Mini-Course 3: Intuitionistic, Sub-intuitionistic, and Super-intuitionistic Logics

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1 Intuitionistic Language: $\mathcal{L}_{\mathsf{I}}^{\mathsf{rt}}$

Syntax for our (sub-/super-)intuitionistic languages is as follows.

1. A set of atomic sentences: $\bot, p, q, r, \ldots$
2. A set of connectives: $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, (,)
3. A set of sentences $\mathcal{S}$ such that:
   (a) All atomics are in $\mathcal{S}$.
   (b) If $A \in \mathcal{S}$ then $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $(A \leftrightarrow B)$ are in $\mathcal{S}$.
   (c) Nothing else is in $\mathcal{S}$.

Intuitionistic frames are identical to normal frames:

Definition 1.1. A *intuitionistic frame* is a triple $\langle \mathcal{W}, \mathcal{R}, \nu \rangle$ s.t.:

1. $\mathcal{W}$ is a non-empty set of worlds;
2. $\mathcal{R}$ is a binary access-relation on $\mathcal{W}$ ($R \subseteq \mathcal{W} \times \mathcal{W}$); and
3. ν is a function from sentence-world pairs into our values 1 and 0 ($\nu : \mathcal{W} \times \mathcal{S} \rightarrow \{1, 0\}$).

Intuitionistic models have certain constraints:

Definition 1.2. An $\mathcal{L}_{\mathsf{I}}^{\mathsf{rt}}$ *model* is an intuitionistic frame s.t.

1. $\mathcal{R}$ is reflexive and transitive.
2. $\nu_w(A) \in \{1, 0\}$ for all atomic A.
3. Absurdity: $\nu_w(\bot) = 0$.
4. Heredity: if $\nu_w(A) = 1$ and $w\mathcal{R}w'$, then $\nu_{w'}(A) = 1$.
5. $\nu_w(A \wedge B) = 1$ iff $\nu_w(A) = 1$ and $\nu_w(B) = 1$.
6. $\nu_w(A \vee B) = 1$ iff $\nu_w(A) = 1$ or $\nu_w(B) = 1$.
7. $\nu_w(A \rightarrow B) = 1$ iff for all w' s.t. $w\mathcal{R}w'$, if $\nu_{w'}(A) = 1$ then $\nu_{w'}(B) = 1$.
8. $\nu_w(A \leftrightarrow B) = 1$ iff for all w' s.t. $w\mathcal{R}w'$, $\nu_{w'}(A) = \nu_{w'}(B)$.

Note: there is no negation. In $\mathcal{L}_{\mathsf{I}}^{\mathsf{rt}}$, negation is *defined*: $\nu_w(\neg A) = \nu_w(A \rightarrow \bot)$. Admissible valuations, satisfaction, and validity are exactly as in normal modal logics.

Exercise 1.3. Prove the Double Negation facts: $A \vDash_{\mathsf{I}}^{rt} \neg\neg A$ but $\neg\neg A \nvDash_{\mathsf{I}}^{rt} A$.

Exercise 1.4. Is $\neg(A \vee B)$ equivalent to $\neg A \wedge \neg B$? What about the equivalence of $\neg(A \wedge B)$ and $\neg A \vee \neg B$? Prove whichever direction(s) you can; give an $\mathcal{L}_{\mathsf{I}}^{\mathsf{rt}}$-countermodel for the direction(s) you can't prove.

Exercise 1.5. Give an $\mathcal{L}_{\mathsf{I}}^{\mathsf{rt}}$-countermodel to Peirce's Law: $\nvDash_{\mathsf{I}}^{rt} ((A \to B) \to A) \to A$

2 Some Sub-Intuitionistic Languages

Sub-intuitionistic languages are languages for which $\mathcal{L}_{\mathsf{I}}^{\mathsf{rt}}$ is an *extension*.

Definition 2.1. Let an $\mathcal{L}_{\mathsf{I}}^{\mathsf{r}}$ *model* be any $\mathcal{L}_{\mathsf{I}}^{\mathsf{rt}}$ model without the transitivity requirement.

Exercise 2.2. Show that $\vDash_{\mathsf{I}}^{rt} (A \to B) \to ((B \to C) \to (A \to C))$ but $\nvDash_{\mathsf{I}}^{r} (A \to B) \to ((B \to C) \to (A \to C))$.

Definition 2.3. Let an $\mathcal{L}_{\mathsf{I}}$ *model* be any $\mathcal{L}_{\mathsf{I}}^{\mathsf{r}}$ model without the reflexivity requirement.

Exercise 2.4. Show that $\vDash_{\mathsf{I}}^{r} (A \wedge (A \to B)) \to B$ but $\nvDash_{\mathsf{I}} (A \wedge (A \to B)) \to B$.

Definition 2.5. Let an $\mathcal{L}_{\mathsf{I}}^{-\mathsf{h}}$ *model* be any $\mathcal{L}_{\mathsf{I}}^{\mathsf{rt}}$ model without the heredity requirement.

Exercise 2.6. Show that $\vDash_{\mathsf{I}}^{rt} A \to (B \to A)$ but $\nvDash_{\mathsf{I}}^{-h} A \to (B \to A)$.

Definition 2.7. Let an $\mathcal{L}_{\mathsf{I}}^{-\mathsf{a}}$ *model* be any $\mathcal{L}_{\mathsf{I}}^{\mathsf{rt}}$ model without the absurdity requirement. Note: this logic is Johannson's Minimal Logic.

Exercise 2.8. Show that $\vDash_{\mathsf{I}}^{rt} \bot \to A$ but $\nvDash_{\mathsf{I}}^{-a} \bot \to A$.

Exercise 2.9. Show that $A, \neg A \nvDash_{\mathsf{I}}^{-a} B$ but $A, \neg A \vDash_{\mathsf{I}}^{-a} \neg B$.

3 A Super-Intuitionistic Language: Gödel-Dummett Logic

Super-intuitionistic languages are sometimes called 'intermediary logics'. They are 'intermediate' in the sense that they are extensions of $\mathcal{L}_{\mathsf{I}}^{\mathsf{rt}}$ and $\mathcal{L}_{\mathsf{cpl}}$ is an extension of them.

Definition 3.1. A relation $\mathcal{R}$ is *linear* iff $\mathcal{R}$ is reflexive, antisymmetric, transitive, and total.

- *Reflexive*: $x\mathcal{R}x$
- *Antisymmetric*: if $x\mathcal{R}y$ and $y\mathcal{R}x$ then $x = y$
- *Transitive*: if $x\mathcal{R}y$ and $y\mathcal{R}z$, then $x\mathcal{R}z$
- *Total*: either $x\mathcal{R}y$ or $y\mathcal{R}x$

Definition 3.2. Let an $\mathcal{L}_{\mathsf{I}}^{\leq}$ *model* be any $\mathcal{L}_{\mathsf{I}}^{\mathsf{rt}}$ model s.t. $\mathcal{R}$ is *linear*.

Exercise 3.3. Show that $\nvDash_{\mathsf{I}}^{rt} (A \to B) \vee (B \to A)$ but $\vDash_{\mathsf{I}}^{\leq} (A \to B) \vee (B \to A)$.

We will give a *many-valued* semantics for this language in a few weeks. Tantalizing fact: the semantic values will be *linearly* ordered.