

2 Session Two - Complex Numbers and Vectors

2.1 What is a Complex Number?

The material on complex numbers should be familiar to all those who have successfully passed through first year mathematics. For our direct entrants, the School of Mathematics plans to cover this stuff early on in the session for your benefit. Complex numbers are widely used in a number of areas of physics, and particularly for their usefulness in describing oscillations and waves, both classical and quantum.

If you were to be asked what is the square root of four, you would readily reply two, and perhaps also minus two. But what is the square root of minus four?

In a quote taken from the MathCentre Revision Leaflet that we will be using:

“We start by introducing a symbol to stand for the square root of -1 . Conventionally this symbol is j . That is $j = \sqrt{-1}$. It follows that $j^2 = -1$. Using real numbers we cannot find the square root of a negative number, and so the quantity j is not real. We say it is imaginary.”

Although MathCentre uses j for the square root of minus one, you should be aware that many authors, including the current one, often use i instead.

Our “normal” numbers are known as “real” numbers, and imaginary numbers are defined as above. A number that is a mix of real and imaginary is known as a complex number, and has the general form of

$$z = a + ib \quad \text{or} \quad z = x + iy .$$

2.2 MathCentre Leaflets (attached in the paper-based version)

If appropriate, please take a look at the MathCentre leaflets on complex numbers, covering the following topics:

- 7.1 What is a complex number?
- 7.2 Arithmetic with complex numbers
- 7.3 The Argand Diagram (interesting for maths, and highly useful for dealing with amplitudes and phases in all sorts of oscillations)
- 7.4 Complex numbers in polar form
- 7.5 Complex numbers as $r [\cos \theta + i \sin \theta]$
- 7.6 Multiplication and division in polar form
- 7.7 Complex numbers in the exponential form

2.3 Examples of Arithmetic

Consider imaginary number ib , with b real. What is its square? We use the normal rules of arithmetic to see that $(ib)^2 = i^2 b^2 = (-1)b^2 = -b^2$.

What is the square of complex number $(a + ib)$? Again, using normal rules of arithmetic,

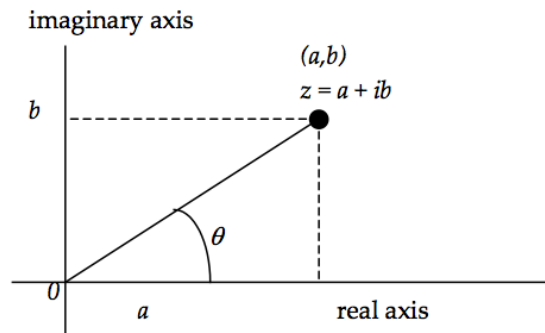
$$(a + ib)^2 = (a + ib) \cdot (a + ib) = a^2 + 2iab + i^2 b^2 = a^2 + 2iab - b^2 = a^2 - b^2 + 2iab .$$

What is $(a + ib) - (c + id)$? We have to handle the real and imaginary parts separately,

$$(a + ib) - (c + id) = (a - c + ib - id) = a - c + i(b - d) .$$

2.4 The Argand Diagram

The Argand diagram is a graphical representation of a complex number. (Its similarity to the way that oscillations and waves can be described diagrammatically is key to the use of complex numbers in the analysis of oscillations.) What would normally be the x -axis of a graph is the real axis, and what would normally be the y -axis is the imaginary axis. We can thus plot complex numbers on the Argand diagram as we would with conventional coordinates.



The length of the line from the origin to the point is known as the **modulus** or magnitude of the complex number. From Pythagoras' theorem you can readily see that it is equivalent to $\sqrt{a^2 + b^2}$. This is often written as $|z|$ and is always positive. The angle between the line and the real axis is known as the **argument** of the complex number, and is shown above as θ . From your existing ideas of trigonometry you can see that $\tan \theta = b/a$, and similar.

In the diagram we see how to describe the complex number as $z = a + ib$. If we label the modulus of the number as r , we see that $a = r \cos \theta$ and $b = r \sin \theta$. Thus we have an alternative way of writing the number, i.e.

$$z = r \cos \theta + ir \sin \theta = r [\cos \theta + i \sin \theta]$$

which is known as the **polar form** with modulus r and argument θ .

2.5 The Exponential Form of a Complex Number

A relation known as Euler's theorem states that (for now, please just accept this)

$$e^{i\theta} = \cos \theta + i \sin \theta .$$

Going back to the polar form, we can see therefore that because we can write

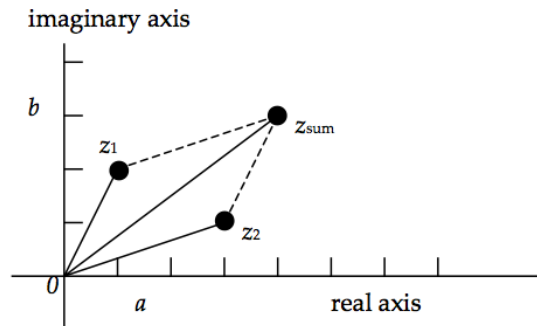
$$z = a + ib = r [\cos \theta + i \sin \theta] ,$$

we can equivalently write our complex number using Euler's theorem as

$$z = r e^{i\theta} .$$

This is considerably more compact than the other forms, and is frequently used. Note that $e^{i\theta}$ contains in general both real and imaginary parts, whereas r is by definition a positive real number.

Example: Plot on the Argand diagram the complex numbers $z_1 = 1 + 2i$ and $z_2 = 3 + i$. Plot also their sum. Determine the modulus and argument of the sum, and express in exponential form.



To plot z_1 we take one unit along the real axis and two up the imaginary axis, giving the left-hand most point on the graph above. Similarly for z_2 we take three units to the right and one up. To find the sum we use the rules given earlier to find that $z_{\text{sum}} = (1 + 2i) + (3 + i) = 4 + 3i$. It is no coincidence that what looks like a parallelogram of vector addition is formed on the graph when we plot the point indicating the sum of the two original complex numbers.

The modulus of the sum is given by the length of the line on the graph, which we can see from Pythagoras is $\sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5$ (positive root taken due to definition of modulus).

The argument of the sum is given by the angle the line makes to the real axis, which we can see as $\tan \theta_{\text{sum}} = 3/4 = 0.75$, so $\theta_{\text{sum}} = 0.64$ radians, or 40 degrees. Just by looking at the Argand diagram we can tell that the arguments of the original complex numbers are different to that of the sum.

To put this into exponential form, we use $r = 5$ and $\theta_{\text{sum}} = 0.64$ radians with $z = r e^{i\theta}$ to get $z = 5 e^{0.64i}$.

In various oscillation and wave problems you are likely to come across this sort of analysis, where the argument of the complex number represents the phase of the wave and the modulus of the complex number the amplitude. The real component of the complex number is then the value of (e.g.) the displacement of the oscillation at any given time. But more of this in your Oscillations and Waves courses.

2.6 The Complex Conjugate

The complex conjugate of z is defined as the (complex) number of same magnitude as z that, when multiplied by z , leaves a purely real result. If we have a complex number $z = a + ib$, its complex conjugate is $z^* = a - ib$ (note that we have replaced i by $-i$).

Example: Show that $a - ib$ is the complex conjugate of $a + ib$.

$$(a + ib) \cdot (a - ib) = a^2 - iab + iab + i^2 b^2 = a^2 - b^2$$

which is purely real, as required.

It can be shown that no matter how a complex number is represented, we can find its complex conjugate by replacing every i by $-i$. Note, however, that sometimes the i might be hidden, and it needs to be brought out in order to get this operation done to it.

Example: Find the complex conjugate of $z = w^{2x+iy}$ where $w = a + ib$.

$$z = (a + ib)^{2x+iy}, \text{ replacing } i \text{ by } -i \text{ gives } z^* = (a - ib)^{2x-iy}.$$

2.7 Dividing Complex Numbers

What do we do to try to evaluate a quotient such as $\frac{a + ib}{c + id}$?

The standard route to dealing with the difficulties of a complex number on the bottom line is to “remove” it. We can do this by multiplying both the top and the bottom line by the complex conjugate of the bottom line:

$$\frac{a + ib}{c + id} = \frac{a + ib}{c + id} \cdot \frac{c - id}{c - id} = \frac{ac + bd + i(bc - ad)}{c^2 + d^2}$$

and now we have separated the quotient into a real part and an imaginary part.

2.8 Scalars and Vectors

Scalars are quantities with magnitude, possibly with units, but without any direction associated with them (beyond plus or minus). Examples include: number of students in the class, mass, speed, potential energy, kinetic energy, electric potential, time, and bank balance. E.g., the train's speed was 40 miles per hour.

Vectors are quantities that have both magnitude and direction. Examples include: weight, velocity, force, electric field, acceleration, and torque. E.g., the train's velocity was 40 miles per hour in a horizontal plane heading at 5 degrees east of north.

Vectors may be expressed in Cartesian coordinates (with unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ pointing along the x, y, z axes) or in various forms of polar coordinates, e.g. $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$. Note that $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ here have nothing directly to do with imaginary numbers, and that some authors use different conventions for unit vectors, such as $\hat{\mathbf{i}}_x, \hat{\mathbf{i}}_y, \hat{\mathbf{i}}_z$ for the three directions in Cartesian coordinates. Vectors may be added up or subtracted by looking at the result on each component separately.

There are various conventions for indicating that a symbol represents a vector. Handwritten work often used a line under the quantity ($\underline{\mathbf{a}}$) to indicate that it was a vector. Recently arrow-like quantities above the symbol have been used to indicate the vector ($\vec{\mathbf{a}}$), as you will see in your course textbook. Printed documents have also used a simple bolding of a symbol (\mathbf{a}) to indicate that it is a vector. That is what we use here.

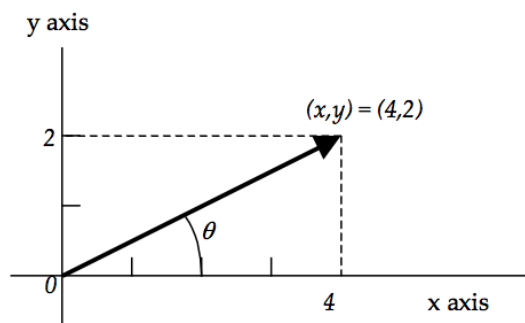
You may wish to look at the MathCentre sheets on vectors, which are in the paper-based version of this handout.

- 6.1 Vectors
- 6.2 The Scalar Product of Vectors
- 6.3 The Vector Product of Vectors

Vectors, and the scalar and vector products will feature strongly in your Mechanics course and your Electricity and Magnetism course this year, amongst others.

Let us look at the vector $\mathbf{A} = 4\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$ to try some examples.

First, we plot this vector on an (x, y) graph.



I have chosen to put one end of the vector at the origin, but this is not necessary. The far end of the vector is in a direction given by the vectors components, i.e. from the start of the vector go four across the way and two up. The arrow shows the vector, which has both direction and length.

The magnitude of a vector \mathbf{A} is written as the modulus, $|\mathbf{A}|$. The magnitude of the vector $\mathbf{A} = 4\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$ using the graph and Pythagoras equals to $|\mathbf{A}| = \sqrt{4^2 + 2^2} = \sqrt{20}$.

The angle θ that the vector makes to the x -axis is given by $\tan \theta = 2/4$.

[Note on the way past the similarities with this and what we have just done with complex numbers.]

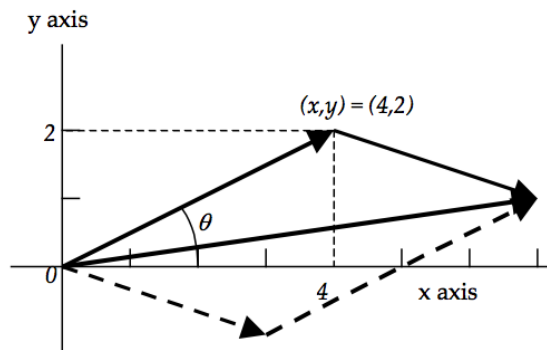
There are times when it is very convenient to use a “unit vector”. This is a vector of unit length (magnitude) and with a well defined direction. The unit vector parallel to the x -axis is $\hat{\mathbf{i}}$.

2.9 Adding Vectors

When adding vectors, we can work with the individual components, e.g.

$$(4\hat{\mathbf{i}} + 2\hat{\mathbf{j}}) + (3\hat{\mathbf{i}} - 1\hat{\mathbf{j}}) = (4 + 3)\hat{\mathbf{i}} + (2 - 1)\hat{\mathbf{j}} = 7\hat{\mathbf{i}} + \hat{\mathbf{j}}.$$

It is often useful to do this diagrammatically



Here we see the two vectors being added with one joining on to the previous one. The final result, shown as the thicker line in the middle of the parallelogram, does not depend on which of the two vectors comes first.

2.10 Subtracting Vectors

If we subtract a vector, we can do the same thing as above with components but using subtraction. In the diagram the process is the same as for adding, but with the vector that is being subtracted being in the reverse direction.

2.11 Multiplying a Vector by a Scalar

If we multiply a vector by a scalar, the direction of the vector stays the same, but the magnitude changes, e.g.

$$3 \times (4\hat{\mathbf{i}} + 2\hat{\mathbf{j}}) = (12\hat{\mathbf{i}} + 6\hat{\mathbf{j}}) .$$

2.12 Scalar Product

If we multiply two vectors together we need to define whether we mean the vector product or the scalar product. The scalar product, given by something like $\mathbf{A} \cdot \mathbf{B}$ gives a scalar as an answer. This is most frequently seen in “work done by a force moving along a line” type problems, where we determine the work by finding the product of the force-strength and the distance moved in the direction of the force. If vectors \mathbf{A} and \mathbf{B} have an angle θ between them, the scalar (or dot) product is

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos \theta .$$

If the vectors are given in Cartesian form such as $(a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}})$ and $(b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}})$, then while their scalar product is still given correctly by the above expression, it is more easily worked out from

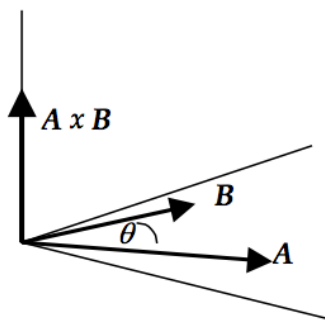
$$\mathbf{A} \cdot \mathbf{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Example:

If $\mathbf{A} = (4\hat{\mathbf{i}} + 2\hat{\mathbf{j}})$ and $\mathbf{B} = (4\hat{\mathbf{i}} - 2\hat{\mathbf{j}})$, then $\mathbf{A} \cdot \mathbf{B} = (4 \times 4) + (2 \times (-2)) = 16 - 4 = 12$.

2.13 Vector Product

This is seen in examples involving torque or angular momentum, for example. The two vectors are manipulated in such a way that the result is also a vector. The vector product is a vector that lies in a plane perpendicular to the two original vectors. Their vector product is perpendicular to the two original vectors, and has a length depending on the magnitudes of the vectors and the angle between them.



If the vectors \mathbf{A} and \mathbf{B} have an angle θ between them, their vector product has magnitude given by

$$|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}| \sin \theta .$$

In the above diagram should the vector product point up or down? One way to work this is out is to think of a screwdriver's action. The screwdriver is put into the slot of the screw that points in the direction of \mathbf{A} (the first in the vector product), and is then turned to make the slot point in the direction of \mathbf{B} (the second of the vector product). Will the screw (and driver) have moved up or down? In this case the screw will have been unscrewed, and will have moved up, hence the vector product pointing in the upwards direction. (Or use the "right-hand-rule" if you are not so familiar with DIY.)

If the vectors are given in Cartesian coordinates such as $(a_1 \hat{\mathbf{i}} + a_2 \hat{\mathbf{j}} + a_3 \hat{\mathbf{k}})$ and $(b_1 \hat{\mathbf{i}} + b_2 \hat{\mathbf{j}} + b_3 \hat{\mathbf{k}})$, then

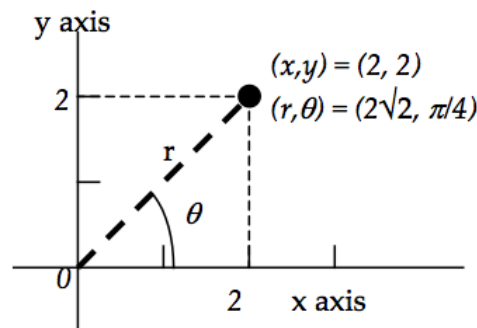
$$\mathbf{A} \times \mathbf{B} = (a_2 b_3 - a_3 b_2) \hat{\mathbf{i}} - (a_1 b_3 - a_3 b_1) \hat{\mathbf{j}} + (a_1 b_2 - a_2 b_1) \hat{\mathbf{k}}$$

This automatically shows the direction of the vector product. The expression is not desperately easy to remember, and is possibly best memorised using the determinant construction form of the above equation

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2) \hat{\mathbf{i}} - (a_1 b_3 - a_3 b_1) \hat{\mathbf{j}} + (a_1 b_2 - a_2 b_1) \hat{\mathbf{k}}$$

2.14 Coordinate Systems

You have been familiar with the Cartesian Coordinate system for some time. Most in the class will have come across polar coordinates more recently. In systems with circular or spherical symmetry, it is often much easier to work with points in space by their distance from the centre of symmetry and the angle that this direction is from defined axes, than it is by the use of Cartesian Coordinates. You will certainly come across polar coordinates in your Mechanics course, and in your Electricity and Magnetism course, as well as courses in mathematics. In Mechanics you will learn (or be reminded) how to describe small length elements in polar coordinates. While dx , dy , dz might be obvious as three length elements in Cartesian coordinates, things are a little less obvious in polars. However, the effort is well worth it for the ease with which (important) problems with circular or spherical symmetry can be tackled using polar coordinates.

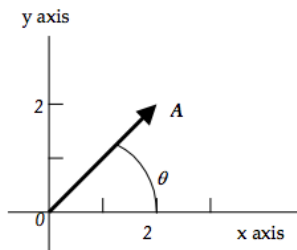


You may wish to take a look at the MathCentre leaflet on polar coordinates.

2.15 Workshop Questions

- Determine the sums (a) $(3 + i) + (3 - 2i)$ and (b) $(-5 + 2i) + (-3 + 4i)$.
- Find the products (a) $(2 + 2i) \cdot (-1 - i)$ and (b) $(2 + 2i) \cdot (2 - 2i)$.
- Plot the following on an Argand diagram, and find the modulus and argument of each
(a) $z = -2$, (b) $z = 2 + 2i$, (c) $z = -1 - i$, (d) $z = 6i$.
- Find the real and imaginary parts of (a) $z = 3e^{5i}$ and (b) $z = 2e^{\pi i}$.
- Express the complex number $(2 + 2i)$ in (a) polar and (b) exponential form.
- Express the complex number $(3 - 4i)$ in (a) polar and (b) exponential form.
- Plot on an Argand diagram the numbers $z_1 = 3$ and $z_2 = 2i$. Determine their sum z_3 and plot this on the same diagram. Find the arguments of z_1 , z_2 , and their sum. Qualitatively, how would the argument of the sum change if the modulus of z_2 was reduced?
- What is the complex conjugate of (a) $z = (4 + 3i)$ and (b) $z = (1 - g)e^{g(2+2i)}$ (g is real)?
- By multiplying together $z_1 = a + ib$ and $z_2 = a - ib$, show that one is the complex conjugate of the other. Do the same with $z_3 = g^{2x+iy}$ and $z_4 = g^{2x-iy}$ to show that one is the complex conjugate of the other (g is real).
- Evaluate the quotients (a) $\frac{2 + 2i}{3 + 4i}$ and (b) $\frac{2 - 4i}{1 - i}$.

11. (a) What is the Cartesian form of the vector \mathbf{A} shown in the diagram below?
 (b) What is the direction this vector makes with the x -axis?
 (c) What is the magnitude of the vector?



12. Sketch the vector $\mathbf{B} = (3\hat{\mathbf{i}} - \hat{\mathbf{j}})$.
13. Sketch the vector $\mathbf{C} = (3\hat{\mathbf{i}} + 2\hat{\mathbf{j}})$.
14. Sketch the vector $\mathbf{D} = (3\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$.
15. Evaluate (a) $\mathbf{B} + \mathbf{C}$ and (b) $\mathbf{A} + \mathbf{B}$.
16. Determine $3\mathbf{C}$.
17. Evaluate (a) $\mathbf{A} - \mathbf{B}$ and (b) $\mathbf{C} - \mathbf{B}$.
18. Find the magnitude and direction of $2\mathbf{B}$.
19. Evaluate (a) $\mathbf{B} \cdot \mathbf{C}$ (b) $\mathbf{C} \cdot \mathbf{B}$ (c) $\mathbf{A} \cdot \mathbf{B}$ (d) $\mathbf{C} \cdot \mathbf{C}$
20. Evaluate (a) $\mathbf{B} \times \mathbf{C}$ (b) $\mathbf{C} \times \mathbf{B}$ (c) $\mathbf{B} \times \mathbf{A}$ (d) $\mathbf{C} \times \mathbf{C}$
21. Evaluate (a) $\mathbf{B} \cdot \mathbf{D}$ (b) $\mathbf{B} \times \mathbf{D}$
22. Express Cartesian coordinates (a) $(3, 4)$ and (b) $(-2, 1)$ in (r, θ) polar coordinates.