Mini-Course 2: Non-Normal Modal Logics

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1 Non-Normal Frames

Syntax for normal modal languages is identical to $\mathcal{L}_K$ and all its extensions. Non-normal frames are a lot like normal frames, but with our worlds divided into two types: *normal* and *non-normal*.

Definition 1.1. A *non-normal frame* is a quadruple $\langle \mathcal{W}, \mathcal{N}, \mathcal{R}, v \rangle$ s.t.:

1. $\mathcal{W}$ is a non-empty set of worlds;
2. $\mathcal{N} \subseteq \mathcal{W}$ is a non-empty set of 'normal' worlds ($\mathcal{W} - \mathcal{N}$ are 'non-normal');
3. $\mathcal{R}$ is a binary access-relation on $\mathcal{W}$ – $R \subseteq \mathcal{W} \times \mathcal{W}$; and
4. v is a function from sentence-world pairs into our values 1 and 0 – $v : \mathcal{W} \times \mathcal{S} \rightarrow \{1, 0\}$.

2 Basic Non-Normal Language: $\mathcal{L}_{NN}$

Definition 2.1. An $\mathcal{L}_{NN}$ *model* is a non-normal frame s.t.

1. If w is normal ($w \in \mathcal{N}$), then v obeys all the clauses for normal modal models.
2. If w is non-normal ($w \in \mathcal{W} - \mathcal{N}$), then v obeys all the clauses for the extensional connectives, and
   (a) $v_w(\Box A) = 0$.
   (b) $v_w(\Diamond A) = 1$.

- A function v is an *admissible valuation* for $\mathcal{L}_{NN}$ iff there is an $\mathcal{L}_{NN}$-model $\langle \mathcal{W}, \mathcal{N}, \mathcal{R}, v \rangle$ and a *normal* world $w \in \mathcal{N}$ s.t. $v = v_w$.
- An admissible valuation v *satisfies* a sentence A iff $v(A) = 1$.
- $\vDash_{NN} A$ iff every $\mathcal{L}_{NN}$-admissible valuation satisfies A.
- $X \vDash_{NN} A$ iff every $\mathcal{L}_{NN}$-admissible valuation that satisfies $\mathcal{X}$ satisfies A.

Exercise 2.2. Prove that Necessitation fails: for some A, $\vDash_{NN} A$ but $\nvDash_{NN} \Box A$.

Definition 2.3. $\mathcal{M}$ is an $\mathcal{L}_{NN}$ *countermodel* to an argument from $\mathcal{X}$ to A iff $\mathcal{M}$ satisfies $\mathcal{X}$ at w, but does not satisfy A at w, for some *normal* $w \in \mathcal{N}$.

Exercise 2.4. Give an $\mathcal{L}_{NN}$ countermodel to $\Box(A \supset \Box(A \vee \neg A))$; show that $\nvDash_{NN} \Box(A \supset \Box(A \vee \neg A))$.

Exercise 2.5. Show that $\mathcal{L}_K$ is an extension of $\mathcal{L}_{NN}$ (and hence that all normal modal logics are too).

Logic S2: $\mathcal{L}^{r}_{\mathsf{NN}}$

Definition 2.6. An $\mathcal{L}^{r}_{\mathsf{NN}}$ *model* is a $\mathcal{L}_{\mathsf{NN}}$ model s.t. $\mathcal{R}$ is *reflexive* over $\mathcal{N}$.

Definition 2.7. C.I. Lewis's Strict Implication: $A \multimap B$ is an abbreviation for $\Box(A \supset B)$.

Exercise 2.8. Show that $A, A \multimap B \nvDash_{\mathsf{NN}} B$ but $A, A \multimap B \vDash^{r}_{\mathsf{NN}} B$.

Exercise 2.9. Prove Lewis's Consistency Postulate; $\vDash^{r}_{\mathsf{NN}} \Diamond(A \wedge B) \supset \Diamond A$

3 Variations on Validity

Logic E2: all-worlds validity for S2

- A function v is an *admissible valuation* for $\mathcal{L}_{\mathsf{E2}}$ iff there is an $\mathcal{L}^{r}_{\mathsf{NN}}$-model $\langle \mathcal{W}, \mathcal{N}, \mathcal{R}, v \rangle$ and any world (*normal* or *non-normal*) $w \in \mathcal{W}$ s.t. $v = v_w$.

Exercise 3.1. Let $\perp$ be a sentence false in all worlds in any model; $v_w(\perp) = 0$ for all $w \in \mathcal{W}$. Show that $\vDash^{r}_{\mathsf{NN}} \perp \multimap A$ but $\nvDash_{\mathsf{E2}} \perp \multimap A$.

Logic S6: standard-worlds validity for S2

Definition 3.2. Let a world w be *standard* iff it is (i) normal and (ii) accesses some non-normal world.

- A function v is an *admissible valuation* for $\mathcal{L}_{\mathsf{S6}}$ iff there is an $\mathcal{L}^{r}_{\mathsf{NN}}$-model $\langle \mathcal{W}, \mathcal{N}, \mathcal{R}, v \rangle$ and *standard* world $w \in \mathcal{N}$ s.t. $v = v_w$.

Exercise 3.3. Show that $\nvDash_{\mathsf{S2}} \Diamond\Diamond A$, but $\vDash_{\mathsf{S6}} \Diamond\Diamond A$.

4 Random Non-Normal Language: $\mathcal{L}_{\mathsf{RNN}}$

There are other ways for modal operators to behave at non-normal worlds. Here is a *random* way of doing it:

Definition 4.1. An $\mathcal{L}^{r}_{\mathsf{RNN}}$ *model* is a non-normal frame s.t.

1. If w is normal ($w \in \mathcal{N}$), then v obeys all the clauses for normal modal models.
2. If w is non-normal ($w \in \mathcal{W} - \mathcal{N}$), then v obeys all the clauses for the extensional connectives, and the modal connectives are randomly assigned:
   (a) $v_w(\Box A) \in \{1, 0\}$.
   (b) $v_w(\Diamond A) \in \{1, 0\}$.

Logic S0.5: $\mathcal{L}^{r}_{\mathsf{RNN}}$

Definition 4.2. An $\mathcal{L}^{r}_{\mathsf{RNN}}$ *model* is a $\mathcal{L}_{\mathsf{RNN}}$ model s.t. $\mathcal{R}$ is *reflexive* over $\mathcal{N}$.

Exercise 4.3. Prove that if A is a tautology of $\mathcal{L}_{\mathsf{cpl}}$ then $\vDash^{r}_{\mathsf{RNN}} \Box A$.

$\mathcal{R}$	Normal	Non-Normal	All-worlds $\models$	Standard-worlds $\models$	Random Non-Normal
None	K	$S2^0$	$E2^0$	$S6^0$	$S0.5^0$
r	T	S2	E2	S6	S0.5
rs	B	$S3^0$	$E3^0$	$S7^0$	S0.5
rt	S4	S3	E3	S7	S0.5
rst	S5	S3.5	E3.5	S7.5	S0.5