Mini-Course 4: Paraconsistent and Paracomplete Logics

Aaron J. Cotnoir
Northern Institute of Philosophy | November 29, 2010

1 Basic Relevant Logic: FDE

The syntax of our basic many-valued languages will be the same as classical logic $\mathcal{L}_{\mathsf{CPL}}$. We will treat $\supset$ and $\equiv$ as defined connectives.

Definition 1.1. The set of values $\mathcal{V}$ for $\mathcal{L}_{\mathsf{FDE}}$ is $\{1, b, n, 0\}$. These values are *ordered* in the following way. A $\mathcal{L}_{\mathsf{FDE}}$ valuation ν is any map from sentences of $\mathcal{L}_{\mathsf{FDE}}$ into $\{1, b, n, 0\}$ ($\nu : \mathcal{S} \to \{1, b, n, 0\}$).

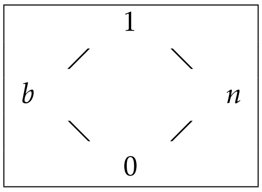

Table 1: Dunn's 4-valued FDE

Definition 1.2. Let $\mathcal{V}$ be any ordered set, and let $x, y \in \mathcal{V}$. The *least upper bound* of x, y ($\mathsf{lub}\{x, y\}$) is the lowest value greater-than-or-equal-to x and y. The *greatest lower bound* of x, y ($\mathsf{glb}\{x, y\}$) is the highest value less-than-or-equal to x and y.

Definition 1.3. A $\mathcal{L}_{\mathsf{FDE}}$ valuation is *admissible* iff it satisfies the following clauses.

1. $\nu(A) \in \{1, b, n, 0\}$ for all atomic A.
2. $\nu(A \vee B) = \mathsf{lub}\{\nu(A), \nu(B)\}$.
3. $\nu(A \wedge B) = \mathsf{glb}\{\nu(A), \nu(B)\}$.
4. $\nu(\neg A) = \begin{cases} 1 \text{ if } \nu(A) = 0 \\ b \text{ if } \nu(A) = b \\ n \text{ if } \nu(A) = n \\ 0 \text{ if } \nu(A) = 1 \end{cases}$

Exercise 1.4. Create truth tables for $\mathcal{L}_{\mathsf{FDE}}$ conjunction and disjunction.

Definition 1.5. Let $\mathcal{D} = \{1, b\}$ be a set of *designated* values.

- A sentence A is *designated* by an $\mathcal{L}_{\mathsf{FDE}}$-admissible valuation ν iff $\nu(A) \in \mathcal{D}$.
- A sentence A is *satisfied* iff it is designated.
- $\vDash_{\mathsf{FDE}} A$ iff A is satisfied by every $\mathcal{L}_{\mathsf{FDE}}$-admissible valuation.
- $\mathcal{X} \vDash_{\mathsf{FDE}} A$ iff every $\mathcal{L}_{\mathsf{FDE}}$-admissible valuation that satisfies $\mathcal{X}$ satisfies A.

Exercise 1.6. Prove the De Morgan Laws for $\mathcal{L}_{\mathsf{FDE}}$.

Exercise 1.7. Show that $\mathcal{L}_{\mathsf{FDE}}$ is paraconsistent; i.e. give a counterexample to Ex Falso Quodlibet: $A \wedge \neg A \nvDash_{\mathsf{FDE}} B$

Exercise 1.8. Show that $\mathcal{L}_{\mathsf{FDE}}$ is paracomplete; i.e. give a counterexample to the Law of Excluded Middle: $B \nvDash_{\mathsf{FDE}} A \vee \neg A$

Exercise 1.9. Give a counterexample to the Deduction Theorem: If $\mathcal{X}, A \vDash_{\mathsf{FDE}} B$ then $\mathcal{X} \vDash_{\mathsf{FDE}} A \supset B$

Fact: Many standard relevant logics arise from adding a (modal) conditional to this logic.

2 Paracomplete Languages: $\mathsf{K}_3, \mathsf{Ł}_3$

Strong Kleene: K_3

Definition 2.1. The set of values $\mathcal{V}$ for $\mathcal{L}_{\mathsf{K}_3}$ is $\{1, n, 0\}$. These values are *ordered* in the following way.

1
|
n
|
0

Table 2: Strong Kleene K_3

Definition 2.2. A $\mathcal{L}_{\mathsf{K}_3}$ valuation is *admissible* iff it satisfies the following clauses.

1. $v(A) \in \{1, n, 0\}$ for all atomic A.
2. $v(A \vee B) = \mathsf{lub}\{v(A), v(B)\}$.
3. $v(A \wedge B) = \mathsf{glb}\{v(A), v(B)\}$.
4. $v(\neg A) = \begin{cases} 1 \text{ if } v(A) = 0 \\ n \text{ if } v(A) = n \\ 0 \text{ if } v(A) = 1 \end{cases}$

Exercise 2.3. Create truth tables for $\mathcal{L}_{\mathsf{K}_3}$ conjunction, disjunction, and the material conditional.

Definition 2.4. A sentence A is *designated* by an $\mathcal{L}_{\mathsf{K}_3}$-admissible valuation v iff $v(A) = 1$. Satisfaction and validity are similar to $\mathcal{L}_{\mathsf{FDE}}$.

Exercise 2.5. Show that $\mathcal{L}_{\mathsf{K}_3}$ is an *extension* of $\mathcal{L}_{\mathsf{FDE}}$.

Exercise 2.6. Show that $\mathcal{L}_{\mathsf{K}_3}$ is *paracomplete* but not paraconsistent.

Exercise 2.7. Prove Modus Ponens is valid: $A, A \supset B \vDash_{K_3} B$.

Exercise 2.8. Give a counterexample to Material Identity: $\nvDash_{K_3} A \supset A$.

Exercise 2.9. Does $\mathcal{L}_{\mathsf{K}_3}$ have any logical truths? Is there any sentence A s.t. $\vDash_{\mathsf{K}_3} A$?

Łukasiewicz's 3-Valued Language: Ł3

$\mathcal{L}_{Ł_3}$ is nearly identical to $\mathcal{L}_{\mathsf{K}_3}$ except that it adds a primitve conditional $\rightarrow$ to the syntax.

Definition 2.10. $\mathcal{L}_{Ł_3}$-admissible valuations are $\mathcal{L}_{\mathsf{K}_3}$-admissible valuations that follow the truth table for $\rightarrow$ given below.

$\rightarrow$	1	n	0
1	1	n	0
n	1	1	n
0	1	1	1

Table 3: Conditional in Ł3

Exercise 2.11. Prove Modus Ponens is valid: $A, A \rightarrow B \vDash_{Ł_3} B$.

Exercise 2.12. Prove Material Identity: $\vDash_{Ł_3} A \rightarrow A$.

3 Paraconsistent Languages: LP, RM3

Logic of Paradox: LP

Definition 3.1. The set of values $\mathcal{V}$ for $\mathcal{L}_{\mathsf{LP}}$ is $\{1, b, 0\}$. These values are *ordered* in the following way.

1
|
b
|
0

Table 4: Logic of Paradox LP

Definition 3.2. A $\mathcal{L}_{\mathsf{LP}}$ valuation is *admissible* iff it satisfies the following clauses.

1. $v(A) \in \{1, b, 0\}$ for all atomic A.
2. $v(A \vee B) = \mathsf{lub}\{v(A), v(B)\}$.
3. $v(A \wedge B) = \mathsf{glb}\{v(A), v(B)\}$.
4. $v(\neg A) = \begin{cases} 1 \text{ if } v(A) = 0 \\ b \text{ if } v(A) = b \\ 0 \text{ if } v(A) = 1 \end{cases}$

Exercise 3.3. Create truth tables for $\mathcal{L}_{\mathsf{LP}}$ conjunction, disjunction, and the material conditional.

Definition 3.4. A sentence A is *designated* by an $\mathcal{L}_{\mathsf{LP}}$-admissible valuation v iff $v(A) \in \{1, b\}$. Satisfaction and validity are similar to $\mathcal{L}_{\mathsf{FDE}}$.

Exercise 3.5. Show that $\mathcal{L}_{\mathsf{LP}}$ is an *extension* of $\mathcal{L}_{\mathsf{FDE}}$, but is not an extension of $\mathcal{L}_{\mathsf{K}_3}$.

Exercise 3.6. Show that $\mathcal{L}_{\mathsf{LP}}$ is *paraconsistent* but not paracomplete.

Exercise 3.7. Give a counterexample to Modus Ponens: $A, A \supset B \nvDash_{LP} B$.

Exercise 3.8. Prove Material Identity: $\vDash_{LP} A \supset A$.

R-Mingle 3-Valued Language: RM_3

$\mathcal{L}_{\mathsf{RM}_3}$ is nearly identical to $\mathcal{L}_{\mathsf{LP}}$ except that it adds a primitve conditional $\rightarrow$ to the syntax.

Definition 3.9. $\mathcal{L}_{\mathsf{RM}_3}$-admissible valuations are $\mathcal{L}_{\mathsf{LP}}$-admissible valuations that follow the truth table for $\rightarrow$ given below.

$\rightarrow$	1	b	0
1	1	0	0
b	1	b	0
0	1	1	1

Table 5: Conditional in RM_3

Exercise 3.10. Prove Modus Ponens is valid: $A, A \rightarrow B \vDash_{\mathsf{RM}_3} B$.

Exercise 3.11. Prove Material Identity: $\vDash_{\mathsf{RM}_3} A \rightarrow A$.

Exercise 3.12. Give a counterexample to Disjunctive Syllogism: $A \vee B, \neg A \vDash_{\mathsf{RM}_3} B$.