(2t+\tau-t'-t'')/2} \\
&= \frac{e^{-i\omega\tau}}{2} \kappa \coth\left(\frac{\beta\omega}{2}\right) \int_{-\infty}^{\min[t, t+\tau]} dt' e^{-\kappa(2t+\tau-2t')/2} \\
&= \frac{e^{-i\omega\tau}}{2} \coth\left(\frac{\beta\omega}{2}\right) e^{-\kappa|\tau|/2}, \tag{6.30}
\end{aligned}$$

from which it follows straightforwardly that

$$C(\Omega) = \coth\left(\frac{\beta\omega}{2}\right) \Im[\alpha(\Omega)]. \tag{6.31}$$

Thus, use of the Markovian approximation means that the fluctuation dissipation theory fails, unless $T \gg (\Omega - \omega) > \kappa$. Such a result can also be seen to be a limit for the use of the quantum regression theorem. In that case, the high temperature (classical) answer is known as the Onsager theorem. The quantum extension of the Onsager theorem is an approximation, that derives from the Markov approximation in the dynamics

6.4 Further reading

The topic of Heisenberg-Langevin equations, and stochastic operator equations is discussed briefly in Scully and Zubairy [10], and more fully in Gardiner and Zoller [14]. The implicit dependence of the quantum regression theorem on the use of the Markovian approximation is highlighted in the papers of Ford and O'Connell [15, 16], and the response by Lax [17]. This topic is addressed again in question 6.2.

Questions

Question 6.1: Comparison of quantum jump and density matrix equations of motion. Consider an initial coherent state,

$$\rho = \exp(-|\alpha|^2) e^{\alpha a^\dagger} |0\rangle\langle 0| e^{\alpha^* a} \quad (6.32)$$

following the equation of motion of the density matrix with $\bar{n} = 0$, i.e.

$$\dot{\rho} = -\frac{\kappa}{2} \left[a^\dagger a \rho + \rho a^\dagger a - 2a \rho a^\dagger \right]. \quad (6.33)$$

6.1.(a) Show that this equation of motion can be satisfied by the ansatz

$$\rho(t) = \exp(-|\alpha(t)|^2) e^{\alpha(t) a^\dagger} |0\rangle\langle 0| e^{\alpha^*(t) a} \quad (6.34)$$

and find the equation satisfied by $\alpha(t)$.

Now consider this question in the *quantum jump* formalism. If the problem can be written as:

$$\rho(t) = \sum_n P_n(t) |\psi_n(t)\rangle\langle \psi_n(t)| \quad (6.35)$$

where $|\psi_n\rangle$ corresponds to the state after having lost exactly n photons.

6.1.(b) Show that $|\psi_n(t)\rangle = |\psi_0(t)\rangle$; (i.e. that losing a photon does not change the state).

6.1.(c) Find $P_n(t)$, and verify that $\sum_n P_n(t) = 1$, and hence show that $\rho(t) = |\psi_0(t)\rangle\langle \psi_0(t)|$.

6.1.(d) Show that at time t , the non-Hermitian evolution leads to

$$|\psi_0(t)\rangle = \frac{1}{\mathcal{N}} \exp(-\kappa t a^\dagger a / 2) |\psi_0(0)\rangle \quad (6.36)$$

where \mathcal{N} is an appropriate normalisation.

6.1.(e) Prove that this result matches that in Eq. (6.34).

Question 6.2: Fluctuation dissipation theorem, and quantum regression

This extended question repeats the analysis in the lecture of the limit of validity of the Markov approximation, but for a damped quantum harmonic oscillator. It shows how the above results do not depend on other approximations that were involved, such as rotating wave approximation. Starting from the Hamiltonian,

$$H = \frac{m\omega_0^2}{2} x^2 + \frac{p^2}{2m} + \sum_j \left[\frac{m_j \omega_j^2}{2} (q_j - x)^2 + \frac{p_j^2}{2m_j} \right] \quad (6.37)$$

One can easily derive the equations of motion:

$$\begin{aligned} \dot{x} &= p/m & \dot{p} &= -m\omega_0^2 x + \sum_j m_j \omega_j^2 (q_j - x) \\ \dot{q}_j &= p_j/m_j & \dot{p}_j &= -m_j \omega_j^2 (q_j - x) \end{aligned}$$

Eliminating p_j and p , one can write:

$$\ddot{q}_j = -\omega_j^2 (q_j - x); \quad \ddot{x} = -\omega_0^2 x + \sum_j \frac{m_j}{m} \omega_j^2 (q_j - x) \quad (6.38)$$

6.2.(a) Show that the equation for $q_j(t)$ is satisfied by:

$$q_j(t) - x(t) = \left[q_j^0 \cos(\omega_j t) + \frac{p_j^0}{m_j \omega_j} \sin(\omega_j t) \right] - \int_{-\infty}^t dt' \cos[\omega_j(t-t')] \dot{x}(t') \quad (6.39)$$

Substituting this in the equation for x one finds:

$$\ddot{x} + \omega_0^2 x = -2 \int_{-\infty}^t dt' \gamma(t-t') \dot{x}(t') + F(t) \quad (6.40)$$

where

$$\begin{aligned} 2\gamma(t-t') &= \sum_j \frac{m_j}{m} \omega_j^2 \cos[\omega_j(t-t')] \\ F(t) &= \sum_j \frac{m_j}{m} \omega_j^2 \left[q_j^0 \cos(\omega_j t) + \frac{p_j^0}{m_j \omega_j} \sin(\omega_j t) \right] \end{aligned}$$

Let us assume a flat spectrum, so $\gamma(t-t') = \gamma \delta(t-t')$. This implies

$$\sum_j \frac{m_j \omega_j^2}{m} \rightarrow \int_0^\infty d\omega \frac{\gamma}{\pi} \quad (6.41)$$

The correlations of F require knowledge of correlations of p_j^0 and q_j^0 , the initial momenta and coordinates. Using the reasonable assumption of uncorrelated, diagonal (thermal) states, one has $\langle p_j^0 q_k^0 \rangle = 0$, and:

$$2m_j \omega_j \langle q_j^0 q_k^0 \rangle = 2 \frac{\langle p_j^0 p_k^0 \rangle}{m_j \omega_j} = \delta_{jk} (2n(\omega_j) + 1) = \delta_{jk} \coth \left(\frac{\omega_j}{2k_B T} \right). \quad (6.42)$$

6.2.(b) Using the above correlations and definition of $F(t)$, show that:

$$\begin{aligned} \frac{1}{2} \langle F(t)F(t') + F(t')F(t) \rangle &= \sum_j \frac{m_j \omega_j^3}{m^2} \cos[\omega_j(t-t')] \coth \left(\frac{\omega_j}{2k_B T} \right) \\ &= \int_0^\infty d\omega \frac{\gamma \omega}{\pi m} \cos[\omega(t-t')] \coth \left(\frac{\omega}{2k_B T} \right). \end{aligned} \quad (6.43)$$

6.2.(c) Show that the general solution to:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = F(t), \quad (6.44)$$

can be written (using $\omega_1 = \sqrt{\omega_0^2 - \gamma^2/4}$) as:

$$x(t) = \int_{-\infty}^t dt' e^{-\gamma(t-t')/2} \frac{\sin[\omega_1(t-t')]}{\omega_1} F(t'). \quad (6.45)$$

6.2.(d) Thus, show that:

$$C(\tau) = \frac{1}{2} \langle x(t+\tau)x(t) + x(t)x(t+\tau) \rangle, \quad (6.46)$$

is given by:

$$C(\tau) = \int_0^\infty d\omega \frac{\gamma\omega}{\pi m} \cos(\omega\tau) \frac{1}{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2} \coth\left(\frac{\omega}{2k_B T}\right). \quad (6.47)$$

Fourier transforming, this gives the correlation function

$$C(\omega) = \int_{-\infty}^\infty dt e^{-i\omega t} C(t) = \frac{\gamma}{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2} \frac{\omega}{m} \coth\left(\frac{\omega}{2k_B T}\right). \quad (6.48)$$

Let us now relate this fluctuation correlation function to the response function. To define the response function $\alpha(t)$, consider the averaged equation motion, responding to a driving force $f(t) = e^{i\omega t} f_0$:

$$\langle \ddot{x} \rangle + \gamma \langle \dot{x} \rangle + \omega_0^2 \langle x \rangle = \frac{f(t)}{m}. \quad (6.49)$$

6.2.(e) Show that the response function, $\langle x \rangle_\omega = \alpha(\omega) f_\omega$ is given by:

$$\alpha(\omega) = \frac{1}{m} \frac{1}{\omega_0^2 - \omega^2 + i\gamma\omega} \quad (6.50)$$

Hence, show one can write the fluctuation-dissipation theorem as:

$$C(\omega) = \Im[\alpha(\omega)] \coth\left(\frac{\omega}{2k_B T}\right). \quad (6.51)$$

Now, let us repeat the above calculations with the Markovian approximation.

6.2.(f) Show that the general time-dependent behaviour of $\langle x(t) \rangle$, following Eq. (6.49) is given by:

$$\langle x(t) \rangle = e^{-\gamma t/2} \left[\langle x_0 \rangle \left(\cos(\omega_1 t) - \frac{\gamma}{2\omega_1} \sin(\omega_1 t) \right) + \frac{\langle p_0 \rangle}{m\omega_1} \sin(\omega_1 t) \right] \quad (6.52)$$

Then, using the Markovian approximation for quantum regression, one can find $C(\tau)$ as defined in Eq. (6.47), by replacing the initial density matrix with $\rho(t) \rightarrow x(t)\rho(t)$. Assuming equilibrium single-time correlations, one has $\langle x(t)p(t) + p(t)x(t) \rangle = 0$, and $\langle x(t)^2 \rangle = (2\bar{n} + 1)/2m\omega_0$. Thus,

$$C(\tau) = \frac{2\bar{n} + 1}{2m\omega_0} e^{-\gamma\tau/2} \left(\cos(\omega\tau) - \frac{\gamma}{2\omega_1} \sin(\omega_1\tau) \right) \quad (6.53)$$

6.2.(g) Taking the equilibrium population $2\bar{n} + 1 = \coth(\omega_0/2k_B T)$, show that this yields

$$C(\omega) = \frac{\gamma}{(\omega^2 - \omega_0^2) + \gamma^2 \omega^2} \frac{\omega^2}{m\omega_0} \coth\left(\frac{\omega_0}{2k_B T}\right) \quad (6.54)$$

6.2.(h) By considering the fluctuation-dissipation theorem, explain why in the classical limit $\omega_0 \gg k_B T$, an exact regression theorem holds. i.e. in this limit, the equations governing two-time and single-time correlation functions are exactly related by the quantum regression theorem, even without Markovian or weak coupling approximations.

Lecture 7

Two-level atom in a cavity: Cavity QED

In the previous few lectures we have discussed a single two-level atom coupled either to a continuum of radiation modes, or considering a cavity, which picks out a particular mode. In this lecture we will consider more carefully the situations in which a cavity will pick out a single mode, by describing how the system changes as one goes from no cavity, via a bad cavity (which modifies the field strength but does not pick out a single mode) to a good cavity, which can pick out a single mode.

To study this problem throughout this crossover, we will consider a two level system coupled both to a cavity pseudo-mode (which itself decays), and also coupled to non-cavity modes, providing incoherent decay. We will show how in this model, a bad cavity can lead to either enhanced decay, and a good cavity can lead to periodic exchange of energy between the cavity and the two-level system, i.e. the Rabi oscillations we previously discussed for a perfect cavity. We will then consider some examples of currently studied experimental cavity QED systems, discussing the values of the relevant parameters describing coupling and decay.

Before discussing the crossover between bad and good cavities, we first investigate a toy model of a 1D cavity, to put into context the meaning of the pseudo-mode description of the cavity mode.

7.1 The Purcell effect in a 1D model cavity

The aim of this section is to describe how a cavity modifies the rate of decay of a two-level system, coupled to one-dimensional radiation modes. Following the previous discussion of system-reservoir coupling, the decay is characterised by the combination of reservoirs density of states and its coupling to the atom, given by:

$$\Gamma(\omega) = 2\pi \sum_k \delta(\omega - ck) \left| \frac{g_k}{2} \right|^2. \quad (7.1)$$

Recalling that $g_k \propto \mathcal{E}_k$, the field strength associated with a single photon, we can find the effect of the cavity on $\Gamma(\omega)$ by determining the spatial

profiles of the modes of the full system, and inserting these into the above sum.

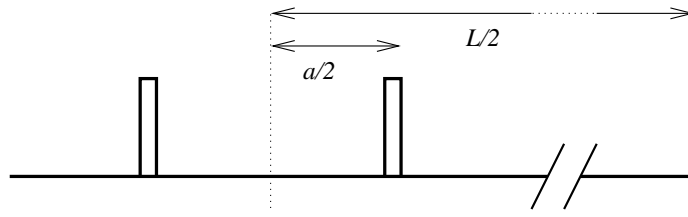


Figure 7.1: Toy model of cavity, length a , in quantisation volume of length L .

We will consider a cavity of size a , embedded in a quantisation volume of length L , where we will take $L \rightarrow \infty$ later on. (See figure 7.1). Our aim is to find the spatial profile of the electric field modes, and thus to calculate

$$\Gamma(\omega) \propto \sum_k \delta(\omega - ck) |\mathcal{E}_k(x=0)|^2. \quad (7.2)$$

Both in the cavity, and outside, the modes will be appropriate combinations of plane waves. Restricting to symmetric solutions (as antisymmetric solutions will vanish at $x=0$), and matching the boundary at $x = \pm L/2$, we may write:

$$\mathcal{E}_k(x) = \begin{cases} f(x) \equiv A_k \cos(kx) & |x| < \frac{a}{2} \\ g(x) \equiv B_k \sin[k(|x| - \frac{L}{2})] & |x| > \frac{a}{2} \end{cases}. \quad (7.3)$$

As $L \rightarrow \infty$, the normalisation condition will approach the simple result $B_k = \sqrt{2/L}$, since the normalisation integral will be dominated by the parts outside the cavity. We now need to find the effect of the imperfect barriers on this mode in order to relate A_k, B_k , and thus find the quantisation condition specifying k , and the mode amplitude, $\mathcal{E}_k(x=0) = A_k$.

As a simple model, let us consider a varying dielectric index, $\varepsilon = \varepsilon_0[1 + \eta\delta(|x| - a/2)]$, giving the equation:

$$\frac{d^2}{dx^2} \mathcal{E}_k(x) = -k^2 \left[1 + \eta\delta\left(|x| - \frac{a}{2}\right) \right] \mathcal{E}_k. \quad (7.4)$$

Let us focus on $x > 0$; then, given the form of $\mathcal{E}_k(x) = \Theta(a/2 - x)f(x) + \Theta(x - a/2)g(x)$ with $f'' = -k^2f$ and $g'' = -k^2g$, we may write:

$$\frac{d^2}{dx^2} \mathcal{E}_k = -k^2 \mathcal{E}_k + 2\delta\left(x - \frac{a}{2}\right) [g'(x) - f'(x)] + \delta'\left(x - \frac{a}{2}\right) [g(x) - f(x)] \quad (7.5)$$

Thus, to solve Eq. (7.4) we require that:

$$f\left(\frac{a}{2}\right) = g\left(\frac{a}{2}\right), \quad g'\left(\frac{a}{2}\right) - f'\left(\frac{a}{2}\right) = -\frac{\eta k^2}{4} \left[f\left(\frac{a}{2}\right) + g\left(\frac{a}{2}\right) \right] \quad (7.6)$$

Substituting the forms of $f(x), g(x)$ into this equation, one may eliminate A_k, B_k to give the eigenvalue condition:

$$\cos\left(\frac{kL}{2}\right) + \frac{k\eta}{2} \cos\left(\frac{ka}{2}\right) \sin\left(\frac{ka}{2} - \frac{kL}{2}\right) = 0. \quad (7.7)$$

For finite L , this provides a restriction on the permissible values of k . As $L \rightarrow \infty$, the permissible values of k become more dense, such that, in this limit, one may instead consider this equation as depending on two separate values kL and ka ; if $L \gg a$, it is possible to significantly change kL without modifying ka . Thus, we will instead rewrite this equation as a condition that specifies the value of kL given a fixed ka :

$$\cos\left(\frac{kL}{2}\right) \left[1 + \frac{k\eta}{2} \sin\left(\frac{ka}{2}\right) \cos\left(\frac{ka}{2}\right)\right] = \sin\left(\frac{kL}{2}\right) \frac{k\eta}{2} \cos^2\left(\frac{ka}{2}\right). \quad (7.8)$$

Using this condition, one may then eliminate kL , and write A_k in terms of $B_k = \sqrt{2/L}$ and functions of ka , i.e.:

$$A_k = -\sqrt{\frac{2}{L}} \frac{1}{\sqrt{1 + k\eta \sin(ka/2) \cos(ka/2) + (k\eta/2)^2 \cos^2(ka/2)}} \quad (7.9)$$

If we define $\Lambda = k\eta/2$, then we may use this formula to give the effective decay rate:

$$\begin{aligned} \Gamma\left(k = \frac{\omega}{c}\right) &= \frac{\Gamma_0}{1 + \Lambda \sin(ka) + \frac{1}{2}\Lambda^2(1 + \cos(ka))} \\ &= \frac{\Gamma_0}{\left(1 + \frac{\Lambda^2}{2}\right) + \Lambda \sqrt{1 + \frac{\Lambda^2}{4}} \cos(ka - \theta_0)}, \quad \tan \theta_0 = \frac{4}{\Lambda}. \end{aligned} \quad (7.10)$$

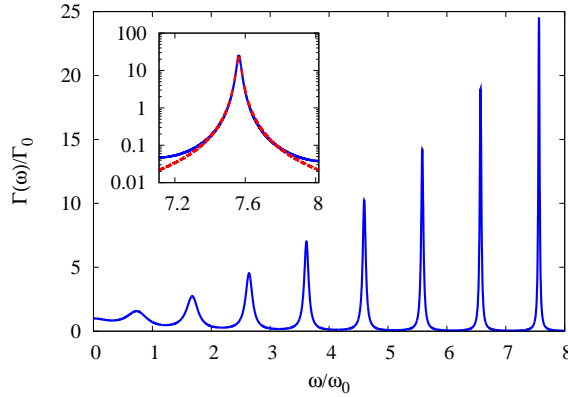


Figure 7.2: Effective decay rate vs frequency for the toy model of a 1D imperfect cavity. Inset compares the exact result, Eq. (7.10) to the Lorentzian approximation of Eq. (7.13)

The form of this function in general is shown in Fig. 7.2. For small k , the effect of the cavity is weak (i.e. $k\eta \ll 1$), and so there is little modification compared to the result without the cavity. For larger k , there are sharp peaks, which can be described by an approximately Lorentzian form (see the inset of Fig. 7.2). This Lorentzian form can be found by expanding Eq. (7.10) near its peaks, which are at:

$$k_0 a = \theta_0 + (2n + 1)\pi. \quad (7.11)$$

We can then expand $\Gamma(\omega)$ near $\omega_0 = ck_0$. If $\Lambda \gg 1$, and if we may neglect its k dependence, then we find:

$$\begin{aligned} \frac{\Gamma}{\Gamma_0} &\simeq \frac{1}{\left(1 + \frac{\Lambda^2}{2}\right) - \frac{\Lambda^2}{2} \left(1 + \frac{2}{\Lambda^2} - \frac{2}{\Lambda^4}\right) \left[1 - \frac{1}{2} \left(\frac{a(\omega - \omega_0)}{c}\right)^2\right]} \\ &\simeq \frac{1}{\frac{1}{\Lambda^2} + \frac{\Lambda^2 a^2}{4c^2} (\omega - \omega_0)^2}. \end{aligned} \quad (7.12)$$

Let us introduce the full width half maximum, κ , such that this expression becomes:

$$\frac{\Gamma}{\Gamma_0} = \frac{P(\kappa/2)^2}{(\omega - \omega_0)^2 + (\kappa/2)^2} \quad (7.13)$$

which leads to the definitions:

$$\frac{\kappa}{2} = \frac{2c}{\Lambda^2 a}, \quad P = \Lambda^2 = \frac{4c}{\kappa a}. \quad (7.14)$$

Here Γ_0 is the result for a 1D system in the absence of a cavity. The factor P describes the maximum value of Γ/Γ_0 at resonance. At resonance, the presence of the cavity enhances the decay rate, and P is the Purcell enhancement factor. It is helpful for comparison to later results to rewrite this as:

$$P = 4 \frac{\omega_0}{\kappa} \frac{c}{\omega_0 a} = \frac{2}{\pi} Q \left(\frac{\lambda}{a}\right), \quad Q \equiv \frac{\omega_0}{\kappa} \quad (7.15)$$

where Q is the quality factor of the cavity mode. The enhancement on resonance therefore depends both on the quality factor, and on the mode volume.

Away from resonance, the decay rate decreases instead of increasing, since there are no modes with significant weight inside the cavity; the minimum value of Γ/Γ_0 occurs when $k_0 a = \theta_0 + 2n\pi$, and is given by $\Gamma/\Gamma_0 = 1/\Lambda^2 = 1/P$, hence for this 1D case, P describes both the enhancement on resonance, and the reduction off resonance.

As η increases further, the Lorentzian density of states becomes sharply peaked, and so the Markovian approximation for decay of the atom no longer holds, instead one can use:

$$\begin{aligned} \frac{d}{dt} \sigma^- &= - \int^t dt' \sigma^-(t') \int \frac{d\omega}{2\pi} \frac{\Gamma_0 P (\kappa/2)^2 e^{i(\epsilon - \omega)(t' - t)}}{(\omega - \omega_0)^2 + (\kappa/2)^2} \\ &= - \frac{\Gamma_0 P \kappa}{4} \int^t dt' \sigma^-(t') e^{i(\epsilon - \omega)(t' - t) - \kappa(t - t')/2}. \end{aligned} \quad (7.16)$$

This form describes the behaviour one would see if the two-level system were coupled to another degree of freedom, with frequency ω and decay rate $\kappa/2$. Hence, we are led in this limit to introduce the cavity pseudo-mode as an extra dynamical degree of freedom. The density of states implies that the cavity mode cannot be treated as a Markovian system, it has a non-negligible memory time $1/\kappa$, and so we must consider its dynamics.

7.2 Weak to strong coupling via density matrices

This section considers the case in which a single cavity mode must be treated beyond a Markovian approximation, by considering the Jaynes-Cummings model, along with relaxation of both the two-level system and the cavity mode. The model is thus given by:

$$\begin{aligned} \frac{d}{dt}\rho = & -i[H, \rho] - \frac{\kappa}{2} \left(a^\dagger a \rho - 2a\rho a^\dagger + \rho a^\dagger a \right) \\ & - \frac{\Gamma'}{2} \left(\sigma^+ \sigma^- \rho - 2\sigma^- \rho \sigma^+ + \rho \sigma^+ \sigma^- \right) \end{aligned} \quad (7.17)$$

$$H = \frac{g}{2} \left(\sigma^+ a + \sigma^- a^\dagger \right) + \epsilon \sigma^z + \omega_0 a^\dagger a. \quad (7.18)$$

One should note that in this equation, a^\dagger is the creation operator for a pseudo-mode of the system, i.e. it does not describe a true eigenmode; the true eigenmodes are instead superpositions of modes inside and outside the cavity, just as in the previous section. Because the pseudo-mode overlaps with a range of true eigenmodes, the probability of remaining in the pseudo-mode will decay (see question 7.1 for the 1D case). This coupling between cavity modes and modes outside the cavity can be described in a Markovian approximation, leading to the decay rate κ in the above density matrix equation.

In addition to the decay of the pseudo-mode, we also include a rate Γ' describing decay of the two level system into modes other than the cavity mode. While in the one dimensional description considered previously, no such other channel exists, in three dimensions, if the cavity is not spherical, then decay into non-cavity directions is possible. In this case, one may consider $\Gamma' = \Gamma\Omega/4\pi$, depending on solid angle. For the experimental systems discussed below, this is generally the case, due to the need to have access to insert atoms into the cavity. More generally Γ' describes the possibility of relaxation into any mode other than the cavity mode; in solid state contexts, other possible reservoirs often exist (e.g phonons, other quasiparticle excitations etc).

Let us solve the equation of motion, Eq. (7.17), starting in an initially excited state $|\uparrow, 0\rangle$. From this initial state, the only possible other subsequent states are $|\downarrow, 1\rangle$ and $|\downarrow, 0\rangle$. As the last of these states cannot evolve into anything else it may be ignored, and we can write a closed set of equations for three elements of the density matrix:

$$P_A = \rho_{\uparrow 0, \uparrow 0}, \quad P_B = \rho_{\downarrow 1, \downarrow 1}, \quad C_{AB} = \rho_{\uparrow 0, \downarrow 1} \quad (7.19)$$

By taking appropriate matrix elements, one finds:

$$\frac{d}{dt}P_A = -i\frac{g}{2}(C_{AB}^* - C_{AB}) - \Gamma'P_A \quad (7.20)$$

$$\frac{d}{dt}P_B = -i\frac{g}{2}(C_{AB} - C_{AB}^*) - \kappa P_B \quad (7.21)$$

$$\frac{d}{dt}C_{AB} = -i(\epsilon - \omega_0)C_{AB} + i\frac{g}{2}(P_A - P_B) - \frac{\kappa + \Gamma'}{2}C_{AB} \quad (7.22)$$

One may further simplify these equations by noting that $P_A P_B - |C_{AB}|^2 = 0$ is conserved, and thus writing $P_A = |\alpha|^2$, $P_B = |\beta|^2$, $C_{AB} = \alpha\beta^*$. Substituting this leads to the simpler equations:

$$\frac{d}{dt} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -\Gamma'/2 & -ig/2 \\ -ig/2 & i\Delta - \kappa/2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad (7.23)$$

where we have written $\Delta = \omega_0 - \epsilon$. In general this leads to two frequencies for decaying oscillations of the excitation probability:

$$\left(i\nu - \frac{\Gamma'}{2}\right) \left(i\nu + i\Delta - \frac{\kappa}{2}\right) + \frac{g^2}{4} = 0. \quad (7.24)$$

Bad cavity limit — Purcell effect in 3D

In the limit of a bad cavity, i.e. $\kappa \ll g, \Gamma$, the two frequencies corresponding to the above equation correspond separately to decay of the excited two-level system and decay of photons. Because these are on very different timescales, one may treat the coupling perturbatively writing $\nu = -i\Gamma'/2 + \delta$, where at $\mathcal{O}(\delta)$ the determinant equation becomes:

$$i\delta \left(i\Delta - \frac{\kappa}{2} + \frac{\Gamma'}{2}\right) + \frac{g^2}{4} = 0, \quad (7.25)$$

and since $\kappa \gg \Gamma$ this bad cavity limit gives:

$$\nu = -\frac{i}{2} \left(\Gamma' + \frac{\kappa g^2}{\kappa^2 + 4\Delta^2}\right) + \frac{\Delta g^2}{\kappa^2 + 4\Delta^2}. \quad (7.26)$$

This describes a cavity-enhanced decay rate, $\Gamma_{\text{eff}} = \Gamma' + \Gamma_{\text{cav}}$, where the cavity decay rate describes the Purcell effect, as discussed above. On resonance we have:

$$\Gamma_{\text{cav}} = 4 \left|\frac{g}{2}\right|^2 \frac{1}{\kappa} \quad (7.27)$$

and recalling the earlier 3D results:

$$\Gamma = \frac{\omega_0^3 |d_{ab}|^2}{3\pi\epsilon_0 c^3}, \quad \left|\frac{g}{2}\right|^2 = \frac{\omega_0 |d_{ab}|^2}{2\epsilon_0 V} \quad (7.28)$$

(where we have assumed resonance, so $\omega_0 = \epsilon$), one may write:

$$\frac{\Gamma_{\text{cav}}}{\Gamma} = P = 4 \frac{3\pi c^3}{2V\omega_0^2} \frac{1}{\kappa} = \frac{3}{4\pi^2} Q \left(\frac{\lambda^3}{V}\right). \quad (7.29)$$

This closely resembles the one dimensional result before; the Purcell enhancement depends both on the quality factor, and also on the mode volume. An estimate of the minimum decay rate off resonance in this case yields $\Gamma_{\text{cav},\text{min}} = \Gamma_{\text{cav},\text{max}} \left(\frac{\kappa}{2\omega_0}\right)^2 = \frac{3}{16\pi^2} \frac{1}{Q} \left(\frac{\lambda^3}{V}\right)$. However, if the background decay rate Γ' also exists, this off-resonant reduction is not as relevant as the on-resonant enhancement.

Strong coupling — Rabi oscillations

If the cavity is sufficiently good, then rather than decay, the excitation probability oscillates. In the resonant case of $\Delta = 0$, one can write the general solution for the oscillation frequency as:

$$\nu = -i\frac{(\Gamma + \kappa)}{4} \pm \sqrt{\frac{g^2}{4} - \left(\frac{\Gamma - \kappa}{4}\right)^2}. \quad (7.30)$$

In order that Rabi oscillations exist, it is thus necessary that $g > (\Gamma - \kappa)/2$. In order that they should be visible before decay has significantly reduced the amplitude, it is necessary that one has $g > \kappa, \Gamma$. In this limit, one has strong coupling, and decay is indeed very strongly modified.

7.3 Examples of Cavity QED systems

Having discussed the role of the parameters g, κ, Γ , and their relation to whether a system is weak or strong coupling, we now consider a variety of experimentally studied cavity QED systems, discussing the values of these parameters, and the advantages/disadvantages of the different systems. A summary of the characteristic values is given in table 7.1; the discussion below highlights the origins of some of these parameters.

System	Atom[1, 18] Optical	Atom[4] Microwave	SC qubit[3] Microwave	Exciton[5] Optical
$\kappa/2\pi$	1 MHz	1 kHz ^a	30 MHz	30 GHz
$\Gamma/2\pi$	3 MHz	30 Hz	3 MHz ^b	0.1 GHz
$g/2\pi$	10 MHz	50 kHz	100 MHz	100 GHz
$\omega/2\pi$	350 THz	50 GHz	10 GHz	400 THz
λ^3/V	10^{-5}	10^{-1}	$\lambda/a = 1$	10^{-1}
Q	10^8	10^8	10^4	10^4
Other t_{\max}	$t_{\text{flight}} \sim 100\mu\text{s}$		$1/T_2 \sim 3\text{MHz}$	$1/T_2 \sim 3\text{GHz}$

^aRecent work reaches 10Hz[19]

^bNot well known, dominated by cavity induced decay

Table 7.1: Characteristic energy scales for different cavity QED realisations, as discussed in the text.

Optical transitions of atoms

We first consider real atoms, and cavities designed to be resonant for optical transitions of these atoms. In this case, the background atomic decay rate Γ' is determined by the intrinsic properties of the atomic transition, reduced by the geometry of the cavity, and so this sets the scale that must be overcome for strong coupling. High quality mirrors at optical wavelengths can be made using dielectric Bragg mirrors. These consist of alternating layers of materials with different dielectric constant, with spatial period $\lambda/4$, so that even if the dielectric contrast between adjacent layers is not

sufficient to cause strong reflection, the interference of multiple reflections will lead to strong overall reflection.

To allow access for atoms to enter the cavity, the cavity length must be much larger than $\lambda \simeq 0.8\mu\text{m}$, and a typical size[1, 18] is a cavity length $L \sim 200\mu\text{m}$, and beam waist $w \sim 20\mu\text{m}$, which gives a ratio $\lambda^3/V \sim 10^{-5}$. This small ratio means that a very high Q factor is needed to reach strong coupling, and $Q \sim 10^8$ is possible in current experiments. Typical values of the corresponding g, Γ, κ are given in Table 7.1. In addition to the timescales included within Eq. (7.17), in some experimental systems, there is another timescale, that of the time for an atom to leave the cavity, however this is typically larger than all other timescales.

Microwave transitions of atoms

Remaining with atoms as the “matter” part of the Jaynes-Cummings Hamiltonian, some of the problematic features above can be removed if one considered instead microwave frequency transitions of the atoms. This increases the wavelength, allowing cavities comparable to the wavelength, and can also significantly improve the cavity quality. For microwave frequencies, superconducting cavities can be used (since the frequency is less than the BCS gap, $\Delta_{BCS} \sim \text{THz}$, so the superconducting mirrors are diamagnetic at these frequencies. The remaining limitation on cavity quality instead comes from scattering off the mirror surface, and any gaps required in the cavity for atom injection. For niobium cavities operating at 50GHz ($\lambda \simeq 6\text{mm}$), values of $\lambda^3/V \simeq 0.1$ and $Q \sim 10^{10}$ are possible[19].

The relevant atomic transitions at these frequencies are transitions between highly excited atomic states (Rydberg states). For states with $n, l \gg 1$, the outermost electron remains far from the nucleus and so sees only the screened charge of +1, so the energy levels of such orbits are almost Hydrogenic. For large n but $l \simeq 0$, the orbit is perturbed by the enhanced potential near the nucleus, which can be incorporated by a quantum defect, δ_l to write:

$$E_{n,l} \simeq -\frac{\mathcal{R}y}{(n - \delta_l)^2}, \quad (7.31)$$

with $\delta_l \rightarrow 0$ for large l . It is thus clear that transitions between $n, n - 1$ will have small energies, $\Delta E_n \propto 1/n^3$, and so for $n \simeq 50$ one can have microwave frequency transitions. Atoms can be prepared in such states by using tuned laser pulses.

As well as allowing better cavities at these frequencies, Rydberg atoms also have significantly reduced spontaneous decay rates without sacrificing coupling to the cavity mode. The total decay rate from atomic state nl is given by:

$$\Gamma_{\text{tot}} \propto \sum_{n'l'} \omega_{nl,n'l'}^3 |d_{nl,n'l'}|^2. \quad (7.32)$$

Selection rules imply that $l' = l \pm 1$, which means that two different limits of decay rates exist, depending on whether $l = n - 1$ or $l \ll n$, as illustrated in Fig. 7.3.

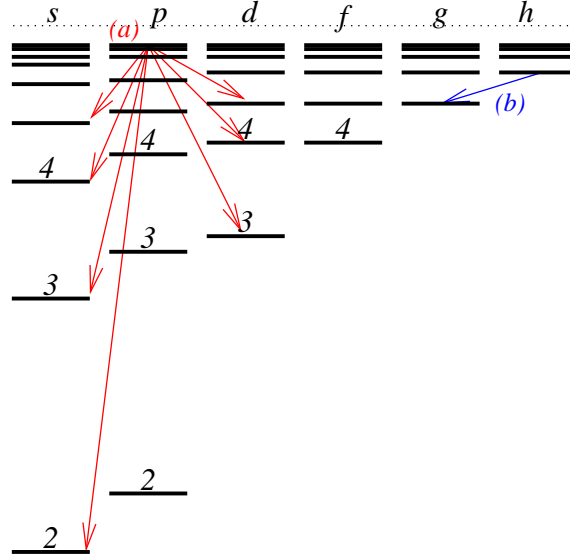


Figure 7.3: Cartoon of atomic level scheme, and transitions from a highly excited state with (a) $l \ll n$, or (b) $l = n - 1$. Note that the actual values used are of the order $n, l \sim 50$, much larger than shown here.

$l \ll n$ In this case, decay to almost all other values $n' < n$ is possible (see arrows (a) in Fig. 7.3). We may consider the characteristic decay rate to nearby levels $n' \simeq n$, or to the ground state $n' = 1$.

- Decay to $n' \simeq n$ has $\omega \propto 1/n^3$, and the dipole matrix element depends on the characteristic size of the orbit for large n , i.e. $d \propto n^2$ (this holds because both initial and final states have similar extensions). These combine to give $\Gamma_{n' \simeq n} \propto n^{-5}$.
- Alternatively, for transitions to the ground state, ω is n independent, being more or less fixed at the the Rydberg energy. In this case the dipole matrix element is however much smaller, as it now involves the overlap between a large Rydberg state and a much smaller ground state orbital. Thus, the dipole matrix element depends on the overlap $d \propto 1/\sqrt{n^3}$, giving $\Gamma_{n' \simeq 1} \propto n^{-3}$.

As such, the latter process dominates for large n , and so the overall decay rate is a factor n^{-3} slower than for low excitation states.

$l \simeq n - 1$ In this case, the only transition allowed by selection rules is to $n' = n - 1$. Thus, this has the decay rate in the first part of the above case, $\Gamma \propto n^{-5}$ and is yet even slower than for $l \ll n$. These states with $l = n - 1$ are known as circular Rydberg states, as they correspond closely to classical circular atomic orbits, as expected from the correspondence principle.

While the decay rates are significantly reduced by the reduced energy of nearby transitions, and reduced overlap of remote transitions, the coupling

to confined radiation for nearby transitions is not reduced in the same way. This is because the continuum density of states does not enter the calculation of g . Instead, one has $g \propto \sqrt{\omega}d$ for resonant transitions, giving $g \propto \sqrt{n}$, which increases with n . However, compared to optical transitions, the value of g is significantly reduced because $g \propto 1/\sqrt{V}$, and the cavity has $V \simeq \lambda^3$. Thus, the value of g in table 7.1 is reduced compared to optical transitions, but is reduced by a smaller factor than the reduction of the atomic decay rate.

Superconducting qubits in microwave resonators

Continuing with microwave frequencies, one can also consider “artificial atoms” coupled to microwave cavities; these have the advantage of considerably increasing the coupling strength, by (in this case) increasing the number of electrons involved in the artificial atom.

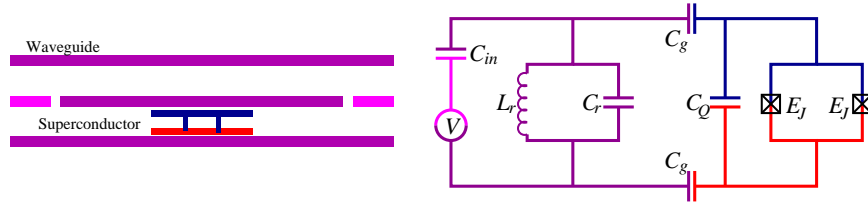


Figure 7.4: Schematic diagram of superconducting qubit capacitively coupled to a stripline microwave resonator (left), and the equivalent circuit (right), coloured for comparison of diagrams — after Koch et al. [20].

Figure 7.4 illustrates one schematic design of a superconducting island capacitively coupled to a stripline resonator. This particular design is known as the “transmon” qubit[20], as the atomic states corresponds to quantised modes plasmon oscillations, modified by the transmission line. The equivalent circuit shows that the artificial atom consists of a pair of Josephson junctions, shunted by a large capacitance. The pair of junctions exists so that one may tune the effective Josephson coupling via a magnetic flux, giving $E_{J,\text{eff}} = E_J \cos(\Phi_B/\Phi_0)$, giving:

$$H_{\text{atom}} = \frac{e^2(n_Q - n_{\text{bias}})^2}{2C_{Q,\text{eff}}} - E_{J,\text{eff}} \cos(\phi_Q), \quad (7.33)$$

where the number and phase operators obey canonical commutation relations. This qubit is capacitively coupled to the resonator mode, i.e. the charge difference across the resonator conductors voltage biases the qubit. This coupling can be engineered to be relatively large, of the order of 100MHz. In other designs of qubit, the coupling is reported to be able to be made yet larger, such that $g \geq \omega$. This limit is referred to as ultra-strong coupling, in which the rotating wave approximation is invalid.

Unlike the microwave system with Rydberg atoms, where atoms eventually leave the cavity and can be measured, the observation of these superconducting systems is typically via the emitted radiation. For this reason

κ is chosen to be larger than physical constraints would require, so that sufficient photons escape to allow measurement. The origin of the artificial atom decay rate in these experiments is not particularly clear, since coupling to bulk radiation modes should be negligible. However, since the system is in a solid state environment, other degrees of freedom exist that can lead to relaxation. These other degrees of freedom certainly lead to dephasing, so that in these systems, $1/T_2$ is a significant rate, arising from charge and flux noise on the superconducting circuit.

Quantum-dot excitons in semiconductor microstructures

Finally, we consider artificial atoms at optical frequencies. One example of this concerns excitons in quantum dots, coupled to semiconductor microstructures. The exciton states can be regarded as Hydrogen-like wavefunctions of electrons and holes, with a reduced binding energy, and increased Bohr radius due to the dielectric screening $\epsilon_{\text{rel}} \simeq 10$ and reduced electronic mass. Although the excitonic binding energy is much less than optical frequency, the relevant transition is the creation of a bound exciton, which corresponds to the band gap less the binding energy. Typical semiconductor band gaps are of the order of 1eV , leading to optical frequencies.

Compared to real atoms, the notable improvement in table 7.1 is the significantly enhanced g ; this arises from the increased Bohr radius, and hence larger dipole matrix element, as well as the much reduced mode volume. As the quantum dots are fixed inside the semiconductor, microstructures can be grown with characteristic sizes comparable to the wavelength of light. The mechanism of light confinement used for such cavities varies, combining one or more of dielectric contrast, Bragg mirrors, or photonic band gap materials. Dielectric contrast can produce reasonable confinement for whispering gallery modes (WGM) in circular resonators; in this case the WGM makes a shallow angle of incidence with the edge of the cavity, and so is confined by total internal reflection. Cavity decay rates are however typically much larger than cavities used for atoms, due to the lower quality of mirrors in integrated semiconductor heterostructures. The exciton decay rate is much larger than for real atoms due to the solid state environment. Just as for superconducting artificial atoms, this again leads both to reduced T_1 and also T_2 dephasing.

7.4 Further reading

For a discussion of the Purcell effect, and the crossover between weak and strong coupling, see e.g. Meystre and Sargent III [12]. The discussion of particular cavity QED systems in this chapter is based on various reviews: for Rydberg atoms, see Raimond et al. [4] or Gallagher [21]; for superconducting qubits, see Blais et al. [3]; for excitons in semiconductor microcavities, see Khitrova et al. [5].

Questions

Question 7.1: Pseudo-mode decay rate.

Consider a standing wave confined within the cavity, i.e. $\psi_0(x) = \sqrt{2/a} \cos[(2n + 1)\pi x/a]$. By considering the decomposition of this wave onto the true eigenmodes of the extended system (as calculated in Sec. 7.1), show that the time evolution of the overlap $|\langle \psi_0 | \psi_0(t) \rangle|$ decays at rate $\kappa/2$, as defined in Eq. (7.14).

Lecture 8

Collective effects of two-level atoms: open systems, superradiance

So far in these lectures we have either considered behaviour of a single two-level atom, or if we have considered ensembles then we have assumed the atoms act independently. However, as first pointed out by Dicke[22], this approach cannot be correct if the atoms are close together, as they see the same electromagnetic modes with sufficiently similar phases, and so emission is a collective process. This collective behaviour can significantly change how decay occurs, even when the electromagnetic modes are treated as a Markovian bath, with short memory times.

In this lecture we will consider the case of many atoms coupled to a continuum of radiation modes (i.e. without a cavity); the next lecture will consider collective effects of many atoms with a cavity.

8.1 Simple density matrix equation for collective emission

We may begin by considering coupling between a collection of atoms and radiation modes:

$$H_{SR} = \sum_k \sum_i \frac{g_k}{2} \left(\sigma_i^- a_k^\dagger e^{i(\omega_k - \epsilon)t - i\mathbf{k}\cdot\mathbf{r}_i} + \sigma_i^+ a_k e^{i(\epsilon - \omega_k)t + i\mathbf{k}\cdot\mathbf{r}_i} \right). \quad (8.1)$$

Repeating the derivation of the system density matrix equation, we recover:

$$\begin{aligned} \frac{d}{dt}\rho(t) = & - \sum_{ij} \int^t dt' \sum_k \left| \frac{g_k}{2} \right|^2 e^{i\mathbf{k}\cdot(\mathbf{r}_i - \mathbf{r}_j)} \\ & \times \left[\sigma_i^+ \sigma_j^- \xi_k^* \rho - \sigma_j^- \rho \sigma_i^+ (\xi_k + \xi_k^*) + \rho \sigma_i^+ \sigma_j^- \xi_k \right]. \quad (8.2) \end{aligned}$$

Two extremes of this equation exist; if $|r_i - r_j| \gg \lambda_0$, then the atoms will act independently. In the other extreme, we may neglect all phase factors,

in which case we can introduce:

$$\mathbf{J} = \sum_i \sigma_i \quad (8.3)$$

and write:

$$\frac{d}{dt}\rho(t) = -\frac{\Gamma}{2} [J^+ J^- \rho - J^- \rho J^+ + \rho J^+ J^-]. \quad (8.4)$$

This second limit is what we will consider in this section. We must however note that when separations are small, one must also consider the effects of Coulomb interactions between the two-level systems. In the Coulomb gauge, this means the explicit Coulomb term will have an effect; in the dipole gauge, the problem is instead that long-wavelength modes (for which phase factors *can not* be neglected) describe the effects of Coulomb interaction, leading to important energy shifts. We will discuss this effect in section 8.2. Putting aside these issues for the moment, let us study the consequences of Eq. (8.4)

Dicke Enhancement of emission rate

A simple understanding of how collective emission differs from independent emission can be found by considering eigenstates of $|J\rangle, J^z$. These are spin states and so obey:

$$J^2|J, M\rangle = J(J+1)|J, M\rangle \quad (8.5)$$

$$J^z|J, M\rangle = M|J, M\rangle \quad (8.6)$$

$$J^-|J, M\rangle = \sqrt{J(J+1) - M(M-1)}|J, M-1\rangle \quad (8.7)$$

We may then use these states as a basis for the density matrix equation, and writing equations for the diagonal elements $P_{MJ} = \rho_{MJ,MJ}$ we find:

$$\begin{aligned} \frac{d}{dt}P_{M,J} &= -\Gamma[J(J+1) - M(M-1)]P_{M,J} \\ &\quad + \Gamma[J(J+1) - M(M+1)]P_{M+1,J} \end{aligned} \quad (8.8)$$

The rate of emission is thus $I = \Gamma[J(J+1) - M(M-1)] = \Gamma(J+M)(J-M+1)$, whereas if incoherent, the total emission rate would be ΓN .

If we consider cases where $J = N/2$, if J takes the maximum value possible consistent with the number of two-level systems, then the radiation rates at characteristic values of M are:

$$\begin{aligned} M &= \frac{N}{2} & I &= \Gamma N \\ M &= 0 & I &= \Gamma \frac{N^2}{4} \\ M &= -\frac{N}{2} + 1 & I &= \Gamma N \end{aligned} \quad (8.9)$$

Thus, near $M = 0$, there is macroscopic enhancement of the emission rate compared to the incoherent case. Even when $M = 1 - N/2$, the emission is much greater than it would be for a single excitation if considering independent atoms.

Solving the simple model

The above estimates indicate the general idea of superradiance; as M decreases, the collective effects increase in importance, leading to an increasing rate of transitions through the super-radiant cascade. Our aim in the following will be to describe this cascade. Before discussing a tractable approximation scheme, we may note that the Laplace transformed equations can be straightforwardly solved to give:

$$\begin{aligned} [s + \Gamma 2J]P_{J,J}(s) &= 1 \\ [s + \Gamma(J + M)(J - M + 1)]P_{M,J}(s) &= \Gamma(J + M + 1)(J - M)P_{M+1,J}(s) \end{aligned}$$

The general result will thus be a sum of exponential decays with different time constants, and prefactors linear in t (degenerate decay rates exist, with $M \leftrightarrow 1 - M$ having the same rate, hence the matrix problem for eigenvalues is defective). While this approach may be appropriate numerically, it does not help to produce a qualitative understanding of the decay, so we will instead concentrate on approximately solving the density matrix equations.

Semiclassical evolution after initial times

As a first approach, we may write a semiclassical approximation for how M evolves. Since J does not change during the time evolution, we will drop the label J , and assume $J = N/2$ throughout.

$$\begin{aligned} \frac{d}{dt}\langle M \rangle &= \sum_M M \frac{d}{dt}P_M = -\Gamma \sum_M [M - (M - 1)](J + M)(J - M + 1)P_M \\ &= -\Gamma \langle (J + M)(J - M - 1) \rangle \end{aligned} \quad (8.10)$$

The semiclassical approximation is to assume that $\langle (M - \langle M \rangle)^2 \rangle \ll \langle M \rangle^2$ so that expectations of products of operators can be represented as products of expectations, and one has:

$$\frac{d}{dt}\langle M \rangle \simeq -\Gamma(J^2 - \langle M \rangle^2) \quad (8.11)$$

We have also assumed here that $J, \langle M \rangle \gg 1$, such an approximation will be seen later to arise naturally whenever the semiclassical approximation is valid.

These equations can be solved by substituting $\langle M \rangle = J \tanh(\chi)$, which lead to $d\chi/dt = -\Gamma J$, hence the general solution is:

$$\langle M \rangle = -J \tanh[\Gamma J(t - t_D)], \quad I = -\Gamma \frac{d\langle M \rangle}{dt} = \Gamma J^2 \operatorname{sech}^2[\Gamma J(t - t_D)] \quad (8.12)$$

Semiclassical results in Heisenberg picture The same results can be recovered in the Heisenberg picture by considering equations of motion for the operators J^z, J^\pm . This approach will be useful later on, when considering an extended system. Using the commutation relations $[J^+, J^-] =$

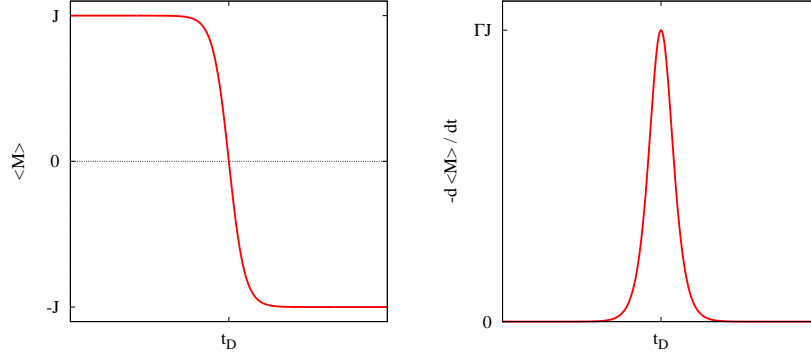


Figure 8.1: Time evolution of $\langle M \rangle$ and the associated rate of radiation calculated semiclassically

$2J^z, [J^z, J^\pm] = \pm J^\pm$, we may write:

$$\frac{d}{dt}\langle J^- \rangle = \text{Tr} \left(J^- \frac{d\rho}{dt} \right) = -\frac{\Gamma}{2} \text{Tr} ([J^- J^+ - J^+ J^-] J^- \rho) = \Gamma \langle J^z J^+ \rangle. \quad (8.13)$$

Similarly, one finds $(d/dt)\langle J^z \rangle = -\Gamma \langle J^+ J^- \rangle$. The semiclassical approximation corresponds again to factorising products of operators, thus yielding:

$$\frac{d}{dt}\langle J^z \rangle = -\Gamma |\langle J^- \rangle|^2, \quad \frac{d}{dt}\langle J^- \rangle = \Gamma \langle J^z \rangle \langle J^- \rangle \quad (8.14)$$

Clearly $\langle J^z \rangle = \langle M \rangle$, and so the previous equations can be recovered by writing $\langle J^z \rangle = J \tanh(\chi)$ along with $\langle J^- \rangle = J \text{sech}(\chi) e^{i\phi}$. The phase ϕ is constant, and is not determined by the equations of motion. While this substitution is most suitable to solve the equations, it is also worth noting (for future reference) that another substitution is more natural in order to understand the behaviour. By noting that the components J^-, J^z represent parts of a collective spin, one is led to write:

$$\langle J^z \rangle = J \cos(\theta), \quad \langle J^- \rangle = J \sin(\theta) e^{i\phi}, \quad \rightarrow \quad \dot{\theta} = \Gamma J \sin(\theta). \quad (8.15)$$

Hence, superradiance corresponds to the collective spin behaving as a damped pendulum, initially in its inverted state.

Early time evolution

In the above description, it is clear that at early times, when fully inverted, the classical equations will not start. However, at early times the semiclassical approximation fails. In order that semiclassics is valid, the distribution of M should not spread too much, and so the M dependence of the time evolution rate should be small compared to the mean evolution, i.e.

$$\left| \frac{d}{dM} (J+M)(J-M+1) \right| = |2M-1| \ll J^2 - M^2. \quad (8.16)$$

It is clear that this is true as long as $|M| \ll J$, thus semiclassical evolution describes the late time evolution, but not the early time. At early times,

M spreads, but at later times no significant spread occurs. This allows us to write:

$$P_M(t) = \int ds P_{J-s}(t_0) \delta \left(M - \langle M(t) \rangle |_{\langle M(t_0) \rangle = J-s} \right). \quad (8.17)$$

i.e., if we find the density matrix at early times, then for each possible value of M arising from it, we may evolve forward semiclassically. The initial state, can be matched to the solution in Eq. (8.12), expanded for small s to give:

$$\langle M(t_0) \rangle = J - s \simeq J \left[1 - 2e^{2\Gamma J(t_0 - t_D)} \right] \quad (8.18)$$

thus the “delay time” appearing in the semiclassical equation is given by:

$$t_D(s) = t_0 + \frac{1}{2\Gamma J} \ln \left(\frac{2J}{s} \right). \quad (8.19)$$

This formula allows one to translate the result of the early time evolution at t_0 into the subsequent evolution of M . Small values of s , describing small deviations from an initially inverted state, correspond to longer delay times; because the inverted state is unstable the initial motion away from this point is exponential growth, hence the logarithmic dependence of delay time on the value of s . To complete the problem, we must find how the distribution of M spreads for early times.

At early times, excitation numbers remain of order $M \simeq J$, i.e. we may write $P_{J=M-s}$ and expand Eq. (8.8) for small s to give:

$$\begin{aligned} \frac{d}{dt} P_s &= -\Gamma(2J - s)(s + 1)P_s + \Gamma(2J - s + 1)sP_{s-1} \\ &\simeq -\Gamma 2J[(s + 1)P_s - sP_{s-1}]. \end{aligned} \quad (8.20)$$

Solving the first two cases, with the initial condition $P_0 = 1, P_{s>0} = 0$, one finds:

$$P_0 = e^{-\Gamma 2Jt}, \quad P_1 = e^{-\Gamma 2Jt}(1 - e^{-\Gamma 2Jt}), \quad \dots \quad (8.21)$$

from which one may guess the general solution:

$$P_s = e^{-\Gamma 2Jt}(1 - e^{-\Gamma 2Jt})^s \quad (8.22)$$

This can be shown to satisfy the equation of motion, i.e.:

$$\begin{aligned} \frac{d}{dt} P_s &= -\Gamma 2J [P_s + s(e^{-\Gamma 2Jt} - 1 + 1)P_{s-1}] \\ &= -\Gamma 2J [(1 + s)P_s - sP_{s-1}]. \end{aligned} \quad (8.23)$$

This is a Bose-Einstein distribution, $P_s \propto z^s$ with $z = 1 - \exp(-\Gamma 2Jt)$ which has mean excitation $\langle s \rangle = z/(1 - z) = e^{\Gamma 2Jt} - 1$, which grows exponentially at early times.

If we choose t_0 such that $2\Gamma Jt_0 \gg 1$, then (for large enough J) it is possible simultaneously to fulfil the condition for semiclassical evolution, and also to expand the above distribution for large t as giving:

$$P_s(t_0) = \exp \left(-\Gamma 2Jt_0 - se^{-\Gamma 2Jt_0} \right) \quad (8.24)$$

Putting together all of the above ingredients, one has the result:

$$P_M(t) = \int ds \exp(-\Gamma 2Jt_0 - se^{-\Gamma 2Jt_0}) \times \delta\left(M + J \tanh\left[\Gamma J\left(t - t_0 - \frac{1}{2\Gamma J} \ln\left(\frac{2J}{s}\right)\right)\right]\right) \quad (8.25)$$

Since t_0 was somewhat arbitrary, this result should not depend on t_0 , and indeed (see question 8.2) it does not. Thus, one can set $t_0 = 0$ and find a full semiclassical approximation of the time evolution of P_M .

Early time evolution for Heisenberg picture The above analysis of P_M for short times has an equivalent manifestation in the Heisenberg picture. For early times, one cannot factorise $\langle J^+ J^- \rangle$ as $|\langle J^- \rangle|^2$, but instead, one needs to worry about correlations of the operators. This can be treated approximately by assuming that the initial value of $\langle J^- \rangle$ should be drawn from a Gaussian distribution of mean 0, and variance $\langle JJ | J^+ J^- | JJ \rangle = 2J$. These quantum fluctuations are then amplified by the semiclassical behaviour, just as the semiclassical evolution of P_J amplifies small initial differences in M via the different classical trajectories they lead to.

8.2 Beyond the simple model

The above discussion assumed a contradictory set of conditions; it assumed that the atoms are close enough that phase differences between their coupling to light could be neglected, and that also assumed that the Coulomb interactions between the atoms could be neglected. As the following will show, one or other of these conditions can hold, but not both, and their violation will modify the superradiant behaviour.

Coulomb interactions and dephasing

If the atoms remain close together, the above treatment of coupling to propagating radiation modes is correct, but Coulomb interactions between atoms will break the indistinguishability that led to the collective enhancements. Indistinguishability is broken as each atom sees a different environment of Coulomb interactions from its neighbours. The Coulomb interaction can be written as:

$$H_{\text{coulomb}} = \sum_{i>j} \Omega_{ij} \left(\sigma_i^+ \sigma_j^- + \sigma_j^+ \sigma_i^- \right). \quad (8.26)$$

Treating atoms in the dipole approximation, the dipole-dipole coupling is given by:

$$\Omega_{ij} = \frac{|d_{ab}|^2}{4\pi\epsilon_0 r_{ij}^3} \left(1 - 3 \frac{(\hat{\mathbf{d}} \cdot \mathbf{r}_{ij})^2}{r_{ij}^2} \right). \quad (8.27)$$

Before discussing the effect of this term, we may first notice that it depends on rather similar parameters to Γ , and in terms of the characteristic

strength, one may write:

$$\frac{\Omega_{ij}}{\Gamma} \simeq \frac{|d_{ab}|^2}{4\pi\epsilon_0 r_{ij}^3} \frac{3\pi\epsilon_0 c^3}{|d_{ab}|^2 \omega_0^3} = \frac{3}{32\pi^3} \left(\frac{\lambda}{r_{ij}} \right)^3 \quad (8.28)$$

Thus, the same condition required in order to neglect phase differences also implies that $\Omega_{ij} \gtrsim \Gamma$.

Superradiance occurred because of constructive interference between the indistinguishable pathways of atomic de-excitation. The Coulomb term makes atoms distinguishable. Physically this means that while the coupling to light in Eq. (8.4) causes transitions between symmetric atomic states, Coulomb interaction make symmetric superpositions evolve into less symmetric superpositions, thus reducing J .

Coulomb interactions do not however change the excitation level M , it merely transfers excitation between atoms. As such, one may estimate the relative size of effect of the Coulomb term vs M by considering the number of possible states with a given M . When $M = N/2$, only one state is possible, the symmetric state with $J = M = N/2$, thus Coulomb interactions may shift the total energy, but not modify the state. For smaller values of M , one may use the number of representations of N spins with modulus J , given by:

$$\nu_N(J) = \frac{(2J+1)N!}{(N/2+J+1)!(N/2-J)!} \quad (8.29)$$

The number of possible states of given M is the number of representations having $J > M$, i.e. $\sum_{J=M}^{N/2} \nu_N(J)$. This behaviour leads to the following scenario. For $M \simeq J$, Coulomb terms do not significantly effect the decay, but as M approaches J (in fact, when $M \simeq \sqrt{J}$), Coulomb terms rapidly dephase the atoms, transferring the system to states with $J \leq \sqrt{N}$, and thus suppressing superradiance.

If one were to work in the electric Dipole gauge, the equations may appear different, as no static Coulomb term exists in the Hamiltonian. However, in this case, the low k part of the photon mediated interaction describes the same physics. This means that even though a Markovian approximation can be made for the sum in Eq. (8.2) for nearly resonant modes, the low k modes must be treated carefully, and contribute an important imaginary part (energy shift). In contrast, the different form of g_k in the Coulomb gauge means that the low k parts of this sum cause no particular problem, as the Coulomb term is already explicitly included.

Extended systems

To avoid the effects of Coulomb interactions, one may instead consider extended systems, where $r_{ij} > \lambda$, and so Coulomb effects become weaker. However, in this case, we can no longer neglect the phase differences seen by different atoms, and so the evolution will be more complicated than Eq. (8.4).

The simplest possible extended system is a one dimensional system¹. In this case, we may replace the phase factors by $\exp[ik(z_i - z_j)]$, and forget about summation over directions of k . Note that in this replacement, we have chosen to specifically consider forward propagating waves in the z direction. An equivalent description of backward propagating waves will exist, however for simplicity we neglect this complication.

In this approximation, our equation of motion becomes:

$$\begin{aligned} \frac{d}{dt}\rho(t) = & - \sum_{ij} \int^t dt' \sum_k \left| \frac{g_k}{2} \right|^2 e^{ik(z_i - z_j)} \\ & \times \left[\sigma_i^+ \sigma_j^- \xi_k^* \rho - \sigma_j^- \rho \sigma_i^+ (\xi_k + \xi_k^*) + \rho \sigma_i^+ \sigma_j^- \xi_k \right]. \end{aligned} \quad (8.30)$$

and we may note that:

$$\int^t dt' \xi_k^* = \frac{1}{i(\omega_k - \epsilon) + 0^+} = -iP \left(\frac{1}{\omega_k - \epsilon} \right) + \pi\delta(\omega_k - \epsilon) \quad (8.31)$$

In the previous discussion of decay, we have only considered the delta function, and have neglected the principal value part. This can sometimes be valid, as the imaginary part gives an energy shift — i.e. it contributes a term that looks like $[H_{\text{eff}}, \rho]$, describing Hamiltonian dynamics. If this Lamb shift is site independent (or if we only have a single atom) then such a shift can be absorbed into renormalisation of the bare Hamiltonian. However, it can also lead to effective interactions between the atoms. We have already mentioned one example of this; Coulomb interactions in the Dipole gauge originate from the low k contribution of these imaginary parts. In the current case, we work in the Coulomb gauge, so these low k terms are not important, however the imaginary parts of the resonant terms (i.e. for $\omega_k \simeq \epsilon$) will play an important role in the extended system. These terms lead to phase shifts that are responsible for causality emerging in the final result.

To see this causality, we consider:

$$\Lambda_{ij} = \sum_k \left| \frac{g_k}{2} \right|^2 \frac{e^{ik(z_i - z_j)}}{i(\omega_k - \epsilon) + 0^+}. \quad (8.32)$$

Concentrating on nearly resonant terms, we may approximate the density of states by $\Gamma/2\pi$, and extend the sum over k to $\pm\infty$, yielding:

$$\Lambda_{ij} \simeq \frac{\Gamma}{2\pi} \int_{-\infty}^{\infty} cdk \frac{e^{ik(z_i - z_j)}}{i(ck - \epsilon) + 0^+} = \Gamma\Theta(z_i - z_j)e^{ik_0(z_i - z_j)} \quad (8.33)$$

where $k_0 = \epsilon/c$. In the case that we may neglect phases, this becomes $\Gamma/2$, due to the step function. If we phase shift the system operators

¹To be able to truly treat a 1D system, one has two opposing constraints; the system must be narrow enough that there are not significant numbers of transverse modes. At the same time, diffraction must be small enough that propagation remains axial. These constraints are best satisfied by a tube of waist $w \simeq \sqrt{L\lambda}$.

$\sigma_i^- \rightarrow \sigma_i^- e^{-ik_0 z_i}$ then we find:

$$\frac{d\rho}{dt} = -\Gamma \sum_{z_i > z_j} \left(\sigma_i^+ \sigma_j^- \rho - \sigma_j^- \rho \sigma_i^+ - \sigma_i^- \rho \sigma_j^+ + \rho \sigma_j^+ \sigma_i^- \right) \quad (8.34)$$

This describes causality, in that operators must always proceed from small z to larger z . Note that the previous factor of $1/2$ is replaced by the restricted range of the sum. Because we have written a Markovian density matrix equation, we have neglected propagation time of electric fields, and so the above equation only applies for intermediate sized systems. For larger systems Heisenberg equations can be used to extend the range of validity.

To describe the behaviour more concretely, let us consider the Heisenberg equations within this Markovian approximation. Following the procedure of Sec. 8.1 we may write:

$$\begin{aligned} \frac{d}{dt} \langle \sigma_k^z \rangle &= -\Gamma \sum_{z_i > z_j} \text{Tr} \left\{ \left([\sigma_k^z, \sigma_i^+] \sigma_j^- - \sigma_j^+ [\sigma_k^z, \sigma_i^-] \right) \rho \right\} \\ &= -\Gamma \sum_{z_j < z_k} \langle \sigma_k^+ \sigma_j^- + \sigma_j^+ \sigma_k^- \rangle \end{aligned} \quad (8.35)$$

$$\frac{d}{dt} \langle \sigma_k^- \rangle = 2\Gamma \sum_{z_j < z_k} \langle \sigma_k^z \sigma_j^- \rangle. \quad (8.36)$$

These equations cannot directly be factorised if they apply to single two-level systems, but a coarse grained version, in terms of:

$$\mathcal{N}(z) = \sum_i \delta(z - z_i) \langle \sigma_i^z \rangle, \quad \mathcal{P}(z) = \sum_i \delta(z - z_i) \langle \sigma_i^- \rangle, \quad (8.37)$$

can be considered instead. These give the semiclassical approximation

$$\frac{d\mathcal{N}}{dt} = -\Gamma \int^z dz' [\mathcal{P}(z) \mathcal{P}^*(z') + \mathcal{P}(z') \mathcal{P}^*(z)] \quad (8.38)$$

$$\frac{d\mathcal{P}}{dt} = 2\Gamma \int^z dz' \mathcal{N}(z) \mathcal{P}(z') \quad (8.39)$$

If we define $\mathcal{E} = \int^z dz' \mathcal{P}(z')$ then this reduces to the simpler set of equations:

$$\frac{d\mathcal{N}}{dt} = -\Gamma [\mathcal{P}\mathcal{E}^* + \mathcal{E}\mathcal{P}^*], \quad \frac{d\mathcal{P}}{dt} = 2\Gamma \mathcal{N}\mathcal{E}, \quad \frac{d\mathcal{E}}{dz} = \mathcal{P}. \quad (8.40)$$

Here, \mathcal{E} can be interpreted as proportional to the electric field strength, resulting from the integrated polarisation of the medium to the left of the current position. (Were we to have instead written Heisenberg equations from the beginning, and were we to have avoided making the Markovian approximation, then a similar set of equations would have been found, but with the retarded time $\tau = t - z/c$ appearing in place of the t .)

Equations (8.40) can then be understood by the same substitution as in Sec. 8.1, mapping them to a generalised damped pendulum problem. Writing $\mathcal{N} = \mathcal{N}_0 \cos(\theta)$, $\mathcal{P} = \mathcal{N}_0 \sin(\theta)e^{i\phi}$ one has the equations:

$$\frac{d\theta}{dt}e^{i\phi} = 2\Gamma\mathcal{E}, \quad \frac{d\mathcal{E}}{dz} = \mathcal{N}_0 \sin(\theta)e^{i\phi} \quad (8.41)$$

giving the result:

$$\frac{d^2\theta}{dzdt} = 2\Gamma\mathcal{N}_0 \sin(\theta), \quad I \propto |\mathcal{E}|^2 \propto \left| \frac{d\theta}{dt} \right|^2. \quad (8.42)$$

Compared to the small system equation, the principal difference here is that this equation is now second order, i.e. it contains both damping and inertia. For sufficiently long samples, one thus has ringing oscillations, as each atom accumulates a different phase depending on where it is. This causes overshoot, and hence oscillations.

8.3 Further reading

A comprehensive review of superradiance may be found in Gross and Haroche [23].

Questions

Question 8.1: Coherent states of spin

Consider an alternative choice of coherent atomic states, known as coherent states of spin:

$$|\psi(\alpha)\rangle = \mathcal{N}e^{\alpha J_+} | -J, J \rangle. \quad (8.43)$$

8.1.(a) Show that normalisation of $|\psi(\alpha)\rangle$ implies $\mathcal{N}(1 + |\alpha|^2)^J = 1$.

8.1.(b) Evaluate $\langle \psi(\alpha) | J_z | \psi(\alpha) \rangle$ and $\langle \psi(\alpha) | J_+ J_- | \psi(\alpha) \rangle$.

8.1.(c) Thus, compare the rate of photon emission from a coherent state of spin and the Dicke states discussed earlier in this section.

Question 8.2: t_0 independence of superradiance $P_s(t)$ Show that equation (8.25) is independent of t_0 , by evaluating the partial derivative with respect to t_0 and showing it vanishes.

Lecture 9

More collective effects: Cavity system and the Dicke model

In lecture 8, collective atomic decay was discussed for coupling to a continuum of modes. This lecture considers the case of collective atomic behaviour in a cavity. Two particular cases are considered: the first section is on the possibility of spontaneous coherence in thermal equilibrium, and the second section discusses the general time evolution starting from a fully excited atomic state. In the first section we ask whether the model in thermal equilibrium can undergo a phase transition in which the two-level systems polarise, and generate an expectation of the photon field.

9.1 Phase transitions, spontaneous superradiance

In this first section, we will consider the Dicke model without the rotating wave approximation. Written in terms of spin operators σ_i for the two-level systems, this is:

$$H = \sum_i \epsilon \sigma_i^z + \omega a^\dagger a + \sum_i \frac{g}{2} (a^\dagger + a) (\sigma_i^+ + \sigma_i^-). \quad (9.1)$$

We wish to consider the thermodynamics of this model, by considering a mean-field theory, in which we assume a coherent mean-field for the photon state, and find the associated free energy. If the free energy is minimised by a non-zero photon field, (i.e. if such a state has a lower free energy than the vacuum state), then there is a phase transition to a spontaneous superradiant phase. For a coherent photon state $\exp(-|\alpha|^2/2 + \alpha a^\dagger)|0\rangle$, the associated partition function can be written as:

$$\mathcal{Z}[\alpha] = e^{-\beta\omega|\alpha|^2} \prod_i \sum_{\sigma_i} \exp(-\beta [2g\alpha' \sigma_i^x + \epsilon \sigma_i^z]). \quad (9.2)$$

Here, we have written real and imaginary parts of α as $\alpha = \alpha' + i\alpha''$, and have used $\sigma_i^+ + \sigma_i^- = 2\sigma_i^x$. One may then evaluate the sum over states of

the two-level system by transforming to the diagonal basis in the presence of field α' . The eigenvalues of the 2×2 matrices are then $\pm E/2$, with $E = \sqrt{\epsilon^2 + 4g^2\alpha'^2}$, and so the free energy is given by:

$$F(\alpha) = \omega|\alpha|^2 - \frac{N}{\beta} \ln \left[\cosh \left(\frac{\beta}{2} \sqrt{\epsilon^2 + 4g^2\alpha'^2} \right) \right]. \quad (9.3)$$

To determine where a non-zero value of α can minimise F , we should find the derivative of F with respect to α , to locate possible stationary points. Clearly, $dF/d\alpha'' = 2\omega\alpha''$, and so for any stationary point we must have $\alpha'' = 0$. For the real part of α we have instead

$$\frac{dF}{d\alpha'} = 2\omega\alpha' - \frac{N}{\beta} \frac{4g^2\alpha'}{2\sqrt{\epsilon^2 + 4g^2\alpha'^2}} \tanh \left(\frac{\beta}{2} \sqrt{\epsilon^2 + 4g^2\alpha'^2} \right). \quad (9.4)$$

Using the eigenvalue energies E defined above, this may be rewritten in the simpler form:

$$\frac{dF}{d\alpha'} = 2\alpha' \left[\omega - \frac{Ng^2}{E} \tanh \left(\frac{\beta E}{2} \right) \right]. \quad (9.5)$$

A non-zero α' solution exists if the term in brackets can be made to vanish. In order that this vanishes at some temperature $T > 0$, we require that the function $\tanh(\beta E/2)/E$ crosses the value ω/Ng^2 . As illustrated in Fig. 9.1, at small temperatures, $\tanh(\beta E/2)/E \simeq 1/E$, while at large temperatures it becomes $1/2T$. Thus, for a crossing to occur, one requires $1/E > \omega/Ng^2$, and since $E > |\epsilon|$, a transition may exist if $\epsilon < Ng^2/\omega$.

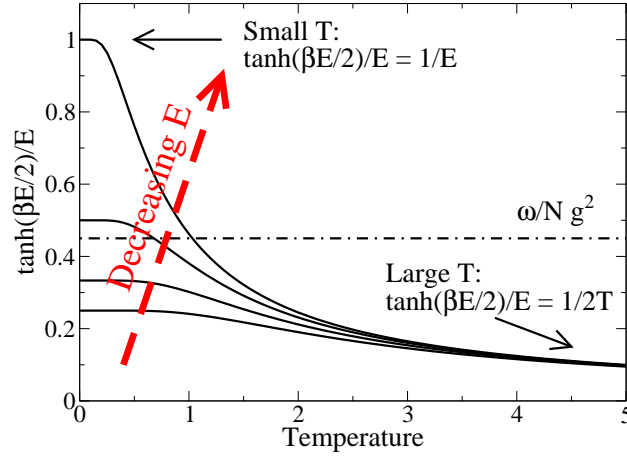


Figure 9.1: Graphical representation of Eq. (9.5). It is clear that for a solution to exist, it is necessary for $1/|\epsilon| > 1/E > \omega/Ng^2$.

Now, recalling the definition of g in terms of microscopic parameters and matrix elements in Eq. (1.39), we have the condition:

$$\epsilon\omega < \frac{N}{V} \left(\frac{2|d_{ab}|^2\epsilon^2}{\epsilon_0\omega} \right). \quad (9.6)$$

Thus, apparently a transition can occur if the density of two-level systems is large enough. This phase transition was originally discussed by Hepp and Lieb [24, 25], and in the form presented here by Wang and Hioe [26].

9.2 No-go theorem: no vacuum instability

If the density of two-level systems is large enough, the Dicke model is invalid, as we have neglected the matter-radiation coupling originating from the A^2 terms. Let us restore these terms, and see at what density of two-level systems they become important. In the same single-mode approximation in which we are treating the quantised radiation, the correction due to these terms can be written following Eq. (1.35) as:

$$\delta H_{A^2} = N\zeta(a + a^\dagger)^2; \quad \zeta = \frac{q^2}{4m\epsilon_0\omega V}. \quad (9.7)$$

(NB, the mass m appearing here for the contribution from a single atom is $1/m = 1/m_e + 1/m_i$, i.e. is the reduced electron-ion mass.) Including this term in the Hamiltonian, the function $F(\alpha)$ acquires an extra dependence on α' , giving

$$F(\alpha) = \omega|\alpha|^2 + \zeta N 4\alpha'^2 - \frac{N}{\beta} \ln \left[\cosh \left(\frac{\beta}{2} \sqrt{\epsilon^2 + 4g^2\alpha'^2} \right) \right]. \quad (9.8)$$

In terms of the condition for a phase transition to occur, this means one should replace $\omega \rightarrow \omega + 4\zeta N$; i.e. the condition is now $\epsilon(\omega + 4\zeta N) < Ng^2$. Physically, this means that the A^2 terms in the Hamiltonian describe a dielectric response of the atoms, opposing large transverse fields; as a result, the energy cost of spontaneous polarisation has increased, and spontaneous effects can arise only if the two-level system susceptibility is larger than it was above. Combining the definitions of ζ and g , we may write the new condition as:

$$\epsilon\omega + \epsilon \frac{N}{V} \left(\frac{q^2}{m\epsilon_0\omega} \right) < \frac{N}{V} \left(\frac{2|d_{ab}|^2\epsilon^2}{\epsilon_0\omega} \right). \quad (9.9)$$

As a function of density of two-level systems, this condition is clearly not satisfied at $N/V = 0$, but will be satisfied for some finite density of two-level systems if and only if:

$$\epsilon \frac{q^2}{m\epsilon_0\omega} < \frac{2|d_{ab}|^2\epsilon^2}{\epsilon_0\omega}. \quad (9.10)$$

Thus, spontaneous polarisation requires that the energy and dipole matrix elements of an atom can satisfy $2|d_{ab}|^2\epsilon > q^2/m$.

Thomas-Reiche-Kuhn sum rule

The energy and dipole matrix elements obey a sum rule, restricting the relative values they may take. This sum rule can be derived from assuming that the internal atomic Hamiltonian depends on internal momentum like

$H_0 \simeq p^2/2m + V(r)$. (Such an assumption was already implicit in our derivation of g .) Then, as before, $p/m = i[H_0, r]$, and so

$$[r, [H_0, r]] = -i\frac{1}{m}[r, p] = \frac{1}{m}. \quad (9.11)$$

Since this identity gives a c-number, the expectation of this identity for any state will match, thus:

$$\frac{q^2}{m} = \langle \psi | [qr, [H_0, qr]] | \psi \rangle. \quad (9.12)$$

Then, inserting a complete set of states $|\phi\rangle$, we have:

$$\begin{aligned} \frac{q^2}{m} &= \sum_{\phi} \left\{ \langle \psi | qr | \phi \rangle \langle \phi | [H_0, qr] | \psi \rangle - \langle \psi | [H_0, qr] | \phi \rangle \langle \phi | qr | \psi \rangle \right\} \\ &= \sum_{\phi} \{ d_{\psi\phi}(E_{\phi} - E_{\psi})d_{\phi\psi} - d_{\phi\psi}(E_{\psi} - E_{\phi})d_{\psi\phi} \} \\ &= 2 \sum_{\phi} |d_{\psi\phi}|^2 (E_{\phi} - E_{\psi}). \end{aligned} \quad (9.13)$$

This is the Thomas-Reiche-Kuhn sum rule, relating the total oscillator strength, weighted by energy, to the charge and mass of particles involved. There is an immediate corollary of this rule; suppose that ψ is chosen to be the ground state. In this case, the right hand side of Eq. (9.13) is a sum of positive terms, and so is greater than any one of its terms. Taking the two lowest levels, one may then write:

$$\frac{q^2}{m} > 2|d_{ab}|^2 \epsilon. \quad (9.14)$$

The sum rule has therefore proved the opposite inequality to the one we require for a phase transition; the sum rule prevents such a phase transition from occurring. This observation was first pointed out by Rzażewski, Wódkiewicz, and Żakowicz [27].

9.3 Radiation in a box; restoring the phase transition

The above result shows that the phase transition by which the vacuum state becomes unstable, and spontaneously polarises is an artefact of that model, and not physical. This however is not the end of the story. There is a simple extension to the Dicke model, which is appropriate in a variety of recent works on confined quantum optical systems, which can restore the phase transition. That extension is to consider a closed system with a density of excitations. This means considering the at-first counterintuitive idea of a chemical potential for photons.

The idea of a chemical potential for photons is not generally considered because photons cannot easily be confined, and so a fixed density of photons

is hard to achieve. Without confinement, photons can be exchanged with the bulk, which acts as a reservoir at zero chemical potential. However, in engineered cavity quantum electrodynamics systems, photon confinement is exactly what is being created, and so for such systems, including polaritons in semiconductor microcavities, Josephson junctions in microwave waveguides, and atoms in superconducting mirror cavities, a chemical potential for photons is a useful concept.

Adding a chemical potential, we have

$$H \rightarrow \tilde{H} = H - \mu M, \quad M = \sum_i \left(\sigma_i^z + \frac{1}{2} \right) + a^\dagger a. \quad (9.15)$$

The net result is to replace $\epsilon \rightarrow \tilde{\epsilon} = \epsilon - \mu$ and $\omega \rightarrow \tilde{\omega} = \omega - \mu$ in the previous Hamiltonian (although the factors of ϵ and ω in the coupling strengths remain unmodified). In this case, it is clear that one can satisfy the condition $\tilde{\epsilon}(\tilde{\omega} + 4\zeta N) < Ng^2$ by ensuring the chemical potential is close enough to the two-level system energy. Since this may be done at any density of two-level systems, we may assume a low density, and so neglect ζ — i.e. neglect the A^2 terms in the Hamiltonian.

Since a transition may occur, it is interesting to find the critical temperature of this transition. From the adapted form of Eq. (9.5), this condition is:

$$(\epsilon - \mu)(\omega - \mu) = Ng^2 \tanh\left(\frac{\epsilon - \mu}{2T}\right). \quad (9.16)$$

This can be combined with the equilibrium expectation of the number of excitations $\langle M \rangle$ in the absence of an expectation of α , which gives.

$$\frac{\langle M \rangle}{N} = \frac{1}{2} \left[1 - \tanh\left(\frac{\epsilon - \mu}{2T}\right) \right]. \quad (9.17)$$

Combining these allows one to plot the phase boundary, as shown in Fig. 9.2. See question 9.1 for further discussion.

9.4 Dynamic superradiance

In this section, we consider the general time evolution of the Dicke model, starting from a fully inverted atomic state. This describes the analog of the superradiance discussed in Sec. 8 in the case where there is a good cavity.

For illustration, we discuss here the simplest case of $\epsilon = \omega$, and so our model is:

$$H = \omega J^z + \frac{g}{2} (a^\dagger J^- + a J^+) + \omega a^\dagger a. \quad (9.18)$$

To study the semiclassical dynamics, we first write the Heisenberg equa-

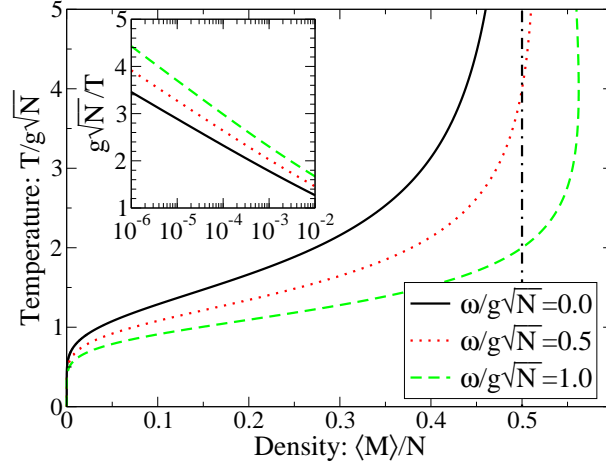


Figure 9.2: Critical temperature vs excitation density, plotted for three different values of ω (measured with respect to ϵ), including the case of $\omega = \epsilon$ discussed in Q. 9.1. The inset shows $1/T$ vs density on a logarithmic scale, illustrating the asymptotic form at low densities.

tions of motion:

$$j^x = i[H, J_x] = -\omega J^y + \frac{g}{2}i(a - a^\dagger)J^z, \quad (9.19)$$

$$j^y = i[H, J_y] = +\omega J^x - \frac{g}{2}(a + a^\dagger)J^z, \quad (9.20)$$

$$j^z = i[H, J_z] = -\frac{g}{2}i(a - a^\dagger)J^x + \frac{g}{2}(a + a^\dagger)J^y, \quad (9.21)$$

$$i\dot{a} = -[H, a] = \frac{g}{2}(J^x - iJ^y) + \omega a. \quad (9.22)$$

The semiclassical approximation is then to replace these equations for non-commuting operators for equations for their commuting expectations, and to factorise the expectations, so $\langle (a + a^\dagger)J^z \rangle \rightarrow \langle a + a^\dagger \rangle \langle J^z \rangle$. Doing this, and writing $\langle a \rangle = \alpha = \alpha' + i\alpha''$ the above equations can be written as:

$$\dot{\mathbf{J}} = \mathbf{h} \times \mathbf{J}, \quad \mathbf{h} = \begin{pmatrix} g\alpha' \\ -g\alpha'' \\ \omega \end{pmatrix}, \quad i\dot{\alpha} = \frac{g}{2}(J_x - iJ_y) + \omega\alpha. \quad (9.23)$$

In order to solve these equations, it is first convenient to transform to a rotating frame. This means substituting $\alpha \rightarrow \alpha e^{-i\omega t}$, and $J^- \rightarrow J^- e^{-i\omega t}$, i.e.

$$\mathbf{J} \rightarrow \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{J}. \quad (9.24)$$

This then gives

$$\dot{\mathbf{J}} = \mathbf{h} \times \mathbf{J}, \quad \mathbf{h} = \begin{pmatrix} g\alpha' \\ -g\alpha'' \\ 0 \end{pmatrix}, \quad i\dot{\alpha} = \frac{g}{2}(J_x - iJ_y). \quad (9.25)$$

At this point, starting in the excited state $\mathbf{J} = J_0 \hat{\mathbf{z}}$, we can make the ansatz:

$$\mathbf{J} = J_0 \begin{pmatrix} 0 \\ \sin(\theta) \\ \cos(\theta) \end{pmatrix}, \quad \dot{\mathbf{J}} = J_0 \dot{\theta} \begin{pmatrix} 0 \\ \cos(\theta) \\ -\sin(\theta) \end{pmatrix}. \quad (9.26)$$

Then, substituting this into the equations of motion, we have:

$$J_0 \dot{\theta} \begin{pmatrix} 0 \\ \cos(\theta) \\ -\sin(\theta) \end{pmatrix} = g J_0 \begin{pmatrix} -\alpha'' \cos(\theta) \\ -\alpha' \cos(\theta) \\ \alpha' \sin(\theta) \end{pmatrix}, \quad (9.27)$$

which implies $\alpha'' = 0$ and $\dot{\theta} = -g\alpha'$. Then, the equation for $\alpha = \alpha'$ gives:

$$\dot{\alpha}' = -\frac{gJ_0}{2} \sin(\theta). \quad (9.28)$$

Thus, the angle defining the Bloch vector obeys the equation of motion:

$$\ddot{\theta} = -g\dot{\alpha}' = \frac{g^2 J_0}{2} \sin(\theta). \quad (9.29)$$

This is the equation of motion for an (initially) inverted pendulum, and so the Bloch vector makes traces out great circles passing through the entirely inverted, and entirely empty states. One should compare this second order equation for dynamical superradiance in a cavity to the first order equation in the decaying case discussed in Sec. 8.1 of the previous lecture.

When non-resonant, i.e. $\epsilon \neq \omega$, the comparable equations are more complicated (see question 9.2), however the general result is similar: after transformation to a rotating frame, the Bloch vector traces out circles (see Fig. 9.3), and can always be mapped to the problem of an inverted pendulum.

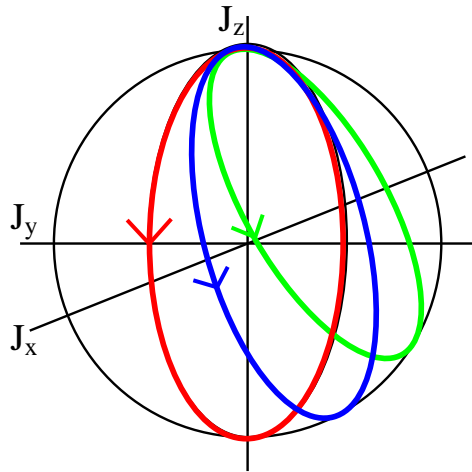


Figure 9.3: Bloch sphere, and paths that Bloch vector takes in the rotating frame, in the general (non-resonant) case.

9.5 Further Reading

The original phase transition of the Dicke model is discussed in Refs.[24–26]. The “no-go” theorem is introduced in Ref.[27] and elaborated further by, e.g. Ref.[28]. The phase transition of a model with a chemical potential is discussed by Eastham and Littlewood [29, 30].

As hinted to in the discussion on dynamic superradiance, the Dicke model is integrable, and this integrability extends even to the case of a distribution of two-level system energies:

$$H = \sum_i \epsilon_i \sigma_i^z + \omega a^\dagger a + \sum_i \frac{g}{2} \left(a^\dagger \sigma_i^- + a \sigma_i^+ \right). \quad (9.30)$$

The integrability of this model can be proven by defining a vector function $\mathbf{L}(z)$, having the following properties:

- The modulus of the vector $\mathbf{L}(z)$ is conserved.
- The Taylor expansion in z contains as many powers of z as there are two-level systems.

Together, these mean that $\mathbf{L}(z)$ contains sufficient conserved quantities that the system is integrable. This method is discussed in specific case of the Dicke model by Yuzbashyan, Kuznetsov, and Altshuler [31]. The history of this integrability, and its relation to a wider class of models is discussed in the review article by Dukelsky, Pittel, and Sierra [32]. The time-dependent solution, starting from the excited state is given by the same method used in question 9.2 by Barankov and Levitov [33].

Questions

Question 9.1: Asymptotic form of phase boundary

9.1.(a) By combining Eq. (9.16) with Eq. (9.17), show that at low densities, in the resonant case ($\epsilon = \omega$), one may write:

$$T_c \simeq \frac{-g}{\ln(\langle M \rangle / N)}. \quad (9.31)$$

9.1.(b) Find the low density asymptote in the more general case $\epsilon \neq \omega$.

Question 9.2: Non resonant Bloch vector precession

Consider the non-resonant case of the Dicke model:

$$H = \epsilon J^z + \frac{g}{2} \left(a^\dagger J^- + a J^+ \right) + \omega a^\dagger a. \quad (9.32)$$

9.2.(a) Show that the semiclassical equations of motion may be written as:

$$i\dot{J}^- = \epsilon J^- - 2g\alpha J^z, \quad (9.33)$$

$$i\dot{J}^z = g(J^+\alpha - J^-\alpha^*), \quad (9.34)$$

$$i\dot{\alpha} = gJ^- + \omega\alpha. \quad (9.35)$$

9.2.(b) By making a change to a rotating basis, but using $\alpha \rightarrow \alpha e^{-i\eta t}$, $J^- \rightarrow J^- e^{-i\eta t}$, show that this changes the above equations by replacing $\omega \rightarrow \tilde{\omega} = \omega - \eta$ and $\epsilon \rightarrow \tilde{\epsilon} = \epsilon - \eta$.

The solution to these equations, analogous to the one discussed in Sec. 9.4 has the special property that α is real. Making the ansatz $J^- = A\alpha + iB\dot{\alpha}$, $J^z = J_0 - C\alpha^2$, and using $\Im(\alpha) = 0$, show the following:

9.2.(c) The imaginary part of the equation for J_- implies $A = \tilde{\epsilon}B$;

9.2.(d) The equation for J_z implies $C = gB/2$;

9.2.(e) The equation for α implies $gB/2 = 1$, and $\tilde{\omega} + \tilde{\epsilon} = 0$

The last of these conditions fixes η , i.e. $\eta = (\omega + \epsilon)/2$. This leaves only the real part of the equation of motion for J_- .

9.2.(f) Show that this equation has a first integral, and that this first integral is equivalent to the condition that $|J|^2 = J^+J^- + (J^z)^2$ is a constant of the motion.

9.2.(g) Using the initial condition $|J|^2 = J_0^2$, show that this final equation may be written as:

$$\dot{\alpha} = \frac{g}{2}\alpha\sqrt{\alpha_0^2 - \alpha^2}, \quad \alpha_0^2 = 2J_0 - 4\left(\frac{\tilde{\epsilon}}{g}\right)^2 \quad (9.36)$$

9.2.(h) Thus, writing $\alpha = \alpha_0 \sin(\theta/2)$, show that the equation of motion has the form::

$$\ddot{\theta} = \frac{g^2}{2} \left[J_0 - 2\left(\frac{\tilde{\epsilon}}{g}\right)^2 \right] \sin(\theta) \quad (9.37)$$

9.2.(i) Find the form of J^z in terms of θ , and compare to the solution in the resonant case.

Lecture 10

Lasers and micromasers

In this lecture, we move to discuss a somewhat more classical system, the laser. The laser is classical in the sense that far above the lasing threshold, it produces large coherent states with classical correlations, and it is based on stimulated emission of radiation, which can be described classically. However, we will discuss not only simple lasers, for which reasonable classical descriptions may exist, but also more quantum mechanical systems, such as the micromaser, in which two-level atoms fall, one at a time, through a cavity, and in which the quantum mechanical evolution of each atom matters.

10.1 Density matrix equations for a micromaser and a laser

The aim of this section will be to write down a density matrix equation describing a laser, and to find the steady state of this equation. The laser will consist of a cavity mode, coupled both to the continuum of modes outside the cavity, and to a gain medium that provides photons to the cavity mode. The modes outside the cavity lead to the standard decay term:

$$\left. \frac{d}{dt} \rho \right|_{\text{decay}} = -\frac{\kappa}{2} (a^\dagger a \rho - 2a \rho a^\dagger + \rho a^\dagger a), \quad (10.1)$$

which can be written in the Fock basis as:

$$\left. \frac{d}{dt} \rho_{nn'} \right|_{\text{decay}} = -\frac{\kappa}{2} \left[(n + n') \rho_{nn'} - 2\sqrt{(n+1)(n'+1)} \rho_{n+1, n'+1} \right]. \quad (10.2)$$

For the gain medium, we will begin our discussion by discussing a formalism that can apply both to few atom lasing systems, and also to the typical incoherent gain medium, which will be introduced in the next section.

Gain from two-level atoms

To give a model of gain that encompasses both coherent dynamics of a single injected atom (for a micromaser), as well as describing incoherent evolution of an externally pumped atom (for a laser), we will describe the

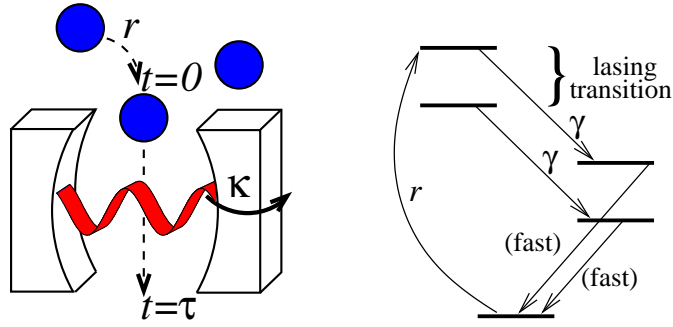


Figure 10.1: Left: Micromaser scheme, atoms in their excited state are injected at a rate r , and fall through the cavity in time τ . Right: Relation between a many-level lasing scheme, with decay via intermediate states to the ground state, and pumping to the excited state, and the quasi two-level description, with injection rate r , and two-level system lifetimes distributed according to the decoherence rate γ .

dynamics of the gain medium, allowing for arbitrary rates of decoherence and pumping. We therefore consider a two-level system, which starts in the excited state, and interacts with the cavity for time τ ; new two-level systems are injected at a rate r . If we take τ to be the same for all atoms, we expect to see periodic dependence on τ , as one allows a given number of Rabi oscillations, transferring energy between the two-level system and the cavity. This model is appropriate for atoms falling through a cavity at a fixed speed. If instead τ is drawn from a Poisson distribution, average rate γ , then one recovers the standard laser result; at a rate γ the two-level system decays to some other states, and then rapidly decays to the ground state, and is then re-excited to the excited state of the two-level manifold at a rate r : the net result is loss of coherence at a rate γ , and injection of energy at a rate r . This relation is illustrated in Fig. 10.1.

To find the gain supplied by the two-level systems, we want to find the evolution of the field density matrix. If interaction between the two-level system and the cavity for time τ causes the change $\Delta\rho_{\text{field}}(\tau)$, and new excited two-level systems are created at rate r , then one can write the evolution of the field for either a fixed τ , or a Poissonian distribution as

$$\left. \frac{d}{dt} \rho_{\text{field}} \right|_{\tau} = r \Delta\rho_{\text{field}}(\tau), \quad \left. \frac{d}{dt} \rho_{\text{field}} \right|_{\gamma} = r \int_0^{\infty} \gamma d\tau \Delta\rho_{\text{field}}(\tau) e^{-\gamma\tau}. \quad (10.3)$$

Thus, we require the change of the field state due to evolution for time τ :

$$\begin{aligned} \Delta\rho_{\text{field}}(\tau) &= \text{Tr}_{\text{atom}} [\rho(t_0 + \tau)] - \rho_{\text{field}}(t_0) \\ &= \rho_{\text{field},\downarrow\downarrow}(t_0 + \tau) + \rho_{\text{field},\uparrow\uparrow}(t_0 + \tau) - \rho_{\text{field},\uparrow\uparrow}(t_0). \end{aligned} \quad (10.4)$$

In the second line we have used the fact that the two-level system is by definition excited at $t = t_0$. Now, using results of Sec. 3.2, we can follow

the evolution of a state $|\uparrow, n\rangle$ with n photons to time τ :

$$\begin{aligned} |\uparrow, n\rangle \rightarrow & \left[\cos\left(\frac{\Omega_{n+1}}{2}\tau\right) + i \cos(2\theta) \sin\left(\frac{\Omega_{n+1}}{2}\tau\right) \right] |\uparrow, n\rangle \\ & + i \sin(2\theta) \sin\left(\frac{\Omega_{n+1}}{2}\tau\right) |\downarrow, n-1\rangle, \end{aligned} \quad (10.5)$$

with $\Omega_n = \sqrt{(\epsilon - \omega)^2 + g^2 n}$, $\tan(2\theta_n) = g\sqrt{n}/(\epsilon - \omega)$ as in chapter 3. For simplicity, let us assume resonance, $\epsilon = \omega$, so $\theta = \pi/4$. Then, using this wavefunction evolution, one finds:

$$\begin{aligned} \Delta\rho_{nn'}(\tau) = \rho_{nn'} & \left[\cos\left(\frac{g}{2}\tau\sqrt{n+1}\right) \cos\left(\frac{g}{2}\tau\sqrt{n'+1}\right) - 1 \right] \\ & + \rho_{n-1, n'-1} \sin\left(\frac{g}{2}\tau\sqrt{n}\right) \sin\left(\frac{g}{2}\tau\sqrt{n'}\right). \end{aligned} \quad (10.6)$$

It is apparent this equation describes gain, since it describes transfer of probability from the state with $n-1$ photons to the state with n photons. In distinction Eq. (10.2) describes evolution from $n+1$ photons to n photons. Note also that as anticipated, the fixed time τ would lead to possible cancellation if integer numbers of Rabi oscillations occur. However, since the oscillation period is different for each number of photons, the general result is very complicated, and can lead to a very non-smooth probability distribution.

Gain medium with decoherence — laser limit

Now let us consider the case where τ is a random variable, distributed according to exponential decay, rate γ . Thus, we should average Eq. (10.6) over such a distribution. For simplicity, we will present here only the calculations for the diagonal elements, $p_n = \rho_{nn}$; for the general case, see Q. 10.1.

$$\left. \frac{dp_n}{dt} \right|_{\text{gain}} = r \left[-\left\langle \sin^2\left(\frac{g}{2}\tau\sqrt{n+1}\right) \right\rangle p_n + \left\langle \sin^2\left(\frac{g}{2}\tau\sqrt{n}\right) \right\rangle p_{n-1} \right], \quad (10.7)$$

where:

$$\begin{aligned} \left\langle \sin^2\left(\frac{\alpha}{2}\tau\right) \right\rangle &= \int \gamma d\tau e^{-\gamma\tau} \sin^2\left(\frac{\alpha}{2}\tau\right) = -\frac{\gamma}{4} \int d\tau e^{-\gamma\tau} (e^{i\alpha\tau} + e^{-i\alpha\tau} - 2) \\ &= -\frac{\gamma}{4} \left(\frac{1}{\gamma + i\alpha} + \frac{1}{\gamma - i\alpha} - \frac{2}{\gamma} \right) = \frac{\alpha^2/2\gamma^2}{1 + \alpha^2/\gamma^2}. \end{aligned} \quad (10.8)$$

Thus, the rate of gain due to a such a distribution of atomic coherence times is given by:

$$\left. \frac{dp_n}{dt} \right|_{\text{gain}} = \frac{r}{2} \left[\frac{(g^2 n/\gamma^2)}{1 + (g^2 n/\gamma^2)} p_{n-1} - \frac{(g^2(n+1)/\gamma^2)}{1 + (g^2(n+1)/\gamma^2)} p_n \right]. \quad (10.9)$$

10.2 Laser rate equations

Combining the gain in Eq. (10.9) with the diagonal part of the decay in Eq. (10.2), one finds the equation of motion for the probability distribution:

$$\frac{dp_n}{dt} = \frac{r}{2} \left[\frac{(g^2 n / \gamma^2)}{1 + (g^2 n / \gamma^2)} p_{n-1} - \frac{(g^2 (n+1) / \gamma^2)}{1 + (g^2 (n+1) / \gamma^2)} p_n \right] - \kappa [np_n - (n+1)p_{n+1}]. \quad (10.10)$$

From this equation we can now determine when lasing will occur, as well as the full probability distribution when either well above or well below threshold.

Laser threshold condition

Lasing occurs when the vacuum state becomes unstable, i.e. when $\partial_t \langle n \rangle > 0$. Thus, let us find:

$$\partial_t \langle n \rangle = \sum_n n \partial_t p_n. \quad (10.11)$$

Let us however assume that $g^2 n / \gamma^2$ is small — i.e. that the average population is small compared to $(\gamma/g)^2$, so that the denominators in the gain can be expanded. Then,

$$\partial_t \langle n \rangle = \sum_n n \left\{ \frac{r}{2} \left[\frac{g^2 n}{\gamma^2} \left(1 - \frac{g^2 n}{\gamma^2} \right) p_{n-1} - \frac{g^2 (n+1)}{\gamma^2} \left(1 - \frac{g^2 (n+1)}{\gamma^2} \right) p_n \right] - \kappa [np_n - (n+1)p_{n+1}] \right\}. \quad (10.12)$$

Now, shifting the various sums, one can rewrite this as

$$\partial_t \langle n \rangle = \sum_n p_n \left\{ \frac{r}{2} \left[\frac{g^2}{\gamma^2} [(n+1)^2 - n(n+1)] - \left(\frac{g^2}{\gamma^2} \right)^2 [(n+1)^3 - n(n+1)^2] \right] - \kappa [n^2 - n(n-1)] \right\}, \quad (10.13)$$

which can be written as:

$$\partial_t \langle n \rangle = \left[\frac{r}{2} \frac{g^2}{\gamma^2} - \kappa \right] \langle n \rangle + \frac{r}{2} \frac{g^2}{\gamma^2} - \frac{r}{2} \left(\frac{g^2}{\gamma^2} \right)^2 \langle (n+1)^2 \rangle. \quad (10.14)$$

This can be understood as three terms; the first describes competition between loss and stimulated emission; the second describes spontaneous emission; the third describes the nonlinear susceptibility of the two-level systems, reducing the gain they provide when $\langle n \rangle$ is large. The first and third of these could be recovered from a semiclassical theory of lasing, however the existence of spontaneous emission requires a fully quantum theory. However, if our aim is the lasing condition, this can easily be identified from the first term: stimulated emission outstrips decay if $g^2 > 2\kappa(\gamma^2/r)$, and lasing then occurs.

Steady state probabilities

The coefficients in Eq. (10.10) depend on n in such a way that one may write:

$$\frac{dp_n}{dt} = -B_n p_n + B_{n-1} p_{n-1} - C_n p_n + C_{n+1} p_{n+1}, \quad (10.15)$$

with excitation rate $B_n = r g^2 (n+1) / [\gamma^2 + g^2 (n+1)]$ and decay rate $C_n = \kappa n$. To find a steady state that is valid for all n , we need to find a relation between terms on the right hand side such that this will always vanish. The only consistent equality is to choose:

$$B_n p_n = C_{n+1} p_{n+1} \iff B_{n-1} p_{n-1} = C_n p_n. \quad (10.16)$$

The steady state distribution is then given by:

$$p_n = \prod_{m=1}^n \frac{B_{m-1}}{C_m} p_0 = \prod_{m=1}^n \frac{r(g^2/\gamma^2)}{2\kappa [1 + (g^2 m/\gamma^2)]} p_0, \quad (10.17)$$

with p_0 set by normalisation. This product can in general be expressed in terms of a hypergeometric function. Let us consider two limits where a simpler expressions result.

Below threshold If we are at small powers, then small values of n will be most probable. In this case, one may neglect the term involving $g^2 m/\gamma^2$ in the denominator. Thus, at low pumping rates, below threshold, one has $p_n \propto (r g^2 / 2\gamma^2 \kappa)^n$, which is a thermal distribution, with $\exp(-\beta E_{\text{eff}}) = (r g^2 / 2\gamma^2 \kappa)$, from which one recovers

$$\langle n \rangle = \frac{1}{e^{\beta E_{\text{eff}}} - 1} = \frac{r(g^2/\gamma^2)}{2\kappa - r(g^2/\gamma^2)}, \quad (10.18)$$

in agreement with the small pumping rate limit of Eq. (10.14). Clearly, such an approximation is only valid below threshold, otherwise p_n cannot be normalised. In this low pump limit, there is an “effective pump rate”, limited by the ratio of g/γ , indicating that emission competes with relaxation.

Far above threshold The opposite limit occurs when $\langle n \rangle$ is large enough that the term 1 in the denominator can be neglected. In this case one has $p_n \propto (r/2\kappa)^n / n!$. This is a Poissonian distribution, as one would expect for a coherent state, with average population $r/2\kappa$, set only by the balance of pump rate and decay rate, since strong fields mean that stimulated emission beats any other relaxation process, thus for every excited atom created, half a photon is emitted.

10.3 Laser Linewidth

Our analysis of the laser so far was based on the density matrix equation and its steady states. Just as in lecture 5, we can also calculate the spectrum, by

using two-time correlation functions via the quantum regression theorem. In the current context, we have a density matrix equation for the cavity field, and so we can proceed directly to evaluate:

$$I(\nu) \propto 2\Re \left\{ \int_0^\infty dt e^{i\nu t} \langle a^\dagger(t) a(0) \rangle \right\} \quad (10.19)$$

As in the case of resonance fluorescence, we can thus find this by determining the expectation of $\langle a^\dagger(t) \rangle$, when the initial state at $t = 0$ is taken to result from acting with a on the equilibrium density matrix.

If we have the time-dependent solution of the equation of motion for the density matrix, then we can write

$$\langle a^\dagger(t) \rangle = \text{Tr} \left[a^\dagger \rho(t) \right] = \sum_n \sqrt{n+1} \rho_{n,n+1}(t). \quad (10.20)$$

We will define the shorthand $\rho_{n,n+k}(t) = \rho_n^k(t)$, and so our task is to find $\rho_n^1(t)$. Before writing its equation of motion, let us anticipate how it should behave. Regardless of the initial conditions, at long times this expectation should decay to zero, as it does not conserve the number of photons. However, since the initial conditions are those set by $a\rho_{\text{eqbm}}$, the expectation of $\langle a^\dagger(t) \rangle$ will be non-zero; we are interested in how this function decays in time.

Equation of motion

Combining the off-diagonal gain found in Eq. (10.31) with the decay term from Eq. (10.2) gives an equation of motion for the off-diagonal density matrix that can be written in the form:

$$\frac{d\rho_n^k}{dt} = -A_n^k \rho_n^k + B_{n-1}^k \rho_{n-1}^k + C_{n+1}^k \rho_{n+1}^k. \quad (10.21)$$

Writing $\eta = g^2/\gamma^2$ for brevity, the coefficients are given by:

$$\begin{aligned} B_n^k &= \frac{r}{2} \left[\frac{\eta \sqrt{(n+1)(n+1+k)}}{1 + \eta(n+1+k/2) + \eta^2 k^2/16} \right], \\ C_n^k &= \kappa \sqrt{n(n+k)}, \\ A_n^k &= \frac{r}{2} \left[\frac{\eta(n+1+k/2) + \eta^2 k^2/8}{1 + \eta(n+1+k/2) + \eta^2 k^2/16} \right] + \kappa \left(n + \frac{k}{2} \right). \end{aligned}$$

(see question 10.1 for derivation of the off-diagonal gain terms). Note that the terms B_n^k, C_n^k have been written in such a way that for $k = 0$ they reduce to the previous B_n, C_n discussed above. Whereas for $k = 0$, the term A_n^k is simply related to the B_n^k, C_n^k by $A_n^0 = B_n^0 + C_n^0$, no such simple relation exists in general. Since it was this simple relation that lead to the detailed balance condition for the probability distribution, no equivalent condition exists with $k \neq 0$.

Approximate solution

Although the previous detailed balance solution does not apply to $k \neq 0$, it motivates a possible extension, which gives a solution decaying in time:

$$\rho_n^k(t) = e^{-D_n^k t} \prod_{m=1}^n \frac{B_{m-1}^k}{C_m^k} \rho_0^k, \quad (10.22)$$

where D_n^k is a parameter to be determined self consistently. Substituting this ansatz for ρ_n^k into Eq. (10.21) gives:

$$\begin{aligned} -D_n^k \rho_n^k &= -A_n^k \rho_n^k + B_{n-1}^k \left[e^{-(D_{n-1}^k - D_n^k)t} \frac{C_n^k}{B_{n-1}^k} \rho_n^k \right] \\ &\quad + C_{n+1}^k \left[e^{-(D_{n+1}^k - D_n^k)t} \frac{B_n^k}{C_{n+1}^k} \rho_n^k \right] \end{aligned} \quad (10.23)$$

The terms in brackets are the expressions for $\rho_{n\pm 1}^k$ in terms of ρ_n^k from Eq. (10.22). If $D_n^k \simeq D_{n\pm 1}^k$, then the exponential terms may be dropped, giving:

$$D_n^k = A_n^k - C_n^k - B_n^k. \quad (10.24)$$

Thus, if we solve Eq. (10.24), and find it varies slowly with n , then $D^k \simeq D_n^k$ defines the decay rate of the off-diagonal density matrix. We therefore want to find D^1 to solve for the linewidth. If we assume we are far above threshold, so $\langle n \rangle \gg 1$, then we can expand the square roots in C_n^1 and B_n^1 , to give:

$$\begin{aligned} D &\simeq \frac{r}{2} \left[\frac{\eta(\langle n \rangle + 1 + 1/2) + \eta^2/8 - \eta(\langle n \rangle + 1 + 1/2)}{\eta\langle n \rangle + 1 + \eta(1 + 1/2) + \eta^2/16} \right] \\ &\quad + \kappa \left(\langle n \rangle + \frac{1}{2} - \langle n \rangle \left[1 + \frac{1}{2\langle n \rangle} - \frac{1}{8\langle n \rangle^2} \right] \right) + \mathcal{O}\left(\frac{1}{\langle n \rangle^2}\right) \\ &\simeq \frac{1}{8\langle n \rangle} \left(\frac{r g^2}{2 \gamma^2} + \kappa \right) \end{aligned} \quad (10.25)$$

Thus, using this time-dependence in the two-time correlation, we have:

$$g_1(\tau) \simeq e^{-D\tau}; \quad (10.26)$$

We have been working in an interaction picture, neglecting free evolution of the photon fields at frequency ω . Restoring this, the frequency dependence of the intensity is then given by:

$$I(\nu) = \frac{D}{(\nu - \omega)^2 + D^2} \quad (10.27)$$

Hence the FWHM of this Lorentzian lineshape is given by:

$$2D = \frac{1}{4\langle n \rangle} \left(\frac{r g^2}{2 \gamma^2} + \kappa \right) \quad (10.28)$$

Interpretation of linewidth

This result for the lineshape has a simple interpretation in terms of phase diffusion. If we consider the radiation field of a laser to be described by an ensemble of coherent states, with effectively fixed magnitude $|\alpha| = \sqrt{\langle n \rangle}$, but different phases, then decay of correlations occurs because of growth of phase differences. Each time a photon is incoherently emitted into, or lost from the cavity the phase changes. This is illustrated in Fig. 10.2.

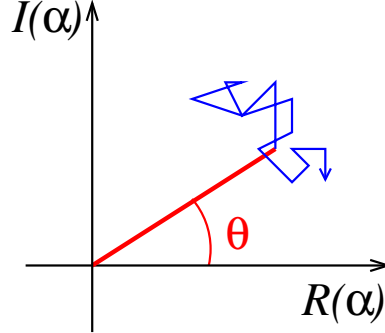


Figure 10.2: Cartoon of coherent state undergoing a random walk due to spontaneous emission events, leading to phase diffusion for the coherent state.

To turn this cartoon into a physical estimate, we start from the rate of such changes per unit time, which is $d = \kappa + rg^2/2\gamma^2$, by adding rates of photon decay and spontaneous emission (hence the appearance of the effective pumping rate, as appearing in the threshold condition, rather than the total pump rate, as in the probability distribution far above threshold). If we assume a random walk, with d events per unit time, the change to the coherent state is governed by the random walk probability distribution:

$$P(\Delta\alpha) = \sqrt{\frac{td}{\pi}} \exp\left(-\frac{(\Delta\alpha)^2}{td}\right) \quad (10.29)$$

However, the variation is entirely phase, so $\Delta\alpha = \sqrt{\langle n \rangle} \Delta\theta$. Using this, the decay of correlations after time τ can be found by averaging the state:

$$\begin{aligned} g_1(t) &= \frac{\langle a^\dagger(t)a(0) \rangle}{|\langle a \rangle|^2} = \langle e^{-i[\theta + \Delta\theta(t)]} e^{i\theta} \rangle \\ &= \int d\theta \sqrt{\frac{td}{\pi}} \exp\left(-i(\Delta\theta) - \frac{\langle n \rangle (\Delta\theta)^2}{td}\right) = \exp\left(-\frac{td}{4\langle n \rangle}\right), \end{aligned} \quad (10.30)$$

which reproduces the previous result, $D = d/4\langle n \rangle$.

10.4 Further reading

The laser rate equations presented here follow closely the presentation in Scully and Zubairy [10]. For a different presentation of laser theory, the review article by Haken [34]. The discussion of laser linewidth in this Chapter is based on Scully and Zubairy [10].

Questions

* **Question 10.1: Off-diagonal matrix elements**

Evaluate the general off-diagonal matrix elements describing gain, described in Eq. (10.6), and thus show that, writing $\rho_{n,n+k} = \rho_n^k$, one has:

$$\begin{aligned} \left. \frac{d\rho_n^k}{dt} \right|_{\text{gain}} &= -\frac{r}{2} \left[\frac{(n+1+k/2)(g^2\gamma^2) + k^2g^4/8}{\gamma^4 + (n+1+k/2)g^2\gamma^2 + k^2g^4/16} \right] \rho_n^k \\ &\quad + \frac{r}{2} \left[\frac{\sqrt{n(n+k)}(g^2\gamma^2)}{\gamma^4 + (n+k/2)g^2\gamma^2 + k^2g^4/16} \right] \rho_{n-1}^k \quad (10.31) \end{aligned}$$

Question 10.2: Second-order correlation functions

Using the quantum regression theorem, find an expression for the second order correlation function:

$$g_2(\tau) = \frac{1}{N} \left\langle a^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a(t) \right\rangle \quad (10.32)$$

in terms of the equations of motion and initial conditions for the variables $p_n = \langle n | a \rho a^\dagger | n \rangle$.

Solve these equations and initial conditions explicitly in the two limits $g \gg \gamma$ and $g \ll \gamma$ (i.e. far above, and far below threshold), and comment on the form of $g_2(\tau)$ in the two cases.

[For values near the threshold, one can solve these equations numerically to find the full evolution of the second order coherence function.]

Lecture 11

More on lasers

In the previous lecture, we developed a quantum theory of the laser and micromaser, in terms of the density matrix of the photon field. The aim of this lecture is firstly to repeat that derivation, this time starting from a more familiar density matrix equation for the coupled system. This will then make clear the approximations required in the results discussed last time. It will also allow one to see the relation between the previous results, and the semiclassical (Maxwell-Bloch) equations.

In discussing the relation of the semiclassical theory, and the results of the full quantum theory, we will see that there is a parameter, β , that controls the ratio of emission into the cavity mode vs all other decay channels of the excited gain medium. When β is small, there is a well defined threshold, and the semiclassical theory can describe this adequately. When β is large (which will occur for strong coupling) the threshold becomes less well defined. To further study the sharpness of threshold, we also consider the uncertainty in photon number, which peaks at threshold. In terms of this quantity, we can also re-analyse the micromaser, looking at its steady states and noise levels, seeing that there are peaks of noise near transitions, but sub-Poissonian noise between them. Armed with these concepts, we will then briefly discuss recent cavity QED realisations of single atom lasers and micromasers, noting the novel features these experiments show.

11.1 Density matrix equation

As in the previous lecture, we will consider an effectively three-level description of the laser gain medium. The description effectively involves only three levels, as we assume recycling to the ground state is fast following any transitions out of the lasing levels, thus only the ground and excited states of the lasing transition, g, e and the true ground state 0 are important. (See Fig. 11.1).

The coherent dynamics in this system is therefore controlled by the simple Hamiltonian:

$$H = \frac{g}{2} \sum_i \left(a \sigma_i^{eg} + a^\dagger \sigma_i^{ge} \right). \quad (11.1)$$

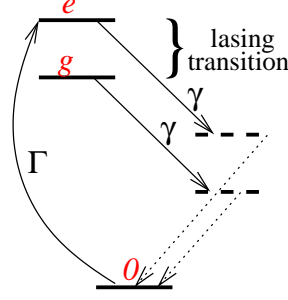


Figure 11.1: Labelling of three level scheme

The notation σ_i^{eg} is the operator that takes atom i from state g to state e . Since we have three active levels, we need this extended notation, rather than the two-level system definitions of σ^\pm, σ^z . In addition to the Hamiltonian, there are incoherent processes describing pumping, decay and dephasing. The description of pumping as purely incoherent would correspond to coupling the $0 \rightarrow e$ transition to an inverted reservoir. Alternatively, it may be seen as an approximation description of coherent pumping to a fourth level — this is discussed further below.

In describing pumping and decay, we assume these processes act on each atom in the gain medium separately. As such, we separate the rate of pumping of an individual atom Γ from the total pumping rate r that appeared in the previous lecture — we will see later what quantity we should identify with r . Also, in distinction to the last lecture, we include both decay of the two lasing levels, and in addition pure dephasing, with a rate γ_{\parallel} . Putting these ingredients together we have:

$$\begin{aligned}
 \partial_t \rho = & -i[H, \rho] - \frac{\kappa}{2} \left(a^\dagger a \rho - 2a \rho a^\dagger + \rho a^\dagger a \right) \\
 & - \sum_i \left\{ \frac{\Gamma}{2} (\sigma_i^{00} \rho - 2\sigma_i^{e0} \rho \sigma_i^{0e} + \rho \sigma_i^{00}) \right. \\
 & + \frac{\gamma}{2} (\sigma_i^{ee} \rho - 2\sigma_i^{0e} \rho \sigma_i^{e0} + \rho \sigma_i^{ee} + \sigma_i^{gg} \rho - 2\sigma_i^{0g} \rho \sigma_i^{g0} + \rho \sigma_i^{gg}) \\
 & \left. - \frac{\gamma_{\parallel}}{4} \left[(\sigma_i^{ee} - \sigma_i^{gg})^2 \rho - 2(\sigma_i^{ee} - \sigma_i^{gg}) \rho (\sigma_i^{ee} - \sigma_i^{gg}) + \rho (\sigma_i^{ee} - \sigma_i^{gg})^2 \right] \right\}.
 \end{aligned} \tag{11.2}$$

In writing this, we have used the fact that each term acts on a single atom to collapse adjacent atomic operators, e.g. $\sigma^{0e} \sigma^{e0} = \sigma^{00}$, hence the simpler form of the pumping and decay terms. In the following, we will drop the subscript i on the atomic degrees of freedom, assuming all atoms are equivalent, and so the sum can be replaced by the number of atoms N_A .

Reduction to photon density matrix

We will now reduce Eq. (8.4), which contains both field and atomic degrees of freedom, to an equation for just the field degrees of freedom. This means

we want to introduce:

$$\rho_\psi = \text{Tr}_{\text{atoms}}(\rho) = \rho_{ee} + \rho_{gg} + \rho_{oo} \quad (11.3)$$

The extra notation ρ_{ij} indicates an operator in the photon space, but a c-number in the atomic degrees of freedom; i.e. we define:

$$\rho_{ij} = \text{Tr}_{\text{atoms}}(\sigma^{ji}\rho) \quad (11.4)$$

where n, m are photon number states. With this notation, taking a trace of Eq. (11.2) over atom states gives:

$$\begin{aligned} \partial_t \rho_\psi = & -i \frac{gNA}{2} \left(a \rho_{ge} + a^\dagger \rho_{eg} - \rho_{ge} a - \rho_{eg} a^\dagger \right) \\ & - \frac{\kappa}{2} \left(a^\dagger a \rho_\psi - 2a \rho_\psi a^\dagger + \rho_\psi a^\dagger a \right). \end{aligned} \quad (11.5)$$

Thus, to proceed, we must also find equations for ρ_{eg} etc. Taking a trace over atoms with a factor of σ_{ge} then gives:

$$\begin{aligned} \partial_t \rho_{eg} = & -\frac{ig}{2} (a \rho_{gg} - \rho_{ee} a) - \frac{\kappa}{2} \left(a^\dagger a \rho_{eg} - 2a \rho_{eg} a^\dagger + \rho_{eg} a^\dagger a \right) \\ & - \frac{\gamma}{2} (1+1) \rho_{eg} - \frac{\gamma_{\parallel}}{4} (1+2+1) \rho_{eg}. \end{aligned} \quad (11.6)$$

At this point, we start to make use of the same approximations as made in the previous lecture; i.e. $\kappa \ll \gamma$. This firstly means we can drop the κ term in the above equation. It secondly means that, for the purpose of substituting into Eq. (11.5), we may find the steady state of the atomic degrees of freedom. Defining $\gamma_t = \gamma + \gamma_{\parallel}$, this gives the condition:

$$0 = \partial_t \rho_{eg} = -\frac{ig}{2} (a \rho_{gg} - \rho_{ee} a) - \gamma_t \rho_{eg}, \quad (11.7)$$

giving the result:

$$\rho_{eg} = -\frac{ig}{2\gamma_t} (a \rho_{gg} - \rho_{ee} a). \quad (11.8)$$

We can now repeat the above logic to find the relation of the diagonal elements $\rho_{ee}, \rho_{gg}, \rho_{oo}$. Dropping the κ terms, and looking for steady states, one find:

$$\begin{aligned} 0 = & -\frac{ig}{2} \left(a^\dagger \rho_{eg} - \rho_{ge} a \right) - \gamma \rho_{gg} \\ = & -\frac{g^2}{4\gamma_t} \left(a^\dagger a \rho_{gg} + \rho_{gg} a^\dagger a - 2a^\dagger \rho_{ee} a \right) - \gamma \rho_{gg} \end{aligned} \quad (11.9)$$

$$\begin{aligned} 0 = & -\frac{ig}{2} \left(a \rho_{ge} - \rho_{eg} a^\dagger \right) - \gamma \rho_{ee} + \Gamma \rho_{oo} \\ = & -\frac{g^2}{4\gamma_t} \left(a a^\dagger \rho_{ee} + \rho_{ee} a a^\dagger - 2a \rho_{gg} a^\dagger \right) - \gamma \rho_{ee} + \Gamma \rho_{oo} \end{aligned} \quad (11.10)$$

$$0 = -\Gamma \rho_{oo} + \gamma (\rho_{ee} + \rho_{gg}). \quad (11.11)$$

Equations (11.9) and (11.10) have made use of the result in Eq. (11.8). Furthermore, Eq. (11.11), along with the definition in Eq. (11.3) implies that:

$$\rho_{00} = \frac{\gamma}{\Gamma + \gamma} \rho_{\psi}, \quad \rho_{ee} + \rho_{gg} = \frac{\Gamma}{\Gamma + \gamma} \rho_{\psi}. \quad (11.12)$$

We then have a closed set of three equations for $\rho_{\psi}, \rho_{ee}, \rho_{gg}$. However, because these equations involve photon operators, it is rather involved to invert them to get a single equation for ρ_{ψ} . To do this inversion, it is easiest to work in the photon number basis. Restricting to the diagonal elements, we have equations for $p_n, p_{e,n}$ and $p_{g,n}$, which become:

$$0 = -\frac{g^2}{4\gamma_t} (2np_{g,n} - 2np_{e,n-1}) - \gamma p_{g,n} \quad (11.13)$$

$$0 = -\frac{g^2}{4\gamma_t} (2(n+1)p_{e,n} - 2(n+1)p_{g,n+1}) - \gamma p_{e,n} + \frac{\Gamma\gamma}{\Gamma + \gamma} p_n. \quad (11.14)$$

Shifting $n \rightarrow n-1$ in Eq. (11.14), it is clear one can combine the two equations, by taking their difference to give:

$$0 = -\frac{g^2}{\gamma_t} n (p_{g,n} - p_{e,n-1}) - \gamma (p_{g,n} - \gamma p_{e,n-1}) - \frac{\Gamma\gamma}{\Gamma + \gamma} p_{n-1}. \quad (11.15)$$

This equation can be solved for $p_{g,n} - p_{e,n-1}$ to give:

$$p_{g,n} - p_{e,n-1} = -\frac{1}{\gamma + g^2 n / \gamma_t} \frac{\Gamma\gamma}{\Gamma + \gamma} p_{n-1}. \quad (11.16)$$

Then, noticing that the atom-photon coupling terms in Eq. (11.9) and Eq. (11.10) add up to give the required term in Eq. (11.5), one has the field equation in the form:

$$\begin{aligned} \partial_t p_n = & -\frac{g^2 N_A}{4\gamma_t} [2n(p_{g,n} - p_{e,n-1}) - 2(n+1)(p_{g,n+1} - p_{e,n})] \\ & - \kappa n p_n + \kappa(n+1)p_{n+1} \end{aligned} \quad (11.17)$$

and substituting Eq. (11.16) recovers the previous field equation:

$$\begin{aligned} \partial_t p_n = & \frac{r}{2} \frac{g^2 n / \gamma_t}{\gamma + g^2 n / \gamma_t} p_{n-1} - \frac{r}{2} \frac{g^2 (n+1) / \gamma_t}{\gamma + g^2 (n+1) / \gamma_t} p_n \\ & - \kappa n p_n + \kappa(n+1)p_{n+1} \end{aligned} \quad (11.18)$$

where $r = N_A \gamma \Gamma / (\gamma + \Gamma)$. Clearly, if $\gamma \gg \Gamma$, then $r \simeq \Gamma N_A$, so Γ is the pumping strength, but if Γ becomes too large, then the pumping rate is limited by the recycling rate γ , as the true ground state becomes depleted. If $\gamma_t = \gamma$, (i.e. if γ_{\parallel} vanishes), the above reduces exactly to the results of the previous lecture.

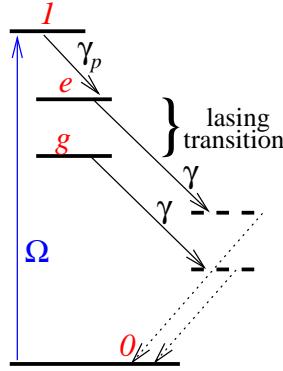


Figure 11.2: Labelling of four level scheme

Four-level scheme

As a brief digression, let us note how a four level scheme, as illustrated in Fig. 11.2 reduces to the above problem in a limit of fast relaxation. In this system, there is also a level 1, with a coherent coupling $H = (\Omega/2)(\sigma^{01} + \sigma^{10})$, and the density matrix equation has an additional term:

$$\partial_t \rho \rightarrow \partial_t \rho - \frac{\gamma_p}{2} (\sigma^{11} \rho - 2\sigma^{e1} \rho \sigma^{1e} + \rho \sigma^{11}). \quad (11.19)$$

The pumping term for ρ_{ee} , which previously was $\Gamma \rho_{00}$ now becomes:

$$\partial_t \rho_{ee}|_{\text{pump}} = \gamma_p \rho_{11} \quad (11.20)$$

If we assume γ_p is very rapid, then in the same spirit as the above derivation we have:

$$\partial_t \rho_{11} = -\frac{i\Omega}{2} (\rho_{01} - \rho_{10}) - \gamma_p \rho_{11}, \quad (11.21)$$

$$\partial_t \rho_{10} = -\frac{i\Omega}{2} (\rho_{00} - \rho_{11}) - \frac{\gamma_p}{2} \rho_{10}, \quad (11.22)$$

and then setting these time derivatives to zero one has:

$$\rho_{10} = \frac{i\Omega}{\gamma_p} (\rho_{11} - \rho_{00}), \quad 0 = -\frac{\Omega^2}{\gamma_p} (\rho_{11} - \rho_{00}) - \gamma_p \rho_{11} \quad (11.23)$$

we can then find ρ_{11}, ρ_{10} in terms of ρ_{00} , which allows us to derive an effective incoherent pump rate of: $\Gamma_{\text{eff}} = \gamma_p \Omega^2 / (\Omega^2 + \gamma_p^2)$. If Ω is small, then $\Gamma_{\text{eff}} \propto \Omega^2 / \gamma_p$, but if pumping is too large, there is an additional constraint on the pump rate, set by the relaxation to e . If γ_p is not large compared to other timescales (i.e. if g were to become very large), then it would no longer be possible to eliminate the dynamics of this level.

Maxwell Bloch equations

Having shown that Eq. (11.2) corresponds to the same physics as discussed in the previous lecture, let us now discuss a different approximate description of Eq. (11.2). This is the semiclassical approximation, which should

be valid when numbers of photons and atoms are large. We will thus use the density matrix equations to write equations for expectations of three operators:

$$\alpha = \langle a \rangle, \quad \mathcal{P} = N_A \langle \sigma^{ge} \rangle, \quad \mathcal{N} = N_A \langle \sigma^{ee} - \sigma^{gg} \rangle, \quad (11.24)$$

corresponding to the electric field, polarisation and inversion respectively. The approximation scheme will be that we replace expectations of products of operators by products of expectations. From the density matrix equations of motion, one immediately finds:

$$\partial_t \alpha = -i \frac{g}{2} \mathcal{P} - \frac{\kappa}{2} \alpha, \quad \partial_t \mathcal{P} = +i \frac{g}{2} \alpha \mathcal{N} - \gamma_t \mathcal{P}. \quad (11.25)$$

For the equation for \mathcal{N} we must write:

$$\begin{aligned} \partial_t \langle \sigma^{ee} \rangle &= -i \frac{g}{2} (\alpha \mathcal{P}^* - \alpha^* \mathcal{P}) + \Gamma \langle \sigma^{00} \rangle - \gamma \langle \sigma^{ee} \rangle \\ \partial_t \langle \sigma^{gg} \rangle &= +i \frac{g}{2} (\alpha \mathcal{P}^* - \alpha^* \mathcal{P}) - \gamma \langle \sigma^{gg} \rangle \\ \partial_t \langle \sigma^{00} \rangle &= -\Gamma \langle \sigma^{00} \rangle + \gamma \langle \sigma^{ee} \rangle + \gamma \langle \sigma^{gg} \rangle \end{aligned}$$

We will make an additional assumption here, that $\Gamma \ll \gamma$, in which case $\langle \sigma^{00} \rangle \simeq 1$, and we can identify $r = \Gamma N_A$. Thus, in this case, the difference of the above equations becomes:

$$\partial_t \mathcal{N} = -ig (\alpha \mathcal{P}^* - \alpha^* \mathcal{P}) + r - \gamma \mathcal{N} \quad (11.26)$$

We may now consider Eq. (11.26) and Eq. (11.24), and look at their steady states, and normal mode spectrum, to understand their behaviour.

Steady states of Maxwell Bloch equations

For a steady state, Eq. (11.25) immediately implies that:

$$\mathcal{P} = i \frac{\kappa}{g} \alpha = \frac{ig}{2\gamma_t} \mathcal{N} \alpha. \quad (11.27)$$

This implies either that $\alpha = 0$ or that $\mathcal{N} = 2\gamma_t \kappa / g^2$. The latter solution (corresponding to lasing) implies that inversion locks at its threshold value; this is known as gain clamping. For the last equation, we have either that:

Below threshold, $\alpha = 0$. In this case, $\partial_t \mathcal{N} = 0$ implies $\mathcal{N} = r/\gamma$ (hence threshold occurs at $r = 2\kappa(\gamma\gamma_t/g^2)$, as found previously).

Above threshold, $\mathcal{N} = \mathcal{N}_{\text{sat}} = 2\gamma_t \kappa / g^2$. In this case, we may use the gain equation as a condition for α , using Eq. (11.27) to eliminate \mathcal{P} , and then writing:

$$0 = -2\kappa |\alpha|^2 + r - \gamma \mathcal{N}_{\text{sat}} \quad \Rightarrow \quad |\alpha|^2 = \frac{\gamma}{2\kappa} \left(\frac{r}{\gamma} - \mathcal{N}_{\text{sat}} \right) \quad (11.28)$$

Thus, above threshold $|\alpha|^2$ increases linearly with pump rate in this mean field theory. This is illustrated in Fig. 11.3.

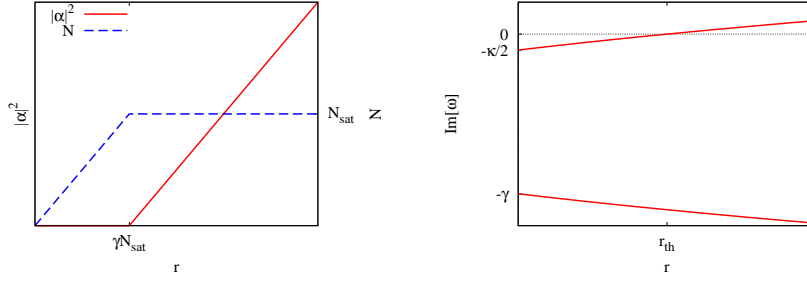


Figure 11.3: Left: Field strength, and inversion arising from mean field theory. Right: Decay rates of fluctuations of Maxwell-Bloch equations.

Fluctuations of Maxwell-Bloch equations

The instability of the non-lasing state at threshold can be understood by considering fluctuations, and linearising the Maxwell-Bloch equations around $\alpha = \mathcal{P} = 0, \mathcal{N} = r/\gamma$. One then finds:

$$\partial_t \begin{pmatrix} \delta\alpha \\ \delta\mathcal{P} \\ \delta\mathcal{N} \end{pmatrix} = \begin{pmatrix} -\kappa/2 & -ig/2 & 0 \\ igr/2\gamma & -\gamma_t & 0 \\ 0 & 0 & -\gamma \end{pmatrix} \begin{pmatrix} \delta\alpha \\ \delta\mathcal{P} \\ \delta\mathcal{N} \end{pmatrix} \quad (11.29)$$

Thus, inversion fluctuations are clearly damped. The photon field and polarisation have collective oscillations at a frequency ω given by:

$$\omega = -i \left(\frac{\kappa}{4} + \frac{\gamma_t}{2} \right) \pm i \sqrt{\left(\frac{\kappa}{4} - \frac{\gamma_t}{2} \right)^2 + \frac{g^2 r}{4\gamma}}. \quad (11.30)$$

This frequency is purely imaginary, so the modes are either growing or decaying¹. If $r > r_{\text{th}}$, one of these roots becomes positive, and such fluctuations will grow. These modes are shown in Fig. 11.3.

If $\gamma_t \gg \kappa$, the two modes are always well separated, and the slow dynamics can be found by eliminating the matter degrees of freedom, writing:

$$\mathcal{P} = \frac{ig}{2\gamma_t} \mathcal{N}, \quad r = \left(\gamma + \frac{g^2}{\gamma_t} |\alpha|^2 \right) \mathcal{N}. \quad (11.31)$$

If we then define $n = |\alpha|^2$, the remaining equation becomes:

$$\partial_t n = \frac{r}{2} \frac{g^2 n}{\gamma\gamma_t + g^2 n} - \kappa n \quad (11.32)$$

This obviously has the same steady states as found above, the fluctuations are not identical, but become so asymptotically if $\gamma_t \gg \kappa$.

¹NB, because of the pumping and decay scheme chosen, inversion is always positive in this system, $\mathcal{N} = r/\gamma$ — a more realistic model, with thermal transitions between e and g might allow for negative inversion. If inversion is large enough and negative, then Rabi oscillations are possible, as in the cavity QED discussion earlier. However, the Rabi splitting collapses when the system is inverted, so these are not relevant in the current model of pumping.

11.2 Spontaneous emission, noise, and β parameter

In the previous section we found a semiclassical equation for n , the mean-field photon number. This equation is valid as long as numbers of photons are large. When numbers of photons are small, spontaneous emission becomes important, and effectively $n \rightarrow n + 1$ in the previous equation. This equation, with $n \rightarrow n + 1$ can also be seen as the result of writing:

$$\partial_t \langle n \rangle = \langle B_n - C_n \rangle \quad (11.33)$$

with B_n, C_n as derived in the previous lecture, and re-derived above. If we make the mean-field assumption $\langle B_n \rangle \simeq B_{\langle n \rangle}$, and we write n for $\langle n \rangle$, then we have:

$$\partial_t n = \frac{r}{2} \frac{g^2(n+1)}{\gamma\gamma_t + g^2(n+1)} - \kappa n \quad (11.34)$$

We will investigate here the effects of this $+1$ — we should be aware however that whenever the effects of this term become significant, so do fluctuations (as we will investigate more fully below), in which case no mean-field theory can be applicable — i.e. it is necessary to study the full probability distribution, not just $\langle n \rangle$. However, if we can understand when the $+1$ term will matter, then we can see the conditions under which semiclassics is a reasonable approximation.

The important question regarding the effects of the $+1$ is to understand whether single photons will affect the dynamics. This in turn means the question of whether one photon has a significant effect on saturating the transition, i.e. what is the ratio of $g^2/\gamma\gamma_t$. It is standard to define the parameter β slightly differently, as:

$$\beta = \frac{g^2}{\gamma\gamma_t + g^2}. \quad (11.35)$$

If $g^2 \ll \gamma\gamma_t$ then single photons have a negligible effect, and this correspond to $\beta \ll 1$. If $g^2 \gg \gamma\gamma_t$, single photons are important, and $\beta \rightarrow 1$. In terms of β , we may rewrite Eq. (11.34) as:

$$\partial_t n = \frac{r\beta}{2} \frac{n+1}{1+\beta n} - \kappa n \quad (11.36)$$

Based on the semiclassical threshold discussed above (valid if $\beta \ll 1$, and so $\beta \simeq g^2/\gamma\gamma_t$) we will define $r_{\text{th}} = 2\kappa/\beta$. In terms of this threshold, we can then solve the steady state number of photons to be given by $r(n+1) = r_{\text{th}}n(1+\beta n)$, with the solution:

$$n = \frac{1}{2\beta} \left[\frac{r}{r_{\text{th}}} - 1 \pm \sqrt{\left(\frac{r}{r_{\text{th}}} - 1\right)^2 + 4\beta \frac{r}{r_{\text{th}}}} \right] \quad (11.37)$$

This equation (normalised by plotting βn) is shown in Fig. 11.4. For small β it is clear that one recovers the sharp step like feature at $r = r_{\text{th}}$. However

as β increases, the step becomes more smeared, and eventually, at $\beta = 1$, one has $n\beta = r/r_{\text{th}}$ for all r , i.e. no step survives at all. This indicates that in the limit of strong coupling, there is no clear “laser threshold”, but rather a thresholdless lasing behaviour.

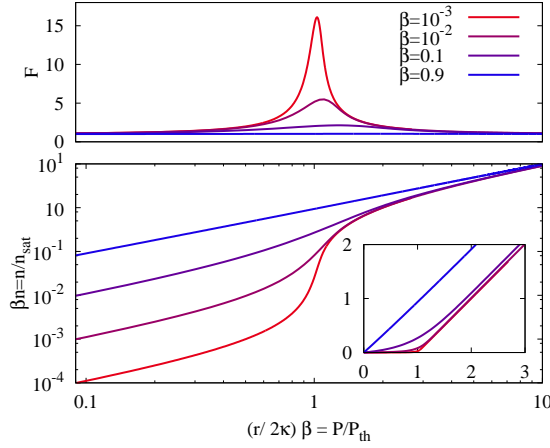


Figure 11.4: Numbers of photons vs pump rate (normalised by r_{th} , and Fano factor, for different values of β

Fano factor

As well as the sharpness of $\langle n \rangle$ vs pump power, further insight into the role of β can be found by considering the full probability distribution, and the relative size of fluctuations. We choose to measure the Fano factor, defined by:

$$F = \frac{\langle n^2 \rangle - \langle n \rangle^2}{\langle n \rangle}. \quad (11.38)$$

This parameter measures deviations from Poissonian statistics, as $F = 1$ for a Poisson distribution. The fluctuations can also equivalently be defined by $g_2(0)$, where:

$$g_2(0) = \frac{\langle a^\dagger a^\dagger a a \rangle}{\langle a^\dagger a \rangle^2} = \frac{\langle n^2 \rangle - \langle n \rangle}{\langle n \rangle^2}. \quad (11.39)$$

Another parameter occasionally quoted is the Mandel Q parameter. The three parameters are related by $Q = F - 1 = \langle n \rangle (g_2(0) - 1)$. We will consider only F below.

Let us now determine F from the full probability distribution as derived in the last lecture, and assuming β is not too large. Far above threshold, the distribution is Poissonian, and so $F \rightarrow 1$. Below the threshold, we may use the Bose-Einstein distribution $p_n \propto (r/r_{\text{th}})^n$ to give:

$$\langle n \rangle = \frac{r/r_{\text{th}}}{1 - r/r_{\text{th}}}, \quad \langle n^2 \rangle = \frac{r/r_{\text{th}}(1 + r/r_{\text{th}})}{(1 - r/r_{\text{th}})^2} \quad (11.40)$$

and so:

$$F = 1 + \frac{r/r_{\text{th}}}{1 - r/r_{\text{th}}} \quad (11.41)$$

which again becomes 1 is far from threshold.

Exactly at threshold, one can make an approximation for small β that:

$$p_n \propto \prod_m (1 + \beta m)^{-1} \simeq \exp\left(-\beta \sum_m m\right) \simeq \exp(-\beta n^2/2). \quad (11.42)$$

This is valid if $\beta \ll 1$, in which case, p_n has decayed to a small value well before $\beta n \simeq 1$. With this one-sided Gaussian distribution, one then finds:

$$\langle n \rangle = \sqrt{\frac{2}{\pi\beta}}, \quad \langle n^2 \rangle = \frac{1}{\beta}, \quad F = \frac{1}{\sqrt{\beta}} \left(\sqrt{\frac{\pi}{2}} - \sqrt{\frac{2}{\pi}} \right). \quad (11.43)$$

Thus, if $\beta \ll 1$, there is a sharp spike of Fano factor at the transition — the uncertainty in photon number becomes very large here in the semiclassical limit. On the other hand, everywhere else, the Fano factor is small. The full behaviour of the Fano factor vs β is shown in Fig. 11.4. It is clear that for $\beta \simeq 1$, there is no clear spike, and instead one has a broad region near the ill-defined threshold where fluctuations are large.

Micromaser and noise

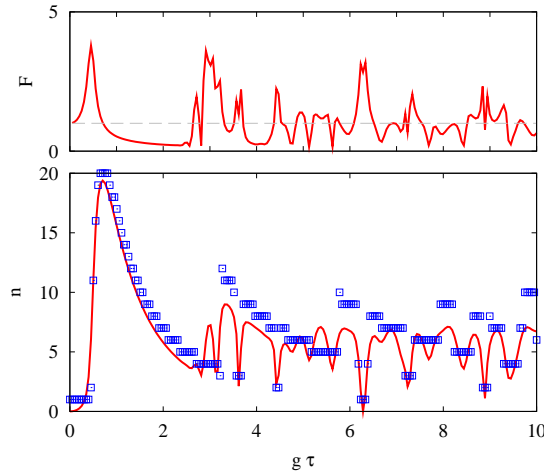


Figure 11.5: Numbers of photons and Fano factor vs coupling $g\tau$ for a micromaser. Blue squares show the most probable number of photons. (r/κ is fixed at 20)

The above results for the laser have $F > 1$ in all cases, meaning that the noise is always at least Poissonian, and is sometimes worse. On the other hand, the micromaser that we discussed previously can have sub-Poissonian noise. Recalling that in that case, the probability distribution is given by:

$$p_n \propto \prod_m \left(\frac{r}{\kappa}\right) \left(\frac{\sin^2 g\tau \sqrt{m}/2}{m}\right). \quad (11.44)$$

There are two important features of this distribution; the first is the irregular dependence on $g\tau$, which was commented on before. This is illustrated

in Fig. 11.5. More importantly, there is the dependence on n , which is what can lead to number squeezing (i.e. sub-Poissonian noise).

The number dependence of the gain means that the distribution is attracted to particular number states. This is illustrated schematically in Fig. 11.6, which plots $r \sin^2(g\tau\sqrt{n+1}/2) - \kappa n$ as a function of n . Because the width of the attraction regions varies in a non-smooth way, the width of the distribution is not simply connected to the mean number of photons. Thus, the variance can be far less than it would be for a Poisson distribution with the same mean number of photons.

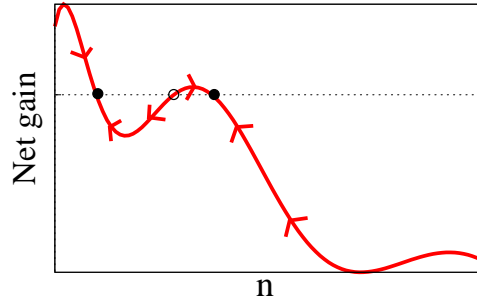


Figure 11.6: Gain vs n for micromaser equation

The number dependence of the gain is also responsible for the discontinuous jumps that can occur on increasing r . As r increases, new attractors appear, i.e. new peaks of $\sin^2 g\tau\sqrt{n+1}/2$ fall within the range $n > r/\kappa$. The photon number and Fano factor vs pump rate is shown in Fig. 11.7. This kind of mean-field-like argument can describe the possible steady states reasonably well (under the standard $n \gg 1$ validity condition of mean field theory). It cannot however describe the points of transition between the different possible steady states. This can be clearly seen in Fig. 11.7.

11.3 Single atom lasers

Armed with the above ideas of the Fano factor, describing noise, and the validity of semiclassics, and the β factor describing how saturation effects occur, we can now consider cavity QED lasing systems, where small numbers of atoms are involved.

Firstly, we should note that the density matrix equation in Sec. 11.1 could reasonably describe a single atom. However, it does rely on the separation of timescales, and if very strong coupling is required, then the approximations used in deriving it will become invalid. To see why this might occur, let us rewrite the threshold condition as:

$$r = \frac{N_A \Gamma \gamma}{\Gamma + \gamma} = \frac{2\kappa}{\beta} \quad (11.45)$$

If $N_A = 1$, and γ cannot be significantly tuned (as it describes intrinsic decay rates of atoms), then to reach threshold with a single atom, it is

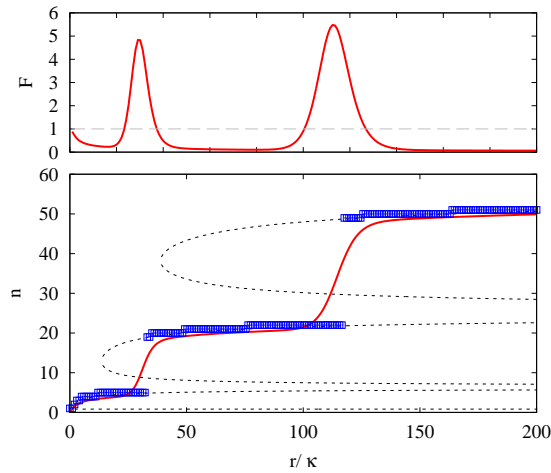


Figure 11.7: Numbers of photons and Fano factor vs coupling $g\tau$ for a micromaser. Blue squares show the most probable number of photons. Dotted lines correspond to the mean-field results, $r \sin^2(g\tau\sqrt{n+1}) = \kappa n$. ($g\tau$ is fixed at 2.5)

generally necessary to increase β , so that one will no longer have $\beta \ll 1$. In this case, fluctuations will matter more, the behaviour will be thresholdless, and if g becomes very strong then no separation of timescales is possible. Thus, while single atom lasing need not automatically imply a breakdown of the above description, reaching threshold with a small number of atoms generally requires strong coupling, and this in term means large β , hence thresholdless lasing.

Optical cavity, real atom

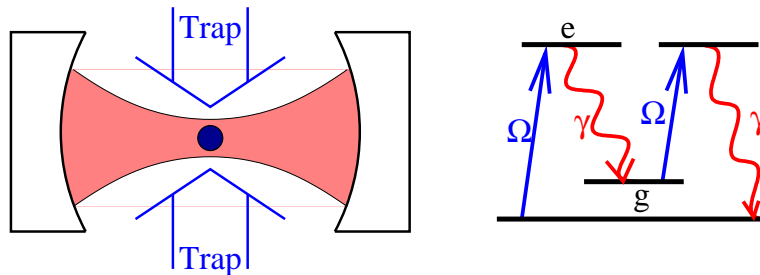


Figure 11.8: Schematic of single atom optical lasing scheme, and level scheme

Lasing from a single atom, using an optical transition and a regular cavity has been achieved by [35]. The system and level scheme are shown schematically in Fig. 11.8. The parameters for this experiment were $(g, \kappa, \gamma) = 2\pi \times (16, 4.2, 2.6)$ MHz. This implies $\beta \simeq 0.97$, and so the lasing will be thresholdless. In addition, because of the strong coupling, the adiabatic elimination of the non-lasing levels is not a good approximation in this system. This allows the possibility of sub-Poissonian behaviour, which was

observed experimentally, and can be described theoretically by modelling the full four level system illustrated in Fig. 11.8.

Josephson qubit micromaser

The other example we will discuss is a rather different system and rather different method of lasing. This system is a superconducting qubit as the artificial atom, coupled to a mechanical resonator, thus this is phonon lasing rather than photon lasing[36, 37]. The circuit is shown schematically in Fig. 11.9

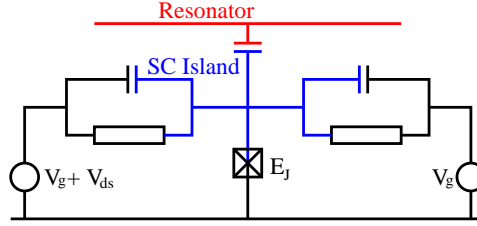


Figure 11.9: Circuit diagram of single artificial atom laser

The two-level “atom” here consists of two charge states of the island, having either zero or two extra electrons on the island (two electrons because of Cooper pairing, and the consequent odd/even energy gap). Thus one has $\sigma^+ = \sigma^{20}$ etc. This gives the Hamiltonian:

$$H = [E_C + \lambda(a + a^\dagger)]\sigma^z + E_J\sigma^x + \omega a^\dagger a \quad (11.46)$$

where E_C depends on the capacitance of the island, and λ describes the shift due to the location of the mechanical resonator. An equivalent model would also describe a capacitively coupled stripline resonator. As well as this Hamiltonian behaviour, the bias V_{ds} leads to a single-electron current through the island, with individual electrons hopping in a two step process. Eliminating the singly charged state (which decays fast), one has the effective density matrix equation:

$$\begin{aligned} \partial_t \rho = -i[H, \rho] - \frac{\kappa}{2} (a^\dagger a \rho - 2a \rho a^\dagger + \rho a^\dagger a) \\ - \frac{\Gamma}{2} (\sigma^+ \sigma^- \rho - 2\sigma^- \rho \sigma^+ + \rho \sigma^+ \sigma^-) \end{aligned} \quad (11.47)$$

Lasing can occur if $E_C + \omega \simeq 0$, in which case, there is a cycle of incoherent transition from $2 \rightarrow 0$, followed by coherent transition from $0 \rightarrow 2$ with emission of a phonon to conserve energy. However, because the photon number affects the energy of the state E_C , this leads to micromaser like behaviour. In the theoretical work [37, 38], the parameters are however in a regime where mean field theory can describe some of the features seen by full density matrix evolution.

11.4 Further reading

Discussion of density matrix equations and Maxwell-Bloch equations for our model laser can be found in many textbooks, e.g. Meystre and Sargent III [12], Yamamoto and Imamoglu [11]. The effect of the β parameter, and the nature of the threshold is discussed in Rice and Carmichael [39]. A review on cavity QED particularly focusing on micromaser behaviour can be found in Walther et al. [40].

Lecture 12

Three levels, electromagnetically induced transparency, and coherent control

Most of the lectures so far has been restricted to two-level atoms. While there many are obvious extensions to the case of more than two levels, this section focuses only on one phenomena which requires three levels: That is the coherent interference of pathways for atomic transitions, and the possibility of suppressed transitions. By using an applied electric field, one can suppress transitions between levels, and thus prevent absorption, leading to *Electromagnetically Induced Transparency*. This lecture starts by discussing the phenomena in terms of the density matrix evolution, allowing incoherent decay of atomic energy levels, and considering radiation semi-classically. In this way, we will see that conditions under which the group velocity of light is significantly reduced while absorption of light remains small is possible in this system — a combination of features that does not occur for a two level system. Then, by considering the atomic wavefunction evolution, it is possible to understand the underlying interference phenomena which lead to the cancellation that prevents absorption. Finally, we return to a quantum optics description of the states that are free from absorption — dark state polaritons — discussing how the nature of these coherent quantum states can be controlled.

12.1 Semiclassical introduction

We consider the level scheme shown in Fig. 12.1. Dipole matrix elements exist between the ground state b and the excited state a , and also between the metastable state c and excited state a . The transition between b and c is however dipole forbidden. The states a and c are coupled by a strong “pumping” radiation field, described only by the effective Rabi splitting it induces, Ω_p . The transition between a and b is then probed by a weaker

field. We allow the possibility of incoherent decay from a to b , and also a (much weaker) decay from c to b . The coherent part of the evolution can

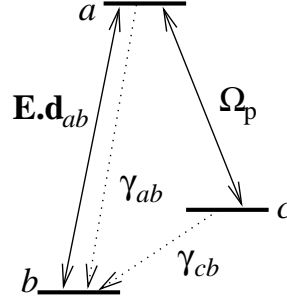


Figure 12.1: Scheme of atomic energy levels and allowed, and driven, transitions between them.

be described by the Hamiltonian:

$$H = \begin{pmatrix} \omega_a & -Ed_{ab}e^{-i\omega t} & -\Omega_p e^{-i\omega_p t} \\ -Ed_{ab}^* e^{i\omega t} & \omega_b & 0 \\ -\Omega_p^* e^{i\omega_p t} & 0 & \omega_c \end{pmatrix}. \quad (12.1)$$

Susceptibility; slow light.

Let us consider the absorption and emission coefficients, by finding the susceptibility of the three level system; i.e. finding how the polarisation, $P = (N_{\text{atoms}}/V)d_{ab}\rho_{ab}$, depends on the field E applied resonantly with the $a \leftrightarrow b$ transition. We assume the field E is weak, so we consider only linear response, and can assume the populations of the a and c levels remain small. We are interested in the evolution of the density matrix, $\dot{\rho} = -i[H, \rho] + \hat{\mathcal{L}}\rho$, where $\hat{\mathcal{L}}$ represented a superoperator, describing the incoherent evolution, described in lecture 4. Since $\text{Tr}\rho = 1$, small populations of a , c imply $\rho_{bb} \approx 1$ and $\rho_{ac} \approx 0$, and Hermiticity means the only relevant components are ρ_{ab} and ρ_{cb} . Thus:

$$\begin{aligned} \dot{\rho}_{ab} &= -i[(H_{aa} \cdot \rho_{ab} + H_{ab} \cdot 1 + H_{ac} \cdot \rho_{cb}) \\ &\quad - (0 \cdot H_{ab} + \rho_{ab} \cdot H_{bb} - 0 \cdot H_{cb})] - \gamma_{ab}\rho_{ab} \\ &= -(i\omega_{ab} + \gamma_{ab})\rho_{ab} + i\Omega_p e^{-i\omega_p t}\rho_{cb} + iEd_{ab}e^{-i\omega t}, \end{aligned} \quad (12.2)$$

$$\begin{aligned} \dot{\rho}_{cb} &= -i[(H_{ca} \cdot \rho_{ab} + 0 \cdot 1 + H_{cc} \cdot \rho_{cb}) \\ &\quad - (0 \cdot H_{ab} + \rho_{cb} \cdot H_{bb} - 0 \cdot H_{cb})] - \gamma_{cb}\rho_{cb} \\ &= -(i\omega_{cb} + \gamma_{cb})\rho_{cb} + i\Omega_p^* e^{i\omega_p t}\rho_{ab}. \end{aligned} \quad (12.3)$$

Let us now assume that the pump is resonant, so that $\omega_p = \omega_{ac} = \omega_a - \omega_c$. Then, rewriting $\rho_{ab} = \tilde{\rho}_{ab}e^{-i\omega t}$, and $\rho_{cb} = \tilde{\rho}_{cb}e^{-i(\omega - \omega_p)t}$, and defining the probe detuning $\Delta = \omega_{ab} - \omega$, we have:

$$\partial_t \begin{pmatrix} \tilde{\rho}_{ab} \\ \tilde{\rho}_{cb} \end{pmatrix} = \begin{pmatrix} -(i\Delta + \gamma_{ab}) & i\Omega_p \\ i\Omega_p^* & -(i\Delta + \gamma_{cb}) \end{pmatrix} \begin{pmatrix} \tilde{\rho}_{ab} \\ \tilde{\rho}_{cb} \end{pmatrix} + \begin{pmatrix} iEd_{ab} \\ 0 \end{pmatrix}. \quad (12.4)$$

This has the form $\dot{X} = MX + A$, and we want the steady state solution (after decay of any initial transients), which is clearly given by $X = -M^{-1}A$. Thus we can extract the polarisation by finding ρ_{ab} :

$$P = \frac{N_{\text{atoms}}}{V} d_{ab} \rho_{ab} = \frac{N_{\text{atoms}}}{V} d_{ab} \frac{(i\Delta + \gamma_{cb})iEd_{ab}}{(i\Delta + \gamma_{ab})(i\Delta + \gamma_{cb}) + |\Omega_p|^2} e^{-i\omega t}. \quad (12.5)$$

Then, noting the time dependence of $E(t) = Ee^{-i\omega t}$, one can easily identify the susceptibility, χ by $P = \chi \varepsilon_0 E$. i.e.

$$\chi = \frac{N_{\text{atoms}}}{V} \frac{|d_{ab}|^2}{\varepsilon_0} \frac{i(i\Delta + \gamma_{cb})}{(i\Delta + \gamma_{ab})(i\Delta + \gamma_{cb}) + |\Omega_p|^2}. \quad (12.6)$$

Let us now discuss the properties of this susceptibility. It is clear that $\chi(-\Delta) = -\chi(\Delta)^*$, thus if $\chi = \chi' + i\chi''$, it is clear that χ' is an odd function of Δ (and vanishes at $\Delta = 0$), and χ'' an even function. However, $\chi(\Delta = 0) \propto \gamma_{cb}$, which we assumed earlier to be small, as this transition is dipole forbidden. To be precise, $\chi''(\Delta = 0) \propto \gamma_{cb}/(\gamma_{ab}\gamma_{cb} + |\Omega_p|^2)$; thus the absorption part is small if $\gamma_{cb} \ll \Omega_p$; the coherent driving field has induced transparency. The full real and imaginary parts of susceptibility are plotted in Fig. 12.2, both with and without a driving field Ω_p .

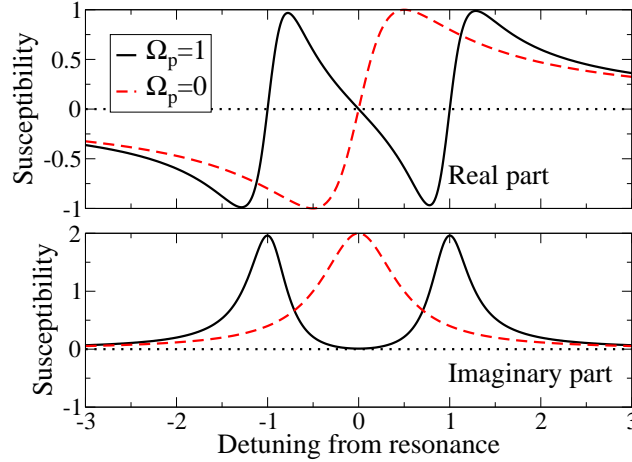


Figure 12.2: Real and imaginary parts of susceptibility, plotted for $\gamma_1 = 0.5$, $\gamma_3 = 0.01$, and two values of Ω_p as indicated.

Since both real and imaginary parts of χ are zero, or very small at $\Delta = 0$, let us consider linear expansion; clearly only the real part exists to linear order. Setting $\gamma_{cb} = 0$, one trivially finds:

$$\chi = -\frac{N_{\text{atoms}}}{V} \frac{|d_{ab}|^2}{\varepsilon_0} \frac{\Delta}{|\Omega_p|^2} + \mathcal{O}(\Delta^2). \quad (12.7)$$

The linear part of susceptibility relates to the dispersion of the probe field; $n(\Delta) = \sqrt{1 + \chi(\Delta)}$, which can be used with $\omega = \omega_{ab} - \Delta$, and $n(\omega)\omega = ck$.

One thus has phase velocity $v_p = c/n(\Delta)$ and group velocity $v_g = c/[n - (\omega_{ab} - \Delta)dn/d\Delta]$. Assuming Δ remains small, one can expand:

$$n \approx 1 - \frac{N_{\text{atoms}}}{V} \frac{|d_{ab}|^2}{\varepsilon_0} \frac{\Delta}{2|\Omega_p|^2}, \quad \frac{dn}{d\Delta} \approx -\frac{N_{\text{atoms}}}{V} \frac{|d_{ab}|^2}{\varepsilon_0} \frac{1}{2|\Omega_p|^2}. \quad (12.8)$$

Thus, to leading order:

$$v_p \approx c, \quad v_g \approx \frac{c}{1 + \frac{N}{|\Omega_p|^2} \frac{\omega_{ab}|d_{ab}|^2}{2\varepsilon_0 V}}, \quad (12.9)$$

where we have rearranged factors to identify the atom-radiation coupling strength $g^2 = \omega_{ab}|d_{ab}|^2/2\varepsilon_0 V$ [cf. the definition in Eq. (1.39), in the case $\omega_k = \epsilon = \omega_{ab}$]. Thus, if $Ng^2/|\Omega_p|^2$ is large, one can have small absorption, yet large enough dispersion to significantly reduce the group velocity, and thus have non-absorbed slow light.

Decay length of slow light

The previous analysis shows that near zero detuning, one can have a small group velocity, but small absorption. Let us now be more specific about how small the absorption is. What matters in practice is the attenuation of the signal as it propagates, following $I(l, \omega) = I(0)e^{-kl\chi''(\omega)}$. Since the attenuation is frequency dependent, this may also lead to distortion of the propagating signal. The frequency dependence of the transmission intensity is shown in Fig. 12.3.

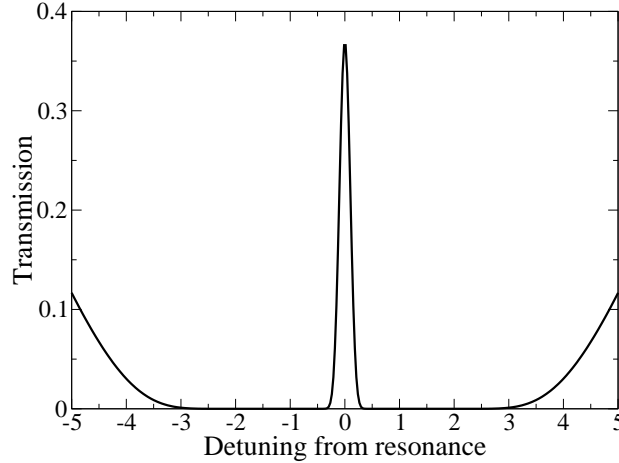


Figure 12.3: Transmission after $L = (50/\pi)\lambda$, for the same parameters as in Fig. 12.2.

Considering the case $\gamma_{cb} = 0$, for small Δ , the second order expansion yields:

$$\chi'' \approx 2i \frac{Ng^2}{ck} \Delta^2 \frac{\gamma_{ab}}{|\Omega_p|^4}, \quad (12.10)$$

where we have written $\omega_{ab} = ck$. One can thus identify a frequency dependent decay length:

$$L = c \frac{1}{2Ng^2} \frac{1}{\Delta^2} \frac{|\Omega_p|^4}{\gamma_{ab}}. \quad (12.11)$$

12.2 Coherent evolution alone; why does EIT occur

Let us now ignore the decay rates, γ_{ab} and γ_{bc} , and look at the evolution of atomic states alone. Thus, since we are not interested in the dielectric response, we may denote $E d_{ab} = \Omega_s$ as the signal field. Considering the fully resonant case, $\omega = \omega_{ab}$ and $\omega_p = \omega_{ac}$, one can write:

$$|\psi\rangle = c_a e^{-i\omega_a t} |a\rangle + c_b e^{-i\omega_b t} |b\rangle + c_c e^{-i\omega_c t} |c\rangle \quad (12.12)$$

$$i\partial_t \begin{pmatrix} c_a \\ c_b \\ c_c \end{pmatrix} = \begin{pmatrix} 0 & -\Omega_s & -\Omega_p \\ -\Omega_s^* & 0 & 0 \\ -\Omega_p^* & 0 & 0 \end{pmatrix} \begin{pmatrix} c_a \\ c_b \\ c_c \end{pmatrix}. \quad (12.13)$$

One can easily find the eigenvalues and vectors of this problem; in terms of $\Omega_0 = \sqrt{|\Omega_s|^2 + |\Omega_p|^2}$ one has:

$$\lambda = 0 : \quad \frac{1}{\Omega_0} \begin{pmatrix} 0 \\ \Omega_p \\ -\Omega_s \end{pmatrix}; \quad \lambda = \pm\Omega : \quad \frac{1}{\sqrt{2}\Omega_0} \begin{pmatrix} \mp\Omega_0 \\ \Omega_s^* \\ \Omega_p^* \end{pmatrix}. \quad (12.14)$$

Thus, if one starts in the state $(\Omega_p |b\rangle - \Omega_s |c\rangle) / \Omega_0$, then the evolution never reaches the excited state $|a\rangle$. Since the only substantial decay constant was from the state $|a\rangle$, this then means one has no absorption: The evolution is trapped in this *dark state*, and shows no decay. One may also describe this process as destructive interference between the transitions $|b\rangle \rightarrow |a\rangle$ and $|b\rangle \rightarrow |a\rangle \rightarrow |c\rangle \rightarrow |a\rangle$ etc.

Since such a dark state exists, from which no further excitation is possible, the full evolution (with decay rates) can be understood as follows: Regardless of the initial state, decay out of states $|a\rangle$ allows one to reach various superpositions $B |b\rangle + C |c\rangle$. Eventually, one will reach dark state, after which no further evolution can occur. [See Q. 12.1].

12.3 Coherent control of quantised radiation: Dark state polaritons

The previous sections show that the existence of the classical driving field, resonant with the $c \leftrightarrow a$ transition modify the response seen by a probe field at the $b \leftrightarrow a$ transition: One can have vanishing absorption, while still having strong dispersion, allowing propagation of “slow light”. In this section, we consider the probe field quantum mechanically, and ask how the classical driving changes the evolution of quantised radiation. As illustrated in Fig. 12.4, we consider the driving as still classical, but the probe field is quantum mechanical.

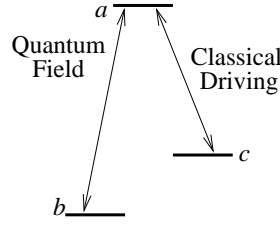


Figure 12.4: Illustration of which transitions are classically driven, and which described quantum mechanically

In order to consider propagation through the EIT medium, we consider a continuum of radiation modes, as well as explicitly including a sum over atomic states. Note that in the earlier treatment, the sum over atoms occurred implicitly via the factor N_{atoms} in Eq. (12.6) Assuming resonance for the driving field, and writing σ_i^{xy} for the three by three matrix describing transitions from level y to level x of atom i , one can write:

$$H = \sum_{k>0} ck a_k^\dagger a_k + \sum_i \left[-g\sigma_i^{ba} e^{-i\omega_{ab}t} \left(\sum_k a_k^\dagger e^{-ikx_i} \right) - \Omega_p^* \sigma_i^{ca} + H.c. \right]. \quad (12.15)$$

Then, making a gauge transform to a rotating frame one can remove the explicit $e^{-i\omega_{ab}t}$ time dependence, in favour of replacing $ck \rightarrow ck - \omega_{ab}$. Then, one may make a Fourier transform of the photon field, defining:

$$a(x) = \sum_k a_k e^{i(k - \omega_{ab}/c)x}. \quad (12.16)$$

By writing $k - \omega_{ab}/c$ rather than k , one recovers the correct energy from the derivative w.r.t. x . Thus, one can write

$$H = \int dx a^\dagger(x) (-ic\partial_x) a(x) + \sum_i \left[-g\sigma_i^{ba} a^\dagger(x_i) e^{-i\omega_{ab}x_i/c} - \Omega_p^* \sigma_i^{ca} + H.c. \right]. \quad (12.17)$$

Note the factor of $e^{-i\omega_{ab}x_i/c}$ appearing in the photon-atom coupling term. Let us introduce collective variables:

$$\begin{aligned} X(x) &= \sum_i \sigma_i^{ca} \delta(x - x_i) \\ P(x) &= \sum_i \sigma_i^{ba} \delta(x - x_i) e^{-i\omega_{ab}x_i/c} \\ S(x) &= \sum_i \sigma_i^{bc} \delta(x - x_i) e^{-i\omega_{ab}x_i/c}. \end{aligned} \quad (12.18)$$

The definitions here are not the most symmetric, but are physically motivated: P represents a polarisation wave, while S represents a “spin wave” — this language is appropriate to cases where the low lying levels b, c arise from opposite spin states. The definitions of $P(x)$ and $X(x)$ are easily understood, as they allow one to rewrite Eq. (12.17) as:

$$H = \int dx a^\dagger(x) (-ic\partial_x) a(x) - \left[gP(x) a^\dagger(x) + \Omega_p^* X(x) + H.c. \right]. \quad (12.19)$$

To understand the definition of $S(x)$, we must consider commutation relations. Writing commutators using $[\sigma^{ij}, \sigma^{kl}] = \delta_{jk}\sigma^{il} - \delta_{li}\sigma^{kj}$, the cross commutators are given by:

$$\begin{aligned} [P(x), X^\dagger(x')] &= S(x)\delta(x-x'), \\ [S(x), X(x')] &= P(x)\delta(x-x'), \\ [P^\dagger(x), S(x')] &= X^\dagger(x)\delta(x-x'). \end{aligned} \quad (12.20)$$

Other than these, and their conjugate forms, all other cross commutators vanish. Thus, it is clear that the phase factors in the definition of $S(x)$ in Eq. (12.18) is required to give these simple forms.

We now have a tractable form of the quantum Hamiltonian, which allows us to consider how an initial state will evolve. Let us solve the simpler problem, of finding the quantum states that are trapped in the absorption-free subspace. It is clear that the atomic ground state, with no photons is an eigenstate, which we will denote $|0\rangle$. Let us consider excited states of the form:

$$|\Phi\rangle = \Phi^\dagger |0\rangle = \int dx \phi(x) \left(\alpha a^\dagger(x) + \beta S^\dagger(x) \right) |0\rangle, \quad (12.21)$$

and ask for the condition under which no excitations to the excited atomic state, a , occur. Note the distinction to the discussion in Sec. 12.2; in that case we wanted superpositions of b and c states so that in the presence of a given probe electric field there were no excitations; in the current case we want a superposition of radiation and spin-wave excitations. Since the trial state in Eq. (12.21) contains no excitations, we require only that the time derivative of number of excitations vanishes. Thus, the condition we require can clearly be written as: $\langle 0 | P[H, \Phi^\dagger] |0\rangle = 0$. Let us consider the commutator, using the results in Eq. (12.20), along with $[a(x), a^\dagger(x')] = \delta(x-x')$:

$$\begin{aligned} \left[H, \left(\alpha a^\dagger(x) + \beta S^\dagger(x) \right) \right] = \\ \alpha \left(ic\partial_x a^\dagger(x) - gP^\dagger(x) \right) + \beta \left(ga^\dagger(x)X(x) - \Omega_p P^\dagger(x) \right). \end{aligned} \quad (12.22)$$

The term proportional to $a^\dagger(x)X(x)$ is second order, involving both atomic and radiation excitations, for low intensities it can be neglected. The number of excitations then vanishes under the condition: $\alpha g + \beta\Omega_p = 0$. If we wish to find a normalised wavefunction, we require:

$$\delta(x-x') = [\Phi(x), \Phi^\dagger(x')] = |\alpha|^2 \delta(x-x') + |\beta|^2 \sum_i \delta(x-x'). \quad (12.23)$$

Thus, one has $1 = |\alpha|^2 + |\beta|^2$ in addition to the previous condition, thus:

$$\alpha = \frac{\Omega_p}{\sqrt{|\Omega_p|^2 + g^2 N}}, \quad \beta = \frac{-g}{\sqrt{|\Omega_p|^2 + g^2 N}}. \quad (12.24)$$

From Eq. (12.22), we find this state evolves according to the Heisenberg equation of motion,

$$-i\partial_t\Phi^\dagger(x) = [H, \Phi^\dagger] = \frac{ic\Omega_p}{\sqrt{|\Omega_p|^2 + g^2N}}\partial_x a^\dagger(x). \quad (12.25)$$

To eliminate $a^\dagger(x)$, we wish to write this in terms of the field $\Phi^\dagger(x)$. The field Φ^\dagger is the linear combination of a^\dagger and S^\dagger which is dark — i.e. never undergoes a transition to the excited atomic state; another orthogonal linear combination of a^\dagger and S^\dagger exists, let us call that Ψ^\dagger . Then, one can write:

$$\begin{pmatrix} \Phi^\dagger \\ \Psi^\dagger \end{pmatrix} = \frac{1}{\sqrt{|\Omega_p|^2 + g^2N}} \begin{pmatrix} \Omega_p & -g\sqrt{N} \\ g\sqrt{N} & \Omega_p^* \end{pmatrix} \begin{pmatrix} a^\dagger \\ \sqrt{N}S^\dagger \end{pmatrix}. \quad (12.26)$$

Here we have included a factor of $1/\sqrt{N}$ required to achieve bosonic commutation relations into the definition of a bosonic spin operator S^\dagger/\sqrt{N} . Inverting this to find a^\dagger in terms of Φ^\dagger and Ψ^\dagger , and assuming an initially dark state so that $\Psi^\dagger \rightarrow 0$, Eq. (12.25) becomes:

$$-i\partial_t\Phi^\dagger(x) = \frac{ic\Omega_p}{\sqrt{|\Omega_p|^2 + g^2N}}\partial_x \frac{\Omega_p^*}{\sqrt{|\Omega_p|^2 + g^2N}}\Phi^\dagger(x), \quad (12.27)$$

i.e.

$$\left(\partial_t + \frac{c}{1 + g^2N/|\Omega_p|^2}\partial_z \right) \Phi^\dagger = 0. \quad (12.28)$$

Thus, we have found a *dark state polariton* [41], a state which is never excited to the radiative atomic state a , and is half light, half spin-wave excitation. It propagates at a reduced velocity, which is the group velocity found in Eq. (12.9). Importantly, one can control the nature of the state by the ratio of Ω_p to $g\sqrt{N}$. In a strong driving field $\Omega_p \gg g\sqrt{N}$, the excitation is almost pure photon; for a vanishing field $\Omega_p \ll g\sqrt{N}$, the excitation becomes pure spin wave. Thus, starting with a large Ω_p , one can inject an arbitrary quantum state of photons. By adiabatically reducing Ω_p , the quantum state is exactly transferred to the spin waves. These propagate slowly, until Ω_p is increased again, and the state transferred back to photons. Hence, coherent control allows the trapping of quantum states of light.

12.4 Further reading

The general topics of EIT, dark states, and coherent control in three level systems are discussed in a review by Fleischhauer et al. [42]. The question of coherently transferring the quantum state of light to “dark state polaritons” is introduced in Ref.[41], and discussed further in Ref.[43]. An interesting application of this idea is described in Ref. [44], where a quantum state of light is transported by transferring atoms between two condensates, and then releasing the state of light from the second condensate.

Questions

Question 12.1: Evolution to dark state

As mentioned in Sec. 12.2, decay should be responsible for projecting into the *dark state*. Considering Eq. (12.4); find the evolution to the dark state if one starts in the ground state, i.e. find the general time dependent solution starting from $\rho_{ab} = \rho_{cb} = 0$. Sketch the time dependence of $\Re[\rho_{cb}]$.

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