

Country Portfolios in Open Economy Macro Models*

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Abstract

This paper develops a simple approximation method for computing equilibrium portfolios in dynamic general equilibrium open economy macro models. The method is widely applicable, simple to implement, and gives analytical solutions for equilibrium portfolio positions in any combination or types of asset. It can be used in models with any number of assets, whether markets are complete or incomplete, and can be applied to stochastic dynamic general equilibrium models of any dimension, so long as the model is amenable to a solution using standard approximation methods. We first illustrate the approach using a simple two-asset endowment economy model, and then show how the results extend to the case of any number of assets and general economic structure.

Keywords: Country portfolios, solution methods.

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1 Introduction

This paper develops a simple and tractable approach to computing equilibrium financial asset portfolios in open economy dynamic stochastic general equilibrium (DSGE) models. To a large extent, existing open economy macroeconomic models ignore portfolio composition, analyzing financial linkages between countries in terms of net foreign assets, with no distinction made between assets and liabilities. But recent research has highlighted the presence of large cross-country *gross* asset and liability positions, with considerable heterogeneity among countries in the composition of portfolios among different classes of assets. Lane and Milesi-Ferretti (2001, 2006) document the rapid growth of country portfolio holdings, particularly in the last decade. They show that even large countries such as the UK hold gross assets and liabilities that are multiples of GDP.

The growth in international financial portfolios raises a number of important questions for open economy macroeconomics. What are the determinants of the size and composition of gross portfolio positions? Can standard theories account for the observed structure of portfolio holdings? Moreover, the large size of gross positions makes it likely that the portfolio composition itself affects macroeconomic outcomes. With gross positions as large or larger than GDP, unanticipated changes in exchange rates or asset prices can generate changes in net external wealth ('valuation effects') that are the same order of magnitude as annual current accounts.¹ This raises questions about how portfolio composition may affect the international business cycle and international transmission of shocks. Finally, by generating significant wealth re-distributions in response to fluctuations in exchange rates and asset prices, international portfolio composition may have significant implications for economic policy. How should monetary and fiscal policies be designed in an environment of endogenous portfolio choice?

While these questions are obviously of interest to open economy macroeconomists and policymakers, current theoretical models and solution methods cannot answer them in any very systematic way. This is because the standard approaches to solving general equilibrium models make it difficult to incorporate portfolio choice. The usual method of analysis in DSGE models is to take a linear approximation around a non-stochastic steady

¹Lane and Milesi-Ferretti (2001) emphasize the quantitative importance of valuation effects on external assets and liabilities. See also subsequent work by Ghironi et al. (2005), Gourinchas and Rey (2005), and Tille (2003, 2004).

state. But optimal portfolios are not uniquely defined in a non-stochastic steady state, so there is no natural point around which to approximate. Moreover, portfolios are also not defined in a first-order approximation to a DSGE model, since such an approximation satisfies certainty equivalence, so that all assets are perfect substitutes. As a result, the analysis of portfolio choice in DSGE models appears to be intractable in all but the most restricted of cases.²

In this paper we develop and analyze an approximation method which overcomes these problems. Our method can be applied to any standard open economy model with any number of assets, any number of state variables, and complete or incomplete markets, so long as the model is amenable to solution using conventional approximation procedures for DSGE models. We find a general formula for asset holdings which can be very easily incorporated into the standard solution approach. The technique is simple to implement and can be used to derive either analytical results (for sufficiently small models) or numerical results for larger models.

A key feature of our approach is to recognize that, at the level of approximation usually followed in open economy macroeconomics, one only requires a solution for the ‘steady-state’ portfolio holdings. The steady state portfolio is defined as the constant (or ‘zero-order’) term in a Taylor series approximation of the true equilibrium portfolio function. Higher-order aspects of portfolio behaviour are not relevant for first-order accurate macro dynamics. Equivalently, time variation in portfolios is irrelevant for all questions regarding the first-order responses of macroeconomic variables like consumption, output, real exchange rates, etc. Therefore, the solution we derive exhausts all the macroeconomic implications of portfolio choice at this level of approximation.

²If there are enough financial assets to allow perfect risk sharing (so that international financial markets are effectively complete) the problem becomes somewhat easier. In this case, it is possible to identify an equilibrium macroeconomic allocation independent of financial structure, and then, given this allocation, one can derive the implied portfolio which supports the equilibrium. Engel and Matsumoto (2005) and Kollmann (2006) represent examples of such an approach. However, when markets are incomplete (in the sense that there are not sufficient assets to allow perfect risk sharing) optimal portfolios and macroeconomic equilibrium must be derived simultaneously. This makes the problem considerably more difficult. Given the lack of empirical support for international risk sharing (see, for instance, Obstfeld and Rogoff, 2000), models with incomplete markets are likely to be of particular interest. Heathcote and Perri (2004) provide one example of an incomplete markets model in which it is possible to derive explicit expressions for equilibrium portfolios. Their model is, however, only tractable for a specific menu of assets and for specific functional forms for preferences and technology.

How do we obtain the zero-order component of the equilibrium portfolio? This is done by combining a second-order approximation of the portfolio selection equation with a first-order approximation to the remaining parts of the model. Of course, these two approximations will be interdependent; the endogenous portfolio weights will depend on the variance-covariance matrix of excess returns produced by the general equilibrium model, but that in turn will depend on the portfolio positions themselves. The key innovation of our approach is to show that this simultaneous system can be solved to give a simple *closed-form* analytical solution for the equilibrium portfolio.

While our solution procedure is novel, its mathematical foundations are already established in the literature, in particular in the work of Samuelson (1970), and in different form by Judd (1998) and Judd and Guu (2001). Samuelson shows how a mean-variance approximation of a portfolio selection problem is sufficient to identify the optimal portfolio in a near-non-stochastic world. Judd and Guu (2001) show how the same equilibrium can be identified by using a combination of a Bifurcation theorem and the Implicit Function Theorem. Our solution approach relies on first-order and second-order approximations of the model, rather than the Implicit Function and Bifurcation Theorems, but the underlying theory described by Judd and Guu (2001) is applicable to our equilibrium solution. In particular, the steady-state portfolio derived using our technique corresponds to a bifurcation point in the set of non-stochastic equilibria. The main contribution of this paper is to show how this solution can easily be derived in standard DSGE models. We note in addition, that there is nothing about the approximation method that restricts its use to open economy models. It can be applied to any heterogeneous agent DSGE model, whether in a closed or open economy context.³

As we have already stated, the steady-state portfolio is all that is needed in order to analyze the first-order properties of a general equilibrium model. But for many purposes, it may be necessary to analyze the dynamics of portfolio holdings themselves. In addition, in order to do welfare analysis, it is usually necessary to analyze a second-order approximation of a model. At the level of second-order approximation, time variation in portfolios becomes relevant for macroeconomic dynamics. But these features can be obtained by an extension of our method to higher-order approximations of the model. In particular, the state-contingent, or first-order aspects of the equilibrium portfolio, can be

³Samuelson (1970) and Judd and Guu (2001) did not develop their results in open economy (or general equilibrium) contexts.

obtained by combining a third-order approximation of the portfolio selection equations, with a second-order approximation to the rest of the model. The current paper focuses on the derivation of steady-state portfolios because this represents a distinct and valuable first-step in the analysis of portfolio choice in open-economy DSGE models. In a final section however, we discuss briefly the extension of the method to higher orders. A companion paper, Devereux and Sutherland (2007), shows how higher-order solutions to portfolios also have an analytical representation.

In the related literature a number of approaches have been developed for analysing portfolio choice in incomplete-markets general equilibrium models. In a recent paper, Tille and Van Wincoop (2007) show how the zero and higher-order components of portfolio behaviour in an open economy model can be obtained numerically via an iterative algorithm. Their approach delivers a numerical solution for steady-state portfolios in manner analogous to the analytical solutions derived in this paper. Judd *et al* (2002) develop a numerical algorithm based on ‘spline collocation’ and Evans and Hnatkovska (2005) present a numerical approach that relies on a combination of perturbation and continuous-time approximation techniques.⁴ The methods developed by Judd *et al* and Evans and Hnatkovska are very complex compared to our approach and they represent a significant departure from standard DSGE solution methods. Devereux and Saito (2005) use a continuous time framework which allows some analytical solutions to be derived, but their approach can not handle general international macroeconomic models with diminishing-returns technology or sticky nominal goods prices.⁵

This paper proceeds as follows. The next section sets out a two-asset portfolio choice problem within a simple two-country endowment model and shows how our method can be applied in this context. Section 3 develops a more general n-asset portfolio problem within a generic two country DSGE model and shows how the method can be generalised to accommodate a wide class of models. In this section also, we extend the basic example model of Section 2 to allow for more shocks and a wider class of assets. Section 4 briefly outlines how the method can be extended to derive a solution for the first-order component

⁴Evans and Hnatkovska (2005) develop an approach similar to that of Campbell and Viceira (2005), who present a comprehensive analysis of optimal portfolio allocation for a single agent.

⁵More recently, a number of papers have started to use the method developed in the current paper to solve for optimal portfolio holdings in DSGE models. See in particular Coeurdacier, Kollmann and Martin (2008), and Coeurdacier (2008).

of the equilibrium portfolio. Section 5 concludes the paper.

2 Example: A Simple Two-Asset Endowment Model

2.1 The Model

We first illustrate how the solution procedure works in a simple two-country example with only two internationally traded assets, where agents consume an identical consumption good, and income takes the form of an exogenous endowment of the consumption good.

Agents in the home country have a utility function of the form

$$U_t = E_t \sum_{\tau=t}^{\infty} \theta_{\tau} u(C_{\tau}) \quad (1)$$

where C is consumption and $u(C) = (C^{1-\rho})/(1-\rho)$. θ_{τ} is the discount factor, which is determined as follows

$$\theta_{\tau+1} = \theta_{\tau} \beta(C_{A\tau}), \quad \theta_0 = 1$$

where C_A is aggregate home consumption and $0 < \beta(C_A) < 1$, $\beta'(C_A) \leq 0$. If $\beta(C_A)$ is a constant (i.e. $\beta'(C_A) = 0$) then the discount factor is exogenous. It is well known that, in this case, incompleteness of financial markets implies a unit root in the first-order approximated model. Although our solution approach works perfectly well in this case, there may be occasions where it is useful to eliminate the unit root. This can be achieved by allowing $\beta'(C_A) < 0$. Endogenising the discount factor in this way has no impact on the applicability of our solution approach.⁶ In what follows we assume

$$\beta(C_A) = \omega C_A^{-\eta}$$

where $0 \leq \eta < \rho$ and $0 < \omega \bar{C}_A^{-\eta} < 1$ where \bar{C}_A is the steady state value of consumption.

The budget constraint for home agents is given by

$$\alpha_{1,t} + \alpha_{2,t} = \alpha_{1,t-1} r_{1,t} + \alpha_{2,t-1} r_{2,t} + Y_t - C_t \quad (2)$$

⁶Following Schmitt Grohe and Uribe (2003), θ_{τ} is assumed to be taken as exogenous by individual decision makers. The impact of consumption on the discount factor is therefore not internalized. Our solution approach can be easily modified to deal with the alternative case where θ_{τ} depends on *individual* consumption, and the effect of consumption plans on θ_{τ} is internalised (see Devereux and Sutherland 2008).

where Y is the endowment received by home agents, $\alpha_{1,t-1}$ and $\alpha_{2,t-1}$ are the real holdings of the two assets (purchased at the end of period $t - 1$ for holding into period t) and $r_{1,t}$ and $r_{2,t}$ are gross real returns. It is assumed that the vector of available assets is exogenous and predefined. The stochastic process determining endowments and the nature of the assets and the properties of their returns are specified below.

Define $W_t = \alpha_{1,t} + \alpha_{2,t}$ to be the total net claims of home agents on the foreign country at the end of period t (i.e. the net foreign assets of home agents). The budget constraint can then be re-written as

$$W_t = \alpha_{1,t-1}r_{x,t} + r_{2,t}W_{t-1} + Y_t - C_t \quad (3)$$

where

$$r_{x,t} = r_{1,t} - r_{2,t}$$

Here asset 2 is used as a numeraire and $r_{x,t}$ measures the "excess return" on asset 1.

At the end of each period agents select the portfolio of assets to hold into the following period. Thus, for instance, at the end of period t home agents select $\alpha_{1,t}$ to hold into period $t + 1$. The first-order condition for the choice of $\alpha_{1,t}$ can be written in the following form

$$E_t [u'(C_{t+1})r_{1,t+1}] = E_t [u'(C_{t+1})r_{2,t+1}] \quad (4)$$

Foreign agents face a similar portfolio allocation problem with a budget constraint given by

$$W_t^* = \alpha_{1,t-1}^*r_{x,t} + r_{2,t}W_{t-1}^* + Y_t^* - C_t^* \quad (5)$$

where an asterisk indicates foreign variables. In equilibrium it follows that $W_t^* = -W_t$. Foreign agents have preferences similar to (1) so the first-order condition for foreign agents' choice of $\alpha_{1,t}^*$ is

$$E_t [u'(C_{t+1}^*)r_{1,t+1}] = E_t [u'(C_{t+1}^*)r_{2,t+1}] \quad (6)$$

Assets are assumed to be in zero net supply, so market clearing in asset markets implies

$$\alpha_{1,t-1} + \alpha_{1,t-1}^* = 0, \quad \alpha_{2,t-1} + \alpha_{2,t-1}^* = 0$$

To simplify notation, in what follows we will drop the subscript from $\alpha_{1,t}$ and simply refer to α_t . It should be understood, therefore, that $\alpha_{1,t} = -\alpha_{1,t}^* = \alpha_t$, $\alpha_{2,t} = W_t - \alpha_t$ and $\alpha_{2,t}^* = W_t^* + \alpha_t$.

Endowments are the sum of two components, so that

$$Y_t = Y_{K,t} + Y_{L,t}, \quad Y_t^* = Y_{K,t}^* + Y_{L,t}^* \quad (7)$$

where $Y_{K,t}$ and $Y_{K,t}^*$ represent ‘capital income’ and $Y_{L,t}$ and $Y_{L,t}^*$ ‘labour income’. The endowments are determined by the following simple stochastic processes

$$\begin{aligned} \log Y_{K,t} &= \log \bar{Y}_K + \varepsilon_{K,t}, & \log Y_{L,t} &= \log \bar{Y}_L + \varepsilon_{L,t} \\ \log Y_{K,t}^* &= \log \bar{Y}_K + \varepsilon_{K,t}^*, & \log Y_{L,t}^* &= \log \bar{Y}_L + \varepsilon_{L,t}^* \end{aligned}$$

where $\varepsilon_{K,t}$, $\varepsilon_{L,t}$, $\varepsilon_{K,t}^*$ and $\varepsilon_{L,t}^*$ are zero-mean i.i.d. shocks symmetrically distributed over the interval $[-\epsilon, \epsilon]$ with $Var[\varepsilon_K] = Var[\varepsilon_K^*] = \sigma_K^2$, $Var[\varepsilon_L] = Var[\varepsilon_L^*] = \sigma_L^2$. We assume $Cov[\varepsilon_K, \varepsilon_K^*] = Cov[\varepsilon_L, \varepsilon_L^*] = 0$ and $Cov[\varepsilon_K, \varepsilon_L] = Cov[\varepsilon_K^*, \varepsilon_L^*] = \sigma_{KL}$.

The two assets are assumed to be one-period equity claims on the home and foreign capital income.⁷ The real payoff to a unit of the home equity in period t is defined to be $Y_{K,t}$ and the real price of a unit of home equity is denoted $Z_{E,t-1}$. Thus the gross real rate of return on home equity is

$$r_{1,t} = Y_{K,t}/Z_{E,t-1} \quad (8)$$

Likewise the gross real return on foreign equity is

$$r_{2,t} = Y_{K,t}^*/Z_{E,t-1}^* \quad (9)$$

where $Z_{E,t-1}^*$ is the price of the foreign equity.

The first-order conditions for home and foreign consumption are

$$C_t^{\eta-\rho} = \omega E_t [C_{t+1}^{-\rho} r_{2,t+1}], \quad C_t^{*\eta-\rho} = \omega E_t [C_{t+1}^{*-\rho} r_{2,t+1}] \quad (10)$$

where the distinction between individual and aggregate consumption has been dropped because, in equilibrium, $C = C_A$ and $C^* = C_A^*$. Finally, equilibrium consumption plans satisfy the resource constraint

$$C_t + C_t^* = Y_t + Y_t^* \quad (11)$$

⁷Notice that we are assuming that, by default, all capital in a country is owned by the residents of that country. This allows us to treat equity claims to capital income as inside assets, i.e. assets in zero net supply. This is purely an accounting convention. Our solution method works equally in the alternative approach, where capital is not included in the definition of Y and Y^* and equity is treated as an outside asset which is in positive net supply. The present approach makes our derivations easier however.

2.2 Zero-order and first-order components

Despite the extreme simplicity of this model, it is only in special cases that an exact solution can be found, e.g. when there is no labour income (in which case trade in equities supports the perfect risk-sharing equilibrium).⁸ Furthermore, the optimal portfolio can not be obtained using first-order approximation techniques, so standard linearisation approaches to DSGE models can not provide even an approximate solution to the general case. Our method, nevertheless, *does* yield an approximate solution to the general case. Before describing the method, it is useful to show why standard solution techniques do not work for this model, and to demonstrate how our method offers a way around the problems.

First, we define some terms relating to the true and approximate portfolio solutions. Notice that agents make their portfolio decisions at the end of each period and are free to re-arrange their portfolios each period. In a recursive equilibrium, therefore, the equilibrium asset allocation will be some function of the state of the system in each period - which is summarised by the state variables. We therefore postulate that the true portfolio (i.e. the equilibrium portfolio in the non-approximated model) is a function of state variables. In the model defined above there is only one state variable, W - so we postulate $\alpha_t = \alpha(W_t)$.⁹

Now consider a first-order Taylor-series expansion of $\alpha(W_t)$ around the point $W = \bar{W}$

$$\alpha(W_t) \simeq \alpha(\bar{W}) + \alpha'(\bar{W})(W_t - \bar{W})$$

This approximation contains two terms: $\alpha(\bar{W})$, which is the *zero-order component* (i.e. α at the point of approximation) and $\alpha'(\bar{W})(W_t - \bar{W})$, which is the *first-order component* (assuming $(W_t - \bar{W})$ is evaluated up to first-order accuracy). Notice that, by definition, the zero-order component of α is non-time varying. The approximate dynamics of the portfolio are captured by the first-order component.

⁸If there is no labour income then equities can be used to trade all income risk. It is easy to show that the equilibrium portfolio is for home and foreign agents to hold portfolios equally split between home and foreign equity. This implies perfect consumption risk sharing. This is a useful benchmark for comparison with the solution yielded by our method.

⁹Optimal portfolio allocation will of course depend on the properties of asset returns generated by the model. In equilibrium, however, the stochastic properties of asset returns will also be a function of state variables, so the impact of asset returns on portfolio allocation is implicit in the function $\alpha(W_t)$.

When analysing a DSGE model up to first-order accuracy the standard solution approach is to use the non-stochastic steady-state of the model as the approximation point, (i.e. the zero-order component of each variable) and to use a first-order approximation of the model's equations to solve for the first-order component of each variable. Neither of these steps can be used in the above model. It is very simple to see why. In the non-stochastic equilibrium equations (4) and (6) imply

$$r_{1,t+1} = r_{2,t+1}$$

i.e. both assets pay the same rate of return. This implies that, for given W , all portfolio allocations pay the same return, so any value for α is consistent with equilibrium. Thus the non-stochastic steady state does not tie down a unique portfolio allocation.

A similar problem arises in a first-order approximation of the model. First-order approximation of equations (4) and (6) imply

$$E_t[r_{1,t+1}] = E_t[r_{2,t+1}]$$

i.e. both assets have the same expected rate of return. Again, any value of α is consistent with equilibrium.

So neither the non-stochastic steady state nor a first-order approximation of the model provide enough equations to tie down the zero or first-order components of α . The basic problem is easy to understand in economic terms. Assets in this model are only distinguishable in terms of their risk characteristics and neither the non-stochastic steady state nor a first-order approximation capture the different risk characteristics of assets. In the case of the non-stochastic steady state there is, by definition, no risk, while in a first-order approximation there is certainty equivalence.

This statement of the problem immediately suggests a solution. It is clear that the risk characteristics of assets only show up in the second-moments of model variables, and it is only by considering higher-order approximations of the model that the effects of second-moments can be captured. This fundamental insight has existed in the literature for many years. It was formalised by Samuelson (1970), who established that, in order to derive the zero-order component of the portfolio, it is necessary to approximate the portfolio problem up to the *second order*. Our solution approach follows this principle. We show that a second-order approximation of the portfolio optimality conditions provides a condition which makes it possible to tie down the zero-order component of α . The

second-order approximation captures the impact of the portfolio choices on the correlation between portfolio returns and the marginal utility of consumption. It therefore captures differences between assets in their ability to hedge consumption risk and thus ties down an optimal portfolio allocation. In this paper we show in detail how to use second-order approximations of the portfolio optimality conditions to solve for the zero-order component of α .¹⁰

Having established this starting point, it is relatively straightforward to extend the procedure to higher-order components on α . Samuelson (1970) in fact states a general principle that, in order to derive the N th-order component of the portfolio, it is necessary to approximate the portfolio problem up to order $N + 2$. In Section 4 we briefly outline how, by following this principle, the solution for the first-order component of α can be derived from third-order approximations of the portfolio optimality conditions. The full details of the solution procedure for the first-order component are given in a companion paper, Devereux and Sutherland (2007).

While Samuelson (1970) was the first to show how solutions for the zero and higher-order components of the portfolio may be derived, more recently Judd and Guu (2001) have demonstrated an alternative solution approach which sheds further light on the nature of the zero-order portfolio. They show how the problem of portfolio indeterminacy in the non-stochastic steady state can be overcome by using a Bifurcation theorem in conjunction with the Implicit Function Theorem. Their approach shows that the zero-order portfolio is a bifurcation point in the set of non-stochastic equilibria. Like Samuelson (1970), our solution approach relies on second-order approximations of the model to identify the zero-order component, but the underlying theory described by Judd and Guu (2001) is also applicable to our equilibrium solution. In particular, the zero-order portfolio derived using our technique corresponds to the solution that emerges from the Judd and Guu approach. Our solution can therefore be rationalised in the same way, i.e. it is a bifurcation point in the set of non-stochastic equilibria.¹¹

¹⁰Note that Samuelson approached the problem by approximating the agent's utility function, while we take approximations of agents' first-order conditions. It is possible to show that the two approaches produce identical results for the zero order component of the portfolio.

¹¹As already explained, in a non-stochastic world all portfolio allocations are equivalent and can be regarded as valid equilibria. A stochastic world on the other hand (assuming independent asset returns and suitable regularity conditions on preferences) has a unique equilibrium portfolio allocation. If one considers the limit of a sequence of stochastic worlds, with diminishing noise, the equilibrium portfolio

The general underlying principles of the solution we derive are thus well established. The main contribution of this paper is to provide a solution approach which can easily be applied to DSGE models.¹² We now demonstrate this by solving for the zero-order component of α in the simple two-asset endowment model described above.

2.3 Solving for the zero-order portfolio

In what follows, a bar over a variable indicates its value at the approximation point (i.e. the zero-order component) and a hat indicates the log-deviation from the approximation point (except in the case of $\hat{\alpha}$, \hat{W} and \hat{r}_x , which are defined below). Notice that the non-stochastic steady state, while failing to tie down α , still provides solutions for output, consumption and rates of return. We therefore use the non-stochastic steady state of the model as the approximation point for all variables except α . For simplicity, we assume that the two countries are identical so that the non-stochastic steady state implies $\bar{W} = 0$. It follows from equations (4) and (6) that $\bar{r}_1 = \bar{r}_2 = 1/\bar{\beta}$ (where $\bar{\beta} = \beta(\bar{C}) = \omega\bar{C}^{-\eta}$) and thus $\bar{r}_x = 0$. Equations (3) and (5) therefore imply that $\bar{Y} = \bar{Y}^* = \bar{C} = \bar{C}^*$. Since $\bar{W} = 0$, it also follows that $\bar{\alpha}_2 = -\bar{\alpha}_1 = -\bar{\alpha}_2^* = \bar{\alpha}_1^* = -\bar{\alpha}$.¹³

As argued above, solving for the zero-order component of α requires a second-order expansion of the portfolio problem. So we start by taking a second-order approximation of the home-country portfolio first-order condition, (4), to yield

$$E_t \left[\hat{r}_{x,t+1} + \frac{1}{2}(\hat{r}_{1,t+1}^2 - \hat{r}_{2,t+1}^2) - \rho\hat{C}_{t+1}\hat{r}_{x,t+1} \right] = O(\epsilon^3) \quad (12)$$

where $\hat{r}_{x,t+1} = \hat{r}_{1,t+1} - \hat{r}_{2,t+1}$ and $O(\epsilon^3)$ is a residual which contains all terms of order higher than two. Applying a similar procedure to the foreign first-order condition, (6), yields

$$E_t \left[\hat{r}_{x,t+1} + \frac{1}{2}(\hat{r}_{1,t+1}^2 - \hat{r}_{2,t+1}^2) - \rho\hat{C}_{t+1}^*\hat{r}_{x,t+1} \right] = O(\epsilon^3) \quad (13)$$

tends towards a limit which correspond to one of the many equilibria in the non-stochastic world. This limiting portfolio is the bifurcation point described by Judd and Guu (2001), i.e. it is the point in the set of non-stochastic equilibria which intersects with the sequence of stochastic equilibria.

¹²Note that both Samuelson (1970) and Judd and Guu (2001) demonstrate their results using static partial equilibrium models of portfolio allocation.

¹³The assumption of symmetry is purely for convenience of exposition. The solution method can easily be adapted to deal with asymmetries across the home and foreign countries.

These expression can now be combined to show that, in equilibrium, the following equations must hold

$$E_t \left[(\hat{C}_{t+1} - \hat{C}_{t+1}^*) \hat{r}_{x,t+1} \right] = 0 + O(\epsilon^3) \quad (14)$$

and

$$E_t [\hat{r}_{x,t+1}] = -\frac{1}{2} E_t [\hat{r}_{1,t+1}^2 - \hat{r}_{2,t+1}^2] + \rho \frac{1}{2} E_t \left[(\hat{C}_{t+1} + \hat{C}_{t+1}^*) \hat{r}_{x,t+1} \right] + O(\epsilon^3) \quad (15)$$

These two equations express the portfolio optimality conditions in a form which is particularly convenient for deriving equilibrium portfolio holdings and excess returns. Equation (14) provides an equation which must be satisfied by equilibrium portfolio holdings. Equation (15) shows the corresponding set of equilibrium expected excess returns.

We will now show that equation (14) provides a sufficient condition to tie down the zero-order component of α . In order to do this we first state two important properties of the approximated model.

Property 1 In order to evaluate the left hand side of equation (14) it is sufficient to derive expressions for the first-order accurate behaviour of consumption and excess returns. This is because the only terms that appear in equation (14) are products, and second-order accurate solutions for products can be obtained from first-order accurate solutions for individual variables.

Property 2 The only aspect of the portfolio decision that affects the first-order accurate behaviour of consumption and excess returns is $\bar{\alpha}$, i.e. the zero-order component of the α . The first-order component, i.e. the deviation of α from the approximation point, does not affect the first-order behaviour of consumption and excess returns. To see why this is true notice that portfolio decisions only enter the model via the portfolio excess return, i.e. via the term $\alpha_{t-1} r_{x,t}$ in the budget constraints. A first-order expansion of this term is $\bar{\alpha} \hat{r}_{x,t} + \bar{r}_x \hat{\alpha}_{t-1}$. But $\bar{r}_x = 0$ so only $\bar{\alpha} \hat{r}_{x,t}$ remains.

It is now straightforward to show that equation (14) provides a condition which ties down $\bar{\alpha}$. Property 2 tells us that it is possible to evaluate the first-order behaviour of $(\hat{C}_{t+1} - \hat{C}_{t+1}^*)$ and $\hat{r}_{x,t+1}$ conditional on a given value of $\bar{\alpha}$. Property 1 tells us that $E_t \left[(\hat{C}_{t+1} - \hat{C}_{t+1}^*) \hat{r}_{x,t+1} \right]$ can therefore also be evaluated conditional on a given value of $\bar{\alpha}$. Equation (14) tells us that a solution for $\bar{\alpha}$ is one which implies $E_t \left[(\hat{C}_{t+1} - \hat{C}_{t+1}^*) \hat{r}_{x,t+1} \right] = 0$.

In order to derive this solution for $\bar{\alpha}$ it is first necessary to solve for the first-order accurate behaviour of $(\hat{C}_{t+1} - \hat{C}_{t+1}^*)$ and $\hat{r}_{x,t+1}$ conditional on a given value of $\bar{\alpha}$. The first-order accurate behaviour of $\hat{r}_{x,t+1}$ is particularly simple in this model. First-order approximations of (8) and (9) imply

$$\hat{r}_{x,t+1} = \hat{Y}_{K,t+1} - \hat{Y}_{K,t+1}^* - (\hat{Z}_{E,t} - \hat{Z}_{E,t}^*) + O(\epsilon^2)$$

where $O(\epsilon^2)$ is a residual which contains all terms of order higher than one, so

$$E_t[\hat{r}_{x,t+1}] = E_t[\hat{Y}_{K,t+1}] - E_t[\hat{Y}_{K,t+1}^*] - (\hat{Z}_{E,t} - \hat{Z}_{E,t}^*) + O(\epsilon^2)$$

Notice that (15) implies that, up to a first-order approximation, $E_t[\hat{r}_{x,t+1}] = 0$ so

$$(\hat{Z}_{E,t} - \hat{Z}_{E,t}^*) = E_t[\hat{Y}_{K,t+1}] - E_t[\hat{Y}_{K,t+1}^*] + O(\epsilon^2)$$

and thus, since Y_K and Y_K^* are i.i.d., $\hat{r}_{x,t+1}$ is given by¹⁴

$$\hat{r}_{x,t+1} = \hat{Y}_{K,t+1} - \hat{Y}_{K,t+1}^* + O(\epsilon^2) \quad (16)$$

The first-order accurate solution for $(\hat{C}_{t+1} - \hat{C}_{t+1}^*)$ is also straightforward to derive. A first-order approximation of the home and foreign budget constraints implies

$$\hat{W}_{t+1} = \frac{1}{\beta} \hat{W}_t + \hat{Y}_{t+1} - \hat{C}_{t+1} + \tilde{\alpha} \hat{r}_{x,t+1} + O(\epsilon^2) \quad (17)$$

$$-\hat{W}_{t+1} = -\frac{1}{\beta} \hat{W}_t + \hat{Y}_{t+1}^* - \hat{C}_{t+1}^* - \tilde{\alpha} \hat{r}_{x,t+1} + O(\epsilon^2) \quad (18)$$

where $\hat{W}_t = (W_t - \bar{W})/\bar{C}$ and $\tilde{\alpha} = \bar{\alpha}/(\beta\bar{Y})$. Combining (17) and (18) with (16) and an appropriate transversality condition implies

$$\begin{aligned} \sum_{i=0}^{\infty} \beta^i E_{t+1}(\hat{C}_{t+1+i} - \hat{C}_{t+1+i}^*) &= \frac{2}{\beta} \hat{W}_t + (\hat{Y}_{t+1} - \hat{Y}_{t+1}^*) \\ &\quad + 2\tilde{\alpha}(\hat{Y}_{K,t+1} - \hat{Y}_{K,t+1}^*) + O(\epsilon^2) \end{aligned} \quad (19)$$

where use has been made of the fact that $E_{t+1}[\hat{Y}_{t+1+i}] = E_{t+1}[\hat{Y}_{t+1+i}^*] = E_{t+1}[\hat{Y}_{K,t+1+i}] = E_{t+1}[\hat{Y}_{K,t+1+i}^*] = 0$ for all $i > 0$.

¹⁴Notice from this derivation that, in this model, r_x is completely independent of α . This makes the application of our solution process particularly simple. In the next section we will show that our method can easily be applied to more general models where the behaviour of r_x may be directly or indirectly influenced by α .

The first-order conditions for consumption, equations (10), imply

$$E_{t+1}[\hat{C}_{t+1+i} - \hat{C}_{t+1+i}^*] = \varsigma(\hat{C}_{t+1} - \hat{C}_{t+1}^*) + O(\epsilon^2) \text{ for all } i > 0 \quad (20)$$

where $\varsigma = (\rho - \eta)/\rho$ so $(\hat{C}_{t+1} - \hat{C}_{t+1}^*)$ is given by

$$\begin{aligned} \hat{C}_{t+1} - \hat{C}_{t+1}^* &= \frac{2(1 - \bar{\beta}\varsigma)}{\bar{\beta}} \hat{W}_t + (1 - \bar{\beta}\varsigma)(\hat{Y}_{t+1} - \hat{Y}_{t+1}^*) \\ &\quad + 2(1 - \bar{\beta}\varsigma)\tilde{\alpha}(\hat{Y}_{K,t+1} - \hat{Y}_{K,t+1}^*) + O(\epsilon^2) \end{aligned} \quad (21)$$

Equations (16) and (21) show the first-order accurate behaviour of $(\hat{C}_{t+1} - \hat{C}_{t+1}^*)$ and $\hat{r}_{x,t+1}$ conditional on a given value of $\bar{\alpha}$. Combining these expression yields

$$\begin{aligned} E_t \left[(\hat{C}_{t+1} - \hat{C}_{t+1}^*) \hat{r}_{x,t+1} \right] &= \\ (1 - \bar{\beta}\varsigma) E_t \left[\left((\hat{Y}_{t+1} - \hat{Y}_{t+1}^*) + 2\tilde{\alpha}(\hat{Y}_{K,t+1} - \hat{Y}_{K,t+1}^*) \right) (\hat{Y}_{K,t+1} - \hat{Y}_{K,t+1}^*) \right] &+ O(\epsilon^3) \end{aligned} \quad (22)$$

It follows from (14) and (22) that the solution for $\tilde{\alpha}$ is

$$\tilde{\alpha} = -\frac{1}{2} \frac{E_t[(\hat{Y}_{t+1} - \hat{Y}_{t+1}^*)(\hat{Y}_{K,t+1} - \hat{Y}_{K,t+1}^*)]}{E_t[(\hat{Y}_{K,t+1} - \hat{Y}_{K,t+1}^*)^2]} + O(\epsilon) \quad (23)$$

or

$$\tilde{\alpha} = -\frac{\delta + (1 - \delta)\sigma_{KL}/\sigma_K^2}{2} + O(\epsilon) \quad (24)$$

where $\delta = \bar{Y}_K/(\bar{Y}_K + \bar{Y}_L) = \bar{Y}_K/\bar{Y}$. Notice that the residual in this expression is a first-order term. The solution for $\bar{\alpha}$ is given by $\bar{\alpha} = \tilde{\alpha}\bar{\beta}\bar{Y}$.

To provide an economic interpretation of our solution it is helpful to re-express (24) in terms of the proportion of home equity held by home residents. The total value of home equity is $\bar{\beta}\bar{Y}_K$, so the proportion held by home residents is given by

$$\frac{\bar{\beta}\bar{Y}_K + \bar{\alpha}}{\bar{\beta}\bar{Y}_K} = \frac{1 - (1 - \delta)\sigma_{KL}/\delta\sigma_K^2}{2} \quad (25)$$

The most obvious benchmark against which to compare (25) is the case where there is no labour income, i.e. where $\delta = 1$ and $\sigma_L^2 = 0$. In this case there is a known exact solution to the model where home and foreign agents hold a balanced portfolio of home and foreign equities. From (25), our solution yields exactly this outcome. i.e. home agents hold exactly half of home equity (and by implication half of foreign equity). Then, we can check from (21) that the equilibrium portfolio yields full consumption risk sharing. More

generally, in cases where this is labour income risk, i.e. $0 < \delta < 1$ and $\sigma_L^2 > 0$, there is no exact solution to the model, but our zero-order solution provides an approximate solution. Equation (25) shows that if $\sigma_{KL} = 0$ (i.e. labour and capital income are uncorrelated) agents continue to hold a balanced portfolio of home and foreign equity, but equation (21) shows that full consumption risk sharing is not achieved in this case. The equilibrium portfolio deviates from an equal balance of home and foreign equity when there is some correlation between capital and labour income. For instance, when there is a negative correlation, i.e. $\sigma_{KL} < 0$, there will be home bias in equity holdings (i.e. home agents will hold more than half of home equity and foreign agents will hold more than half of foreign equity).¹⁵

Before showing how the solution procedure can be applied to a more general model, we use (24) to address a number of potentially puzzling issues. First, notice that despite the presence of time subscripts, all the terms in (23), including the conditional second-moments, are constant. So our solution for $\bar{\alpha}$ is non-time-varying (which is consistent with our definition of the zero-order component). At first sight it may seem contradictory that portfolio allocations are non-time varying while net wealth, in the form of \hat{W}_t , is time varying. But this is to confuse orders of approximation. $\bar{\alpha}$ is the zero-order component of the portfolio, and should be compared to the zero-order component of net wealth, \bar{W} , which, like $\bar{\alpha}$, is non-time varying. \hat{W}_t on the other hand, is the first-order component of net wealth, and this should be compared to the first-order component of portfolios, $\hat{\alpha}_t$. Both \hat{W}_t and $\hat{\alpha}_t$ are time varying. But by Property 2 it is possible to solve for the dynamics of \hat{W} without having to know the behaviour of $\hat{\alpha}$. As explained above, having solved for $\bar{\alpha}$ it is possible to solve for $\hat{\alpha}_t$ by analysing a third-order approximation of the portfolio problem. This is discussed below in Section 4.

A more general implication of Property 2, which is worth emphasising, is that it is not necessary to solve for the first-order behaviour of $\hat{\alpha}$ in order to solve for the first-order behaviour of other variables in the model. It is therefore possible to analyse the implications of the above model for the first-order behaviour of all variables other than α without having to solve for $\hat{\alpha}$.

The logic presented above implies that the zero-order component of the portfolio, $\bar{\alpha}$, is analogous to the zero-order component of the other variables in the model. At first sight this may also seem contradictory, since the zero-order components of other variables are

¹⁵Conversely, when $\sigma_{KL} > 0$, we have a bias *against* home assets, as in Baxter and Jermann (1997).

derived from the non-stochastic steady state, while our solution for $\bar{\alpha}$ is derived from an explicitly stochastic analysis. The way to resolve this apparent contradiction is to interpret $\bar{\alpha}$ as the equilibrium for portfolio holdings in a world with an arbitrarily small amount of stochastic noise, i.e. the equilibrium in a ‘near-non-stochastic’ world. If one considers the limit of a sequence of stochastic worlds, with diminishing noise, the equilibrium portfolio tends towards a limit which correspond to one of the many portfolio equilibria in the non-stochastic world. This limiting portfolio is a bifurcation point described by Judd and Guu (2001), i.e. it is the point in the set of non-stochastic equilibria which intersects with the sequence of stochastic equilibria. Our solution for $\bar{\alpha}$ corresponds to the portfolio allocation at this bifurcation point.¹⁶

3 Generalising to an n-Asset Model

3.1 A general framework

We now show how the solution method can be extended to a much more general model with many assets. The model is general enough to encompass the range of structures that are used in the recent open economy macro literature. However, only those parts of the model directly necessary for understanding the portfolio selection problem need to be explicitly described. Other components of the model, such as the labour supply decisions of households and the production and pricing decisions of firms, are not directly relevant to the portfolio allocation problem, so these parts of the model are suppressed. The solution approach is consistent with a wide range of specifications for labour supply, pricing and production. Thus, the non-portfolio parts of the model may be characterised by endogenous or exogenous employment, sticky or flexible prices and wages, local cur-

¹⁶Suppose that the covariance matrix of the innovations is given by $\Sigma = \zeta \Sigma_0$ where $\zeta > 0$ is a scalar and Σ_0 is a valid covariance matrix. Notice that the solution for $\tilde{\alpha}$ given in (24) is independent of ζ . So the value of $\tilde{\alpha}$ given by (24) (and therefore the value of $\bar{\alpha}$) is equivalent to the value that would arise in the case of an arbitrarily small, but non-zero, value of ζ - i.e. the value of $\tilde{\alpha}$ that would arise in a world which is arbitrarily close to a non-stochastic world. Furthermore, notice that as ϵ tends to zero (which is equivalent to ζ tending to zero) the size of the residual in (24) tends to zero. So, as the amount of noise tends to zero, the value of $\tilde{\alpha}$ becomes arbitrarily close to the true value of portfolio holdings in the non-approximated model. Our solution for $\bar{\alpha}$ can therefore be thought of as the true portfolio equilibrium in a world which is arbitrarily close to the non-stochastic equilibrium.

rency pricing or producer currency pricing, perfect competition or imperfect competition, etc.¹⁷

We continue to assume that the world consists of two countries. The home country is assumed to produce a good (or a differentiated bundle of goods) with aggregate quantity denoted Y_H (which can be endogenous) and aggregate price P_H . Similarly the foreign country produces quantity Y_F of the foreign good (or bundle of goods) at price P_F^* . In what follows foreign currency prices are denoted with an asterisk.

Agents in the home country now have a utility function of the form

$$U_t = E_t \sum_{\tau=t}^{\infty} \theta_{\tau} [u(C_{\tau}) + v(\cdot)] \quad (26)$$

where C is an aggregator of the home and foreign good consumption, and $u(\cdot)$ is a twice continuously differentiable and concave period utility function. The function $v(\cdot)$ captures those parts of the preference function which are not relevant for the portfolio problem.¹⁸ The aggregate consumer price index for home agents is denoted P . θ_t is again the discount factor, which may be endogenous in the way previously described.

There are n assets and a vector of n returns (for holdings of assets from period $t - 1$ to t) given by $r'_t = [r_{1,t} \ r_{2,t} \ \dots \ r_{n,t}]$. Asset payoffs and asset prices are measured in terms of the aggregate consumption good of the home economy (i.e. in units of C). Returns are defined to be the sum of the payoff of the asset and capital gains relative to the asset price. As before, it is assumed that the vector of available assets is exogenous and predefined.

The budget constraint for home agents is given by

$$\sum \alpha_{i,t} = \sum r_{i,t} \alpha_{i,t-1} + Y_t - C_t \quad (27)$$

where $[\alpha_{1,t-1}, \alpha_{2,t-1} \dots \alpha_{n,t-1}]$ are the holdings of the n assets purchased at the end of period $t - 1$ for holding into period t . Y is the total disposable income of home agents expressed

¹⁷Our method can therefore easily be applied to the sticky-price models commonly employed in the open economy macro literature, e.g. as developed in Obstfeld and Rogoff (1995), Devereux and Engel (2003), Corsetti and Pesenti (2005) and Benigno and Benigno (2003).

¹⁸For these other aspects of the preference function to be irrelevant for portfolio selection it is necessary to assume utility is additively separable in $u(C)$ and $v(\cdot)$. Extensions to cases of non-additive separability (e.g. habit persistence in consumption) are straightforward, as will become more clear below. Using (26) allows us to illustrate the method with minimal notation.

in terms of the home consumption good. Thus, Y may be given by $Y_H P_H / P + T$ where T is a fiscal transfer (or tax if negative).

Using the definition of net wealth (net foreign assets)

$$W_t = \sum_i \alpha_{i,t} \quad (28)$$

the budget constraint may be re-written in the form

$$W_t = \alpha'_{t-1} r_{x,t} + r_{n,t} W_{t-1} + Y_t - C_t \quad (29)$$

where

$$\alpha'_{t-1} = \left[\alpha_{1,t-1} \quad \alpha_{2,t-1} \quad \dots \quad \alpha_{n-1,t-1} \right]$$

and

$$r'_{x,t} = \left[(r_{1,t} - r_{n,t}) \quad (r_{2,t} - r_{n,t}) \quad \dots \quad (r_{n-1,t} - r_{n,t}) \right] = \left[r_{x,1,t} \quad r_{x,2,t} \quad \dots \quad r_{x,n-1,t} \right]$$

Here the n th asset is used as a numeraire and $r_{x,t}$ measures the "excess returns" on the other $n - 1$ assets.

There are $n - 1$ first-order conditions for the choice of the elements of α_t which can be written in the following form

$$\begin{aligned} E_t [u'(C_{t+1})r_{1,t+1}] &= E_t [u'(C_{t+1})r_{n,t+1}] \\ E_t [u'(C_{t+1})r_{2,t+1}] &= E_t [u'(C_{t+1})r_{n,t+1}] \\ &\vdots \\ E_t [u'(C_{t+1})r_{n-1,t+1}] &= E_t [u'(C_{t+1})r_{n,t+1}] \end{aligned} \quad (30)$$

Foreign-country agents face a similar portfolio allocation problem with a budget constraint given by

$$\frac{1}{Q_t} W_t^* = \frac{1}{Q_t} [\alpha'^*_{t-1} r_{x,t} + r_{n,t} W_{t-1}^*] + Y_t^* - C_t^* \quad (31)$$

where $Q_t = P_t^* S_t / P_t$ is the real exchange rate. The real exchange rate enters this budget constraint because Y^* and C^* are measured in terms of the foreign aggregate consumption good (which may differ from the home consumption good) while asset holdings and rates of return are defined in terms of the home consumption good.

Foreign agents are assumed to have preferences similar to (26) so the first-order conditions for foreign-country agents' choice of α_t^* are

$$\begin{aligned}
E_t [Q_{t+1}^{-1} u'(C_{t+1}^*) r_{1,t+1}] &= E_t [Q_{t+1}^{-1} u'(C_{t+1}^*) r_{n,t+1}] \\
E_t [Q_{t+1}^{-1} u'(C_{t+1}^*) r_{2,t+1}] &= E_t [Q_{t+1}^{-1} u'(C_{t+1}^*) r_{n,t+1}] \\
&\vdots \\
E_t [Q_{t+1}^{-1} u'(C_{t+1}^*) r_{n-1,t+1}] &= E_t [Q_{t+1}^{-1} u'(C_{t+1}^*) r_{n,t+1}]
\end{aligned} \tag{32}$$

The two sets of first-order conditions, (30) and (32), and the market clearing condition $\alpha_t = -\alpha_t^*$, provide $3(n-1)$ equations which determine the elements of α_t , α_t^* and $E_t[r_{x,t+1}]$.

Clearly, in any particular general equilibrium model, there will be a set of first-order conditions relating to intertemporal choice of consumption, labour supply, etc., for the home and foreign consumers, and a set of first-order conditions for price setting and factor demands for home and foreign producers. Taken as a whole, and combined with an appropriate set of equilibrium conditions for goods and factor markets, this full set of equations will define the general equilibrium of the model. As already explained, the details of these non-portfolio parts of the model are not necessary for the exposition of the solution method, so they are not shown explicitly. In what follows these omitted equations are simply referred to as the "non-portfolio equations" of the model.

The non-portfolio equations of the model will normally include some exogenous forcing variables. In the typical macroeconomic model these take the form of AR1 processes which are driven by zero-mean i.i.d. innovations. We assume that there are m such disturbances, summarised in a vector, x , which is determined by the following process

$$x_t = Nx_{t-1} + \varepsilon_t \tag{33}$$

where ε is a vector of zero-mean i.i.d. innovations with covariance matrix Σ . It is assumed that the innovations are symmetrically distributed over the interval $[-\epsilon, \epsilon]$.¹⁹

3.2 A general expression for the zero-order portfolio

Again we use the non-stochastic steady state of the model as the approximation point for non-portfolio variables. And again, for convenience of exposition, we assume that

¹⁹Clearly there must be a link between Σ and ϵ . The value of ϵ places an upper bound on the elements of Σ . So an experiment which involves considering the effects of reducing ϵ implicitly involves reducing the magnitude of the elements of Σ .

the home and foreign countries are identical so that $\bar{W} = 0$, $\bar{Y} = \bar{Y}^* = \bar{C} = \bar{C}^*$ and $\bar{r}_1 = \bar{r}_2 \dots = \bar{r}_n = 1/\bar{\beta}$ where $\bar{\beta} = \beta(\bar{C}) = \omega \bar{C}^{-\eta}$. Note again that this implies $\bar{r}_x = 0$.

As before we proceed by taking second-order approximations of the home and foreign portfolio first-order conditions. For the home country this yields

$$\begin{aligned} E_t \left[(\hat{r}_{1,t+1} - \hat{r}_{n,t+1}) + \frac{1}{2}(\hat{r}_{1,t+1}^2 - \hat{r}_{n,t+1}^2) - \rho \hat{C}_{t+1}(\hat{r}_{1,t+1} - \hat{r}_{n,t+1}) \right] &= O(\epsilon^3) \\ E_t \left[(\hat{r}_{2,t+1} - \hat{r}_{n,t+1}) + \frac{1}{2}(\hat{r}_{2,t+1}^2 - \hat{r}_{n,t+1}^2) - \rho \hat{C}_{t+1}(\hat{r}_{2,t+1} - \hat{r}_{n,t+1}) \right] &= O(\epsilon^3) \\ &\vdots \\ E_t \left[(\hat{r}_{n-1,t+1} - \hat{r}_{n,t+1}) + \frac{1}{2}(\hat{r}_{n-1,t+1}^2 - \hat{r}_{n,t+1}^2) - \rho \hat{C}_{t+1}(\hat{r}_{n-1,t+1} - \hat{r}_{n,t+1}) \right] &= O(\epsilon^3) \end{aligned} \quad (34)$$

where $\rho \equiv -u''(\bar{C})\bar{C}/u'(\bar{C})$ (i.e. the coefficient of relative risk aversion). Re-writing (34) in vector form yields

$$E_t \left[\hat{r}_{x,t+1} + \frac{1}{2}\hat{r}_{x,t+1}^2 - \rho \hat{C}_{t+1}\hat{r}_{x,t+1} \right] = O(\epsilon^3) \quad (35)$$

where

$$\hat{r}_{x,t+1}^{2'} \equiv \begin{bmatrix} \hat{r}_{1,t+1}^2 - \hat{r}_{n,t+1}^2 & \hat{r}_{2,t+1}^2 - \hat{r}_{n,t+1}^2 & \dots & \hat{r}_{n-1,t+1}^2 - \hat{r}_{n,t+1}^2 \end{bmatrix}$$

Applying a similar procedure to the foreign first-order conditions yields

$$E_t \left[\hat{r}_{x,t+1} + \frac{1}{2}\hat{r}_{x,t+1}^2 - \rho \hat{C}_{t+1}^* \hat{r}_{x,t+1} - \hat{Q}_{t+1} \hat{r}_{x,t+1} \right] = 0 + O(\epsilon^3) \quad (36)$$

The home and foreign optimality conditions, (35) and (36), can be combined to show that, in equilibrium, the following conditions must hold

$$E_t \left[(\hat{C}_{t+1} - \hat{C}_{t+1}^* - \hat{Q}_{t+1}/\rho) \hat{r}_{x,t+1} \right] = 0 + O(\epsilon^3) \quad (37)$$

and

$$E[\hat{r}_x] = -\frac{1}{2}E[\hat{r}_x^2] + \rho \frac{1}{2}E_t \left[(\hat{C}_{t+1} + \hat{C}_{t+1}^* + \hat{Q}_{t+1}/\rho) \hat{r}_{x,t+1} \right] + O(\epsilon^3) \quad (38)$$

These equations are equivalent to (14) and (15) in the example from before. There we showed that equation (14) provided a sufficient condition to tie down the zero-order component of the portfolio allocation. We now show that equation (37) provides a sufficient condition to tie down the zero-order component of the portfolio in the general model.

Properties 1 and 2 played a central role in deriving the solution to the example above. These properties also hold for the general model, and remain central in the derivation of the solution. Clearly, Property 1 applies in the general model. The left hand side

of equation (37) consists entirely of products of variables and can thus be evaluated to second-order accuracy using first-order accurate expressions for $\hat{C} - \hat{C}^* - \hat{Q}/\rho$ and \hat{r}_x . Likewise, Property 2 holds in the general model. Again, the portfolio allocation enters only via the excess portfolio return, $\alpha' r_x$. And, just as in the simple model, $\bar{r}_x = 0$, so the first-order approximation of the excess portfolio return is $\bar{\alpha} \hat{r}_x$. Thus only the zero-order component of α enters the first-order approximated model.

The general outline of the solution strategy is the same as that described for the simple model. First we solve for the first-order accurate behaviour of $\hat{C} - \hat{C}^* - \hat{Q}/\rho$ and \hat{r}_x in terms of $\bar{\alpha}$. Then we solve for the $\bar{\alpha}$ that ensures (37) is satisfied.

But now things are somewhat more complicated because the behaviour of $\hat{C} - \hat{C}^* - \hat{Q}/\rho$ and \hat{r}_x is determined by a potentially complex set of first-order dynamic equations. Indeed, at first sight, the general model may seem too complex to be solved explicitly, and it may appear that a numerical approach is necessary to solve for the $\bar{\alpha}$. We show, however, that it is possible to derive a closed-form analytical solution for $\bar{\alpha}$ in the general model. In fact, we derive a formula for $\bar{\alpha}$ which is applicable to any model with the same general features as the one described above.

To see why it is possible to obtain a closed-form solution, it is necessary to state a further important property of the approximated model.

Property 3 To a first-order approximation, the portfolio excess return, $\bar{\alpha} \hat{r}_{x,t+1}$, is a zero mean i.i.d. random variable. This follows from equation (38), which shows that the equilibrium expected excess return contains only second-order terms. So, up to a first order approximation, $E_{t-1} [\hat{r}_{x,t+1}]$ is zero, i.e. there is no predictable element in $\hat{r}_{x,t+1}$. The first-order approximation of the portfolio excess return, $\bar{\alpha} \hat{r}_{x,t+1}$, is therefore a linear function of the i.i.d. innovations, ε_{t+1} , and must therefore itself be an i.i.d. random variable.

Property 3 greatly simplifies the solution process because it implies that $\bar{\alpha}$ affects the first-order behaviour of the economy in a very simple way. In particular, $\bar{\alpha}$ *does not affect the eigenvalues* of the first-order system. Thus, in any given period (e.g. period t) the dynamic properties of the expected path of the economy from period $t + 1$ onwards are independent of $\bar{\alpha}$. The period t behaviour of the economy is affected by $\bar{\alpha}$ only through its effect on the size and sign of the i.i.d. innovations to wealth that arise from the portfolio excess return, $\bar{\alpha} \hat{r}_{x,t}$.

The only potential complication is that $\hat{r}_{x,t}$ may *itself* depend on period t innovations to wealth (and therefore $\bar{\alpha}$). This complication is, however, easily overcome by breaking the solution process for $\hat{C} - \hat{C}^* - \hat{Q}/\rho$ and \hat{r}_x into two stages. In the first stage we treat the portfolio excess return, $\bar{\alpha}\hat{r}_x$, as an exogenous i.i.d. random variable, and solve the first-order model to yield an expression for \hat{r}_x in terms of exogenous innovations to wealth and the other exogenous innovations to the model. In the second stage we use this expression to solve out for the behaviour of $\hat{C} - \hat{C}^* - \hat{Q}/\rho$ and \hat{r}_x in terms of ε (i.e. the true exogenous innovations of the model). This provides the expressions required to evaluate (37) and thus to solve for $\bar{\alpha}$.²⁰

A full description of the derivation of the solution is given in the Appendix. Here we simply state the resulting expression for equilibrium portfolios. To do this, first note that a first-order approximation of the home budget constraint is given by

$$\hat{W}_t = \frac{1}{\beta}\hat{W}_{t-1} + \hat{Y}_t - \hat{C}_t + \tilde{\alpha}'\hat{r}_{xt} + O(\epsilon^2) \quad (39)$$

where $\hat{W}_t = (W_t - \bar{W})/\bar{Y}$ and $\tilde{\alpha} = \bar{\alpha}/(\beta\bar{Y})$. The budget constraint can now be written in the form

$$\hat{W}_t = \frac{1}{\beta}\hat{W}_{t-1} + \hat{Y}_t - \hat{C}_t + \xi_t + O(\epsilon^2) \quad (40)$$

where $\tilde{\alpha}'\hat{r}_{xt}$ has been replaced by ξ_t . We temporarily treat ξ as an exogenous i.i.d. variable²¹.

Equation (40) can be combined with first-order approximations of all the other non-portfolio equations of the model and solved using any standard linear rational expectations solution procedure to yield a state-space solution for all endogenous variables in terms of the state variables of the model and ε_{t+1} and ξ_{t+1} . It is then possible to write the following

²⁰Notice from equation (15) that, in the example, r_x does not depend on $\bar{\alpha}$, so this two-step process was not necessary.

²¹We note that this step is necessary so as to deliver the analytical expression for the equilibrium portfolio (43) below. If we wished to simply obtain a numerical solution for $\tilde{\alpha}$, we could substitute the numerical solution to the linearized model into (35) for an arbitrary initial assumption for $\tilde{\alpha}$, and then iterate to find the equilibrium $\tilde{\alpha}$ which satisfies (35) as an equality. See Tille and Van Wincoop (2007). The appeal of using the approach is that one may use formula (43) as an addition on to any model solved (analytically or numerically) by first order approximation. By using (43), the researcher avoids the need to do any higher order approximation - thus, the procedure is quite useful also in numerical analysis of DSGE economies.

expressions for the first-order accurate behaviour of \hat{r}_{xt+1} , and $\hat{C}_{t+1} - \hat{C}_{t+1}^* - \hat{Q}_{t+1}/\rho$

$$\hat{r}_{xt+1} = R_1 \xi_{t+1} + R_2 \varepsilon_{t+1} + O(\epsilon^2) \quad (41)$$

$$\hat{C}_{t+1} - \hat{C}_{t+1}^* - \hat{Q}_{t+1}/\rho = D_1 \xi_{t+1} + D_2 \varepsilon_{t+1} + D_3 \begin{bmatrix} x_t \\ s_{t+1} \end{bmatrix} + O(\epsilon^2) \quad (42)$$

where R_1 , R_2 , D_1 , D_2 and D_3 are coefficient matrices extracted from the first-order state-space solution of the model and x and s are respectively vectors of the exogenous and endogenous state variables of the model.²²

The Appendix shows that a solution for $\tilde{\alpha}$ is

$$\tilde{\alpha} = [R_2 \Sigma D_2' R_1' - D_1 R_2 \Sigma R_2']^{-1} R_2 \Sigma D_2' + O(\epsilon) \quad (43)$$

where Σ is the covariance matrix of ε . Notice that the residual in this expression is a first-order term. The solution for $\bar{\alpha}$ is simply given by $\bar{\alpha} = \tilde{\alpha} \bar{\beta} \bar{Y}$.²³

It should be emphasised that applying the procedure to any given model requires *only* that the user apply (43), which needs only information from the first-order approximation of the model in order to construct the D and R matrices (which will obviously depend on the particulars of the model under study). So long as the model satisfies the general properties described above, the other details of the model, such as production, labour supply, and price setting can be varied without affecting the implementation. The derivations used to obtain (43) (as presented in the Appendix) do not need to be repeated. In summary, the solution for equilibrium $\tilde{\alpha}$ has three steps:

1. Solve the non-portfolio equations of the model with the budget constraint in the form of (40) to yield a state space solution.

²²Notice that, as follows from Property 3, \hat{r}_{xt+1} does not depend on the values of the state variables.

²³Note that a much simpler version of (43) arises in the case where there are sufficient assets to replicate the complete markets equilibrium. In general in this case there must be at most $n - 1$ independent shocks and n independent assets. $R_2 \Sigma$ is therefore a square invertible matrix and the expression for $\tilde{\alpha}$ can be simplified to $\tilde{\alpha} = [D_2' R_1' - D_1 R_2']^{-1} D_2' + O(\epsilon)$. In this case, with complete markets, optimal portfolios do not depend on the variance-covariance matrix of shocks, as shown, for instance, in Coeurdacier (2008) and Kollmann (2006). Note also that (43) is not necessarily the only solution which satisfies (37) when there are more than two assets, i.e. $n > 2$. However, as the Appendix shows, the solution in (43) has an obvious claim to be the first focus of analysis since this solution always exists and it corresponds to the unique solution in the two-asset case. We leave the further analysis of multiple equilibria to future research.

2. Extract the appropriate rows from this solution to form R_1 , D_1 , R_2 and D_2 .
3. Calculate $\tilde{\alpha}$ using (43).

3.3 Extending the example model

We illustrate the use of the general solution procedure by applying it to a modified and extended version of our earlier example model. In this modified model we introduce more general shock processes and allow for a wider range of assets to be traded between home and foreign households. Assume that total endowments are given by AR1 processes of the form

$$\hat{Y}_t = \psi \hat{Y}_{t-1} + \varepsilon_{Y,t}, \quad \hat{Y}_t^* = \psi \hat{Y}_{t-1}^* + \varepsilon_{Y^*,t} \quad (44)$$

where $0 \leq \psi < 1$ and a hat indicates log deviations from the non-stochastic steady state. We drop the distinction between capital and labour income endowments but extend the model to allow for AR1 fiscal policy shocks of the form

$$\hat{G}_t = \psi \hat{G}_{t-1} + \varepsilon_{G,t}, \quad \hat{G}_t^* = \psi \hat{G}_{t-1}^* + \varepsilon_{G^*,t} \quad (45)$$

where G and G^* are home and foreign government purchases of the consumption good. In (44) and (45) $\varepsilon_{Y,t}$, $\varepsilon_{G,t}$, $\varepsilon_{Y^*,t}^*$ and $\varepsilon_{G^*,t}^*$ are zero-mean i.i.d. shocks which are symmetrically distributed over the interval $[-\epsilon, \epsilon]$ with $Var[\varepsilon_Y] = Var[\varepsilon_Y^*] = \sigma_Y^2$, $Var[\varepsilon_G] = Var[\varepsilon_G^*] = \sigma_G^2$. We assume $Cov[\varepsilon_Y, \varepsilon_Y^*] = Cov[\varepsilon_G, \varepsilon_G^*] = 0$, $Corr[\varepsilon_Y, \varepsilon_G] = Corr[\varepsilon_Y^*, \varepsilon_G^*] = \nu$. We assume $\bar{G} = \bar{G}^*$ and $\bar{Y} = \bar{Y}^*$ and define $g = \bar{G}/\bar{Y} = \bar{G}^*/\bar{Y}^*$ to be the steady state share of government spending in output.

The resource constraint is now

$$C_t + C_t^* + G_t + G_t^* = Y_t + Y_t^*$$

The utility function for home households takes the form of (1) with $U(C) = \log C$. Government spending does not yield utility. Government spending is assumed to be financed via lump-sum taxes.

In order to tie down nominal goods prices we introduce quantity theory equations of the form

$$M_t/P_t = C_t + G_t \quad M_t^*/P_t^* = C_t^* + G_t^*$$

where M and M^* are home and foreign money supplies. These equations may be interpreted as simplified cash-in-advance constraints. We assume money supplies follow AR1 process of the form

$$\hat{M}_t = \psi \hat{M}_{t-1} + \varepsilon_{M,t}, \quad \hat{M}_t^* = \psi \hat{M}_{t-1}^* + \varepsilon_{M^*,t}$$

where $\varepsilon_{M,t}$ and $\varepsilon_{M^*,t}$ are zero-mean i.i.d. shocks which are symmetrically distributed over the interval $[-\epsilon, \epsilon]$ with $Var[\varepsilon_M] = Var[\varepsilon_M^*] = \sigma_M^2$. Monetary shocks are assumed to be uncorrelated with endowment and government spending shocks.

Consider two alternative menus of assets. In the first case assume that only equity shares in the two endowment streams, Y and Y^* can be traded between the two economies. We refer to this as the EQ economy. In the second case we assume that both equity shares and nominal bonds denominated in the currencies of the two countries can be traded. We refer to this as the EB economy.

The budget constraint for home agents is given by

$$\sum \alpha_{i,t} = \sum r_{i,t} \alpha_{i,t-1} + Y_t - C_t - G_t \quad (46)$$

where $[\alpha_{1,t-1}, \alpha_{2,t-1} \dots \alpha_{n,t-1}]$ are asset holdings and n is the number of assets. In the EQ economy $n = 2$ and in the EB economy $n = 4$.

Equities issued in period t are assumed to represent a claim on endowments in period $t + 1$. The real rates of return on equities are therefore given by

$$r_{E,t+1} = Y_{t+1}/Z_{E,t} \quad r_{E^*,t+1} = Y_{t+1}^*/Z_{E^*,t}$$

where $Z_{E,t}$ and $Z_{E^*,t}$ are the prices of home and foreign equities in period t . Nominal bonds issued in period t are claims on a unit of currency in period $t + 1$. The real rates of return on nominal bonds are therefore given by

$$r_{B,t+1} = 1/(P_{t+1}Z_{B,t}) \quad r_{B^*,t+1} = 1/(P_{t+1}^*Z_{B^*,t})$$

where $Z_{B,t}$ and $Z_{B^*,t}$ are the prices of home and foreign nominal bonds in period t .

The first-order conditions for home and foreign consumption imply

$$C_t^{\eta-1} = \omega E_t [C_{t+1}^{-1} r_{E^*,t+1}], \quad C_t^{*\eta-1} = \omega E_t [C_{t+1}^{*-1} r_{E^*,t+1}] \quad (47)$$

The first-order conditions for portfolio allocation in the EQ economy are

$$E_t [C_{t+1}^{-1} r_{E,t+1}] = E_t [C_{t+1}^{-1} r_{E^*,t+1}], \quad E_t [C_{t+1}^{*-1} r_{E,t+1}] = E_t [C_{t+1}^{*-1} r_{E^*,t+1}] \quad (48)$$

	EQ-Economy	EB-Economy
Home Equities	$-\frac{1-gv\sigma_G/\sigma_Y}{2(1-\beta\psi)}$	$-\frac{(1-\bar{\beta}\psi)(1-gv\sigma_G/\sigma_Y)\sigma_M^2+g^2\bar{\beta}(1-v^2)(1-\psi-\eta)\sigma_G^2}{2(1-\bar{\beta}\psi)[(1-\beta\psi)\sigma_M^2+g^2\bar{\beta}(1-v^2)(1-\psi-\eta)\sigma_G^2]}$
Home Nominal Bonds	-	$\frac{\bar{\beta}(1-v^2)(1-\psi-\eta)g^2\sigma_G^2}{2(1-\bar{\beta}\psi)[(1-\beta\psi)\sigma_M^2+g^2\bar{\beta}(1-v^2)(1-\psi-\eta)\sigma_G^2]}$

(Note: $\bar{\beta} = \omega\bar{C}^{-\eta} = \omega\bar{C}^{*-\eta}$, by assumption $0 < \bar{\beta} < 1$.)

Table 1: Steady state asset holdings of home households

while in the EB economy the corresponding conditions are

$$\begin{aligned}
E_t [C_{t+1}^{-1}r_{E,t+1}] &= E_t [C_{t+1}^{-1}r_{E^*,t+1}] & E_t [C_{t+1}^{*-1}r_{E,t+1}] &= E_t [C_{t+1}^{*-1}r_{E^*,t+1}] \\
E_t [C_{t+1}^{-1}r_{B,t+1}] &= E_t [C_{t+1}^{-1}r_{E^*,t+1}] & E_t [C_{t+1}^{*-1}r_{B,t+1}] &= E_t [C_{t+1}^{*-1}r_{E^*,t+1}] \\
E_t [C_{t+1}^{-1}r_{B^*,t+1}] &= E_t [C_{t+1}^{-1}r_{E^*,t+1}] & E_t [C_{t+1}^{*-1}r_{B^*,t+1}] &= E_t [C_{t+1}^{*-1}r_{E^*,t+1}]
\end{aligned} \tag{49}$$

The above model can be solved to yield first-order accurate solutions for $(\hat{C}_{t+1} - \hat{C}_{t+1}^*)$ and $\hat{r}_{x,t+1}$ conditional on ξ_t . These solutions yield expressions for the coefficient matrices D_1 , D_2 , R_1 and R_2 which, together with the expression for Σ from this example, can be substituted into equation (43) to yield expressions for steady state asset holdings. The solutions for asset holdings are reported in Table 1. The details of the first-order approximated model and the resulting expression for D_1 , D_2 , R_1 and R_2 are reported in the Appendix.

3.3.1 The EQ economy

The first column in Table 1 reports the value of $\tilde{\alpha}_E$, the holdings by home households of home country equities in the EQ economy. In a symmetric equilibrium home holdings of foreign equities, $\tilde{\alpha}_{E^*}$, are simply given by $\tilde{\alpha}_{E^*} = -\tilde{\alpha}_E$.

Notice the similarity between the expression for $\tilde{\alpha}_E$ and the solution for equity holdings in our previous example model (see equation (24)). Clearly, fiscal policy shocks in this model play a very similar role to labour income shocks in the previous model, so the implications for equity holdings are very similar. The main difference between the two cases is the term $(1 - \bar{\beta}\psi)$ in the denominator of the expression in Table 1. This reflects

the fact that shocks in the modified model are persistent (with AR1 coefficient ψ) so holdings of one-period equity must be larger than in the i.i.d. case to offset the impact of current shocks on the discounted value of the future endowment streams.

Notice from Table 1 that monetary shocks are irrelevant for equity holdings in the EQ economy. Real equity returns, and thus the hedging properties of equities, are independent of nominal magnitudes, so the optimal equity portfolio is independent of monetary shocks.

It is possible to use the solutions for asset holdings to judge the implications of portfolio holdings for consumption risk sharing. This can be done by considering the behaviour of consumption differences, $C - C^*$. Table 2 reports the impact effects of shocks on $C - C^*$ for the EQ and EB economics.²⁴ It is immediately clear that, in general, risk sharing is not perfect in the EQ economy. Both government spending and endowment shocks affect $C - C^*$. However, a number of special cases can be identified where perfect risk sharing is achieved. For instance, if there are no government spending shocks (either $g = 0$ or $\sigma_G = 0$) then the expressions in Table 2 show that the $C - C^*$ will be independent of all shocks. Alternatively, if government spending and endowment shocks are perfectly correlated ($v = 1$, or $v = -1$) then again the expressions in Table 2 show that $C - C^*$ will be independent of all shocks (to see this, just add the first and second cells of the EQ-Economy column in Table 1).

In all the special cases just listed, trade in equities is sufficient to eliminate consumption risk. Note however, that this result only applies to the symmetric version of the model considered here, and even in the symmetric case it only applies to consumption risk measured at the first-order level. Full replication of the complete markets equilibrium (in the $g = 0$, $\sigma_G = 0$, $v = 1$, or $v = -1$ cases) in an asymmetric version of the model, or at higher orders of approximation, can only be achieved with the addition of a third asset (which may be a non-contingent bond). This ensures that there are at least as many independent excess returns as independent shocks.

3.3.2 The EB economy

Now consider the case where both equities and nominal bonds can be traded. Table 1 reports the equilibrium home-country holdings of home equity, $\tilde{\alpha}_E$, and home nominal bonds, $\tilde{\alpha}_B$. Symmetry implies $\tilde{\alpha}_{E^*} = -\tilde{\alpha}_E$ and $\tilde{\alpha}_{B^*} = -\tilde{\alpha}_B$.

²⁴The model is entirely symmetric, so it is sufficient to report the impact effect of shock differentials, i.e. $(\varepsilon_{Y,t} - \varepsilon_{Y^*,t})$, $(\varepsilon_{G,t} - \varepsilon_{G^*,t})$ and $(\varepsilon_{M,t} - \varepsilon_{M^*,t})$.

Shocks	EQ-Economy	EB-Economy
Endowment ($\varepsilon_{Y,t} - \varepsilon_{Y^*,t}$)	$\frac{[1-\bar{\beta}(1-\eta)]g\nu\sigma_G/\sigma_Y}{2(1-\bar{\beta}\psi)(1-g)}$	$\frac{[1-\bar{\beta}(1-\eta)](1-\bar{\beta}\psi)g\nu\sigma_M^2\sigma_G/\sigma_Y}{(1-g)[(1-\bar{\beta}\psi)^2\sigma_M^2+g^2\bar{\beta}^2(1-\nu^2)(1-\psi-\eta)^2\sigma_G^2]}$
Gov spending ($\varepsilon_{G,t} - \varepsilon_{G^*,t}$)	$-\frac{[1-\bar{\beta}(1-\eta)]g}{2(1-\bar{\beta}\psi)(1-g)}$	$-\frac{[1-\bar{\beta}(1-\eta)](1-\bar{\beta}\psi)\sigma_M^2}{(1-g)[(1-\bar{\beta}\psi)^2\sigma_M^2+g^2\bar{\beta}^2(1-\nu^2)(1-\psi-\eta)^2\sigma_G^2]}$
Money supply ($\varepsilon_{M,t} - \varepsilon_{M^*,t}$)	0	$-\frac{[1-\bar{\beta}(1-\eta)]\bar{\beta}(1-\nu^2)(1-\psi-\eta)g^2\sigma_G^2}{(1-g)[(1-\bar{\beta}\psi)^2\sigma_M^2+g^2\bar{\beta}^2(1-\nu^2)(1-\psi-\eta)^2\sigma_G^2]}$

Table 2: Impact effects of shocks on C-C*

The first point to note from these expressions is that monetary shocks now have an impact on equilibrium portfolio holdings. This is because monetary shocks have an impact on the real returns on nominal bonds and thus affect their hedging properties.

The implications of monetary shocks are illustrated more clearly in Table 2, which shows how all shocks affect $C - C^*$ in the EB economy. It is immediately obvious that, in general, risk sharing is not perfect in the EB economy. This is despite the addition of two new assets compared to the EQ economy. Nominal bonds provide an extra means for hedging government spending and endowment shocks. But their efficacy as hedging instruments is partly undermined by monetary shocks.

Table 2 shows, however, that there are a number of special cases where perfect risk sharing can be achieved in the EB economy. The first case obviously arises when there are no monetary shocks ($\sigma_M = 0$). In this case nominal bonds offer a good hedge for government spending shocks, while equities provide a good hedge for endowment shocks. Other special cases are similar to those identified for the EQ economy (i.e. the absence of government spending shocks or perfect correlation between government spending and endowment shocks). These special cases arise even when there are monetary shocks.

Note again that, in all these special cases, perfect risk sharing is only achieved in the symmetric version of the model considered here, and even in the symmetric case it is only achieved when consumption risk measured at the first-order level. Full replication of the

complete markets equilibrium (in the $\sigma_M = 0$, $g = 0$, $\sigma_G = 0$, $v = 1$, or $v = -1$ cases) requires the addition of a fifth asset (which again may be a non-contingent bond). This ensures that there are at least as many independent excess returns as independent shocks.

4 Solving for the First-Order Portfolio

The analysis presented above shows how a second-order approximation of the portfolio optimality condition provides a sufficient condition to tie down the zero-order component of the portfolio, $\bar{\alpha}$. We have shown that, from Property 2, the solution for $\bar{\alpha}$ is all that is required to derive first-order accurate solutions for all other variables of a model. Thus, if the objective is to analyse the impulse responses of variables such as output or consumption (or indeed any variable other than α), or if one is primarily interested in the business cycle properties of a model, then there is no need to go any further than obtaining a solution for $\bar{\alpha}$. It is likely however that the first-order dynamic behaviour of α will also prove to be an interesting topic of research in its own right. For instance, we might like to analyze the separate movement in different types of assets and gross portfolio positions following macro shocks. In addition, to conduct welfare analysis, we would generally need to evaluate the model up to a second-order approximation, which would require incorporating the dynamic properties of α . We therefore now briefly outline how the solution approach can be extended to solve for the first-order component of α .

The general principles that underlie an extension of the procedure are simply stated. In line with Samuelson (1970) it is necessary to approximate the portfolio problem up to the third order. In the context of the simple model this involves a third-order approximation of the portfolio optimality condition, as follows

$$E_t \begin{bmatrix} -\rho(\hat{C}_{t+1} - \hat{C}_{t+1}^*)\hat{r}_{x,t+1} + \frac{\rho^2}{2}(\hat{C}_{t+1}^2 - \hat{C}_{t+1}^{*2})\hat{r}_{x,t+1} \\ -\frac{\rho}{2}(\hat{C}_{t+1} - \hat{C}_{t+1}^*)(\hat{r}_{1,t+1}^2 - \hat{r}_{2,t+1}^2) \end{bmatrix} = 0 + O(\epsilon^4) \quad (50)$$

It is now possible to show, using modified versions of Properties 1 and 2, that (50) provides a sufficient condition to tie down the first-order component of α .

A modified version of Property 1 states that the expression on the left hand side of (50) can be evaluated up to third-order accuracy using first and second-order accurate expressions for $\hat{C} - \hat{C}^*$, \hat{r}_1 , \hat{r}_1 and \hat{r}_x . Thus it is, at most, necessary to evaluate these variables up to second order.

A modified version of Property 2 states that only the zero and first-order components of α enter a second-order approximation of the model. This is simple to show by taking a second-order approximation of the portfolio excess return, $\alpha_{1,t-1}r_{x,t}$, as follows

$$\bar{\alpha}\hat{r}_{x,t} + \bar{r}_x\hat{\alpha}_{1,t-1} + \frac{1}{2}\bar{\alpha}(\hat{r}_{1,t}^2 - \hat{r}_{2,t}^2) + \hat{\alpha}_{t-1}\hat{r}_{x,t} + O(\epsilon^3) \quad (51)$$

where $\hat{\alpha}_t = (\alpha_t - \bar{\alpha})$. As before $\bar{r}_x = 0$, so only the zero and first-order components of α are necessary to evaluate (51).

The general solution strategy can now be described. First, postulate that, up to first-order accuracy, $\hat{\alpha}_t$ is a linear function of the state variables of the model. Thus postulate $\hat{\alpha}_{t-1} = \gamma'z_t$ where z is the vector of state variables and γ is a vector of coefficients which are to be determined. The modified version of Property 2 shows that it possible to evaluate the first and second-order behaviour of $\hat{C} - \hat{C}^*$, \hat{r}_1 , \hat{r}_2 and \hat{r}_x conditional on a value for γ , and hence, from the modified version of Property 1, it is possible to evaluate the left hand side of (50) conditional on γ . The equilibrium γ is the one which ensures (50) is satisfied.²⁵

The details of the solution procedure for γ are presented in Devereux and Sutherland (2007), where we derive a closed-form solution which is applicable to a wide class of models.

5 Conclusion

Portfolio structure has become a central issue in open economy macroeconomics and international finance. Despite this, existing models and solution methods are not well-suited to analyzing portfolio choice in policy-relevant general equilibrium environments. This paper develops a simple approximation method for portfolio choice problems in dynamic general equilibrium models. Our approach is extremely easy to implement and can be used in any of the existing models that rely on first-order approximation methods. If the researcher is primarily interested in the implications of portfolio choice for the first-order properties of macro variables (such as GDP, consumption, or the real exchange rate), either through impulse response analysis or by computing second moments so as to describe

²⁵Note that the conditional third moments in (50) are time varying and depend on state variables. The fact that (50) must be satisfied for all values of state variables and in all time periods provides just enough equations to tie down all the elements of γ .

volatility and comovement, then the solution method outlined here allows a full answer to these questions. Since the overwhelming majority of the research in international finance and macroeconomics is carried out at the level of first-order approximation, the method is widely applicable. It can be used to study many empirical questions in the interface between international finance and macroeconomics. Moreover, the method allows us to study the macroeconomic determinants of optimal steady-state portfolio holdings for any asset or combination of assets, whether markets are complete or incomplete.

We note that, although the motivation and applications discussed in the paper pertain to open economy macro models, there is nothing inherent in the solution approach which restricts the application to open economies. The method applies to portfolio choice in any heterogeneous agent models dynamic general equilibrium models. This is true for both the zero-order portfolio solution, as well as the first-order solution for portfolio dynamics. Taken in combination, the methods described here offer a tractable approach to incorporating financial structure into a wide class of stochastic dynamic general equilibrium models.

Appendix 1: Solving the n-asset model

In this Appendix, we show how to derive (43) in the text. The first-order approximation of the model can be summarised in a matrix equation of the form

$$A_1 \begin{bmatrix} s_{t+1} \\ E_t [c_{t+1}] \end{bmatrix} = A_2 \begin{bmatrix} s_t \\ c_t \end{bmatrix} + A_3 x_t + B \xi_t + O(\epsilon^2) \quad (52)$$

where s is the vector of predetermined variables, c is the vector of jump variables, x is defined in (33) and B is a column vector with unity in the row corresponding to (40) and zero in all other rows. The state-space solution to (52) can be derived using any standard solution method for linear rational expectations models. It can be written as follows

$$\begin{aligned} s_{t+1} &= F_1 x_t + F_2 s_t + F_3 \xi_t + O(\epsilon^2) \\ c_t &= P_1 x_t + P_2 s_t + P_3 \xi_t + O(\epsilon^2) \end{aligned} \quad (53)$$

By extracting the appropriate rows from (53) it is possible to write the following expression for the first-order accurate relationship between excess returns, \hat{r}_{xt+1} , and ε_{t+1} and ξ_{t+1}

$$\hat{r}_{xt+1} = R_1 \xi_{t+1} + R_2 \varepsilon_{t+1} + O(\epsilon^2) \quad (54)$$

where the matrices R_1 and R_2 are formed from the appropriate rows of (53). Equation (54) shows how first-order accurate realised excess returns depend on exogenous i.i.d. shocks, ε_{t+1} and ξ_{t+1} . In particular, it shows how \hat{r}_{xt+1} depends on i.i.d. shocks to wealth. This completes the first stage in solving for the first-order behaviour of $\hat{C} - \hat{C}^* - \hat{Q}/\rho$ and \hat{r}_x .

Now we impose the condition that, rather than being exogenous, the innovations to wealth, ξ_{t+1} , are endogenously determined by excess portfolio returns via the relationship

$$\xi_{t+1} = \tilde{\alpha}' \hat{r}_{xt+1} \quad (55)$$

where the vector of portfolio allocations, $\tilde{\alpha}$, is yet to be determined. This equation, together with (54), can be solved to yield expressions for ξ_{t+1} and \hat{r}_{xt+1} in terms of the exogenous innovations as follows

$$\xi_{t+1} = \tilde{H} \varepsilon_{t+1} \quad (56)$$

$$\hat{r}_{xt+1} = \tilde{R} \varepsilon_{t+1} + O(\epsilon^2) \quad (57)$$

where

$$\tilde{H} = \frac{\tilde{\alpha}' R_2}{1 - \tilde{\alpha}' R_1}, \quad \tilde{R} = R_1 \tilde{H} + R_2 \quad (58)$$

Equation (57), which shows how realised excess returns depend on the exogenous i.i.d. innovations of the model, provides one of the relationships necessary to evaluate the left-hand side of (37). The other relationship required is the link between $\hat{C}_{t+1} - \hat{C}_{t+1}^* - \hat{Q}_{t+1}/\rho$ and the vector of exogenous innovations, ε_{t+1} . This relationship can be derived in a similar way to (57). First extract the appropriate rows from (53) to yield the following

$$\hat{C}_{t+1} - \hat{C}_{t+1}^* - \hat{Q}_{t+1}/\rho = D_1 \xi_{t+1} + D_2 \varepsilon_{t+1} + D_3 \begin{bmatrix} x_t \\ s_{t+1} \end{bmatrix} + O(\epsilon^2) \quad (59)$$

where the matrices D_1 , D_2 and D_3 are formed from the appropriate rows of (53). After substituting for ξ_{t+1} using (56) this implies

$$\hat{C}_{t+1} - \hat{C}_{t+1}^* - \hat{Q}_{t+1}/\rho = \tilde{D} \varepsilon_{t+1} + D_3 \begin{bmatrix} x_t \\ s_{t+1} \end{bmatrix} + O(\epsilon^2) \quad (60)$$

where

$$\tilde{D} = D_1 \tilde{H} + D_2 \quad (61)$$

Equations (57) and (60) show the first-order accurate behaviour of \hat{r}_{xt+1} and $\hat{C}_{t+1} - \hat{C}_{t+1}^* - \hat{Q}_{t+1}/\rho$ and they can be used to evaluate the second-order accurate behaviour of

the left hand side of equation (37), as follows

$$E_t \left[(\hat{C}_{t+1} - \hat{C}_{t+1}^* - \hat{Q}_{t+1}/\rho) \hat{r}_{x,t+1} \right] = \tilde{R} \Sigma \tilde{D}' + O(\epsilon^3) \quad (62)$$

where Σ is the covariance matrix of ε .²⁶ The equilibrium value of $\tilde{\alpha}$ satisfies the following equation

$$\tilde{R} \Sigma \tilde{D}' = 0 \quad (63)$$

This matrix equation defines $(n - 1)$ equations in the $(n - 1)$ elements of $\tilde{\alpha}$.

To solve for $\tilde{\alpha}$ first substitute for \tilde{R} and \tilde{D} in (63) and expand to yield

$$R_1 \tilde{H} \Sigma \tilde{H}' D'_1 + R_2 \Sigma \tilde{H}' D'_1 + R_1 \tilde{H} \Sigma D'_2 + R_2 \Sigma D'_2 = 0 + O(\epsilon^3) \quad (64)$$

Substituting for \tilde{H} and \tilde{H}' and multiplying by $(1 - \tilde{\alpha}' R_1)^2$ yields

$$\begin{aligned} R_1 \tilde{\alpha}' R_2 \Sigma R'_2 \tilde{\alpha} D'_1 + R_2 \Sigma R'_2 \tilde{\alpha} D'_1 (1 - \tilde{\alpha}' R_1) \\ + R_1 \tilde{\alpha}' R_2 \Sigma D'_2 (1 - \tilde{\alpha}' R_1) + R_2 \Sigma D'_2 (1 - \tilde{\alpha}' R_1)^2 = 0 + O(\epsilon^3) \end{aligned} \quad (65)$$

Note that $\tilde{\alpha}' R_1$, $(1 - \tilde{\alpha}' R_1)$ and D_1 are all scalars. It therefore follows that $\tilde{\alpha}' R_1 = R'_1 \tilde{\alpha}$ and $D'_1 = D_1$. Using these facts (65) simplifies to

$$\Lambda [D_1 R_2 \Sigma R'_2 \tilde{\alpha} - R_2 \Sigma D'_2 R'_1 \tilde{\alpha} + R_2 \Sigma D'_2] = 0 + O(\epsilon^3) \quad (66)$$

where $\Lambda = R_1 \tilde{\alpha}' + I(1 - R'_1 \tilde{\alpha})$ and I is the identity matrix. In the case where there are two assets ($n = 2$) R_1 and $\tilde{\alpha}$ are scalars and Λ is equal to unity. In this case equation (66) can be solved to yield the following expression for the equilibrium $\tilde{\alpha}$

$$\tilde{\alpha} = [R_2 \Sigma D'_2 R'_1 - D_1 R_2 \Sigma R'_2]^{-1} R_2 \Sigma D'_2 + O(\epsilon) \quad (67)$$

This is also a solution in the case where $n > 2$. However, while the solution given in (67) is unique in the $n = 2$ case, other solutions for $\tilde{\alpha}$ may exist if $n > 2$. It is a well-established result that multiple equilibria may exist when markets are incomplete, so it is not surprising that (66) may have multiple solutions. However, the solution given in (67) has an obvious claim to be the first focus of analysis, since it always exists and it corresponds to the unique solution in the two-asset case. Equation (67) is reported as equation (43) in the main text.

²⁶Notice D_3 does not appear in this expression because, by assumption, $E_t(\varepsilon_{t+1} x_t) = E_t(\varepsilon_{t+1} s_{t+1}) = 0$.

Appendix 2: Solving the example model

Here we present the details for the derivation of the example in Section 3. In order to derive the solution for $\tilde{\alpha}$ it is necessary to solve for the first-order accurate behaviour of $(\hat{C}_{t+1} - \hat{C}_{t+1}^*)$ and $\hat{r}_{x,t+1}$ conditional on ξ_t . This solution yields the coefficient matrices D_1 , D_2 , R_1 and R_2 .

First-order approximation of the consumption optimality conditions, the resource constraint and the quantity theory equations imply

$$E_{t+1}[\hat{C}_{t+1} - \hat{C}_{t+1}^*] = (1 - \eta)(\hat{C}_{t+1} - \hat{C}_{t+1}^*) + O(\epsilon^2) \quad (68)$$

$$(1 - g)\hat{C}_t + (1 - g)\hat{C}_t^* + g\hat{G}_t + g\hat{G}_t^* = \hat{Y}_t + \hat{Y}_t^* + O(\epsilon^2) \quad (69)$$

$$\hat{M}_t - \hat{P}_t = (1 - g)\hat{C}_t + g\hat{G}_t + O(\epsilon^2) \quad \hat{M}_t^* - \hat{P}_t^* = (1 - g)\hat{C}_t^* + g\hat{G}_t^* + O(\epsilon^2) \quad (70)$$

A first-order approximation of the budget constraint for home agents is given by

$$\hat{W}_t = \frac{1}{\beta}\hat{W}_{t-1} + \hat{Y}_t - (1 - g)\hat{C}_t - g\hat{G}_t + \tilde{\alpha}'\hat{r}_{xt} + O(\epsilon^2) \quad (71)$$

where $\tilde{\alpha} = \bar{\alpha}/(\beta\bar{Y})$ and $\bar{\alpha}' = [\bar{\alpha}_E]$ and $\hat{r}'_x = [(\hat{r}_E - \hat{r}_{E^*})]$ for the EQ economy and $\bar{\alpha}' = [\bar{\alpha}_B, \bar{\alpha}_{B^*}, \bar{\alpha}_E]$ and $\hat{r}'_x = [(\hat{r}_E - \hat{r}_{E^*}), (\hat{r}_B - \hat{r}_{B^*}), (\hat{r}_{B^*} - \hat{r}_{E^*})]$ for the EB economy. The budget constraint can now be written in the form

$$\hat{W}_t = \frac{1}{\beta}\hat{W}_{t-1} + \hat{Y}_t - (1 - g)\hat{C}_t - g\hat{G}_t + \xi_t + O(\epsilon^2) \quad (72)$$

where ξ_t is treated as an exogenous i.i.d. random variable.

First-order approximations for asset returns satisfy

$$\begin{aligned} \hat{r}_{E,t+1} &= \hat{Y}_{t+1} - \hat{Z}_{E,t} + O(\epsilon^2) & \hat{r}_{E^*,t+1} &= \hat{Y}_{t+1}^* - \hat{Z}_{E^*,t} + O(\epsilon^2) \\ \hat{r}_{B,t+1} &= -\hat{P}_{t+1} - \hat{Z}_{B,t} + O(\epsilon^2) & \hat{r}_{B^*,t+1} &= -\hat{P}_{t+1}^* - \hat{Z}_{B^*,t} + O(\epsilon^2) \end{aligned}$$

while asset prices are given by

$$\begin{aligned} \hat{Z}_{E,t} &= E_t[\hat{Y}_{t+1}] - E_t[\hat{r}_{E,t+1}] + O(\epsilon^2) & \hat{Z}_{E^*,t} &= E_t[\hat{Y}_{t+1}^*] - E_t[\hat{r}_{E^*,t+1}] + O(\epsilon^2) \\ \hat{Z}_{B,t} &= -E_t[\hat{P}_{t+1}] - E_t[\hat{r}_{B,t+1}] + O(\epsilon^2) & \hat{Z}_{B^*,t} &= -E_t[\hat{P}_{t+1}^*] - E_t[\hat{r}_{B^*,t+1}] + O(\epsilon^2) \end{aligned}$$

Note also that, up to a first-order approximation, the following holds

$$E_t[\hat{r}_{E,t+1}] = E_t[\hat{r}_{E^*,t+1}] = E_t[\hat{r}_{B,t+1}] = E_t[\hat{r}_{B^*,t+1}]$$

The above equations can be solved to yield expressions for the first-order behaviour of $(\hat{C}_{t+1} - \hat{C}_{t+1}^*)$ and $\hat{r}_{x,t+1}$. The solution for $(\hat{C}_{t+1} - \hat{C}_{t+1}^*)$ implies that, for both the EQ and EB economies, D_1 and D_2 are given by

$$D_1 = \left[\frac{2[1 - \bar{\beta}(1 - \eta)]}{1 - g} \right], \quad D_2 = \left[\frac{\Delta}{(1-g)} \quad -\frac{\Delta}{(1-g)} \quad -\frac{\Delta g}{(1-g)} \quad \frac{\Delta g}{(1-g)} \quad 0 \quad 0 \right]$$

where $\Delta = [1 - \bar{\beta}(1 - \eta)] / (1 - \bar{\beta}\psi)$. The solution for $\hat{r}_{x,t+1}$ implies that R_1 and R_2 for EQ economy are given by

$$R_1 = [0], \quad R_2 = \left[1 \quad -1 \quad 0 \quad 0 \quad 0 \quad 0 \right]$$

while for EB economy R_1 and R_2 are given by

$$R_1 = \left[\begin{array}{c} 0 \\ 1 - \beta(1 - \eta) \\ -[1 - \beta(1 - \eta)] \end{array} \right], \quad R_2 = \left[\begin{array}{cccccc} 1 & -1 & 0 & 0 & 0 & 0 \\ 1 - \Theta & \Theta - 1 & g\Theta & -g\Theta & -1 & 0 \\ \Theta & -\Theta & -g\Theta & g\Theta & 0 & -1 \end{array} \right]$$

where $\Theta = (1/2)\bar{\beta}(1 - \psi - \eta) / (1 - \bar{\beta}\psi)$.

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