Mini-Course 5: Fuzzy Logics

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1 Łukasiewicz infinite-valued language: $\mathcal{L}_{Ł_\aleph}$

The syntax of our fuzzy languages will be the same as classical logic $\mathcal{L}_{\mathsf{CPL}}$ but with two added connectives: the Łukasiewicz conditional $\rightarrow$ and t-norm conjunction $\otimes$.

Definition 1.1. The set of values $\mathcal{V}$ for $\mathcal{L}_{Ł_\aleph}$ is $[0,1]$ – the closed unit interval in the real numbers. These values are *ordered* in the usual (linear) way with 1 as the only designated value. A $\mathcal{L}_{Ł_\aleph}$ valuation ν is any map from sentences of $\mathcal{L}_{Ł_\aleph}$ into $[0,1]$ ($\nu : \mathcal{S} \rightarrow [0,1]$).

Definition 1.2. A $\mathcal{L}_{Ł_\aleph}$ valuation is *admissible* iff it satisfies the following clauses.

1. $\nu(A) \in [0,1]$ for all atomic A.
2. $\nu(A \vee B) = \mathsf{max}\{\nu(A), \nu(B)\}$.
3. $\nu(A \wedge B) = \mathsf{min}\{\nu(A), \nu(B)\}$.
4. $\nu(\neg A) = 1 - \nu(A)$
5. $\nu(A \rightarrow B) = \begin{cases} 1 \text{ if } \nu(A) \leq \nu(B) \\ 1 - \nu(A) + \nu(B) \text{ otherwise} \end{cases}$
6. $\nu(A \otimes B) = \mathsf{max}\{0, \nu(A) + \nu(B) - 1\}$

Satisfaction and validity are defined as in $\mathcal{L}_{\mathsf{K}_3}$ and $\mathcal{L}_{Ł_3}$.

Exercise 1.3. Prove the Modus Ponens for $\mathcal{L}_{Ł_\aleph}$.

Exercise 1.4. Show that $\mathcal{L}_{Ł_\aleph}$ is paracomplete; i.e. give a counterexample to the Law of Excluded Middle: $B \not\models_{Ł_\aleph} A \vee \neg A$

Exercise 1.5. Let the value n in $\mathcal{L}_{Ł_3}$ be treated as $\frac{1}{2}$. Show that the $\mathcal{L}_{Ł_3}$ conditional and the $\mathcal{L}_{Ł_\aleph}$ conditional agree on the values they share.

Exercise 1.6. Show that $\rightarrow$ is the *residual* of t-norm conjunction. That is, prove that

$$A \otimes B \models_{Ł_\aleph} C \text{ iff } A \models_{Ł_\aleph} B \rightarrow C$$

Exercise 1.7. Let t-norm disjunction be defined thus: $A \oplus B$ iff $\neg(\neg A \otimes \neg B)$. Give a semantic clause for $\oplus$ akin to the one given above for $\otimes$.

2 Variations

Gödel Logic: G

Definition 2.1. A $\mathcal{L}_{\mathsf{G}}$ valuation is *admissible* iff it satisfies the clauses (1)–(3) for admissible $\mathcal{L}_{\mathsf{L}_{\aleph}}$ valuations, and:

4. $\nu(\neg A) = \begin{cases} 1 \text{ if } \nu(A) = 0 \\ 0 \text{ if } \nu(A) > 0 \end{cases}$
5. $\nu(A \to B) = \begin{cases} 1 \text{ if } \nu(A) \leq \nu(B) \\ \nu(B) \text{ otherwise} \end{cases}$
6. $\nu(A \otimes B) = \mathsf{min}\{\nu(A), \nu(B)\}$

Satisfaction and validity are defined as in $\mathcal{L}_{\mathsf{L}_{\aleph}}$.

Exercise 2.2. Prove *Dummett's Law*: $\vDash_{\mathsf{G}} (A \to B) \vee (B \to A)$

Exercise 2.3. Show that $\to$ is the *residual* of *t*-norm conjunction. That is, prove that

$$A \otimes B \vDash_{\mathsf{G}} C \text{ iff } A \vDash_{\mathsf{G}} B \to C$$

Exercise 2.4. Give a semantic clause for $\oplus$ akin to the one given above for $\otimes$.

Exercise 2.5. While $\mathcal{L}_{\mathsf{G}}$ and $\mathcal{L}_{\mathsf{I}}^{\leq}$ have very different semantics, $\vDash_{\mathsf{G}} = \vDash_{I}^{\leq}$. Why do you think that is?

Product Logic: Π

Definition 2.6. A $\mathcal{L}_{\Pi}$ valuation is *admissible* iff it satisfies the clauses (1)–(3) for admissible $\mathcal{L}_{\mathsf{L}_{\aleph}}$ valuations, and:

4. $\nu(\neg A) = \begin{cases} 1 \text{ if } \nu(A) = 0 \\ 0 \text{ if } \nu(A) > 0 \end{cases}$
5. $\nu(A \to B) = \begin{cases} 1 \text{ if } \nu(A) \leq \nu(B) \\ \nu(B)/\nu(A) \text{ otherwise} \end{cases}$
6. $\nu(A \otimes B) = nu(A) \times \nu(B)$

Satisfaction and validity are defined as in $\mathcal{L}_{\mathsf{L}_{\aleph}}$.

Exercise 2.7. Show that $\nu(\neg A) = \nu(A \to \bot)$ whenever $\nu(\bot) = 0$.

Exercise 2.8. Show that $\nu(A \wedge B) = \nu(A \otimes (A \to B)$.

Exercise 2.9. Show that $\to$ is the *residual* of *t*-norm conjunction. That is, prove that

$$A \otimes B \vDash_{\Pi} C \text{ iff } A \vDash_{\Pi} B \to C$$

Exercise 2.10. Give a semantic clause for $\oplus$ akin to the one given above for $\otimes$.