Mini-Course 1: Normal Modal Logics

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1 Preliminaries: Formal Languages

Definition 1.1. A *propositional language* $\mathcal{L}$ comprises:

1. a *sentential syntax*, and
2. a *sentential semantics*.

Definition 1.2. A *sentential syntax* is:

1. A set of atomic sentences;
2. A set of n-ary connectives, including punctuation marks (and);
3. A set of sentences $\mathcal{S}$ such that:
   (a) All atomics are in $\mathcal{S}$.
   (b) If φ is an n-ary connective and $A_1,\ldots,A_n$ are in $\mathcal{S}$, then $\varphi(A_1,\ldots,A_n)$ is in $\mathcal{S}$.
   (c) Nothing else is in $\mathcal{S}$.

Definition 1.3. A *sentential semantics* comprises:

1. a set of *admissible valuations* of the sentential syntax, and
2. a definition of *satisfaction*.

Example: $\mathcal{L}_{\mathsf{cpl}}$

- Atomics: $p, q, r, \ldots$
- Connectives: $\neg$, $\wedge$, $\vee$, $\supset$, $\equiv$, (,)
- Admissible valuations, v:
  - $v(A) \in \{1, 0\}$ for all atomic A.
  - $v(\neg A) = 1$ iff $v(A) = 0$.
  - $v(A \wedge B) = 1$ iff $v(A) = 1$ and $v(B) = 1$.
  - $v(A \vee B) = 1$ iff $v(A) = 1$ or $v(B) = 1$.
  - $v(A \supset B) = 1$ iff $v(A) = 0$ or $v(B) = 1$.
  - $v(A \equiv B) = 1$ iff $v(A) = v(B)$.
- Satisfaction:
  - v satisfies A iff $v(A) = 1$
  - v satisfies X iff $v(A) = 1$ for all A in X.
  - $\vDash A$ iff for all v, $v(A) = 1$.
  - $X \vDash A$ iff every v that satisfies X satisfies A.

2 Normal Modal Languages

Syntax for normal modal languages is identical to $\mathcal{L}_{\mathsf{cpl}}$ except we add two unary connectives ($\Box, \Diamond$) and the corresponding clauses for $\mathcal{S}$.

Basic Normal Modal Language: $\mathcal{L}_{\mathsf{K}}$

Instead of immediately interpreting the syntax via admissible valuations, we will first define structures called 'frames'. Admissible frames are called 'models'. Models will, in turn, specify our admissible valuations.

Definition 2.1. A *frame* is a triple $\langle \mathcal{W}, \mathcal{R}, \nu \rangle$ s.t.:

1. $\mathcal{W}$ is a non-empty set of points (i.e. 'worlds');
2. $\mathcal{R}$ is a binary 'access'-relation on $\mathcal{W}$ – $R \subseteq \mathcal{W} \times \mathcal{W}$; and
3. ν is a function from sentence-world pairs into our values 1 and 0 – $\nu : \mathcal{W} \times \mathcal{S} \to \{1, 0\}$.

Definition 2.2. An $\mathcal{L}_{\mathsf{K}}$ *model* is a frame s.t.

1. $\nu_w(A) \in \{1, 0\}$ for all atomic A.
2. $\nu_w(\neg A) = 1$ iff $\nu_w(A) = 0$.
3. $\nu_w(A \wedge B) = 1$ iff $\nu_w(A) = 1$ and $\nu_w(B) = 1$.
4. $\nu_w(A \vee B) = 1$ iff $\nu_w(A) = 1$ or $\nu_w(B) = 1$.
5. $\nu_w(A \supset B) = 1$ iff $\nu_w(A) = 0$ or $\nu_w(B) = 1$.
6. $\nu_w(A \equiv B) = 1$ iff $\nu_w(A) = \nu_w(B)$.
7. $\nu_w(\Box A) = 1$ iff for all w' s.t. $w\mathcal{R}w'$, $\nu_{w'}(A) = 1$.
8. $\nu_w(\Diamond A) = 1$ iff for some w' s.t. $w\mathcal{R}w'$, $\nu_{w'}(A) = 1$.

- A function ν is an $\mathcal{L}_{\mathsf{K}}$-*admissible valuation* iff there is an $\mathcal{L}_{\mathsf{K}}$-model $\langle \mathcal{W}, \mathcal{R}, \nu \rangle$ and a world $w \in \mathcal{W}$ s.t. $\nu = \nu_w$.
- An $\mathcal{L}_{\mathsf{K}}$-admissible valuation ν *satisfies* a sentence A iff $\nu(A) = 1$.
- $\nvDash_{\mathsf{K}} A$ iff every $\mathcal{L}_{\mathsf{K}}$-admissible valuation satisfies A.
- $X \nvDash_{\mathsf{K}} A$ iff every $\mathcal{L}_{\mathsf{K}}$-admissible valuation that satisfies $\mathcal{X}$ satisfies A.

Exercise 2.3. Show that $\Box\neg A$ is equivalent to $\neg\Diamond A$.

Definition 2.4. $\mathcal{M}$ is an $\mathcal{L}_{\mathsf{K}}$ *countermodel* to an argument from $\mathcal{X}$ to A iff $\mathcal{M}$ satisfies $\mathcal{X}$ at w, but does not satisfy A at w, for some $w \in \mathcal{W}$.

Exercise 2.5. Give an $\mathcal{L}_{\mathsf{K}}$ countermodel to $\Box A \supset A$; show that $\nvDash_{\mathsf{K}} \Box A \supset A$.

Definition 2.6. Where two languages $\mathcal{L}$, $\mathcal{L}^*$ have the same syntax, we say $\mathcal{L}^*$ is an *extension* of $\mathcal{L}$ iff every $\mathcal{L}^*$-model is an $\mathcal{L}$-model. $\vDash_{\mathcal{L}^*}$ is an *extension* of $\vDash_{\mathcal{L}}$ iff $\mathcal{X} \vDash_{\mathcal{L}^*} A$ if $\mathcal{X} \vDash_{\mathcal{L}} A$.

A modal language is called *normal* whenever it is an extension of $\mathcal{L}_{\mathsf{K}}$.

Exercise 2.7. Show that $\vDash_{\mathcal{L}^*_{\mathsf{K}}} \Box(A \supset B) \supset (\Box A \supset \Box B)$ where $\mathcal{L}^*_{\mathsf{K}}$ is any normal modal language.

Logic D: $\mathcal{L}_{\mathsf{K}}^{\mathsf{ser}}$

Definition 2.8. A relation $\mathcal{R}$ is *serial* iff for every x, there is some y s.t. $x\mathcal{R}y$.

Definition 2.9. An $\mathcal{L}_{\mathsf{K}}^{\mathsf{ser}}$ *model* is a $\mathcal{L}_{\mathsf{K}}$ model s.t. $\mathcal{R}$ is serial.

Exercise 2.10. Show that $\vDash_{\mathcal{L}_{\mathsf{K}}^{\mathsf{ser}}} \Box A \supset \Diamond A$.

Logic T: $\mathcal{L}_{\mathsf{K}}^{\mathsf{r}}$

Definition 2.11. A relation $\mathcal{R}$ is *reflexive* iff for every x, $x\mathcal{R}x$.

Definition 2.12. An $\mathcal{L}_{\mathsf{K}}^{\mathsf{r}}$ *model* is a $\mathcal{L}_{\mathsf{K}}$ model s.t. $\mathcal{R}$ is reflexive.

Exercise 2.13. Show that $\vDash_{\mathcal{L}_{\mathsf{K}}^{\mathsf{r}}} \Box A \supset A$.

Logic B: $\mathcal{L}_{\mathsf{K}}^{\mathsf{rs}}$

Definition 2.14. A relation $\mathcal{R}$ is *symmetric* iff for every x and y, $x\mathcal{R}y$ iff $y\mathcal{R}x$.

Definition 2.15. An $\mathcal{L}_{\mathsf{K}}^{\mathsf{rs}}$ *model* is a $\mathcal{L}_{\mathsf{K}}$ model s.t. $\mathcal{R}$ is reflexive and symmetric.

Exercise 2.16. Show that $\vDash_{\mathcal{L}_{\mathsf{K}}^{\mathsf{rs}}} A \supset \Box\Diamond A$.

Logic S4: $\mathcal{L}_{\mathsf{K}}^{\mathsf{rt}}$

Definition 2.17. A relation $\mathcal{R}$ is *transitive* iff for every x, y, and z, if $x\mathcal{R}y$ and $y\mathcal{R}z$, then $x\mathcal{R}z$.

Definition 2.18. An $\mathcal{L}_{\mathsf{K}}^{\mathsf{rt}}$ *model* is a $\mathcal{L}_{\mathsf{K}}$ model s.t. $\mathcal{R}$ is reflexive and transitive.

Exercise 2.19. Show that $\vDash_{\mathcal{L}_{\mathsf{K}}^{\mathsf{rt}}} \Box A \supset \Box\Box A$.

Logic S5: $\mathcal{L}_{\mathsf{K}}^{\mathsf{rst}}$

Definition 2.20. A relation $\mathcal{R}$ is an *equivalence* relation iff $\mathcal{R}$ is reflexive, symmetric, and transitive.

Definition 2.21. An $\mathcal{L}_{\mathsf{K}}^{\mathsf{rst}}$ *model* is a $\mathcal{L}_{\mathsf{K}}$ model s.t. $\mathcal{R}$ is an equivalence relation.

Exercise 2.22. Show that $\vDash_{\mathcal{L}_{\mathsf{K}}^{\mathsf{rst}}} \Diamond A \supset \Box\Diamond A$.

In S5, the access-relation $\mathcal{R}$ plays no significant role. We could equivalently give a semantics for modal operators thus:

- $v_w(\Box A) = 1$ iff for all w', $v_{w'}(A) = 1$.
- $v_w(\Diamond A) = 1$ iff for some w', $v_{w'}(A) = 1$.

Exercise 2.23. Show the following equivalences. (Hint: Utilize earlier exercises!)

1. $\vDash_{\mathcal{L}_{\mathsf{K}}^{\mathsf{rst}}} \Diamond A \equiv \Box\Diamond A$
2. $\vDash_{\mathcal{L}_{\mathsf{K}}^{\mathsf{rst}}} \Diamond A \equiv \Diamond\Diamond A$
3. $\vDash_{\mathcal{L}_{\mathsf{K}}^{\mathsf{rst}}} \Box A \equiv \Diamond\Box A$
4. $\vDash_{\mathcal{L}_{\mathsf{K}}^{\mathsf{rst}}} \Box A \equiv \Box\Box A$

Exercise 2.24. Show that in $\mathcal{L}_{\mathsf{K}}^{\mathsf{rst}}$ there are only six non-equivalent modalities: A, $\neg A$, $\Diamond A$, $\neg\Diamond A$, $\Box A$, $\neg\Box A$. (Hint: Utilize exercises 2.3 and 2.23.)