NIP Summer School 2011
‘Truth and Paradox’ Lecture #1

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1. Tarskianism
2. Basic Paracomplete Theories
3. Basic Paraconsistent Theories

Definition 1.1 (Formal Languages)

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- For any sentence A of PA, we use '$\ulcorner A \urcorner$' to stand for the canonical term referring to *the code number of A*. We get the following result.

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For any sentence $\varphi(x)$ of PA open in one variable, there is a sentence S of PA such that the equivalence $S \leftrightarrow \varphi(\ulcorner S \urcorner)$ is a theorem of PA.

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> *Gödel showed. . . that elementary syntax can be interpreted in number theory. In this way, Gödel put the issue of the legitimacy of self-referential sentences beyond doubt; he showed that they are as incontestably legitimate as arithmetic itself. (Kripke, 1975, p.692)*

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Definition 1.2 (Having Self-reference)

Language $\mathcal{L}$ *has self-reference* iff every sentence A of $\mathcal{L}$ has a standard name $\ulcorner A \urcorner$ in $\mathcal{L}$ and every predicate $\varphi(x)$ of $\mathcal{L}$ has a Gödel Sentence in $\mathcal{L}$.

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- He then demonstrated a case in which this is impossible.

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Proof Sketch

Fix a language that has self-reference and is underwritten by classical logic, and suppose for *reductio* that it can formulate its own adequate, non-trivial truth theory using the unary predicate $\mathrm{T}(x)$. By self-reference there exists a sentence L that is the Gödel Sentence for the predicate $\neg\mathrm{T}(x)$, i.e. (crudely) a Liar sentence which 'says that' it is not true. We reason, in this language:

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And this entails everything—*triviality*—by classical logic (EFQ). □

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Tarski's Theorem is really far more than an undefinability theorem; it says that full truth can't appear in [a non-trivial] language even as a primitive predicate. . . Of course, there's something odd about this conclusion: it should lead us to suspect that we don't even understand a concept of full truth that obeys both the Tarski schema and classical logic. (Field, 2008, p.30)

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- This perfectly captures the usual interpretation of Tarski's Theorem as a *dilemma*: revise the concept of truth, or revise classical logic.

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- Since a classical language with self-reference cannot contain its own truth predicate, perhaps some *other* language could do so.
- This was exactly Tarski's proposal: ascend from language $\mathcal{L}_k$ to define theory X in a richer language $\mathcal{L}_{k+1}$ such that for all sentences A of $\mathcal{L}_k$:

$$X \vDash_{k+1} \mathrm{T_k}(\ulcorner A \urcorner) \leftrightarrow A$$

(think of 'T_k' as 'is a true sentence of $\mathcal{L}_k$')

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- It rules out the *existence* of Liar sentences (as ungrammatical).
- And the method iterates, allowing us to re-apply it to metalangauge $\mathcal{L}_{k+1}$ by ascending to a richer metametalanguage $\mathcal{L}_{k+2}$, and so on.

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    If A is ‘a true sentence of $\mathcal{L}_k$’ for some level k of the ‘hierarchy’ of languages, then A is *simply true*, but this predicate is not contained in any language.

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- We focus on the *propositional* fragment of the logics we consider because this is where most of the 'revisionary' work needs to be done.

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Definition 1.3 (Transparency)

Let β be any sentence in which sentence α occurs. Then the result of substituting $\mathrm{T}(\ulcorner\alpha\urcorner)$ for any occurrence of α in β has the same semantic value or same semantic status as β. (Beall, 2009, p.15)

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- We will call sentences that are counterexamples to LEM GAPS.
- It is tempting to think of gaps as being ‘neither true nor false’. We shall see later why this temptation might need to be resisted. For now, it is useful to think of gaps simply as counterexamples to LEM.

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- Suppose she sorts through every sentence of her language and applies these instructions in all the cases that she can (in some idealized sense).
- With the Liar and other self-referential sentences, she will suspend judgment; the procedure never tells her to assert or deny them.

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- And now we have a counterexample to LEM.
- Let us make this a bit more precise.

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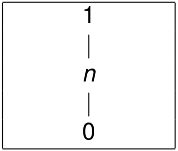

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Representation in Truth Tables

$\neg$	
1	0
n	n
0	1

$\wedge$	1	n	0
1	1	n	0
n	n	n	0
0	0	0	0

$\vee$	1	n	0
1	1	1	1
n	1	n	n
0	1	n	0

$\rightarrow$	1	n	0
1	1	n	0
n	1	n	n
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$\leftrightarrow$	1	n	0
1	1	n	0
n	n	n	n
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NB: the conditional here is the material conditional $(A \rightarrow B) =_{\text{df.}} (\neg A \vee B)$

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- As such, K_3 is *paracomplete*.

K_3TT

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- We want to extend this language with a transparent truth predicate T.
- Kripke (1975) showed us how to do this, one ‘stage’ at a time.

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- Continue on!

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- But there will be some sentences – the 'ungrounded' ones – which will never resolve to a classical value. These sentences, like the Liar, will be stuck with value *n* forever.
- Kripke also showed, although we won't prove it here, that the resulting interpretation for T is *transparent*. That is a great achievement!

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- Note connection with *Conditional Proof*, and the *Deduction Theorem*: if $A \models A$ then $\models A \rightarrow A$
- As a result, there are counterexamples to the *Material T-Biconditionals*! $\not\models_{K_3} \mathrm{T}(\ulcorner A \urcorner) \leftrightarrow A$

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- But in the logics we have considered (and will consider in lecture 2) this is equivalent to $\neg\mathrm{T}(\ulcorner A \urcorner) \wedge \mathrm{T}(\ulcorner A \urcorner)$, which is the assertion of a glut.
- Furthermore, when the sentence in question, $\mathrm{T}(\ulcorner A \urcorner)$, fails to receive a classical value in any fixed point, so does $\neg\mathrm{T}(\ulcorner A \urcorner)$, and thus so do the sentences $\mathrm{T}(\ulcorner A \urcorner) \wedge \neg\mathrm{T}(\ulcorner A \urcorner)$ and $\neg(\mathrm{T}(\ulcorner A \urcorner) \vee \neg\mathrm{T}(\ulcorner A \urcorner))$.

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- Which means that a truth theory on this logic can have T-biconditionals.

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- Contraction is (often) thought to be the problematic feature of conditionals that lead to Curry's paradox.
- The *advanced* paracomplete theories to be discussed in lecture 2 attempt to address these problems.

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- We will call sentences that are counterexamples to EFQ GLUTS.
- We will speak of gluts as being 'both true and false'. This is accurate, but in the logics we consider $\mathrm{T}(\ulcorner A \urcorner) \wedge \neg\mathrm{T}(\ulcorner A \urcorner)$ and $\neg(\mathrm{T}(\ulcorner A \urcorner) \vee \neg\mathrm{T}(\ulcorner A \urcorner))$ are equivalent, so gluts are 'gappy' in the sense we *couldn't* say of GAPS.

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- That's suggestive, but a more interesting route is to argue that we can *prove* that the Liar is a glut with minimal resources.

$$
\dfrac{\dfrac{\overline{\mathrm{T}(\ulcorner L \urcorner) \rightarrow \neg\mathrm{T}(\ulcorner L \urcorner)}\ (\text{TS})}{\neg\mathrm{T}(\ulcorner L \urcorner)}\ (\neg\text{E}) \qquad \dfrac{\dfrac{\dfrac{\vdots}{\neg\mathrm{T}(\ulcorner L \urcorner)}}{L}\ (\text{def. } L)}{\mathrm{T}(\ulcorner L \urcorner)}\ (\text{Nec})}{\mathrm{T}(\ulcorner L \urcorner) \wedge \neg\mathrm{T}(\ulcorner L \urcorner)}\ (\wedge\text{I})
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- With the Liar and other self-referential sentences, she will go back and forth so she will never have decisive reason to *only* assert or deny it.
- She won’t as a result of this indecisiveness accept *everything*, so we have a counterexample to EFQ. Let us make this a bit more precise.

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$$\begin{array}{c} 1 \\ | \\ b \\ | \\ 0 \end{array}$$

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NB: same as K_3, with only difference which elements are designated

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- It is truth-functional, and has classical logic as an extension.
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- As such, *LP* is *paraconsistent*.

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  - We assume everything is either in the extension or anti-extension.

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- This process continues just as before up through the stages.

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- But there will be some sentences – the very same 'ungrounded' ones – which will never resolve to a classical value. They will be stuck with value *b* forever, always in both the extension and anti-extension of T.
- Likewise, the resulting interpretation for T is *transparent*. Great news!

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1	1	b	0
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- Note connection with *'Converse' Conditional Proof* and the *Deduction Theorem*: if $A \rightarrow B \vDash A \rightarrow B$ then $A \rightarrow B, A \vDash B$
- Notice also that some of the *Material T-Biconditionals*, the ones instantiantiated on gluts, are *false* (though they are also true).

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- This patches one obvious weakness of *LP*, MPP: $A \rightarrow B, A \vDash_{RM} B$

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- Again, this leads to the complicated trouble of Curry's paradox.
- The *advanced* paraconsistent theories to be discussed in lecture 3 attempt to address these problems.

References

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