Inferentialism, revision, harmony and (intuitionist) negation

"To deny the doctrine is to change the subject" (Quine)

The doctrine that logic is unrevisable is an old one. In one form, it reflects the notion that basic logic is supremely certain, and our basic logical faculties supremely reliable (Frege). In another, more recent version the root idea is deflationary: logic is immune to revision not because it is supremely certain but because, roughly, (with certain qualifications) there is no sense in the idea of its being mistaken:

Examples of latter anti-revisionist tendency:
Carnap (conventionalism)
Quine (basic logical laws are constitutive principles of correct radical interpretation)
Wittgenstein (logical laws are among the "rules of the language game", they belong to "grammar". The idea of entrenched logic being wrong makes as little sense, and for the same reasons, as the idea that we play Chess by the wrong rules.)

A revision of a logic, for our purposes, is any change — enlargement, modification, or rejection — in its underived principles whose effect is to change the stock of derivable theorems expressible in the original vocabulary. (Thus enlargements e.g. of sentential logic by principles for modal operators don't — rightly — count as revisions unless they change the stock of non-modally expressible theorems.)
Informal logic long preceded formal logic, and the formalisation of logic proceeded by partial steps (Aristotle to Frege, and since). Intermediate stages were characterised by omissions, and incomplete generality, which were later rectified. These were revisions in the above sense. So we can, and have, erred by omission. Why can’t we also err by inclusion?

Anti-revisionist reply: errors of codification of practice are certainly possible - what is not possible is error in the logical principles which we basically accept : the principles which codification aims to codify.
This puts a lot of strain on "basic acceptance" — how do we tell which are the basically accepted principles? It also sits ill alongside the fact that formal logic—and certainly classical sentential and quantificational logic — introduces its own conceptual logical vocabulary — e.g. its own conditional—and therefore makes no official pretence of codifying preformal logical practice.

**Meaning-constitution**

The basic anti-revisionist thought, however, —common, with variations, to each of Carnap, Wittgenstein and Quine (surprising in the last case)— is that basic logic cannot go wrong simply because which principles we accept is what determines the meanings of the connectives and other operators they configure. So they are incorrigible in something like the way a definition is incorrigible. We do not accept, e.g., the principle of *Affirming the Consequent*

\[
A_1, \ldots, A_n \Rightarrow A \Rightarrow B ; \ B_1, \ldots, B_n \Rightarrow B
\]

\[
A_1, \ldots, A_n, B_1, \ldots B_n \Rightarrow A
\]

But if we had, then we should simply have adopted a different meaning for "\(\Rightarrow\)" (that which we now assign to "\(\Leftarrow\)" maybe, or maybe just an inversion of the 'direction' of the arrow - it would depend what else we did.)

Our problem: shouldn’t Inferentialism buy into something like this conservatism? Inferentialism holds, crudely, that the content of the logical operators is fixed by the rules governing them. Doesn’t that invite Wittgenstein’s point? Yet some inferentialists (Prawitz, Tennant, sometimes Dummett) are revisionists. How are we to understand this? Who is right?

(Qualified anti-revisionism (Quine): There could still be *pragmatic* reasons for preferring one practice to another (maybe once upon a time the Queen had the additional power of moving in the style of the Knight...))

This meaning-constitutive anti-revisionist line is vulnerable to three *prima facie* objections. (i) The analogy with regular definition needs qualification to correct for the assimilation of cases where a *new symbol* is annexed to a prior concept—strict explicit
definitions—to cases where a **new concept** is fixed by the determination of a pattern of use for an expression of it. Nothing can go wrong in the former kind of case if the concept in question is in prior good standing. But even if one is sympathetic to the basic concept-constitutive idea, one may still want to allow that the attempt to fix a concept in a certain way may misfire, with only a defective, or even no clear concept at all, resulting.

(ii) If the meaning-constitutive anti-revisionist were correct, there would be as little sense in the idea of challenging, say, Modus Ponens as challenging the definition of ‘father’ as a male parent. But we understand e.g. the McGee challenge very well, and whatever is right about the idea of the concept of the conditional being somehow constitutively related to our basic inferential practice with it had better be consistent with our doing so.

(iii) Having a connective, e.g., with its associated rules of inference, will allow us to establish connections between statements in which it does not occur. Doesn’t there have to be some kind of conservativeness constraint? General form of the point:

Suppose we introduce a connective "" as governed by an introduction rule:

\[(^I)\]

\[
\begin{array}{c}
A_1,\ldots, A_n \Rightarrow \text{Stuff} (A) ; B_1,\ldots, B_n \Rightarrow \text{Blah} (B) \\
A_1,\ldots, A_n, B_1,\ldots, B_n \Rightarrow A^B
\end{array}
\]

where "Stuff..." and :"Blah..." are previously understood types of context; and an elimination rule

\[(^E)\]

\[
\begin{array}{c}
C_1,\ldots, C_n \Rightarrow A^B ; D_1,\ldots, D_n \Rightarrow A \\
C_1,\ldots, C_n, D_1,\ldots, D_n \Rightarrow \text{Buzz} (B)
\end{array}
\]

where "Buzz..." is a previously understood type of context.

Well, in any case where \(D_1,\ldots, D_n \Rightarrow A\), having these rules may get us into trouble unless \(\{A_1,\ldots, A_n , B_1,\ldots, B_n , D_1,\ldots, D_n\}\) are independently collectively sufficient to ensure Buzz(B).

A minimal, very familiar example where things go wrong in this kind of way is Pryor's connective, *tonk*

It's rules are

\[(\text{Tonk I})\]

\[
\begin{array}{c}
A_1,\ldots, A_n \Rightarrow A \\
A_1,\ldots, A_n \Rightarrow A \text{ tonk B} \\
A_1,\ldots, A_n \Rightarrow A \text{ tonk B}
\end{array}
\]

\[
\begin{array}{c}
A_1,\ldots, A_n \Rightarrow A \text{ tonk B} \\
A_1,\ldots, A_n \Rightarrow A \text{ tonk B} \\
A_1,\ldots, A_n \Rightarrow B
\end{array}
\]

\[(\text{Tonk E})\]
Another simple case: the Universal Quonkifier:

**(Quonk I)** \[ A_1,\ldots, A_n \Rightarrow A_t \] where ‘t’ is a term

\[ A_1,\ldots, A_n \Rightarrow (Qx) A_x \]

**(Quonk E)** \[ A_1,\ldots, A_n \Rightarrow (Qx) A_x \]

\[ A_1,\ldots, A_n \Rightarrow A_t \] where ‘t’ is any term

**Pandemic holism**

Anti-revisionist response to the above: concession – the plasticity of meaning. OK: logical rules can go wrong by introducing inconsistency. But they cannot go wrong at least so long as a consistent practice is enjoined. Provided that a consistent practice is enjoined, both the logical expressions the rules concern, and the statements free of their special vocabulary which they enable us to connect via inference, will simply take on meaning in such a way as to be both intelligible and validly so connected respectively. Since they are constitutive rules of practice, moves in the practice will mean what the rules require them to mean—there is no space for them to be somehow objectionable (or virtuous but unrecognised.) (This is, I imagine, something close to what Wittgenstein means when he speaks in the Remarks on the Foundations of Mathematics s of logic as "antecedent to truth").

**Observations:**

(i) A quibble but an important one: presumably the important feature for this line is not consistency but non-explosiveness. Paraconsistent logics constrain a determinate practice—in the sense the holist intends. And a negation-free logic augmented by "tonk" need not be inconsistent.

(ii) Very important: determinacy of practice in the intended sense does not seem to be sufficient for determinacy of content. There are

(a) Cases where a determinate (non-explosive) practice generated by inferential rules is intuitively insufficient for the logical operators they concern to take on an intelligible meaning:

(a) **Tunk**

**(Tunk I)** \[ A_1,\ldots, A_n \Rightarrow A ; B_1,\ldots, B_n \Rightarrow B \]

\[ A_1,\ldots, A_n, B_1,\ldots, B_n \Rightarrow A \text{ tunk } B \]
(Tunk E) \[ A_1, \ldots, A_n \Rightarrow A \text{ tunk } B \; ; \; \{B_1, \ldots, B_n, A\} \Rightarrow C \; ; \; \{C_1, \ldots, C_n, B\} \Rightarrow C \]
\[ A_1, \ldots, A_n, B_1, \ldots, B_n, C_1, \ldots, C_n \Rightarrow C \]

(b) The Existential Bunkifier

(Bunk I) \[ A_1, \ldots, A_n \Rightarrow \text{At}, \quad \text{where } A_1, \ldots, A_n \text{ are 't'- free} \]
\[ A_1, \ldots, A_n \Rightarrow (Bx)Ax \]

(Bunk E) \[ A_1, \ldots, A_n \Rightarrow (Bx)Ax \; ; \; B_1, \ldots, B_n, A_t \Rightarrow C, \quad \text{where } A_1, \ldots, A_n, C \text{ are 't'- free} \]
\[ A_1, \ldots, A_n, B_1, \ldots, B_n \Rightarrow C \quad \text{‘t’-free} \]

What can "(Bx)Ax" mean - what is the quantity of elements in the domain whose being A is necessary and sufficient for "(Bx)Ax" to be true?

Note: these examples do, of course, admit of coherent interpretation—but only at the price of revisionary criticism.

(b) There are cases where a determinate (non-explosive) practice generated by inferential rules does intuitive violence to the meanings of statements as determined prior to the introduction of the novel operator.
Consider e.g. the numerically definite quantifiers, "there are at least n F’s", characterised recursively as follows:

\[ (E_{1x})Fx : (\exists x)Fx \]
\[ (E_{n+1x})Fx : (\exists x)[Fx & (E_ny)(Fy & y \neq x)] \]

These clauses allow us to translate all "at least n" quantifications into a regular first-order quantificational language and to handle their proof theory accordingly without subjecting them to special rules. But now suppose we add infinitely many Intro.-rules which are instances of the following schema:

\[ (E_{m+1 I}) \quad A_1, \ldots, A_n \Rightarrow (E_{n+1x})Fx \]
\[ A_1, \ldots, A_n \Rightarrow (E_{n+2x})Fx \]

This won’t introduce inconsistency/explosion since the rules in question are all satisfiable provided the domain is infinite. But if the pandemic holist were right, meaning should just flow into the new quantifiers without the need for any special assumptions; statements of the form, \((E_mx)Fx\), should just take on a meaning that fits both the extended Intro. rules and their initial definitions and the liaisons thereby established with the normal existential
quantifier, as governed by its regular rules. Commonsense recoils: there are no such meanings to mean—the only way to understand the extended practice is as incorporating an axiom of infinity. So any reason to doubt such an axiom will be reason to think the practice wrong. (This, by the way, is the model for the intuitionist-inferentialist critique of classical logic)

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Interlude: Revising a logic: eleven potentially good reasons

(Exercise: identify and review the defensibility of the values—"Logic should be V"—variously presupposed in the following historical forms of revisionary criticism.)

1. *L is explosive*

2. *L is inconsistent* (pace Dialetheism)

3. *L allows the derivation of falsehoods from truths.*
   This is dialetheism's objection to CL's rule of *ex falso quodlibet*:
   \[ A_1, \ldots, A_n \Rightarrow B & \neg B \]
   \[ A_1, \ldots, A_n \Rightarrow C \]
   (Intuitionist logic, as often formulated, also accepts this rule. But see below.) It is also Field's objection to the classical logical derivation of conjunctive contradiction from \( B \iff \neg B \), where \( B \) is the Liar sentence.

**4. *L fosters the derivation of unjustified conclusions* from what it takes to be theses of logic—\( L \) is epistemically non-conservative.** This is one important kind of intuitionistic motivation for revision of classical logic, principally targeted against LEM. The development of the objection usually involves reasoning by cases and supplementary non-logical theses.

5. *L misrepresents the proof-theoretic behaviour of the natural language logical constants* to which its logical primitives correspond. Advocates of Relevance and Paraconsistent logics have usually had in mind the classical conditional.

6. *L plays a devil's part in the generation of paradox.* (Jc, Field on CL and the Liar/Curry. This is also a possible line on the Sorites—see Peacocke on MPP.)

**7. *L's logical operators are assigned no defensible meaning by its standard semantics* So e.g. Dummett's objections to classically understood quantification over indefinitely extensible totalities come under this heading. Obviously such objections, even if well, taken, issue in a motive for revision only if one holds (a) that logics stand in need of semantic justifications and (b) no unobjectionable semantics validates \( L \). (Overarching issue: how can meanings be objectionable?)

**8. Similar to 7 — *L's logical operations are assigned no intelligible meaning by the standard semantics* e.g. Dummett's considerations on the manifestation and acquisition of understanding of evidence-transcendent truth-conditions as an objection to the principle of Bivalence. Similar qualifications as in 7 apply.
**9. L's rules fail to establish any intelligible meaning for its logical operators.**
Cf. "tunk" and "quonk", etc. One intuitionistic objection to CL concerns the (alleged) disharmony of the classical negation rules.

10. *L gets in the way of a coherent theoretical treatment of some important phenomenon.* There are a whole cluster of issues under this heading concerning vagueness/indeterminacy/paradox. For instance:
   - CL allows the derivation of indeterminate conclusions from true premisses (QM and the classical distributivity laws.) (But NB Schiffer on vagueness.)
   - CL allows the derivation of false conclusions from indeterminate premisses (Field and the Liar.)
   - CL and the importation of the dubious metaphysics of Epistemicism.. This comes back to 4.
   - CL introduces unwelcome determinacy into matters of taste, etc.

11. *Revision of L promises to lead to improvements in empirical theory* (Quine).

Notice that at least four types of consideration these marked **: 4, 7, 8 and 9 — have historically been presented as motivations for broadly intuitionistic revisions of classical logic. There is no single intuitionistic critical standpoint. I now focus on number 9.

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**Harmony and Revisionary Strategy #9**

The basic idea of Harmony is that the rules of inference which govern a logical operator should fit with — 'make sense' in terms of — each other: the E-rules should not enable one to over-extend — derive more from a statement S in which that operator is dominant than is justified by the premises which sanction S via the I-rules. *Tonk* and *Quonk* offend in this respect. Likewise they should not force one under-extend — restrict one to deriving less than is justified by the premises which sanction S via the I-rules. *Tunk* and *Bunk* offend in that respect.

In order to assess controversial cases, we shall need to refine this characterisation. But first ask:

(I) *Why is Disharmony a Bad Thing?*

The obvious concerns are epistemic:

   Overextensive E-rules risk explosion (tonk), inconsistency, unsoundness (consistent but non-truth-preserving patterns of inference), or rules which are as a matter of (even a priori) fact truth-preserving, courtesy of certain collateral conditions being met, but reliance on which is unjustified because we have no reason to think those conditions are met (cf. example of numerical quantifiers above). So there are both safety and rationality concerns.
Underextensive E-rules seem a less serious matter. We will always be able to prove — by other means — the things we ‘ought’ but are not able to prove via such rules. Still, we might be prone to miss this in particular cases. So there is a possible utility concern. (But I’ll argue in a moment that there is more.)

Pandemic holism replies that explosion (and maybe inconsistency) apart, these concerns are exaggerated — in fact depend on untenable ideas about meaning. Provided the rules determine a genuine practice, — not anything goes, — the licensed conclusions of inferences will have their content in part determined by their accessibility as conclusions of those very inferences — that fact will enter into their truth/assertibility conditions. So there will be no sense in the idea that they are really false, or true only courtesy of some unmonitored collateral circumstance. (Non-explosive) logics cannot but be safe because the meanings of the statements they allow us to connect in inferences will (qua ‘plastic’) assume whatever character is needed for the inferences to be safe!

This idea can come in two foreseeable forms. The rich form says that there will always be an available interpretation of its distinctive vocabulary under which a (non-explosive) inferential practice will be (a priori) truth-preserving. The deflated form says that coherence of practice looks after itself — that for sentences to have meanings/truth-conditions with respect to which an inferential practice is sound is just for that practice to be operationally coherent and to link them in the way it does. There is no higher (interpretational or model-theoretic) court of appeal.

[An issue for the rich form: can the claim be justified for some interesting notion of interpretation—where e.g. an interpretation is constrained to be something that actual thinkers might mean?]

Above, we illustrated how there is another set of issues, brought out well by underextensive cases like the connective Tunk and the quantifier Bunk(x). Here there are no issues of coherence of practice, safety, or unjustified collateral assumptions. But there is still a vivid issue of making sense — intelligibility. The practices are indeed intelligibly interpretable but only at the cost of criticism of them — of finding, precisely, the under-extension that they involve.

An important datum in the phenomenology of logic, which a satisfying epistemology must respect: the obviousness of the basic rules. Compare and contrast three types of case: "Nothing can look red
and green all over", MPP, and :"The Knight moves one space front, back or sideways, and one space diagonally". We want this obviousness in propositions of basic logic. It's tempting to say that it is integral to its status as logic. [Big issue: relation of this point to Inferentialism]

**Harmony and the Revisionist's Fork:** inharmonious rules either await validation by interpretation, in which case the best interpretation of them may be one which enforces revision or they will raise an issue of intelligibility—the virtue of harmony is that it seems to go with the ability of rules of inferential practice to *create a concept* in such a way that the phenomenology of obviousness kicks in, and we feel that we ‘know what we are doing’, that the practice ‘makes sense’. But this is the crucial issue for the revisionist inferentialist. Much more needs to be said.

Next ask:

*(II) How best to characterise the idea of Harmony and make it tractable as a criterion in controversial cases?*

Consider a binary connective +. Then what we want is for the E-consequences for A+B to be exactly—no more and no less—than they should be in the light of its I-gounds.

One standard type of characterisation is given by Tennant *(AR&L* ch. 9 p. 94 and *Natural Logic* (1978)). Applied to our envisaged connective +, it comes to this:

**Condition (i):** any binary connective * for which the pattern of +I is valid may be shown by +E to be such that A+B => A*B (so A+B is the strongest statement justified by the +I premises)

and

**Condition (ii):** any binary connective * for which the pattern of +E is valid may be shown by +I to be such that A*B => A+B (so, in effect, +E is the strongest E-rule justified by the I-rule.)

This looks as if it is barking up the right sort of tree. But, a little surprisingly, it proves to misfire completely. Suppose + is *tonk*. So +I is VI, and +E is &E. Then, to establish Condition (i), assume A => A*B, and B => A*B. We need to show via +E that A+B => A*B. Assume A+B. Then both A and B follow by +E. Either will then suffice for A*B, by the assumption.

As for Condition (ii), assume A*B. By hypothesis, the pattern of +E is valid for *, so we have both A and B. Either will suffice via +I for the proof of A+B. So A*B => A+B.
Note: The objection is not that an explosive operator has passed the tests for Harmony. (Consider the abstraction operator for courses-of-values and Frege's Basic Law V. Notice that this forces a distinction between Harmony and conservative extension.) Rather the objection is that tonk is out of Harmony if anything is - the E-rule is manifestly too strong for the I-rule.

(If looks as if this problem for Tennant’s account will afflict any connective whose E-rule over-extends by entailing satisfaction of all the premises for the I-rule and then some more stuff.

Let's try a different tack. Consider &I and &E schematised as follows

Intro. premises:  \( X => A ; Y => B \)
Intro. conclusion: \( X,Y => A&B \)

Elim. premise \( Z => A & B \)
Elim. conclusions \( Z => A \) and \( Z => B \)

Intuitively the Harmony here involves the fact that knowledge that the (only) Elim. premise was accessed in a way permitted by the Intro. rule independently suffices for all the permitted Elim. conclusions (take \( Z = \{X,Y\} \) and use Augmentation of premises.)

That though only gets at avoidance of over-extension. Under-extension is intuitively avoided by the fact that the Elim. conclusions are the strongest that are justified if one knows only that the Elim premise was accessed in a way permitted by the Intro. rule and has no further information about Z.

Try Disjunction:

Intro. premises: \( X => A \) \( Y => B \)
Intro. conclusions: \( X=> AVB \) \( Y=> AVB \)

Elim. premises \( Z => AVB ; W, A => C ; U, B => C \)
Elim. conclusion \( \{Z, W, U\} => C \)
Again, over-extension is pre-empted because knowledge of the fact that the V-involving Elim. premise was accessed in a way permitted by the Intro. rule plus the other Elim. premises independently suffice for the Elim. conclusion (take $Z = \{X\}$ or $Z = \{Y\}$ and use Cut).

Under-extension for the stated form of conclusion is addressed by the consideration that if one knows only that the V-involving Elim. premise was accessed in some way permitted by the Intro. rule and has no further information about $Z$, the remaining Elim. premises are the weakest set that suffice, in conjunction with that knowledge, to yield the Elim. conclusion.

(Note (i) this formulation also works for Conjunction: the null set is indeed the weakest such set. (ii) There is a relativity imported here to the form of the conclusion, which needs to be addressed.)

One more: try the conditional:

Intro. premise: $X, A \Rightarrow B$
Intro. conclusion: $X \Rightarrow A \rightarrow B$

Elim. premises $Y \Rightarrow A \rightarrow B; Z \Rightarrow A$
Elim. conclusions $\{Z, Y\} \Rightarrow B$

And the same formula works again. Over-extension is pre-empted by the fact that knowledge of the fact that the $\rightarrow$ -involving Elim. premise was accessed in a way permitted by the Intro. rule plus the other Elim. premise(s) independently suffices for the Elim. conclusion (take $Y = X$ and use Cut); under-extension is addressed by the consideration that the remaining Elim. premise is the weakest that suffices for the stated conclusion if one knows in addition only that the Elim. premise was accessed in some way permitted by the Intro. rule and has no further information about $Z$.

Resulting rubric: An Elim-rule for an operation # does not over-extend iff knowledge that the # Elim premise was accessed in a way permitted by the relevant I-rule, plus the other Elim premises if any, independently suffices for the Elim conclusion.
An Elim-rule for an operation * does not under-extend if, in the presence of knowledge just that the
# Elim. premise was accessed in a way permitted by the # Intro rule, knowledge of the remaining
Elim premises (if any) is the weakest knowledge that independently suffices for the Elim conclusion
Issues: Tacit relativity to a background proof-theory, invoked by the idea of 'independent
sufficiency'; sharpness and decidability worries.

How does this handle the problem cases? First tonk:

Intro. premises: \( X \Rightarrow A \quad Y \Rightarrow B \)
Intro. conclusions: \( X \Rightarrow A \text{ tonk } B \quad Y \Rightarrow A \text{ tonk } B \)

Elim. premise \( Z \Rightarrow A \text{ tonk } B \)
Elim. conclusions \( Z \Rightarrow A \text{ and } Z \Rightarrow B \)

Tonk should turn out to over-extend. So the question is whether knowledge just that the Elim.
premise was accessed in a way permitted by the Intro. rule (there are no other Elim. premises to
consider) independently suffices for each of the permitted Elim. conclusions. Well, obviously not: if
\( Z \Rightarrow A \text{ tonk } B \) was accessed, for example, from \( Z \Rightarrow B \), then that won't suffice in the general run of
cases for \( Z \Rightarrow A \).

Now tunk:

Intro. premises: \( X \Rightarrow A \quad Y \Rightarrow B \)
Intro. conclusion: \( \{X,Y\} \Rightarrow A \text{ tunk } B \)

Elim. premises \( Z \Rightarrow A \text{ tunk } B \quad W, A \Rightarrow C \quad U, B \Rightarrow C \)
Elim. conclusion \( \{Z, W, U\} \Rightarrow C \)

Tunk should under-extend. So the question is whether if one knows just that \( Z \Rightarrow A \text{ tonk } B \) was
accessed in a way permitted by the Intro. rule, the remaining Elim. premises are the weakest set that
suffices for the stated conclusion. And clearly they are not. Either will suffice on its own.]
So that's the basic idea of harmony — at least as I think it should be intuitively understood. Certainly the characterisation can be improved on. — there are obvious concerns about the effectiveness of this very intuitive formulation of the under-extension clause. But it will suffice for our immediate philosophical purpose.

**Introducing Intuitionist Negation**

Here is one standard Intro. rule

Not Intro. (i) \[X, A \implies B ; Y \implies -B\]
\[\{X,Y\} \implies -A\]

Problem: the premises already contain \('-'\), so this Intro. rule cannot be explanatory except in cases where \(Y \implies -B\) can be established by negation-neutral means.

Possible solution: it seems very intuitive that the negation of \(A\) is that proposition which is true whenever any proposition incompatible with \(A\) is true. (There is a presupposition here which I will return to.) It is plausible, moreover, — though a Big Point — that appreciation of incompatibilities (e.g. between determinate properties under a determinable property) is antecedent in the order of understanding to grasp of an explicit concept of negation. So a natural first stab, where \(B\) and \(C\) are any incompatible propositions, would be

Not Intro. (ii) \[\{X, A\} \implies B ; Y \implies C\]
\[\{X,Y\} \implies -A\]

What should the matching Elimination rule be? Well, any harmonious suggestion should have this effect:

Not Elim \[\{X,Y\} \implies -A\]
\[\{X, Y, A\} \implies \text{both of some pair of incompatible propositions}\]

Let's express the situation when a set of premises entails both of some pair of incompatible propositions (i.e are incompatible) in this familiar way:
Then the Not Elim. rule may be written as

Not Elim (i)  
\[ X \Rightarrow -A \]
\[ \{X, A\} \Rightarrow \Lambda \]

And the Intro. rule, correspondingly, becomes

Not Intro (iii)  
\[ \{X, A\} \Rightarrow \Lambda \]
\[ X \Rightarrow -A \]

And these are intuitively as harmonious as can be.

**EFQ**

Let’s however consider the situation in standard systems of classical and intuitionist sentential logic without the \( \Lambda \)-constant. One common formulation of the intuitionist rules in the text books (references) is:

Not Intro. (i)  
\[ X, A \Rightarrow B ; Y \Rightarrow -B \]
\[ \{X,Y\} \Rightarrow -A \]

Not Elim (ii)  
\[ X \Rightarrow -A \]
\[ X \Rightarrow A \rightarrow B \]

(In the presence of the normal conditional rules, the latter is equivalent to *ex falso quodlibet*

\[ X \Rightarrow A \& -A \]
\[ X \Rightarrow B \]

Or more perspicuously

EFQ  
\[ X \Rightarrow -A \]
\[ \{X, A\} \Rightarrow B \]

Are the Intuitionist negation rules, as standardly formulated, in harmony? Well, are Not Elim (i) and EFQ interderivable in the presence of Not Intro (iii) and standard (single conclusion) structural rules? EFQ to Not Elim (i) is trivial. What about the converse direction? No go as far as I can see.
We have to show that given \( \{X, A\} \Rightarrow \Lambda \), it follows that \( \{X, A\} \Rightarrow B \). Weakening on the left gives us \( \{X, A, B\} \Rightarrow \Lambda \), whence by Not Intro (iii) we have \( \{X, A\} \Rightarrow -B \). \( \{X, A\} \Rightarrow --B \) is similarly obtained. But that’s it. You cannot get an unnegated conclusion. *Discussion point.*

**Against DNE, and what is needed to for its global defence**

What happens when we replace the intuitionistic elimination rule with the classical:

Not Elim (iii) \[
\begin{align*}
X & \Rightarrow --A \\
X & \Rightarrow A
\end{align*}
\]

The nice harmonious negation rules proposed above:

\[
\begin{align*}
\{X, \{A\}\} & \Rightarrow \Lambda \\
X & \Rightarrow -A
\end{align*}
\]

\[
\begin{align*}
X & \Rightarrow -A \\
\{X, \{A\}\} & \Rightarrow \Lambda,
\end{align*}
\]

have the effect, as proposed, that the negation of a proposition is identified with the weakest proposition incompatible with it—since it follows from any set \( X \) of propositions which are conjointly so incompatible. Given that conception of negation, what could justify the *additional strength* of the disharmony-introducing elimination rule

DNE \[
\begin{align*}
X & \Rightarrow --A \\
X & \Rightarrow A
\end{align*}
\]

Whatever the justifying train of thought is, it has to entail that if the state of the world is incompatible with there being a true statement incompatible with the truth of \( A \), it must be such as to make \( A \) true.

Here are three reservations that are each different form the usual intuitionist reservations. One kind of obstacle is if there is (possible) indeterminacy of one particular kind (*type 1 indeterminacy*): where, for instance, some indeterminate cases of a property have a nature that is incompatible with any
incompatible property (borderline cases of "red" e.g. that are red enough to be determinately of no other colour.)

So: one claim that needs to be defended if DNE is to be valid for the concept of negation introduced by Not-Intro above, is that

I: there are no cases of type 1 indeterminacy — no borderline cases of F that are nothing else, G, incompatible with F

Now recall the standard kind of derivation of LEM via DNE; for instance, we can do this

(i) \(- (A \lor -A) \Rightarrow - (A \lor -A)\)
(ii) \(A \Rightarrow A\)
(iii) \(A \Rightarrow (A \lor -A)\)
(iv) \(- (A \lor -A) \Rightarrow -A\)
(v) \(-A \Rightarrow -A\)
(vi) \(-A \Rightarrow A \lor -A\)
(vii) \(- (A \lor -A) \Rightarrow A \lor -A\)
(viii) \(\Rightarrow - (A \lor -A)\)
(ix) \(\Rightarrow A \lor -A\)

[Note for enthusiasts:]
The RAA step at line iv is of the form

RAA(I) \[X \Rightarrow -A, ; Y, B \Rightarrow A\] \[X, Y \Rightarrow -B\]

The instance at line iv (leaving out technically correct but unhelpful brackets) is

\[- (A \lor -A \Rightarrow - (A \lor -A), ; A \Rightarrow A \lor -A \Rightarrow -A\]

RAA (I) and its companion rule

RAA(II) \[X \Rightarrow A, ; Y, B \Rightarrow -A\] \[X, Y \Rightarrow -B\]
are straightforwardly justified via Not-Intro, Not-Elim, and Cut. Exercise.

The form of RAA appealed to at line viii is stronger: it's

RAA(III) \[ X, B \Rightarrow A ; Y, B \Rightarrow \neg A \]
\[ X, Y \Rightarrow \neg B \]

(in the proof, take B as -(AV-A), A as AV-A, and let X and Y be empty). This is stronger than RAA(II) since its left-hand premise is weaker. But it too is straightforwardly justified via Not-Intro, Not-Elim, and Cut. Exercise.]

Note an aspect of significance of the derivation up to line viii. The issues about DNE are all to do with the legitimacy of augmenting the harmonious negation rules with an additional rule which either enforces - as it seems - an unperspicuous departure from the concept of negation as fixed by the harmonious rules or demands supplementary justification in terms of that same concept. But if you want to reject both LEM and its Double negation in tandem, you will need a reason to look askance at the harmonious negation rules themselves, appealed to up to line viii, before DNE gets into the picture.

[Paradox is one possible such reason. Cases where harmonious rules seem to be tarnished with paradox include Basic Law V (course-of-values Intro. and Elim.) and Curry (the conditional rules and truth-intersubstitutivity.).]

OK. So unless we question, not the sufficiency, but the acceptability of the harmonious negation rules, we are forced at least to the double negation of LEM, and DNE will then stick us with LEM itself. But the standard disjunction rules impose a conception of disjunction such that A V B is good iff at least one of the disjuncts is good. So the defensibility of DNE when negation is viewed as characterised by Not-Intro and Not-Elim depends on the defensibility of the idea that for each proposition P, either it or its negation should hold good. Since, assuming the harmonious rules, we have it that the negation of a proposition is good just if some proposition incompatible with it is good, the issue about DNE accordingly boils down to the question whether, for each proposition, there are just two possibilities: either it holds good, or something incompatible with it does.
This holds out hostages both on the propositional end of the bargain, as it were, and on the worldly end. One traditional semantic conception of indeterminacy will query the propositional end: the semantics of a predicate (say "pearl") may leave a gap between situations given as making an application of it true and situations incompatible with its truthful application (though this is not a good model of vagueness in general).

In any case: another claim that needs to be defended if DNE is to be valid for the concept of negation introduced by the harmonious negation rules, is

II: That there are no cases of semantic indeterminacy of this second kind (type 2 indeterminacy)

But there is a set of metaphysical issues too: why cannot the world just come short, as it were, in relevant respects—stop short of either making a proposition true or making something incompatible with it true?* This is something like the picture suggested by distributivity failures in QM. The idea draws strength if we take it that, in order for any proposition incompatible with P to be true, some proposition of the same level must be true - a proposition which entails but is not entailed by the supposition that some proposition incompatible with P is true (equivalent idea: a negation cannot be barely true.) Why cannot reality just fail to (or only indeterminately) fill up the grid of possibilities, as it were, so that a complete description of the world, while compatible with certain definite propositions about it, falls short of (determinately) making them true

III. A third claim that needs to be defended if DNE is to be valid for the concept of negation introduced by the harmonious negation rules, is thus that there are no cases of shortfall of this structure.

NB. These issues all arise within a framework of thoughts about truth and truth-making (broadly correspondence-type thoughts) which are classically perfectly congenial. Reservations on any of points I-III are not intuitionistic reservations, broadly understood.

Summary:

We can conjecture that classical thought characteristically takes two implicit steps to the justification of LEM when disjunction and negation are constrained by their respective harmonious pairs of rules: (i) all legitimate propositions divide all possible states of the world into just two categories: those compatible with them and those not.
(ii) if no actual state of the world obtains incompatible with a proposition, then that is for it to be true. This way of thinking about things is very much part and parcel of the vision of the Tractatus (and non-compulsory).

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*Unless we are to espouse dialetheism, such a short-fall might seem to be independently forced in cases like the Liar. For if the world made Q, =

    Some proposition incompatible with Q is true,
true, it would thereby make something incompatible with Q true as well. So the world must somehow fail to deliver a truth-maker either for Q or for any proposition incompatible with it.
But this nice thought is overshadowed by the threat of Revenge: let "P is sold short" describe the situation of any proposition for which the world so fails to deliver, and consider Q+, =

    Q+ is sold short

.........

**Defending Intuitionist Negation**

The harmonious rules proposed belong with the idea that the negation of a proposition is that proposition which is entailed by any proposition incompatible with A, that it is the weakest proposition incompatible with A. But how do we know that there is any weakest proposition incompatible with A in general — a proposition entailed by any proposition incompatible with A?
Why should there not be a range of inequivalent such propositions, or an infinitely weakening chain?

It’s a curiously interesting question. Can we just stipulate that there is such a proposition? No – cf. entailment.

Can we prove there is such a proposition? What about this candidate:

NegA: Some proposition incompatible with A is true

We have to show
(i) That \( \neg A \) is incompatible with \( A \)

(ii) That for any \( B \), if \( B \) is incompatible with \( A \), \( B \) entails \( \neg A \).

Proof of (i) Suppose \( \neg A \) is true. So — what it says — some true proposition is incompatible with \( A \). Suppose \( B \) witnesses that. Then \( B \) is true and \( B \) is incompatible with \( A \). But this principle looks good:

\[ \text{(Inc): for all } A, B, \text{ any two of } \quad \{ B \text{ is true}, A \text{ is true}, B \text{ is incompatible with } A \} \text{ are incompatible with the third.} \]

So if \( \neg A \) is true, that is incompatible with \( A \)’s truth, ergo with \( A \).

Proof of (ii) Suppose \( B \). Then \( B \) is true. Suppose \( B \) is incompatible with \( A \). So some proposition incompatible with \( A \) is true. So \( \neg A \) follows from our two assumptions. But if \( B \) is incompatible with \( A \), it is so necessarily. Given the principle:

\[
\begin{align*}
X, D &\Rightarrow C; \quad \Box D \\
X &\Rightarrow C
\end{align*}
\]

We have that \( \neg A \) follows in the assumption of \( B \) alone.

QED?

Not so fast: principle Inc is subject to what I will call

**Field's Paradox:**

**Definition:** let \( B_0 \) be the sentence:

\[ B_0 \text{ is incompatible with } A, \]

for some arbitrarily chosen \( A \).

Then

(i) \( \{ B_0, A, B_0 \text{ is incompatible with } A \} \text{ are incompatible} \) - by I

So

(ii) \( \{ B_0, A, B_0 \} \text{ are incompatible} \) - Def \( B_0 \)

So

(iii) \( \{ B_0, A \} \text{ are incompatible} \) - simplifying (ii)

So

(iv) \( B_0 \) - from (iii), meaning of \( B_0 \)

So

(v) \( B_0 \text{ is true} \) - (iv), Truth-I

So

(vi) \( \text{Some true proposition is incompatible with } A, \) - from (iii) and (v)
Since we can get a version of (vi) for any proposition $A$, just by selecting a suitable $B_0$, it seems fair to conclude that

(vii) Any proposition is incompatible with some true proposition

and hence

(viii) Any true proposition is incompatible with some true proposition.

So we have a reductio of the putative necessary truth (Inc). And lemma (i) of the proof that $\neg A$ cuts the intuitionist mustard collapses.

Points:

• No use of any negation rules is involved.

• No use of any conditional rules is involved.

• Only Truth-I is involved. (No truth rule is involved at viii - we have equally well shown that e.g., any boring proposition is incompatible with some true proposition.)

• The fact that the paradox involves no play with negation inferences should give one pause before drawing any conclusions about the acceptability of the proposed negation rules.

• It seems premature to suppose that the paradox refutes (Inc). But what is clear is that a proponent of intuitionist negation who wants to stick to the reasoning above that $\neg A$ successfully addresses the existence problem, owes some other solution to the paradox. What might it be?

• We knew that harmony didn’t guarantee coherence. It merits emphasis that it doesn’t guarantee existence even when coherence is not in doubt. The intuitionistic negation rules are (harmonious and) coherent enough. But the question whether there is any such proposition as the intuitionistic negation of $A$ appears to be open.