# A Simple Second-Order Solution Method for Dynamic General Equilibrium Models[*]

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## Abstract

This paper describes a simple method for the calculation of second-order solutions to dynamic general equilibrium models. The method relies on standard linear solution procedures and does not require any new numerical algorithm. As an illustration, the method is used to derive a full second-order approximation for aggregate utility in a sticky price model. In an open economy example the method is used to calculate the welfare gains from international coordination of monetary policy.

Keywords: second-order approximation, monetary policy, welfare.

JEL: C63, E50, F41

# 1  Introduction

A recent theme in the literature on monetary policy is the evaluation of policy using a welfare criterion based on aggregate utility. Except in rather restrictive cases (see Obstfeld and Rogoff (1998, 2002), Devereux and Engel (2001) and Corsetti and Pesenti (2001)) it is not possible to derive explicit and exact expressions for utility-based welfare functions. Woodford (1999), however, has shown how to derive and evaluate a second-order approximation for aggregate welfare in more general dynamic models. The method has proved useful for a wide range of applications but nevertheless relies on special assumptions which mean that it can not be used in some important and interesting cases.

The central problem can be summarised as follows. A second-order expansion of aggregate utility includes both first-order and second-order terms in the variables of the model. Expected welfare therefore includes terms in the expected levels of variables and the second moments of variables. It is relatively easy to evaluate the second moments in a dynamic model. Second-order accurate solutions for second moments can be derived by simulating a log linear approximation of the model. The problem for the Woodford method arises in evaluating the terms in the expected levels of variables. Such terms themselves depend on second moments and therefore require the solution of a second-order approximation of the model.

Woodford shows that the problem can be avoided by assuming that the deterministic steady state of the model is equal to the first-best equilibrium. This can be achieved by subsidising away any monopolistic distortions. This results in all first-order terms dropping out of the welfare calculation. While restrictive, this assumption is generally not too problematic in models of closed economies.

However, in open economy models it is much more difficult to eliminate the first-order terms from the welfare calculation. In such cases it is only possible to derive a measure for world aggregate welfare. It is not possible to derive measures for individual country welfare. Benigno and Benigno (2001) show that, under some further restrictive assumptions, it is possible to derive some results regarding optimal policy for individual countries when acting as Nash players in a non-cooperative policy game. In these cases it is possible to evaluate individual country welfare. But even here it is not possible to evaluate welfare when both policy makers deviate from the Nash equilibrium policy. The welfare comparison between cooperative and noncooperative policy making is a topic of obvious interest in monetary policy analysis in open economies. There is therefore much to be gained by deriving a solution technique which allows the Woodford method to be applied more generally than currently possible.

This paper outlines and illustrates one possible solution technique. The technique is described with reference to two particular models. The structure of the models used here is quite typical of dynamic general equilibrium models and the solution method is stated in a form which allows it to be applied to other models

which have the same general structure.[1]

There are a number of advantages of the method described in this paper. First, it is a very intuitive generalisation of the methods used to solve simple analytical models of the type analysed by Obstfeld and Rogoff (1998, 2002), Devereux and Engel (2001) and Corsetti and Pesenti (2001). Second, it exploits the economic structure of a model in such a way that valuable intuitions and insights may be revealed that are obscured by more general numerical methods. Third, no new numerical methods are required. The solution can be generated using standard linear solution methods (such as that described in Blanchard and Kahn (1980)).[2]

Recently Sims (2000) and Schmitt-Grohé and Uribe (2001) have described algorithms for deriving second-order accurate solutions to dynamic rational expectations models. The method described in this paper is somewhat less general than those methods. The Sims and Schmitt-Grohé and Uribe methods can be used to derive second-order accurate solutions for the actual sample path of variables. The method described here yields solutions for the expected path of variables. However, the expected path is all that it required to evaluate welfare. And, more generally, the expected path is likely to be the main focus of interest in many applications, so the method described in this paper is likely to be sufficient across a broad range of cases.

Section 2 explains and illustrates the method in the context of a simple closed economy model where agents are monopolistic price setters in a Calvo (1983) price setting structure. As explained above, the Woodford method for deriving a welfare measure works well in such a model so the extra insight gained by deriving a second-order solution is relatively minor. The full solution, however, does suggest that price stability is the welfare maximising monetary policy for any level of monopoly distortion. Without a second-order solution to the model, such welfare results can only be derived when the monopolistic distortion is fully offset with a production subsidy.[3]

Section 3 discusses a two county model. The solution method extends to the two country case in a straightforward way. In the two country case the solution

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[1]Kim and Kim (2000) discuss a number of solution concepts in the context of a stochastic general equilibrium model. One technique they propose and describe is a 'bias correction method.' This adjusts the mean values of endogenous variables to reflect the difference between the deterministic and stochastic steady states. Kim and Kim demonstrate this method using both a static and a dynamic general equilibrium model. The terminology and presentation adopted in this paper is somewhat different from that used by Kim and Kim but the underlying logic of the method is identical to that described by Kim and Kim.

[2]At first sight it may seem surprising that a second-order solution can be generated using a linear solution method. The intuition behind this result is that the first-order terms in a second-order solution are linear in the second moments of the model. The second moments can be generated from a first-order approximation of the model. And a linear solution method can be used to solve for the relationship between the first-order terms and the second moments.

[3]In fact the welfare maximising policy can be deduced without appealing to a second-order approximation of the welfare function, so even to this extent the full second-order solution is not adding much to existing results in the closed economy case. The solution in this simple example does, however, illustrate how the solution method can be used to gain some insight into the workings of a model.

method makes it possible to analyse the gains from policy coordination between the two policy makers. This is illustrated with a simple numerical example. As already explained, except in special cases, such an exercise is not possible without a second-order solution to the model.[4]

It is useful to note that second-order solutions derived for both the models in this paper have been compared to the numerical solutions generated by the Schmitt-Grohé and Uribe method and have been found to match exactly (allowing for small computer rounding errors).[5]

# 2    A Closed Economy Example

The basic method is illustrated with reference to the following simple dynamic general equilibrium model of a closed economy. The economy is populated by a continuum of agents who are each the monopoly producer of a single differentiated product. Agents consume all products produced in the economy. Agents are indexed on the unit interval and the preferences of agent $j$ are given by

$$U_t(j) = E_t \left[ \sum_{\tau=t}^{\infty} \beta^{\tau-t} \left( \log C_\tau(j) + \chi \log \frac{M_\tau(j)}{P_\tau} - \frac{K_\tau}{\mu} y_\tau^\mu(j) \right) \right] \qquad (1)$$

where $C$ is a consumption index, $M$ denotes nominal money holdings, $P$ is the aggregate price level, $y(j)$ is the output of good $j$ and $K$ is a stochastic shock. The consumption index $C$ is defined as follows

$$C = \left[ \int_0^1 c(h)^{\frac{\phi-1}{\phi}} \, dh \right]^{\frac{\phi}{\phi-1}} \qquad (2)$$

where $c(h)$ is the consumption of good $h$. Insurance markets are assumed to exist which allow agents to share all consumption risk. Hence all agents have the same level of $C$.

It will prove useful to define aggregate output in period $t$ as follows

$$Y_t = \left[ \int_0^1 y_t(h)^{\frac{\phi-1}{\phi}} \, dh \right]^{\frac{\phi}{\phi-1}} \qquad (3)$$

where $y(h)$ is the output of good $h$. Note that $C_t = Y_t$ at all times in a closed economy.

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[4]The purpose of this paper is to explain the solution method. It is not to conduct a full analysis of the two country model or the issue of policy coordination. The discussion of the more general economic implications of the results is consequently brief.

[5]As a further check, the method described in this paper has been applied to the neoclassical growth model used as an example in Schmitt-Grohé and Uribe (2001) and the results match (again, allowing for small computer rounding errors). The Appendix gives details of the computations involved.

Price setting follows the Calvo (1983) structure. In any given period proportion $(1 - \gamma)$ of agents are allowed to reset their prices. The first-order condition for price setting in period $t$ is[6]

$$E_t \left\{ \sum_{\tau=t}^{\infty} (\beta\gamma)^{\tau-t} \left[ (1 - \Phi) \left( \frac{X_t}{P_\tau} \right)^{1-\phi} - K_\tau Y_\tau^\mu \left( \frac{X_t}{P_\tau} \right)^{-\phi\mu} \right] \right\} = 0 \qquad (4)$$

where $X_t$ is the price set by agents who are allowed to set prices in period $t$ and $1 - \Phi = (1 + \alpha)(\phi - 1)/\phi$ where $\alpha$ is a production subsidy. This subsidy may be used to offset the monopolistic distortion by setting $(1 + \alpha) = \phi/(\phi - 1)$ implying $\Phi = 0$. The aggregate price index in period $t$ can be written as

$$P_t = \left[ \sum_{\tau=0}^{\infty} (1 - \gamma) \gamma^\tau X_{t-\tau}^{1-\phi} \right]^{\frac{1}{1-\phi}} \qquad (5)$$

where $X_{t-\tau}$ is the price set by agents who last set prices in period $t - \tau$.

The shock variable $K$ is assumed to evolve according to

$$\log K_t = \delta \log K_{t-1} + \varepsilon_t \qquad (6)$$

where $\varepsilon$ is an i.i.d. shock with zero mean and variance $\sigma_\varepsilon^2 > 0$.

For the purposes of illustrating the solution method it proves useful to assume that the monetary authority sets monetary policy in terms of a (possibly state-contingent) target for nominal expenditure. Thus it is assumed that the monetary authority adjusts the money stock so that $P_t C_\tau = \bar{N} K_\tau^\psi$ where $\bar{N}$ is some base target level of nominal expenditure and $\psi$ is a feedback coefficient which may be optimally chosen by the monetary authority.[7]

Denote the deterministic steady state value of a variable with a bar. It is clear that $\bar{K} = 1$. It is also clear that $\bar{P} = \bar{X} = \bar{N}/\bar{C}$. It then follows from (4) that $\bar{C} = \bar{Y} = (1 - \Phi)^{1/\mu}$. It proves useful to choose $\bar{N} = \bar{C}$ which implies $\bar{P} = \bar{X} = 1$.

On the assumption that the utility from real balances is small, expected aggregate flow utility in period $\tau$ can be written as

$$w_\tau = E_0 \left\{ \log C_\tau - \int_0^1 \frac{K_\tau}{\mu} y_\tau^\mu (j) \, dj \right\} \qquad (7)$$

where welfare is being evaluated from the point of view of period 0. Using the methods described in Woodford (1999) a second-order approximation to this expression

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[6]In deriving this expression use is made of the fact that the demand for goods produced by agents who set prices in period $t$ is given by $Y_\tau (X_t/P_\tau)^{-\phi}$.

[7]By specifying monetary policy in this way it is possible to avoid explicit consideration of the nominal interest rate, the consumption Euler equation and the money demand and supply relationships. This formulation of policy is chosen only to provide a simple model for the purposes of illustrating the solution method. Of course, in many applications the interaction between the nominal interest rate and monetary policy may be a central focus of analysis. The solution method can equally well be applied in these more general cases.

can be written as follows

$$w_\tau - \bar{w} = E_0 \left\{ \Phi \hat{C}_\tau - \frac{(1-\Phi)}{2} \mu \left( \hat{Y}_\tau + \frac{1}{\mu} \hat{K}_\tau \right)^2 \right.$$

$$\left. - \frac{(1-\Phi)}{2} \phi \left( 1 + \phi \left( \mu - 1 \right) \right) \Pi_\tau \right\} + O \left( \|\xi\|^3 \right) \qquad (8)$$

where

$$\Pi_\tau = \sum_{i=0}^{\infty} (1 - \gamma) \gamma^i \left( \hat{X}_{\tau-i} - \hat{P}_\tau \right)^2$$

where a hat indicates the log deviation of a variable from its deterministic steady state value and the term $O \left( \|\xi\|^3 \right)$ includes all the terms of third order and above in the log deviation of variables from their deterministic steady state values.

If one is prepared to assume that the net monopolistic distortion (measured by $\Phi$) is zero then it is apparent that flow welfare depends only on second-order terms. When the expectations operator is taken through the expression, the second-order terms become terms in the second moments of the variables of the system. Second-order accurate calculations of second moments can be generated by simulating a first-order approximation of the model.[8] Hence, provided $\Phi$ is zero (or itself is no larger than third order) it is possible to obtain a measure of welfare by simulating the log linearised model.

If, on the other hand, $\Phi$ is not small a correct second-order approximation for welfare can only be obtained if an expression for $E_0[\hat{C}_\tau]$ is first derived. This requires a second-order accurate solution to the full model. The method for obtaining such a solution described here involves first replacing each equation of the model with a second-order expansion. The resulting second-order system is solved using a two step process. The solution process is surprisingly simple. The feature of the problem which makes things very easy is the fact that one only needs to derive a solution for the *expected* path of variables, not the *actual realised* path, i.e. a solution for $E_0[\hat{C}_\tau]$ is being sought. This implies that the set of equations to be solved can be simplified by applying the expectations operator $E_0$.

First it is useful to transform the model into a form which makes it easier to derive second-order expansions. Using the definition of the monetary policy rule the price-setting condition can be written as

$$(1 - \Phi) X_t^{1-\phi(1-\mu)} Q_t = B_t \qquad (9)$$

where

$$Q_t = E_t \left\{ \sum_{\tau=t}^{\infty} (\beta\gamma)^{\tau-t} q_\tau \right\}, \quad B_t = E_t \left\{ \sum_{\tau=t}^{\infty} (\beta\gamma)^{\tau-t} b_\tau \right\} \qquad (10)$$

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[8]The Appendix describes a simple algorithm for generating the second moments of a dynamic model.

and $q_\tau \equiv P_\tau^{\phi-1}$, $b_\tau \equiv (1-\Phi)K_\tau^{1+\psi\mu}P_\tau^{\mu(\phi-1)}$. It is useful to think of $q_\tau$ as a composite of factors which affect marginal utility of the contract price in period $\tau$, while $Q_t$ is the discounted value of $q$ over the expected term of a contract starting in period $t$. Similarly $b_\tau$ can be thought of as a composite of factors which affect the marginal disutility of work effort in period $\tau$ and $B_t$ is the discounted value of $b$ over the expected term of a contract starting in period $t$.

Notice that equation (9) is linear in logs so

$$(1 - \phi(1-\mu))\hat{X}_t + \hat{Q}_t = \hat{B}_t \tag{11}$$

Similarly the definitions of $q$ and $b$ are linear in logs so

$$\hat{q}_t = (\phi - 1)\hat{P}_\tau \tag{12}$$

and

$$\hat{b}_t = (1 + \psi\mu)\hat{K}_t + \mu(\phi - 1)\hat{P}_t \tag{13}$$

The two equations (10) however need to be approximated with second-order expansions as follows[9]

$$\hat{Q}_t + \frac{1}{2}\hat{Q}_t^2 = (1 - \beta\gamma)E_t\left\{\sum_{\tau=t}^{\infty}(\beta\gamma)^{\tau-t}\left[\hat{q}_\tau + \frac{1}{2}\hat{q}_\tau^2\right]\right\} + O\left(\|\xi\|^3\right) \tag{14}$$

$$\hat{B}_t + \frac{1}{2}\hat{B}_t^2 = (1 - \beta\gamma)E_t\left\{\sum_{\tau=t}^{\infty}(\beta\gamma)^{\tau-t}\left[\hat{b}_\tau + \frac{1}{2}\hat{b}_\tau^2\right]\right\} + O\left(\|\xi\|^3\right) \tag{15}$$

These equations can be rewritten in difference form as follows

$$\hat{Q}_t - \beta\gamma E_t\left[\hat{Q}_{t+1}\right] = (1 - \beta\gamma)\hat{q}_t + \lambda_{Q,t} + O\left(\|\xi\|^3\right) \tag{16}$$

$$\hat{B}_t - \beta\gamma E_t\left[\hat{B}_{t+1}\right] = (1 - \beta\gamma)\hat{b}_t + \lambda_{B,t} + O\left(\|\xi\|^3\right) \tag{17}$$

where

$$\lambda_{Q,t} = \frac{1}{2}\left\{(1 - \beta\gamma)\hat{q}_t^2 - \hat{Q}_t^2 + \beta\gamma E_t\left[\hat{Q}_{t+1}^2\right]\right\} \tag{18}$$

$$\lambda_{B,t} = \frac{1}{2}\left\{(1 - \beta\gamma)\hat{b}_t^2 - \hat{B}_t^2 + \beta\gamma E_t\left[\hat{B}_{t+1}^2\right]\right\} \tag{19}$$

A similar procedure can be applied to the equation for aggregate prices (5). A second-order expansion yields

$$\hat{P}_t + (1 - \phi)\frac{1}{2}\hat{P}_t^2 = \sum_{\tau=0}^{\infty}(1 - \gamma)\gamma^\tau\left[\hat{X}_{t-\tau} + (1 - \phi)\frac{1}{2}\hat{X}_{t-\tau}^2\right] + O\left(\|\xi\|^3\right) \tag{20}$$

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[9]Considerable care needs to be taken in rearranging equations such as (4) which include expectational terms. It is important that any rearrangement preserves the form of the expectational terms.

Transforming into difference form yields

$$\hat{P}_t - \gamma \hat{P}_{t-1} = (1 - \gamma) \hat{X}_t + \lambda_{P,t} + O\left(\|\xi\|^3\right) \tag{21}$$

where

$$\lambda_{P,t} = \frac{1 - \phi}{2} \left[(1 - \gamma) \hat{X}_t^2 - \hat{P}_t^2 + \gamma \hat{P}_{t-1}^2\right] \tag{22}$$

It is now possible to write the dynamics of the model in the following matrix form

$$A_1 \begin{bmatrix} Z_{1,t+1} \\ E_t\left[Z_{2,t+1}\right] \end{bmatrix} = A_2 \begin{bmatrix} Z_{1,t} \\ Z_{2,t} \end{bmatrix} + A_3 \Lambda_t + A_4 \varepsilon_t + O\left(\|\xi\|^3\right) \tag{23}$$

where $Z_{1,t} = [\hat{P}_{t-1},\ \hat{K}_{t-1}]'$, $Z_{2,t} = [\hat{B}_t,\ \hat{Q}_t]'$, $\Lambda_t = [\lambda_{P,t},\ \lambda_{B,t},\ \lambda_{Q,t}]'$ and $A_1$, $A_2$, $A_3$ and $A_4$ are matrices of coefficients drawn from equations (6), (16), (17) and (21). Equations (11), (12) and (13) have been used to eliminate $\hat{X}$, $\hat{q}$ and $\hat{b}$.

At this point that it becomes useful to apply the $E_0$ operator.[10] The law of iterated expectations implies that the $E_t$ operators in the above equations become $E_0$. The dynamic system can be rewritten as follows

$$A_1 E_0 \begin{bmatrix} Z_{1,t+1} \\ Z_{2,t+1} \end{bmatrix} = A_2 E_0 \begin{bmatrix} Z_{1,t} \\ Z_{2,t} \end{bmatrix} + A_3 E_0 \left[\Lambda_t\right] + O\left(\|\xi\|^3\right) \tag{24}$$

Stating the model in this form illustrates the core of the problem which has to be solved. The objective is to derive the time path of $E_0\left[Z_{1,t}\right]$ and $E_0\left[Z_{2,t}\right]$. But it can be seen from (24) that this can only be achieved if the time path for $E_0\left[\Lambda_t\right]$ is known. $E_0\left[\Lambda_t\right]$ contains conditional second moments of the variables of the model so it is necessary to generate the time path of conditional second moments.

But a solution to this problem is also now readily apparent. Notice that it is only necessary to derive second-order accurate solutions for conditional second moments. And notice further that second-order accurate solutions for second moments can be obtained by considering first-order accurate solutions to realised values. (This follows because terms of order two and above in the behaviour of realised values become terms of order three and above in the squares and cross products of realised values.) Thus it is possible to generate second-order accurate solutions for second moments by considering the following first-order system

$$A_1 \begin{bmatrix} Z_{1,t+1} \\ E_t\left[Z_{2,t+1}\right] \end{bmatrix} = A_2 \begin{bmatrix} Z_{1,t} \\ Z_{2,t} \end{bmatrix} + A_4 \varepsilon_t + O\left(\|\xi\|^2\right) \tag{25}$$

which is derived from the first-order terms in (23). The term $O\left(\|\xi\|^2\right)$ contains all terms of order two and above.

It is now simple to state the two step solution process.

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[10] In a previous version of this paper the expectations operator was applied at an earlier stage of the analysis. The interpretation of the $\lambda$ terms is consequently slightly different in this version of the paper when compared to the previous version. The current presentation allows a clearer and more thorough explanation of the solution procedure.

**Step 1:** *Use the first-order dynamic system (25) to generate the path of conditional second moments and use these second moments to generate the future time path for $E_0[\Lambda_t]$. (The Appendix describes a simple algorithm for generating the time path of second moments.)*

**Step 2:** *Use the time path of $E_0[\Lambda_t]$ and second-order dynamic system (24) to generate the time path for $E_0[Z_{1,t}]$ and $E_0[Z_{2,t}]$. Any standard method for solving a linear rational expectations model can be used, such as that described in Blanchard and Kahn (1980).*

The time path for $E_0[\hat{C}_t]$ is a simple linear transformation of the elements of $E_0[Z_{1,t}]$ and $E_0[Z_{2,t}]$.

The only remaining issue which needs some comment is the choice of initial conditions for solving (24). Notice that values are required for $\hat{K}_0$, $\hat{P}_0$ and $\hat{P}_0^2$. In a policy analysis exercise where the objective is to evaluate the impact of policy on future welfare it is convenient to choose initial conditions which are neutral from the point of view of future welfare. The obvious choice is to set $\hat{K}_0 = \hat{P}_0 = \hat{P}_0^2 = 0$.

The Appendix provides a numerical example of the solution generated by the two-step method just described.[11]

Some insight into the nature of the solutions yielded by this method can be gained by considering the steady state solution to (24) (i.e. the limit as $t$ tends to infinity). Firstly notice that the steady state values of $E_0[\lambda_Q]$, $E_0[\lambda_B]$ and $E_0[\lambda_P]$ are given by

$$E_0[\lambda_{Q,\infty}] = \frac{(1-\beta\gamma)}{2}E_0\left\{\hat{q}_\infty^2 - \hat{Q}_\infty^2\right\} \tag{26}$$

$$E_0[\lambda_{B,\infty}] = \frac{(1-\beta\gamma)}{2}E_0\left\{\hat{b}_\infty^2 - \hat{B}_\infty^2\right\} \tag{27}$$

$$E_0[\lambda_{P,\infty}] = \frac{(1-\phi)(1-\gamma)}{2}E_0\left\{\hat{X}_\infty^2 - \hat{P}_\infty^2\right\} \tag{28}$$

Seen in this form $E_0[\lambda_Q]$ can be interpreted as the variance in $\hat{q}$ relative to its average discounted value over the period of a typical price contract. Likewise, $E_0[\lambda_B]$ can be interpreted as the variance in $\hat{b}$ relative to its average discounted value over the period of a typical price contract. Thus $E_0[\lambda_Q]$ measures the uncertainty in factors which affect the marginal utility of the contract price while $E_0[\lambda_B]$ measures the uncertainty in factors which affect the marginal disutility of work. Notice that $E_0[\lambda_Q] > 0$ and $E_0[\lambda_B] > 0$. But notice also that $E_0[\lambda_P] < 0$. The expectational term in $E_0[\lambda_P]$ can be interpreted as the variance of contract prices relative to aggregate prices, which can be interpreted as a measure of the dispersion of prices across the population. The fact that $E_0[\lambda_P]$ depends negatively on this measure of price dispersion implies that the expected aggregate price level is pushed down by price volatility. This occurs because, when goods are sufficiently substitutable

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[11]The Appendix also shows how the method can be applied to the neoclassical growth model.

in consumption, agents switch their consumption basket towards whichever goods happen to cheapest in any given period.

It is easy to show that

$$E_0\left[\hat{X}_\infty\right] = \frac{(\phi - 1)(\mu - 1)}{(1 - \phi(1 - \mu))}E_0\left[\hat{P}_\infty\right] + \frac{(\lambda_{B,\infty} - \lambda_{Q,\infty})}{(1 - \phi(1 - \mu))(1 - \beta\gamma)} \qquad (29)$$

Thus the steady state contract price includes a "risk premium" which depends on the difference between the uncertainty in the marginal disutility of work and uncertainty in the marginal utility of the contract price. This is a dynamic generalisation of the risk premium in prices derived in models such as those analysed by Obstfeld and Rogoff (1998, 2002), Devereux and Engel (2001) and Corsetti and Pesenti (2001). Notice that, from inspection of the definitions of $q$ and $b$ in (12) and (13), it seems likely that $E_0[\lambda_B] > E_0[\lambda_Q]$, so the risk premium is likely to be positive.

It is also simple to show that

$$E_0\left[\hat{C}_\infty\right] = -\frac{E_0[\lambda_{B,\infty}] - E_0[\lambda_{Q,\infty}]}{\mu(1 - \beta\gamma)} - \frac{(1 + \phi(\mu - 1))E_0[\lambda_{P,\infty}]}{\mu(1 - \gamma)} \qquad (30)$$

Thus the steady state level of consumption is negatively affected by the risk premium in contract prices (the first term) and positively affected by the dispersion of contract prices across the population (the second term).

It is difficult to draw any firm conclusion about the effects of monetary policy on $E_0[\hat{C}_\infty]$. It is possible to confirm that setting the policy parameter $\psi$ to $-1/\mu$ implies stable aggregate prices *and* yields $E_0[\lambda_B - \lambda_Q] = 0$. On the other hand price stability increases $E_0[\lambda_P]$ to its maximum, so the overall effect on $E_0[\hat{C}_\infty]$ is ambiguous. However, numerical experiments (not reported) suggest that $E_0[\hat{C}_\infty]$ is maximised when prices are stabilised. Woodford (1999) has already shown that a policy of price stability maximises aggregate utility when $\Phi = 0$. The fact that $E_0[\hat{C}_\infty]$ is maximised when prices are stable implies that a policy of price stability maximises aggregate utility for any value of $\Phi$.

# 3 An Open Economy Example

In this section the solution method is applied to a two country version of the model just described. In this case the world economy is populated by a continuum of agents indexed on the unit interval. Agents on the interval [0,n) live in the home economy and agents on the interval [n,1] live in the foreign economy. Preferences are the same as in the closed economy model except that the consumption index $C$ is defined as

$$C = \left[n^{\frac{1}{\theta}}C_H^{\frac{\theta-1}{\theta}} + (1 - n)^{\frac{1}{\theta}}C_F^{\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}} \qquad (31)$$

where $C_H$ and $C_F$ are indices of home and foreign produced goods defined as

$$C_H = \left[\left(\frac{1}{n}\right)^{\frac{1}{\phi}}\int_0^n c(h)^{\frac{\phi-1}{\phi}}dh\right]^{\frac{\phi}{\phi-1}}, \quad C_F = \left[\left(\frac{1}{1-n}\right)^{\frac{1}{\phi}}\int_n^1 c(f)^{\frac{\phi-1}{\phi}}df\right]^{\frac{\phi}{\phi-1}}$$

The preferences of foreign agents over consumption are assumed to be identical to those of home agents. The price index for home agents is

$$P = \left[ nP_H^{1-\theta} + (1-n)\left(SP_F^*\right)^{1-\theta} \right]^{\frac{1}{1-\theta}} \quad (32)$$

where $S$ is the nominal exchange rate and $P_H$ and $P_F^*$ are the price indices for home and foreign goods respectively. The law of one price is assumed to hold for each individual good so purchasing power parity holds for overall prices i.e. $P = SP^*$. Also define $P_F = SP_F^*$. Asset markets are assumed to be sufficiently complete to allow full consumption risk sharing, thus, given the structure of preferences, consumption levels in both countries are identical i.e. $C = C^*$.

The first-order conditions for consumption allocation imply the following aggregate world demands for home and foreign goods

$$Y_H = C \left[ \frac{P_H}{P} \right]^{-\theta}, \quad Y_F = C^* \left[ \frac{P_F^*}{P^*} \right]^{-\theta} \quad (33)$$

where the PPP condition and complete-markets assumption have been used to simplify these expressions.

Price setting follows the Calvo (1983) structure. The first-order condition for price setting for home agents in period $t$ is

$$E_t \left\{ \sum_{\tau=t}^{\infty} (\beta\gamma)^{\tau-t} \left[ (1-\Phi) \frac{X_{H,t}}{C_\tau P_\tau} - K_\tau y_{H,\tau}^{\mu-1} \right] y_{H,\tau} \right\} = 0 \quad (34)$$

where $y_{H,\tau} = Y_{H,\tau} \left( X_{H,t}/P_{H,\tau} \right)^{-\phi}$ is the period $\tau$ output of a home agent who last set his price in period $t$. The equivalent first-order condition for foreign agents is

$$E_t \left\{ \sum_{\tau=t}^{\infty} (\beta\gamma)^{\tau-t} \left[ (1-\Phi) \frac{X_{F,t}^*}{C_\tau^* P_\tau^*} - K_\tau^* y_{F,\tau}^{\mu-1} \right] y_{F,\tau} \right\} = 0 \quad (35)$$

where $y_{F,\tau} = Y_{F,\tau} \left( X_{F,t}^*/P_{F,\tau}^* \right)^{-\phi}$ and where $K^*$ is a foreign shock which is assumed to follow a process similar to (6) but is independent from the home shock. The aggregate price indices for home and foreign produced goods in period $t$ can be written as

$$P_{H,t} = \left[ \sum_{\tau=0}^{\infty} (1-\gamma)\gamma^\tau X_{H,t-\tau}^{1-\phi} \right]^{\frac{1}{1-\phi}} \quad (36)$$

and

$$P_{F,t}^* = \left[ \sum_{\tau=0}^{\infty} (1-\gamma)\gamma^\tau X_{F,t-\tau}^{*1-\phi} \right]^{\frac{1}{1-\phi}} \quad (37)$$

where $X_{H,t-\tau}$ and $X_{F,t-\tau}^*$ are the prices set by the home and foreign agents respectively in period $t-\tau$.

10

The combination of PPP and complete asset markets implies the following expression for the nominal exchange rate in period $t$

$$S_t = \frac{P_t C_t}{P_t^* C_t^*} \tag{38}$$

As in the closed economy case, using the methods described in Woodford (1999) a second-order approximation to flow welfare for the home economy can be written as follows

$$w_\tau - \bar{w} = E_0 \left\{ \hat{C}_\tau - (1 - \Phi) \hat{Y}_{H,\tau} - \frac{(1 - \Phi)}{2} \mu \left( \hat{Y}_{H,\tau} + \frac{1}{\mu} \hat{K}_\tau \right)^2 \right.$$
$$\left. - \frac{(1 - \Phi)}{2} \phi \left( 1 + \phi \left( \mu - 1 \right) \right) \Pi_\tau \right\} + O \left( \|\xi\|^3 \right) \tag{39}$$

where

$$\Pi_\tau = \sum_{i=0}^{\infty} (1 - \gamma) \gamma^i \left( \hat{X}_{H,\tau-i} - \hat{P}_{H,\tau} \right)^2$$

A similar expression can be derived for the foreign economy. Notice, in an open economy $C$ and $Y$ can not be assumed to be equal, so it is now necessary to evaluate both $E_0[\hat{C}_t]$ and $E_0[\hat{Y}_{H,t}]$ in order to evaluate flow welfare.[12]

The procedure for deriving solutions for $E_0[\hat{C}_t]$, $E_0[\hat{C}_t^*]$, $E_0[\hat{Y}_{H,t}]$ and $E_0[\hat{Y}_{F,t}]$ in this open economy model follows the closed economy case. The only additional complication arises in equation (32). For $\theta \neq 1$ this equation is not linear in logs. It is therefore necessary to replace it with a second-order expansion of the following form

$$\hat{P} = n\hat{P}_H + (1 - n) \hat{P}_F + \lambda_{CPI} + O \left( \|\xi\|^3 \right) \tag{40}$$

where

$$\lambda_{CPI} = (1 - \theta) n (1 - n) \left( \hat{P}_H - \hat{P}_F \right)^2 \tag{41}$$

The term $\lambda_{CPI}$ therefore enters as an additional second-moment term in the solution of the model. Notice that it reflects the impact of relative price variations on the aggregate price index. For $\theta > 1$ relative price volatility has a negative impact on the aggregate price level. This is because, when home and foreign goods are good substitutes, in each period agents can divert their consumption towards the set of goods with lower prices.

It is straightforward to derive equations similar to (16), (17) and (21) for the home and foreign economy where

$$q_\tau \equiv \frac{Y_{H,\tau} P_{H,\tau}^\phi}{C_\tau P_\tau}, \quad b_\tau \equiv K_\tau Y_{H,\tau}^\mu P_{H,\tau}^{\phi\mu} \tag{42}$$

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[12]Benigno and Benigno (2001) derive some conditions on $\Phi$ and policy which eliminate these first-order terms. Benigno (2001) shows that it is possible to evaluate world aggregate welfare without using a second-order solution method. This follows because, as in the closed economy case, the first-order terms cancel from world aggregate world welfare.

for the home economy and

$$q_\tau^* \equiv \frac{Y_{F,\tau} P_{F,\tau}^{*\phi}}{C_\tau^* P_\tau^*}, \quad b_\tau^* \equiv K_\tau^* Y_{F,\tau}^\mu P_{H,\tau}^{*\phi\mu} \tag{43}$$

for the foreign economy. All other equations of the model are linear in logs. The model can therefore be written in the form (24) where

$$Z_{1,t} = [\hat{K}_{t-1}, \ \hat{K}_{t-1}^*, \ \hat{P}_{H,t-1}, \ \hat{P}_{F,t-1}^*]'$$

$$Z_{2,t} = [\hat{B}_t, \ \hat{B}_t^*, \ \hat{Q}_t, \ \hat{Q}_t^*]'$$

$$\Lambda_t = [\lambda_{P_H,t}, \ \lambda_{P_F^*,t}, \ \lambda_{B,t}, \ \lambda_{B^*,t}, \ \lambda_{Q,t}, \ \lambda_{Q^*,t}, \ \lambda_{CPI}]'$$

The resulting system can be solved in the way described in the previous section to generate time paths for $E_0[\hat{C}_t]$, $E_0[\hat{C}_t^*]$, $E_0[\hat{Y}_{H,t}]$ and $E_0[\hat{Y}_{F,t}]$.

As an illustration of the use that can be made of this procedure Table 1 reports a simple comparison between coordinated and uncoordinated monetary policy in this two country model.[13] Monetary policy is represented by rules for nominal expenditure of the form $\hat{P}_\tau + \hat{C}_t = \psi_{H,1}\hat{K}_\tau + \psi_{H,2}\hat{K}_\tau^* + \psi_{H,3}\hat{P}_{H,\tau} + \psi_{H,4}\hat{P}_{F,\tau}^*$ for the home economy and $\hat{P}_\tau^* + \hat{C}_t^* = \psi_{F,1}\hat{K}_\tau + \psi_{F,2}\hat{K}_\tau^* + \psi_{F,3}\hat{P}_{H,\tau} + \psi_{F,4}\hat{P}_{F,\tau}^*$ for the foreign economy. In the coordinated case the parameters of these rules are assumed to be chosen to maximise the weighted sum of steady state flow welfare across the two countries.[14] In the uncoordinated case the parameters are chosen by national monetary authorities acting as players in a Nash game. Each national monetary authority attempts to maximise steady state flow welfare for its own population.[15]

Table 1 reports some results from numerical simulations of the above model. The parameter values chosen are taken from Rotemberg and Woodford (1998) and Benigno (2001) and are as follows: $\gamma = 0.75$, $\beta = 0.99$, $\phi = 7.66$, $\mu = 1.47$, $\delta = 0.9$, $n = 0.5$ and $\sigma_\varepsilon^2 = \sigma_{\varepsilon*}^2 = 1$. In addition the production subsidy is chosen so that $\Phi = 0$ for both countries. In Table 1 the column headed $w_N - w_M$ reports the difference between flow welfare in the Nash case and flow welfare when both policy makers follow fixed nominal expenditure targets.[16] This can be thought of as the gain from following an active (but uncoordinated) stabilisation policy. The column headed $w_C - w_N$ reports the difference between flow welfare in the coordinated case and flow welfare in the Nash case. This represents the welfare gain to coordination.

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[13]This is an exercise which can not be performed without a second-order solution method. In the absence of a second-order solution method it is not possible to evaluate the first-order terms in the welfare function for individual countries so it is not possible to calculate the Nash equilibrium.

[14]Woodford (1999) argues that a more suitable welfare measure is the sum of flow welfare discounted to time 0. For the sake of illustration the exercise reported here focuses on the simpler problem of maximising steady state flow utility.

[15]It is assumed that policy makers in both the coordinated and uncoordinated cases are able to commit to their chosen policy rules.

[16]Flow welfare is a linear combination of the second moments of the model so its absolute magnitude is proportional to the variances of the shocks.

| $\theta$ | $w_N - w_M$ | $w_C - w_N$ | Ratio | $\sigma^2_{P^N_H}/\sigma^2_{P^M_H}$ |
|---|---|---|---|---|
| 1 | 2.995 | 0.0 | 0.0 | 0.0 |
| 3 | 2.534 | 0.223 | 0.088 | 0.058 |
| 5 | 2.212 | 0.357 | 0.161 | 0.075 |
| 7 | 2.178 | 0.253 | 0.116 | 0.045 |
| 9 | 2.168 | 0.156 | 0.072 | 0.024 |

Table 1: Welfare Gains from Cooperation

The column headed "Ratio" is the ratio of the welfare gain from coordination to the welfare gain from stabilisation. The column headed $\sigma^2_{P^N_H}/\sigma^2_{P^M_H}$ reports the ratio between the variance of producer prices in the Nash case and the variance of producer prices in the nominal income targeting case.

Table 1 shows that when $\theta = 1$ the coordinated and Nash solutions coincide. There are therefore no gains to coordination. It is also the case that both policy regimes imply stable producer prices in each country. But as $\theta$ is increased from unity, the coordinated and Nash solutions diverge. At first the welfare gains from coordination are small, but for larger values of $\theta$ the welfare gains can be quite large.[17] Producer price stability is optimal in the coordinated regime for all values of $\theta$ but notice that uncoordinated policy produces some price fluctuations for $\theta > 1$. The departure from producer price stability is however never very large, even for very high values of $\theta$. The variance of producer prices is never higher than about 8% of the variance that would arise if monetary policy followed a nominal income targeting rule.[18]

# 4  Concluding Comments

This paper has described a method for deriving a full second-order accurate expected solution to a dynamic general equilibrium model. It should be possible to extend

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[17]In Sutherland (2002) the comparison between coordinated and Nash policy is considered in more detail in a static model. It is shown that the structure of financial markets can have a significant impact on the size of the potential welfare gains from coordination. A distinction is made between the case where risk trading takes place before policy rules are chosen and the case where risk trading takes place after policy rules are chosen. It is shown that the welfare gains from coordination can be considerably higher in the first of these two cases. The simple model used to generate the results in Table 1 is based on the assumption that risk trading takes place before policy rules are chosen.

[18]It is apparent that the welfare gains from coordination are not monotonically increasing in $\theta$. This is an interesting issue which is not explored further in this short paper. It should also be noted that the Nash equilibrium may not be unique. If fact, for some parameter combinations it is possible to identify at least two Nash equilibria. The results reported in Table 1 are simply intended to be an illustration of the use of the solution method so a thorough investigation of multiple Nash equilibria is not directly relevant to this paper. However, this issue may also be an interesting topic for future research.

and apply the method to a wide range of cases. It is not the only method for deriving such solutions and it may not be the most general. But it has a number of useful features when compared to the numerical algorithms developed by Sims (2000) and Schmitt-Grohé and Uribe (2001). In particular the method can reveal some useful economic insights about the structure of the solution to models. Such insights may not be so apparent when using a general numerical algorithm.

# Appendix

**The Time Path of Second Moments**   A simple method for calculating the forward-looking time path for the second moments of a model is the following. Write the solution path for the first-order system in the form

$$Z_{1,t} = GZ_{1,t-1} + D\eta_t \tag{A1}$$

$$Z_{2,t} = FZ_{1,t} \tag{A2}$$

where $\eta$ is the vector of shocks. If $\Theta$ is the covariance matrix of $\eta$ then

$$E_0\left[Z_{1,t}Z_{1,t}'\right] = GE_0\left[Z_{1,t-1}Z_{1,t-1}'\right]G' + D\Theta D' \tag{A3}$$

$$E_0\left[Z_{2,t}Z_{2,t}'\right] = FE_0\left[Z_{1,t}Z_{1,t}'\right]F' \tag{A4}$$

With the initial condition $E_0[Z_{1,1}Z_{1,1}'] = D\Theta D'$ these relationships can be used to generate the entire time path for $E_0[Z_{1,t}Z_{1,t}']$ and $E_0[Z_{2,t}Z_{2,t}']$.

**A Closed Economy Example**   The closed economy model in the main text was summarised in the form of the vectors $Z_{1,t} = [\hat{P}_{t-1},\ \hat{K}_{t-1}]'$, $Z_{2,t} = [\hat{B}_t,\ \hat{Q}_t]'$. Using the parameter values $\gamma = 0.5$, $\beta = 0.95$, $\phi = 2$, $\mu = 2$, $\delta = 0.5$ and $\psi = -0.1$, the first-order approximation for the realised path of $Z_{1,t}$ and $Z_{2,t}$ can be written as

$$\left[\begin{array}{c} \hat{P}_t \\ \hat{K}_t \end{array}\right] = G\left[\begin{array}{c} \hat{P}_{t-1} \\ \hat{K}_{t-1} \end{array}\right] + D\varepsilon_t \tag{44}$$

$$\left[\begin{array}{c} \hat{Q}_t \\ \hat{B}_t \end{array}\right] = F\left[\begin{array}{c} \hat{P}_t \\ \hat{K}_t \end{array}\right] \tag{45}$$

where

$$G = \left[\begin{array}{cc} 0.7938 & 0.0326 \\ 0 & 0.5 \end{array}\right] \qquad F = \left[\begin{array}{cc} 1.0506 & 0.4242 \\ 0.5253 & 0.0335 \end{array}\right] \qquad D = \left[\begin{array}{c} 0 \\ 1 \end{array}\right]$$

This system can be used to generate the time path of second moments using the method described in the previous section of this appendix. The time path of $E_0\left[\lambda_{P,t}\right]$, $E_0\left[\lambda_{B,t}\right]$ and $E_0\left[\lambda_{Q,t}\right]$ can then be constructed from these second moments. Table 2 reports some values for the time path of $E_0\left[\lambda_{P,t}\right]$, $E_0\left[\lambda_{B,t}\right]$ and

| $t$ | $E_0\left[\lambda_{P,t}\right]$ | $E_0\left[\lambda_{B,t}\right]$ | $E_0\left[\lambda_{Q,t}\right]$ | $E_0\left[\hat{P}_t\right]$ | $E_0\left[\hat{B}_t\right]$ | $E_0\left[\hat{Q}_t\right]$ | $E_0\left[\hat{C}_t\right]$ |
|---|---|---|---|---|---|---|---|
| 1 | -0.001590 | 0.103334 | 0.000399 | 0.038975 | 0.580513 | 0.093726 | -0.038975 |
| 2 | -0.001728 | 0.114755 | 0.000413 | 0.071088 | 0.638272 | 0.115257 | -0.071088 |
| 3 | -0.001728 | 0.118316 | 0.000438 | 0.096995 | 0.677391 | 0.132500 | -0.096995 |
| 4 | -0.001759 | 0.119526 | 0.000463 | 0.117669 | 0.706390 | 0.146212 | -0.117669 |
| 5 | -0.001806 | 0.119980 | 0.000482 | 0.134082 | 0.728707 | 0.157079 | -0.134082 |
| 6 | -0.001851 | 0.120170 | 0.000497 | 0.147083 | 0.746147 | 0.165682 | -0.147083 |
| 7 | -0.001886 | 0.120258 | 0.000506 | 0.157374 | 0.759866 | 0.172490 | -0.157374 |
| 8 | -0.001911 | 0.120302 | 0.000513 | 0.165520 | 0.770692 | 0.177878 | -0.165520 |
| 9 | -0.001929 | 0.120326 | 0.000517 | 0.171970 | 0.779251 | 0.182145 | -0.171970 |
| 10 | -0.001940 | 0.120339 | 0.000520 | 0.177079 | 0.786024 | 0.185525 | -0.177079 |
| 11 | -0.001947 | 0.120347 | 0.000522 | 0.181128 | 0.791388 | 0.188203 | -0.181128 |
| 12 | -0.001952 | 0.120352 | 0.000523 | 0.184336 | 0.785639 | 0.190326 | -0.184336 |
| 13 | -0.001955 | 0.120355 | 0.000523 | 0.186880 | 0.799008 | 0.192009 | -0.186880 |
| 14 | -0.001957 | 0.120357 | 0.000524 | 0.188898 | 0.801680 | 0.193344 | -0.188898 |
| 15 | -0.001958 | 0.120358 | 0.000524 | 0.190498 | 0.803798 | 0.194403 | -0.190498 |
| 16 | -0.001959 | 0.120359 | 0.000524 | 0.191768 | 0.805479 | 0.195243 | -0.191768 |
| 17 | -0.001959 | 0.120359 | 0.000524 | 0.192775 | 0.806812 | 0.195910 | -0.192775 |
| 18 | -0.001960 | 0.120360 | 0.000524 | 0.193574 | 0.807870 | 0.196438 | -0.193574 |
| 19 | -0.001960 | 0.120360 | 0.000524 | 0.194208 | 0.808710 | 0.196858 | -0.194208 |
| 20 | -0.001960 | 0.120360 | 0.000525 | 0.194712 | 0.809376 | 0.197191 | -0.194712 |

Table 2: The Closed Economy Model

$E_0\left[\lambda_{Q,t}\right]$. Having generated a sufficiently long time path for $E_0\left[\lambda_{P,t}\right]$, $E_0\left[\lambda_{B,t}\right]$ and $E_0\left[\lambda_{Q,t}\right]$ and thus for $E_0\left[\Lambda_t\right]$ it is straightforward to solve (24) for the time path of $E_0\left[Z_{1,t}\right]$ and $E_0\left[Z_{2,t}\right]$. Table 2 also reports some values for $E_0\left[\hat{P}_t\right]$, $E_0\left[\hat{B}_t\right]$ and $E_0\left[\hat{Q}_t\right]$. It is possible to use these values to calculate expected time paths for the other variables of the model. Table 2 shows some values for $E_0\left[\hat{C}_t\right]$.

**The Neoclassical Growth Model** Schmitt-Grohé and Uribe (2001) illustrate their solution method using the following simple neoclassical growth model:

$$c_t^{-\gamma} = \alpha\beta k_{t+1}^{\alpha-1} E_t c_{t+1}^{-\gamma} A_{t+1} \tag{46}$$

$$c_t + k_{t+1} = A_t k_t^{\alpha} \tag{47}$$

$$\ln A_{t+1} = \rho \ln A_t + \sigma\epsilon_{t+1} \tag{48}$$

where, for simplicity, capital is assumed to depreciate fully after one period (this last assumption corresponds to the numerical example presented by Schmitt-Grohé and Uribe).

It is useful to introduce a new variable $H_t \equiv E_t c_{t+1}^{-\gamma} A_{t+1}$. A second-order expansion of (46) in terms of log deviations from the deterministic steady state can therefore be written as follows

$$-\gamma \hat{c}_t = (\alpha - 1)\hat{k}_{t+1} + \hat{H}_t \tag{49}$$

and a second-order expansion of the definition of $H_t$ yields

$$\hat{H}_t + \frac{1}{2}\hat{H}_t^2 = E_t\left[-\gamma \hat{c}_{t+1} + \hat{A}_{t+1} + \frac{1}{2}\left(-\gamma \hat{c}_{t+1} + \hat{A}_{t+1}\right)^2\right] + O\left(\|\xi\|^3\right) \tag{50}$$

Thus, after substituting for $H_t$, (49) can be written as

$$-\gamma \hat{c}_t = (\alpha - 1)\hat{k}_{t+1} - \gamma E_t\left[\hat{c}_{t+1}\right] + E_t\left[\hat{A}_{t+1}\right] + \lambda_{c,t} + O\left(\|\xi\|^3\right) \tag{51}$$

where

$$\lambda_{c,t} = E_t\left[\frac{1}{2}\left(-\gamma \hat{c}_{t+1} + \hat{A}_{t+1}\right)^2\right] - \frac{1}{2}\left(E_t\left[-\gamma \hat{c}_{t+1} + \hat{A}_{t+1}\right]\right)^2 \tag{52}$$

A second-order expansion of (47) yields

$$\hat{c}_t + \frac{1}{2}\hat{c}_t^2 + \hat{k}_{t+1} + \frac{1}{2}\hat{k}_{t+1}^2 = \hat{A}_t + \alpha \hat{k}_t + \frac{1}{2}\left(\hat{A}_t^2 + \alpha \hat{k}_t^2 + 2\alpha \hat{A}_t \hat{k}_t\right) + O\left(\|\xi\|^3\right) \tag{53}$$

which can be written as

$$\hat{c}_t + \hat{k}_{t+1} = \hat{A}_t + \alpha \hat{k}_t + \lambda_{k,t} + O\left(\|\xi\|^3\right) \tag{54}$$

where

$$\lambda_{k,t} = \frac{1}{2}\left(\hat{A}_t^2 + \alpha \hat{k}_t^2 + 2\alpha \hat{A}_t \hat{k}_t - \hat{c}_t^2 - \hat{k}_{t+1}^2\right) \tag{55}$$

Finally (48) can be written in log deviation form as follows

$$\hat{A}_{t+1} = \rho \hat{A}_t + \sigma \epsilon_{t+1} \tag{56}$$

The model can now be written as a matrix equation of the form (24) where $Z_{1,t} = [\hat{k}_t, \hat{A}_t]'$, $Z_{2,t} = [\hat{c}_t]$ and $\Lambda_t = [\lambda_{k,t}, \lambda_{c,t}]'$.

Using the parameter values in Schmitt-Grohé and Uribe (2001)[19] the first-order approximation for the realised path of $Z_{1,t}$ and $Z_{2,t}$ can be written as

$$\left[\begin{array}{c} \hat{k}_t \\ \hat{A}_t \end{array}\right] = G\left[\begin{array}{c} \hat{k}_{t-1} \\ \hat{A}_{t-1} \end{array}\right] + D\epsilon_t \tag{57}$$

$$\hat{c}_t = F\left[\begin{array}{c} \hat{k}_t \\ \hat{A}_t \end{array}\right] \tag{58}$$

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[19] $\beta = 0.95$, $\alpha = 0.3$, $\rho = 0$ and $\gamma = 2$.

where
$$G = \begin{bmatrix} 0.4191 & 1.3970 \\ 0 & 0 \end{bmatrix} \qquad F = \begin{bmatrix} 0.2525 \\ 0.8417 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This system can be used to generate the time path of second moments and the time path of $E_0[\lambda_{k,t}]$ and $E_0[\lambda_{c,t}]$. Note from (52) that $\lambda_{c,t}$ contains terms in the second moments of $E_t[\hat{c}_{t+1}]$ and $E_t[\hat{A}_{t+1}]$. These second moments can be constructed by observing from (57) and (58) that

$$E_t[\hat{c}_{t+1}] = 0.2525 E_t[\hat{k}_{t+1}] + 0.8417 E_t[\hat{A}_{t+1}]$$

$$E_t[\hat{k}_{t+1}] = 0.4191 \hat{k}_t + 1.3970 \hat{A}_t$$

$$E_t[\hat{A}_{t+1}] = 0$$

Table 3 reports some values for the time path of $E_0[\lambda_{k,t}]$ and $E_0[\lambda_{c,t}]$. It is straightforward to solve (24) for the time path of $E_0[Z_{1,t}]$ and $E_0[Z_{2,t}]$. Table 3 also reports some values for $E_0[\hat{c}_t]$ and $E_0[\hat{k}_t]$.

Schmitt-Grohé and Uribe (2001) report second-order expressions which can be used to generate the realised time path of $\hat{c}_t$ and $\hat{k}_{t+1}$. Applying the $E_0$ operator to the Schmitt-Grohé and Uribe expressions yields expressions for $E_0[\hat{c}_t]$ and $E_0[\hat{k}_{t+1}]$ and turns the second-order terms into second moments. It is then simply to use the Schmitt-Grohé and Uribe expressions to generate time paths for $E_0[\hat{c}_t]$ and $E_0[\hat{k}_t]$ using the second moments generated from (57) and (58). This procedure yields numerical values which are almost identical to those reported in Table 3.

| $t$ | $E_0\left[\lambda_{k,t}\right]$ | $E_0\left[\lambda_{c,t}\right]$ | $E_0\left[\hat{c}_t\right]$ | $E_0\left[\hat{k}_t\right]$ |
|---|---|---|---|---|
| 1 | -0.031416 | 0.233577 | -0.124504 | 0.0 |
| 2 | -0.036934 | 0.233577 | -0.078458 | 0.202121 |
| 3 | -0.037904 | 0.233577 | -0.059670 | 0.279999 |
| 4 | -0.038074 | 0.233577 | -0.051885 | 0.311438 |
| 5 | -0.038104 | 0.233577 | -0.048638 | 0.324404 |
| 6 | -0.038109 | 0.233577 | -0.047280 | 0.329801 |
| 7 | -0.038110 | 0.233577 | -0.046711 | 0.332056 |
| 8 | -0.038110 | 0.233577 | -0.046473 | 0.333000 |
| 9 | -0.038110 | 0.233577 | -0.046373 | 0.333396 |
| 10 | -0.038110 | 0.233577 | -0.046331 | 0.333561 |
| 11 | -0.038110 | 0.233577 | -0.046314 | 0.333631 |
| 12 | -0.038110 | 0.233577 | -0.046306 | 0.333660 |
| 13 | -0.038110 | 0.233577 | -0.046303 | 0.333672 |
| 14 | -0.038110 | 0.233577 | -0.046302 | 0.333677 |
| 15 | -0.038110 | 0.233577 | -0.046301 | 0.333679 |
| 16 | -0.038110 | 0.233577 | -0.046301 | 0.333680 |
| 17 | -0.038110 | 0.233577 | -0.046301 | 0.333680 |
| 18 | -0.038110 | 0.233577 | -0.046301 | 0.333681 |
| 19 | -0.038110 | 0.233577 | -0.046301 | 0.333681 |
| 20 | -0.038110 | 0.233577 | -0.046301 | 0.333681 |

Table 3: The Neoclasical Growth Model

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