

## Cotunneling in the $\nu = 5/2$ fractional quantum Hall regime

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We show that cotunneling in the  $5/2$  fractional quantum Hall regime allows us to test the Moore-Read wave function, proposed for this regime, and to probe the nature of the fractional charge carriers. We calculate the cotunneling current for electrons that tunnel between two quantum Hall edge states via a quantum dot and for quasiparticles with fractional charges  $e/4$  and  $e/2$  that tunnel via an antidot. While electron cotunneling is strongly suppressed, the quasiparticle tunneling shows signatures characteristic of the Moore-Read state. For comparison, we also consider cotunneling between Laughlin states, and find that electron transport between Moore-Read states and between Laughlin states at filling factor  $1/3$  have identical voltage dependences.

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### I. INTRODUCTION

Fractional quantum Hall (FQH) states are intriguing states of matter because elementary collective excitations behave as quasiparticles with fractional charge and statistics. The FQH state at filling factor  $5/2$  ( $5/2$ -FQHS) has become of special interest because it has been identified in several proposals as a state in which the elementary excitations obey non-Abelian fractional statistics.<sup>1-7</sup> Numerical simulations testing these proposals have remained inconclusive,<sup>8-13</sup> mainly due to finite size limitations. A proof of the nature of the  $5/2$ -FQHS should therefore come from experiments.<sup>14</sup> As a first indicator, evidence for a chiral Luttinger liquid at the edges of the FQH sample was obtained in Ref. 15, demonstrating the fractional nature of the quantum state, yet not its Abelian/non-Abelian statistics. It has also been shown that quantum point contacts and interferometers can be constructed in the samples,<sup>16,17</sup> allowing for the implementation of the interferometer-based tests proposed in Refs. 18–21. A thermoelectric probing of different FQH states on quantum dots has been proposed in Ref. 22. Yet further proposals for tests are desirable to obtain conclusive evidence. Theoretical FQH studies have shown that quantum dot (QD) and quantum antidot (AD) structures with corresponding excitations, electrons, and quasiparticles (QPs), exhibit similar physics.<sup>23,24</sup> In the cotunneling regime the number of particles on the dot is conserved and second order tunneling processes dominate transport.<sup>25,26</sup> Elastic (inelastic) processes conserve (change) the state of the dot. The inelastic process leads to an excitation of the dot for bias voltages larger than the level spacing on the dot. Since the implementation of dot structures has become a mature experimental technique, we propose in the present paper a QD based setup for an alternative test of the nature of the  $5/2$ -FQHS. In particular, we show that the cotunneling current strongly depends on the nature of the elementary QP excitations that can contribute to the current that is allowed to tunnel through the dots.

Possible charge carriers are electrons and fractionally charged QPs with non-Abelian statistics. The most prominent candidate QPs are excitations of charge  $e/4$  and  $e/2$  with  $e$  being the electron charge.<sup>27-32</sup> We investigate the existence of signatures of the  $5/2$ -FQHS according to the theoretical description proposed by Moore and Read (MR)<sup>1</sup> in simple transport measurements, e.g., conductance through a QD.

Figure 1 shows the two situations of interest: (a) two different FQH samples at filling factor  $\nu = 5/2$  which are weakly tunnel coupled to a QD, and (b) one single FQH sample whose edge states are weakly tunnel coupled to an AD in the bulk. The tunneling particles in the latter are non-Abelian QPs instead of electrons. The edge states are modeled by a chiral Luttinger liquid theory<sup>33-37</sup> corresponding to the MR state whose eigenmodes are fractional excitations. In this work we focus on cotunneling in the Coulomb blockade regime close above a sequential tunneling peak such that it is energetically favorable to first remove a particle from the dot rather than first adding another particle. An interesting outcome of our calculation is that electron cotunneling via a QD between both Laughlin edge states at filling factor  $\nu = 1/3$  and MR edge states shows the same bias voltage dependence.

### II. RESULT

The cotunneling current from lead  $l$  to lead  $l'$  in lowest order in the bias  $V$  is given by  $I = I_{\text{el}} + I_{\text{inel}}$ , with  $I_{\text{el}}$  being the elastic cotunneling current given by

$$I_{\text{el}} = \frac{2\pi}{\hbar} \left(\frac{V}{\Lambda}\right)^{2\kappa-2} \frac{\gamma_l \gamma_{l'} V}{\Gamma(2\kappa) (\mu_l - \varepsilon_{nq})^2} \theta(V), \quad (1)$$

and  $I_{\text{inel}}$  the inelastic cotunneling current given by

$$I_{\text{inel}} = \frac{2\pi}{\hbar} \left(\frac{V-\Delta}{\Lambda}\right)^{2\kappa-2} \frac{\gamma_l \gamma_{l'} (V - \Delta)}{\Gamma(2\kappa) (\mu_l - \varepsilon_{nq})^2} (1 + \rho^*) \theta(V - \Delta), \quad (2)$$

with bias  $V = \mu_l - \mu_{l'}$ , chemical potential  $\mu_l$  of edge state  $l$ , single-particle level spacing on the dot  $\Delta = \varepsilon_{n'q} - \varepsilon_{nq}$ , dot level  $\varepsilon_{nq}$ , effective bandwidth of the leads  $\Lambda = \hbar u / \alpha$  (bounded by the gap of the  $5/2$ -FQHS) defining the length  $\alpha$ , velocity of the bosonic (fermionic) edge excitations  $u$  ( $v_n$ ), Heaviside step function  $\theta$ , and tunneling rate  $\gamma_l = |A_{ln'n}|^2 / (2\pi v_n \hbar)$  with  $A_{ln'n} = \sum_p t_{lp} \langle n' | d_{pq} | n \rangle$  (see below). The renormalized dot occupation  $\rho^* \propto (V + \Delta)^{2\kappa-1} - (V - \Delta)^{2\kappa-1}$  accounts for the overshooting of the conductance close to the transition from the elastic to the inelastic regime and will be given in detail below. The parameter  $\kappa$  is determined by the type of tunneling particles (see below). For cotunneling of  $e/2$  and  $e/4$  QPs in setup (b), we replace the step function  $\theta$  by the Fermi function

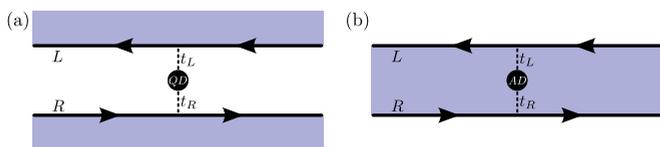


FIG. 1. (Color online) Sketch of the two setups considered in this work: Moore-Read edge states  $L$  and  $R$ , at chemical potentials  $\mu_{L,R}$ , tunnel coupled to (a) a quantum dot (QD) and (b) a quantum antidot (AD). The shaded (blue) region represents the bulk of the FQH samples. In setup (a) tunneling is limited to electrons, while in (b) the quasiparticles (QP) that can tunnel are determined by the filling factor of the AD, and, for the MR state both, charge- $e/2$  and  $-e/4$ , QPs are possible.

$f(V) = (1 + e^{V/k_B T})^{-1}$  (for a temperature  $T < \Delta/k_B$  and the Boltzmann constant  $k_B$ ) which smooths out the discontinuities at  $V = 0, \Delta$  for  $T = 0$ , and we use the  $V \rightarrow \sqrt{V^2 + (k_B T)^2}$  regularization for cotunneling of  $e/4$  QPs.

### III. MODEL

The system is modeled by the Hamiltonian  $H = H_0 + H_T$ , where  $H_0 = H_L + H_R + H_D$  describes the uncoupled FQH edges and the dot, and  $H_T$  the tunneling between them. In the considered systems the leads are fractional quantum Hall edge states modeled according to the MR state and described by the Hamiltonian for lead  $l$  at chemical potential  $\mu_l$ <sup>1,27</sup>

$$H_l = \frac{u\hbar}{2\pi} \int dx (\partial_x \phi_l(x))^2 - i v_n \hbar \int dx \eta_l(x) \partial_x \eta_l(x), \quad (3)$$

where  $\phi_l$  is a chiral boson field, the Majorana field  $\eta_l$  is the zero mode, and  $u$  ( $v_n$ ) is the velocity of the bosonic (neutral fermionic) excitations. For the MR state the lower filled Landau level acts as a background potential and causes a shift of the energy levels which is not important in our discussion. The fermion operator in lead  $l$  is given by<sup>27,35</sup>

$$\psi_{l_e}(x, t) = \frac{e^{-ik_1 x}}{\sqrt{2\pi\alpha}} \eta_l(x, t) e^{i\phi_l(x, t)/\sqrt{2}}, \quad (4)$$

with  $k_1$  proportional to the particle density in the leads. Analogously the  $e/2$  and  $e/4$  QP operators are<sup>27,35,38</sup>

$$\psi_{l_{\frac{e}{2}}}(x, t) = \frac{e^{-ik_1 x}}{\sqrt{2\pi\alpha}} e^{i\phi_l(x, t)/\sqrt{2}}, \quad (5)$$

$$\psi_{l_{\frac{e}{4}}}(x, t) = \frac{e^{-ik_1 x}}{\sqrt{2\pi\alpha}} \sigma_l(x, t) e^{i\phi_l(x, t)/2\sqrt{2}}, \quad (6)$$

where  $\sigma_l$  is a chiral Ising spin field. The dot is modeled by  $H_D = \sum_n \varepsilon_{nq} d_{nq}^\dagger d_{nq}$ , where  $d_{nq}$  is the electron or QP operator for the discrete particle level  $n$  on the dot similar to Eqs. (4), (5), and (6) with particle charge  $q = e$  for electrons on the QD and  $q = e/2, e/4$  for QPs on the AD. The Coulomb repulsion of the particles lifts a possible degeneracy of the energy levels including both charging and interaction energies. Tunneling between the leads and the dot is described by the perturbation  $H_T = \int dx \sum_{l,n} t_{ln} \psi_{lq}^\dagger(x) d_{nq} + \text{H.c.}$ , where  $\psi_{lq}(x)$  and  $d_{nq}$  annihilate particles of charge  $q$  in lead  $l$  at  $x$  and in the dot level  $n$ , respectively, and with tunneling amplitude  $t_{ln}$ . From Ref. 27 we know that QP tunneling processes are dominant in the MR state.

In setup (a) (Fig. 1) the leads are independent as they belong to different FQH systems. Independence between the edges is also assumed in setup (b), which requires that the length of the edges be larger than the coherence length of the excitations, estimated as  $2.3l_0$  with  $l_0 \sim 4 \mu\text{m}$  being the magnetic length.<sup>39,40</sup> Tunneling in the leads is assumed to be limited to the positions closest to the dot, denoted by  $x = 0$  with width  $\Delta x \ll k_1^{-1}$ , because  $t_{ln}$  depends exponentially on the tunneling distance<sup>41</sup> and consequently no special effects arise from the difference in velocities between bosons and Majorana states as opposed to Mach-Zehnder interferometers.<sup>18-21</sup> We focus on the Coulomb blockade regime, where the particle number on the dot is fixed,  $N_q = \sum_n d_{nq}^\dagger d_{nq}$ . The charging energy of the dot is much larger than the single-particle level spacing  $\Delta$  and particularly larger than  $\mu_l - \varepsilon_{nq}$ . Tunneling processes through energetically distant dot levels are suppressed. The dot forms an effective two-level system that contains one particle. The persistence of the ground state in the elastic regime is ensured by charge conserving cotunneling processes which relax the state of the dot. The charge of the tunneling QPs is set by the filling factor of the AD such that there is no mixing of  $e/4$  and  $e/2$  QPs. We consider a regime in which the applied bias and the level spacing on the dot are larger than the temperature, allowing us to essentially neglect temperature effects. Since the MR state is spin polarized, spin is neglected in the model<sup>8,9,11,42-44</sup> and the fermion operator  $\psi_{l_e}$  is identified with the spin-polarized electron.

### IV. TRANSITION RATES

The transition rate  $W_{l'l}(n', n)$  of transferring a particle from lead  $l$  to lead  $l'$  and shifting the particle on the dot from level  $n$  to level  $n'$  is determined by the golden rule

$$W_{l'l}(n', n) = \frac{2\pi}{\hbar} \sum_{|F\rangle \neq |I\rangle} |\langle F | \hat{T} | I \rangle|^2 \delta(E_I - E_F), \quad (7)$$

with the  $T$  matrix  $\hat{T} = H_T(E_I - H_0 + i0^+)^{-1} \hat{T}$ , final,  $|F\rangle$ , and initial,  $|I\rangle$ , states for the two leads and the dot, and energies  $E_F, E_I$  respectively. Level broadening in the leads (reservoirs) is neglected in state  $|F\rangle$ . We focus on the second order in the tunneling Hamiltonian  $H_T$  and the cotunneling regime. The linear order of  $H_T$  in the  $T$  matrix leading to sequential tunneling is suppressed in the Coulomb blockade regime<sup>25</sup> such that only the next order, cotunneling, matters. The final and initial states are given by  $|F\rangle = |F_l\rangle \otimes |F_{l'}\rangle \otimes |F_D\rangle$  and  $|I\rangle = |I_l\rangle \otimes |I_{l'}\rangle \otimes |I_D\rangle$ , with  $|F_l\rangle = \psi_{lq}^\dagger |I_l\rangle$ ,  $|F_{l'}\rangle = \psi_{l'q}^\dagger |I_{l'}\rangle$ ,  $|F_D\rangle = d_{n'q}^\dagger d_{nq} |I_D\rangle$ , and initial lead  $l$  and dot states  $|I_l\rangle$  and  $|I_D\rangle$ , respectively.

From Eqs. (4), (5), and (6) we obtain the correlation functions for electrons,  $q = e/2$ , and  $q = e/4$  QPs, respectively,<sup>35</sup>

$$\langle \psi_{l_e}^\dagger(t) \psi_{l_e}(0) \rangle = \frac{\langle \eta_l(t) \eta_l(0) \rangle}{2\pi\alpha} e^{2J(t)} \propto t^{-(\kappa=3)}, \quad (8)$$

$$\langle \psi_{l_{\frac{e}{2}}}^\dagger(t) \psi_{l_{\frac{e}{2}}}(0) \rangle = \frac{1}{2\pi\alpha} e^{J(t)/2} \propto t^{-(\kappa=1/2)}, \quad (9)$$

$$\langle \psi_{l_{\frac{e}{4}}}^\dagger(t) \psi_{l_{\frac{e}{4}}}(0) \rangle = \frac{\langle \sigma_l(t) \sigma_l(0) \rangle}{2\pi\alpha} e^{J(t)/8} \propto t^{-(\kappa=1/4)}, \quad (10)$$

with  $J(t) = -\ln((-ut + i\alpha)/i\alpha)$ . The indicated power laws define the coefficient  $\kappa$ . The noninteracting case,  $\kappa = 1$ ,

corresponds to the Fermi liquid (FL) limit. We note that, since  $x = 0$ , the Majorana fermion field  $\eta_l$  (Ising spin field  $\sigma_l$ ) results in a simple correlator, increasing the lead correlation exponent  $\kappa$  by 1 (1/8) such that  $\kappa = 3, 1/2, 1/4$  for  $q = e, e/2, e/4$ , which is in contrast to the Mach-Zehnder interference proposals<sup>18–21</sup> and leads to simpler propagators.

In the inelastic cotunneling regime,  $V > \Delta$ , processes exist which excite the dot and change the occupation probabilities of the dot levels. The steady state occupation probabilities are then determined by the master equation  $W^\uparrow \rho(1) - W^\downarrow \rho(2) = 0$ , with  $W^\uparrow = \sum_{l,l'} W_{l'l}(2,1)$ ,  $W^\downarrow = \sum_{l,l'} W_{l'l}(1,2)$ , and  $\rho(n)$  the occupation probability of level  $n = 1, 2$ . The cotunneling current from lead  $l$  to lead  $l'$  is then given by

$$I_{l'l} = \sum_{n,n'} W_{l'l}(n',n) \rho(n). \quad (11)$$

The renormalized dot occupation in Eq. (2) reads  $\rho^* = [W_{l'l}(1,2) - W_{l'l}(2,1)] / (W^\uparrow + W^\downarrow)$ . We evaluate Eq. (7) by means of Fourier integration over time to reexpress the  $T$  matrix in terms of the correlators given in Eqs. (8)–(10). We then find for the cotunneling rates for charge transfer between FQH edge states and a dot

$$W_{l'l}(n',n) = \frac{2\pi}{\hbar} \Gamma_{l'l'n'n}^\kappa \frac{\gamma_l \gamma_{l'} (V - \Delta)}{(\mu_l - \varepsilon_{nq})^2} \theta(V - \Delta), \quad (12)$$

with the real coefficients

$$\Gamma_{l'l'n'n}^\kappa = \frac{\left(\frac{V-\Delta}{\Lambda}\right)^{2\kappa-2}}{\Gamma(2\kappa)} \Big|_{l \neq l'} + \delta_{l',l} \Xi_{l'n'n}^\kappa, \quad (13)$$

where  $\Gamma$  is the Gamma function and  $\Xi_{l'n'n}^\kappa$  results from energy renormalization in the leads. In the limit of noninteracting leads,  $\Gamma_{l'l'n'n}^\kappa = 1$  and hence  $\Xi_{l'n'n}^\kappa = 1$ , the FL result of Ref. 45 close to a sequential tunneling resonance is recovered. For charge conserving processes of  $e/2$  ( $e/4$ ) QPs,  $\Xi_{l'n'n}^{1/2}$  ( $\Xi_{l'n'n}^{1/4}$ ) can be approximated by their most singular contributions, the branch points (branch cuts), such that  $\Xi_{l'n'n}^{1/2} = -\Lambda/\Delta$  ( $\Xi_{l'n'n}^{1/4} = |\Lambda/\Delta|^{3/2}/\sqrt{\pi}$ ), whereas for electron cotunneling processes  $\Xi_{l'n'n}^3$  results in a lengthy expression with, however, negligible effect. From Eqs. (11)–(13) we obtain the results of Eq. (1) (for  $\Delta = 0$ ) and Eq. (2).

## V. DISCUSSION

We have calculated the cotunneling current of the MR state in both setups of electron and quasiparticle tunneling via a dot. The resulting line shapes differ significantly, especially from the line shape of the FL regime.<sup>26</sup> The cotunneling current for electrons shows a power law dependence on both the bias applied to the edge states and the effective bandwidth of the leads,  $V^{2\kappa-1}$  and  $\Lambda^{2-2\kappa}$ . The effective bandwidth  $\Lambda$  is on the order of the effective Landau level gap size in the  $5/2$ -FQHS,  $\sim 100$ – $500$  mK. We note that  $\tilde{\gamma}_l = \gamma_l \Lambda^{1-\kappa}$  can be considered an effective cotunneling amplitude from which it is obvious that electron cotunneling between MR edge states via a QD is highly suppressed by the fourth inverse power of the effective bandwidth,  $\tilde{\gamma}_L \tilde{\gamma}_R = \gamma_L \gamma_R \Lambda^{-4}$ , due to the fact that four  $e/4$  QPs are forced to tunnel simultaneously, in agreement with earlier findings that electron tunneling is least relevant in the MR state.<sup>27</sup> Figure 2 shows the differential

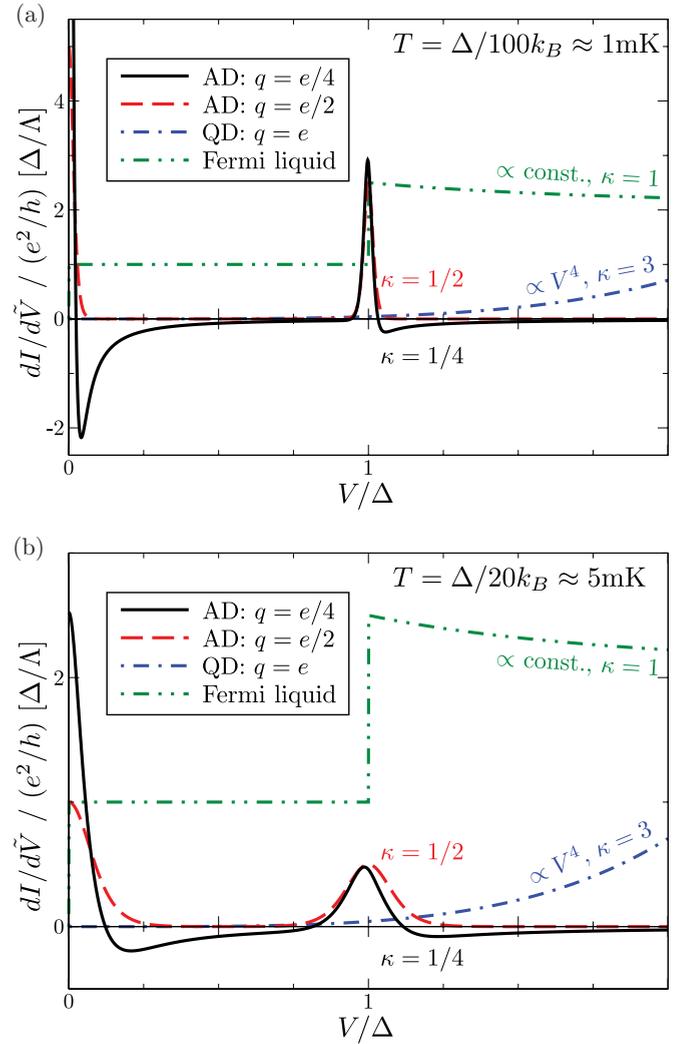


FIG. 2. (Color online) Cotunneling conductance  $dI/d\tilde{V}$  with  $\tilde{V} = V/e$ , and  $I = I_{el} + I_{inel}$  given by Eqs. (1) and (2) for  $e/4$  and  $e/2$  QP transport between MR edge states of the same FQH sample through an AD, electron transport between MR edge states of separate FQH samples through a QD, and electron transport between FL leads through a QD. The  $e/2$  and  $e/4$  QP conductance is shown for the temperatures (a)  $T = \Delta/100k_B \approx 1$  mK and (b)  $T = \Delta/20k_B \approx 5$  mK, respectively.

conductance,  $dI/dV$ , in the cotunneling regime, for electron tunneling in the FL regime and in the MR state via a QD, and for  $e/4$  and  $e/2$  QP tunneling in the MR state via an AD at experimentally achievable temperatures, (a)  $T = \Delta/100k_B \approx 1$  mK and (b)  $T = \Delta/20k_B \approx 5$  mK, respectively. In the FL regime, which corresponds to noninteracting excitations of the FQH edge, previous results<sup>26,45</sup> are recovered. Fig. 3 shows the temperature dependence of the  $e/4$  and  $e/2$  QP cotunneling conductance at the transition from the elastic to the inelastic regime,  $V = \Delta$ . The inset gives both the  $e/4$  QP conductance dip depth,  $-dI/d\tilde{V}$  at the minimum with respect to  $V$ , and the temperature range for which the negative differential conductance of  $e/4$  QPs is observed. The MR state reveals its special signature in the line shape of the cotunneling conductance. For  $e/4$  QP tunneling regions of negative differential conductance appear. On the other hand,

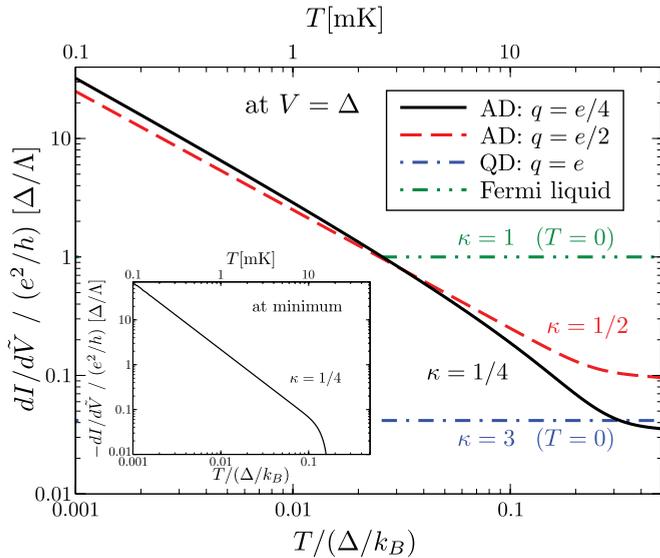


FIG. 3. (Color online) Temperature dependence of the  $e/4$  and  $e/2$  QP cotunneling conductance  $dI/d\tilde{V}$  with  $I = I_{el} + I_{inel}$  given by Eqs. (1) and (2) and  $\tilde{V} = V/e$  fixed at the transition from the elastic to the inelastic regime,  $V = \Delta$ . Values for electron transport between MR edge states of separate FQH samples through a QD and electron transport between FL leads through a QD are shown for zero temperature. On the upper scales  $T$  is given in mK for  $\Delta/k_B = 100$  mK. The inset shows the conductance dip depth,  $-dI/d\tilde{V}$ , for  $e/4$  QP tunneling at the minimum in the nonfixed bias  $V$ .

both  $e/2$  and  $e/4$  QPs show pronounced conductance peaks at the opening of a new transport channel. These special peaks corroborate the findings of Ref. 27 of  $e/2$  and  $e/4$  QP tunneling being relevant. In the cotunneling regime, however,

the renormalization group flow is cut off by bias  $V$  and temperature  $T$  such that the perturbative result is accurate. Our calculations show that the different charge carriers can be clearly distinguished by standard transport measurements.

Our approach is also applicable to a setup of two separate FQH samples with common Laughlin FQH edge states at filling factor  $\nu$  weakly coupled through a QD, similar to the setup in Fig. 1(a). The cotunneling current is then given by Eqs. (1) and (2) with  $\kappa = 1/\nu$ .<sup>35</sup> In this scenario,  $1/\nu$  QPs of fractional charge  $\nu$  combine to a full electron charge when tunneling through the QD, such that not one particle has to tunnel but  $\frac{1}{\nu} - 1$  additional particles. Due to the necessity of simultaneous tunneling the electron cotunneling current is suppressed by  $\Lambda^{2-2/\nu}$ . It is interesting to note that electron tunneling via a QD in both the MR state and the Laughlin state at filling factor  $\nu = 1/3$  show the same voltage dependence. However, the two states differ in the velocities of the excitations due to the bosonic and fermionic nature of the lead eigenmodes.

In conclusion, we have shown that electron cotunneling via a quantum dot is strongly suppressed in the Moore-Read state compared to  $e/4$  and  $e/2$  quasiparticle cotunneling via an antidot. The line shape of the differential conductance reveals the special signature of the Moore-Read state. Both the Moore-Read wave function can be verified and the charge carrying excitation can be determined by measuring the cotunneling current in the setups depicted in Fig. 1.

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