NIP Summer School 2011
'Truth and Paradox' Lecture #2

Aaron Cotnoir

Curry & Contraction

Theorem 1.1 (Curry's Paradox)

No language with self-reference underwritten by a logic satisfying (among other things) Contraction, the rule $A \rightarrow (A \rightarrow B) \vdash A \rightarrow B$, can formulate an adequate non-trivial truth theory that applies to itself.

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(7) $\bot$ [MPP, (3), (6)]

$\square$

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- $Ł_4$ expands the semantic values, has counterexamples to 2-1 contraction.
- However, $Ł_4$ validates 3-2 contraction. Similar Curry problems result.
- Likewise, $Ł_5$ invalidates 3-2 contraction, but validates 4-3 contraction.

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- . . . unless we expand to *infinitely-many* values.
- $Ł_\omega$ is a well-known fuzzy logic. It is *robustly contraction-free*.

Determinate Truth

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- As a result, we can characterize the Liar as 'gappy': $\neg DL \wedge \neg D\neg L$ comes out true!
- If we introduce a determinate-Liar, $Q_1 := \neg D\mathrm{T}(\ulcorner Q_1 \urcorner)$, this can also be characterized as $\neg DDQ_1 \wedge \neg DD\neg Q_1$.
- This process can continue for D^n-Liars, $Q_n := \neg D^n\mathrm{T}(\ulcorner Q_n \urcorner)$, each type characterizable by the D^{n+1} operator.

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- This results in the ω-inconsistency of arithmetic in $Ł_\omega$: $D^\omega A$ fails even when $D^n A$ holds for each n.
- Hajek, Paris, and Shepherdson (2000) extend this result, showing how this leads to outright inconsistency when one adds universal generalizations claiming that truth commutes with negation, etc.

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A09 $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$

A10 $(A \rightarrow B) \rightarrow ((C \rightarrow A) \rightarrow (C \rightarrow B))$

A11 $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$

A12 $A \rightarrow ((A \rightarrow B) \rightarrow B)$

A13 $A \vee \neg A$

A14 $(A \rightarrow \neg A) \rightarrow \neg A$

A15 $A \rightarrow (B \rightarrow A)$

A16 $A \rightarrow (A \rightarrow A)$

A17 $((A \rightarrow B) \rightarrow B) \rightarrow A \vee B$

R1 $A, A \rightarrow B \vdash B$

R2 $A, B \vdash A \wedge B$

Some Axioms and Rules (Priest 2001)

A01 $A \to A$

A02 $A \to A \vee B$ and $B \to A \vee B$

A03 $A \wedge B \to B$ and $A \wedge B \to A$

A04 $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$

A05 $((A \to B) \wedge (A \to C)) \to (A \to B \wedge C)$

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A07 $\neg\neg A \to A$

A08 $(A \to B) \to (\neg B \to \neg A)$

A09 $(A \to B) \to ((B \to C) \to (A \to C))$

A10 $(A \to B) \to ((C \to A) \to (C \to B))$

A11 $(A \to (A \to B)) \to (A \to B)$

A12 $A \to ((A \to B) \to B)$

A13 $A \vee \neg A$

A14 $(A \to \neg A) \to \neg A$

A15 $A \to (B \to A)$

A16 $A \to (A \to A)$

A17 $((A \to B) \to B) \to A \vee B$

R1 $A, A \to B \vdash B$

R2 $A, B \vdash A \wedge B$

R3 $A \to B \vdash (C \to A) \to (C \to B)$

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A12 $A \to ((A \to B) \to B)$

A13 $A \vee \neg A$

A14 $(A \to \neg A) \to \neg A$

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A16 $A \to (A \to A)$

A17 $((A \to B) \to B) \to A \vee B$

R1 $A, A \to B \vdash B$

R2 $A, B \vdash A \wedge B$

R3 $A \to B \vdash (C \to A) \to (C \to B)$

R4 $A \to B \vdash (B \to C) \to (A \to C)$

R5 $A \to B \vdash \neg B \to \neg A$

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A14 $(A \to \neg A) \to \neg A$

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R2 $A, B \vdash A \wedge B$

R3 $A \to B \vdash (C \to A) \to (C \to B)$

R4 $A \to B \vdash (B \to C) \to (A \to C)$

R5 $A \to B \vdash \neg B \to \neg A$

- NB: A08 makes R5 redundant.

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A01 $A \to A$

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A12 $A \to ((A \to B) \to B)$

A13 $A \vee \neg A$

A14 $(A \to \neg A) \to \neg A$

A15 $A \to (B \to A)$

A16 $A \to (A \to A)$

A17 $((A \to B) \to B) \to A \vee B$

R1 $A, A \to B \vdash B$

R2 $A, B \vdash A \wedge B$

R3 $A \to B \vdash (C \to A) \to (C \to B)$

R4 $A \to B \vdash (B \to C) \to (A \to C)$

R5 $A \to B \vdash \neg B \to \neg A$

- NB: A08 makes R5 redundant.
- As does A09 for R4; and A10 for R3

Some Axioms and Rules (Priest 2001)

A01 $A \to A$

A02 $A \to A \lor B$ and $B \to A \lor B$

A03 $A \land B \to B$ and $A \land B \to A$

A04 $A \land (B \lor C) \to (A \land B) \lor (A \land C)$

A05 $((A \to B) \land (A \to C)) \to (A \to B \land C)$

A06 $((A \to C) \land (B \to C)) \to (A \lor B \to C)$

A07 $\neg\neg A \to A$

A08 $(A \to B) \to (\neg B \to \neg A)$

A09 $(A \to B) \to ((B \to C) \to (A \to C))$

A10 $(A \to B) \to ((C \to A) \to (C \to B))$

A11 $(A \to (A \to B)) \to (A \to B)$

A12 $A \to ((A \to B) \to B)$

A13 $A \lor \neg A$

A14 $(A \to \neg A) \to \neg A$

A15 $A \to (B \to A)$

A16 $A \to (A \to A)$

A17 $((A \to B) \to B) \to A \lor B$

R1 $A, A \to B \vdash B$

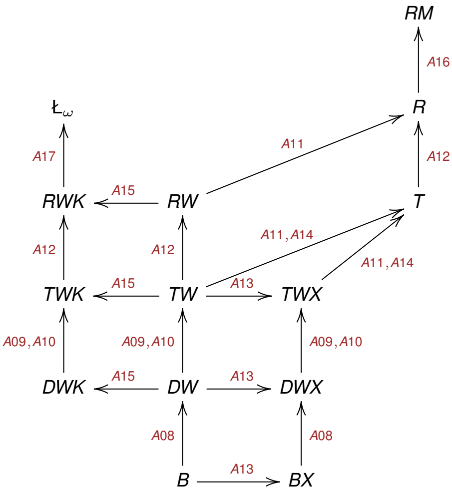
R2 $A, B \vdash A \land B$

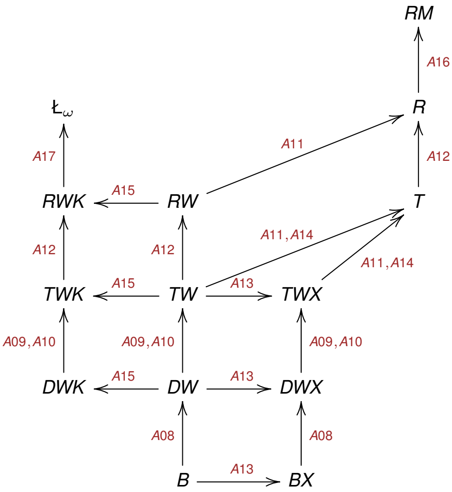
R3 $A \to B \vdash (C \to A) \to (C \to B)$

R4 $A \to B \vdash (B \to C) \to (A \to C)$

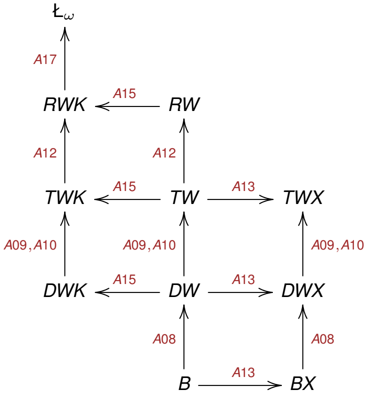
R5 $A \to B \vdash \neg B \to \neg A$

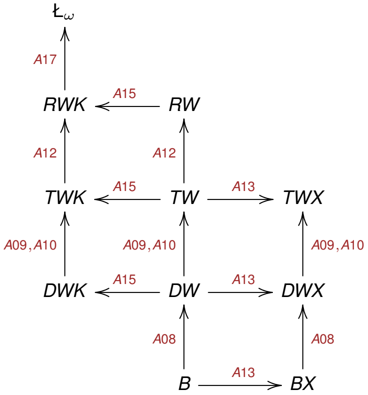
- NB: A08 makes R5 redundant.
- As does A09 for R4; and A10 for R3
- Our *Basic* logic, *B*, is axiomatized by A01-A07 and rules R1-R5.


RM
A16
Ł$_\omega$
R
A11
A17
A12
RWK
A15
RW
T
A12
A12
A11, A14
A11, A14
TWK
A15
TW
A13
TWX
A09, A10
A09, A10
A09, A10
DWK
A15
DW
A13
DWX
A08
A08
B
A13
BX

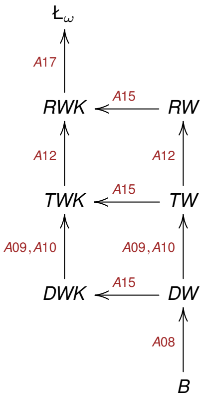


Reminder: A11 is a form of *Contraction*! So none of *T*, *R*, and *RM* are acceptable logics for our purposes.


Ł$_\omega$
A17
RWK
A15
RW
A12
A12
TWK
A15
TW
A13
TWX
A09, A10
A09, A10
A09, A10
DWK
A15
DW
A13
DWX
A08
A08
B
A13
BX



B is *paraconsistent* (as are all relevant logics) and *paracomplete*. In lecture 3, we will discuss the purely paraconsistent logics: e.g. *BX*, *DWX*, *TWX*, as well as *DL*, *DJ*, and *DK*.


Ł$_\omega$
A17
RWK
A15
RW
A12
A12
TWK
A15
TW
A09, A10
A09, A10
DWK
A15
DW
A08
B

- For this lecture, we will discuss the purely paracomplete logics. Which are those?

- For this lecture, we will discuss the purely paracomplete logics. Which are those?

- From *Contraposition* (A08), *Weakening* (A15), together with *Modus Ponens* (R1), one can prove EFQ.

$$\dfrac{\neg A \qquad \dfrac{\dfrac{A \quad \dfrac{}{A \to (B \to A)}}{B \to A} \qquad \dfrac{}{(B \to A) \to (\neg A \to \neg B)}}{\neg A \to \neg B}}{\neg B}$$

- For this lecture, we will discuss the purely paracomplete logics. Which are those?

- From *Contraposition* (A08), *Weakening* (A15), together with *Modus Ponens* (R1), one can prove EFQ.

$$\dfrac{\neg A \qquad \dfrac{\dfrac{A \qquad \overline{A \to (B \to A)}}{B \to A} \qquad \overline{(B \to A) \to (\neg A \to \neg B)}}{\neg A \to \neg B}}{\neg B}$$

- We have $A, \neg A \vdash \neg B$, which given DNE (A07) is equivalent to EFQ.

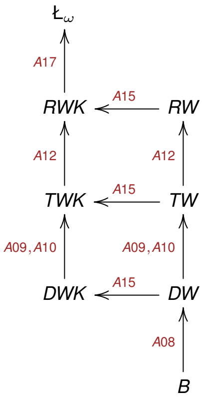
- For this lecture, we will discuss the purely paracomplete logics. Which are those?

- From *Contraposition* (A08), *Weakening* (A15), together with *Modus Ponens* (R1), one can prove EFQ.

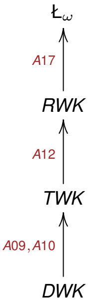
$$\dfrac{\neg A \qquad \dfrac{\dfrac{A \qquad \overline{A \to (B \to A)}}{B \to A} \qquad \overline{(B \to A) \to (\neg A \to \neg B)}}{\neg A \to \neg B}}{\neg B}$$

- We have $A, \neg A \vdash \neg B$, which given DNE (A07) is equivalent to EFQ.

- So, any extension of B with A08 and A15 is not paraconsistent.


Łω
A17
RWK
A15
RW
A12
A12
TWK
A15
TW
A09, A10
A09, A10
DWK
A15
DW
A08
B

Advanced Paracomplete Logics



$Ł_{\omega}$ Axioms

Ł$_\omega$ Axioms

Ł$_\omega$ Axioms

A01 $A \to A$

$Ł_\omega$ Axioms

A01 $A \to A$

A02 $A \to A \vee B$ and $B \to A \vee B$

$Ł_\omega$ Axioms

A01 $A \rightarrow A$

A02 $A \rightarrow A \vee B$ and $B \rightarrow A \vee B$

A03 $A \wedge B \rightarrow B$ and $A \wedge B \rightarrow A$

$Ł_\omega$ Axioms

A01 $A \rightarrow A$

A02 $A \rightarrow A \vee B$ and $B \rightarrow A \vee B$

A03 $A \wedge B \rightarrow B$ and $A \wedge B \rightarrow A$

A04 $A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)$

$Ł_\omega$ Axioms

A01 $A \rightarrow A$

A02 $A \rightarrow A \vee B$ and $B \rightarrow A \vee B$

A03 $A \wedge B \rightarrow B$ and $A \wedge B \rightarrow A$

A04 $A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)$

A05 $((A \rightarrow B) \wedge (A \rightarrow C)) \rightarrow (A \rightarrow B \wedge C)$

$Ł_\omega$ Axioms

A01 $A \rightarrow A$

A02 $A \rightarrow A \vee B$ and $B \rightarrow A \vee B$

A03 $A \wedge B \rightarrow B$ and $A \wedge B \rightarrow A$

A04 $A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)$

A05 $((A \rightarrow B) \wedge (A \rightarrow C)) \rightarrow (A \rightarrow B \wedge C)$

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A01 $A \rightarrow A$

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A07 $\neg\neg A \rightarrow A$

$Ł_\omega$ Axioms

A01 $A \rightarrow A$

A02 $A \rightarrow A \vee B$ and $B \rightarrow A \vee B$

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$Ł_\omega$ Axioms

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A02 $A \to A \vee B$ and $B \to A \vee B$

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A07 $\neg\neg A \to A$

A08 $(A \to B) \to (\neg B \to \neg A)$

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A01 $A \rightarrow A$

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A12 $A \to ((A \to B) \to B)$

A15 $A \to (B \to A)$

A17 $((A \to B) \to B) \to A \vee B$

- Closed under $R1$ and $R2$.

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A10 $(A \rightarrow B) \rightarrow ((C \rightarrow A) \rightarrow (C \rightarrow B))$

A12 $A \rightarrow ((A \rightarrow B) \rightarrow B)$

A15 $A \rightarrow (B \rightarrow A)$

A17 $((A \rightarrow B) \rightarrow B) \rightarrow A \vee B$

- Closed under $R1$ and $R2$.
- Adding A11, A13, or A14 gives *Classical Logic*.

$Ł_\omega$ Axioms

A01 $A \to A$

A02 $A \to A \vee B$ and $B \to A \vee B$

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A08 $(A \to B) \to (\neg B \to \neg A)$

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A10 $(A \to B) \to ((C \to A) \to (C \to B))$

A12 $A \to ((A \to B) \to B)$

A15 $A \to (B \to A)$

A17 $((A \to B) \to B) \to A \vee B$

- Closed under *R*1 and *R*2.
- Adding A11, A13, or A14 gives *Classical Logic*.
- RWK drops A17; TWK also drops A12; DWK also drops A09 and A10.

$Ł_\omega$ Axioms

A01 $A \to A$

A02 $A \to A \vee B$ and $B \to A \vee B$

A03 $A \wedge B \to B$ and $A \wedge B \to A$

A04 $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$

A05 $((A \to B) \wedge (A \to C)) \to (A \to B \wedge C)$

A06 $((A \to C) \wedge (B \to C)) \to (A \vee B \to C)$

A07 $\neg\neg A \to A$

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- Closed under $R1$ and $R2$.
- Adding A11, A13, or A14 gives *Classical Logic*.
- RWK drops A17; TWK also drops A12; DWK also drops A09 and A10.
- How far down does one have to go to avoid the problems with $Ł_\omega$? That's a hard question.

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- Here is his algebraic semantics for his logic (pp. 231–234).
- $\mathcal{V}$ is an infinite set of values partially ordered by $\leq$.
- $Ł_\omega$ has infinitely-many *linearly* ordered values; Field generalizes to partially ordered values.

Conjunction & Disjunction

(1) A binary operator, $\sqcup$, on values corresponds to *lub* relative to $\leq$.

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- Not needed in linearly-ordered value spaces; but required here to guarantee *Reasoning By Cases*.
- Also implies that $\langle \mathcal{V}, \sqcup, \sqcap, 1, 0 \rangle$ is not Boolean unless the only elements in $\mathcal{V}$ are 1 and 0.

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- This constraint has been added by Field in recent work on restricted quantification.

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What Logic?

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(5) 1 is 'join irreducible'.

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(8) $\exists z$ s.t. $z = z*$

(I) $a \Rightarrow b = 1$ iff $a \leq b \rightsquigarrow$ A01, R1

(IIa) If $a \leq b$ then $c \Rightarrow a \leq c \Rightarrow b. \rightsquigarrow$ R3

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  - If $a \leq c$ and $b \leq c$, then $a \sqcup b \leq c$. (This follows from the fact $\sqcup$ is lub.)
- Thus, Field's logic is a slight weakening of DWK.

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- Field also shows how to define a 'determinate truth' operator from his conditional: $DA := A \wedge \neg(A \rightarrow \neg A)$.
- The resulting operator has many of the desirable features of the $Ł_\omega$ operator – including being able to say of any gap that it is 'gappy' – without ω-inconsistency.

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- Finally, Field has recently started toying with adding A12 – or at least its rule form – as it seems desirable for restricted quantification. Adding the full version would bring us up to RWK.
- RWK is the logic identified by Restall (1992) as the place to start. But how does one prove it avoids the problems with $Ł_\omega$?

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- We also have a parallel version of Priest's worry with the inexpressibility of 'determinately true at all levels'. It seems we understand such a notion, and even perhaps Field's model-theory depends on it (i.e. semantic value 1).
- Notably, dialetheic theories don't need to stratify to characterize 'defective' sentences. So, let's turn to those theories next.