Expectations and the Distribution of Wealth

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Abstract

This paper addresses the distributional effects of shocks to expectations in a model of incomplete markets with uninsured idiosyncratic risk and aggregate uncertainty (Krusell and Smith (1998)). I find that small expectational shocks can significantly increase wealth inequality due to the different implications for consumption of the capital-rich and the capital-poor households. I also study the transitional dynamics of this economy under least-squares learning and I find that following a positive expectational shock, the standard deviation of the distribution of capital increases significantly and persistently, and that the effect on aggregate capital is significant but less persistent.

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1 Introduction

Recent research considered representative-agent dynamic general-equilibrium models in which shocks to expectations about future economic fundamentals affect macroeconomic aggregates. This paper addresses the distributional effects of shocks to expectations in a model of incomplete markets with uninsured idiosyncratic risk and aggregate uncertainty (Krusell and Smith (1998)). In this framework, households face a forecasting problem concerning aggregate investment and the main objective of this paper is to examine the effects of changes in expectations on the endogenous distribution of wealth. I find that (small) expectational shocks can significantly increase wealth inequality due to the different implications for consumption of the capital-rich and the capital-poor households. I also study the transitional dynamics of this economy when households use least squares to fine-tune the parameters of their forecasting equation. Least squares learning is an appropriate modeling device in this context along two dimensions. First, the numerical procedure used by Krusell and Smith (1998) to solve for the equilibrium is itself an adaptive-learning algorithm based on linear regressions. In this sense, least squares learning satisfies a cognitive consistency principle that requires that the simulated agents are computationally as capable as the modelers themselves. Second, the results of Evans and Honkapohja (2001) can be conveniently applied to make sure that an equilibrium is dynamically stable. I find that, following an expectational shock of small magnitude (i) the standard deviation of the distribution of capital increases significantly, (ii) this increase in inequality is very persistent, taking six times longer to subside than to peak, and (iii) aggregate capital is also significantly affected by such shocks in a way that is consistent with the literature that links business cycles to expectational shocks.

The notion of macroeconomic fluctuations driven by expectations dates back to Pigou (1926) whose analysis provides an interesting and intuitive link between expectational shocks and investment dynamics and has recently received renewed attention (e.g. Beaudry and Portier (2004)). The main idea is that an expected productivity gain causes aggregate investment to rise which in turn boosts economic activity. This compelling insight of Pigou’s hinges on the assumption that all agents in the economy respond to an expectational shock in an identical fashion – in other words, the argument applies to a representative agent framework. Yet, when there is a non-degenerate distribution of wealth in the economy, it is not straightforward to apply Pigou’s logic to understand the consequences of an expectational shock regarding aggregate investment on economic
outcomes. To see this point, consider the optimal consumption-investment decision of a household whose income depends on wages and the rental rate of capital. With competitive factor markets, diminishing marginal returns, and complementary factors of production, an increase in expected aggregate investment produces two distinct effects on the budget constraint of a non-representative household. First, the complementarity of production factors implies that the increase in the stock of capital causes wages to increase, thus expanding the household’s intertemporal budget set. This is a positive impact from the point of view of the household. Second, higher investment increases the available stock of capital causing a decrease in the rental rate of capital. Thus there is also a negative impact on the household’s budget set.\footnote{This discussion assumes that the household has positive net worth. This issue will be further addressed in the following.} Which one of these two effects dominates depends (among other factors) on the amount of capital owned by a particular household: the economic significance of the second negative effect is higher for capital-rich than for capital-poor households, everything else equal. This observation, and the consumption-smoothing motive, imply that otherwise-identical households save more if they are capital-rich and less if they are capital-poor as a consequence of optimistic expectations about aggregate investment (and pessimistic expectations about the rate of return to capital.) Consequently, the second moment of the distribution of wealth increases, while the first moment may either increase or decrease, depending on the initial distribution of capital. I develop and demonstrate this intuition using a model of incomplete markets with aggregate uncertainty as in Krusell and Smith (1998), in which a non-degenerate distribution of wealth arises endogenously.

This paper is also related to two strands in the learning literature. First, Packalén (2000) and Evans and Honkapohja (2001) study the learnability of RBC-type models, and Evans and Honkapohja (2003), and Bullard and Mitra (2002) study the learnability of new-Keynesian models. These studies use a representative-agent framework, and the learnability of models with heterogeneous agents has not been established before. In this paper, I show that the equilibrium of the heterogeneous-agents economy with aggregate shocks studied by Krusell and Smith (1998) is learnable. This paper is also related to the line of research in the learning literature, which studies the real effects of learning on macroeconomic aggregates in general equilibrium models. Bullard and Duffy (2001) and Williams (2003) find that least squares learning in several DSGE models does not produce strong effects on various log-linearized representative-agent models. This paper contributes
to this literature by showing that heterogeneity across households can matter, as aggregate capital is affected significantly during the learning dynamics.\textsuperscript{2}

Finally, I find that the baseline Krusell and Smith (1998) model can produce relatively high levels of wealth inequality under expectational shocks and learning. The baseline model – without learning and expectational shocks – is characterized by a tight distribution of wealth. To match the empirical distribution of wealth in the US, Krusell and Smith (1998) extend their baseline model by introducing heterogeneity across households’ discount factors. By contrast, Chang and Kim (2006) obtain high wealth inequality –without this additional layer of heterogeneity– by calibrating the households’ income process on the basis of the Panel Study of Income Dynamics. Here, I find that expectational shocks regarding aggregate investment are also capable of producing persistent wealth inequality.

The paper proceeds as follows: section 2 presents the model, and section 3 illustrates the extreme sensitivity of the model’s equilibrium to expectational shocks. In section 4 learning is introduced, and after having established E-stability in section 4.1, the main quantitative results are reported in section 4.2. Section 5 concludes.

2 The Model

The model-economy follows closely Krusell and Smith (1998). There is a continuum (measure one) of infinitely-lived households with constant-relative-risk-aversion time-separable utility function $U_0 = E \sum_{t=0}^{\infty} \frac{c_{1-t}}{1-\eta}$. The aggregate production function is $Y_t = z_t K_t^{\alpha} L_t^{(1-\alpha)}$, where $K_t$ is time-$\alpha$ aggregate capital, $L_t$ is aggregate labor, and $z_t$ is the aggregate productivity parameter. Factors markets are competitive so that wages and real interest rates are

\begin{equation}
    w_t(\mu_t, l_t, z_t) = (1-\alpha) z_t \left( \frac{\mu_t}{l_t} \right)^{\alpha}
\end{equation}

\textsuperscript{2} Beaudry and Portier (2007) study Pigou cycles under the assumption of a representative household and show that, to obtain Pigou cycles in a neo-classical framework, one needs a description of the productive side of the economy richer than assumed here. In particular they identify two settings in which Pigou cycles are possible: one is the case with more than two sectors in the economy, while the second is the case in which there exists a costly distribution system. Pigou cycles in the Krusell and Smith (1998) economy are outside the scope of this paper, which only studies the differential effect of expectations on consumption of households of different wealth.
\[ r_t(\mu_t, l_t, z_t) = \alpha z_t \left( \frac{\mu_t}{L_t} \right)^{(\alpha-1)} \]  

(1b)

where \( \mu_t \) denotes average capital holdings and \( l_t \) denotes the employment rate at time \( t \). Household \( i \in [0, 1] \) is exposed to uninsurable idiosyncratic Markov shocks \( \varepsilon^i_t \in \{0, 1\} \) where \( \varepsilon^i_t = 0 \) means that \( i \) is unemployed at time \( t \). Aggregate productivity follows a two-valued Markov process \( (z_t \in \{z_g, z_b\}) \) with known probabilities. The idiosyncratic shocks are correlated with aggregate productivity and the laws of large numbers imply that the measure of the employed is perfectly correlated with the aggregate shock \( (l_t \in \{l_g, l_b\}) \). The distribution of households over capital holdings and employment states at time \( t \) is trivially defined over a sigma algebra containing all the possible realizations of \( k^i_t \) (individual capital holdings) and \( \varepsilon^i_t \).

Following Krusell and Smith (1998), average capital ownership \( \mu_t \) is used as a sufficient statistic to determine the optimal consumption plan. The household’s value function problem is

\[
v^i(k^i_t, \varepsilon^i_t; \mu_t, z_t) = \max_{k^i_{t+1}, c^i_t} \left\{ \frac{c^i_t - \eta}{1 - \eta} + \beta E \left[ v^i(k^i_{t+1}, \varepsilon^i_{t+1}; \mu_{t+1}, z_{t+1}) | \varepsilon^i_t, z_t \right] \right\} \tag{2}
\]

subject to:

\[
c^i_t + k^i_{t+1} = r(\mu_t, l_t, z_t)k^i_t + w(\mu_t, l_t, z_t)\varepsilon^i_t + (1 - \delta)k^i_t \tag{2a}
\]

\[
\log \mu_{t+1} = \begin{cases} a_0 + a_1 \log \mu_t, & \text{if } z_t = z_g \\ c_0 + c_1 \log \mu_t, & \text{otherwise} \end{cases} \tag{2b}
\]

\[
k^i_{t+1} \geq \kappa \tag{2c}
\]

the transition probabilities

where \( c^i_t \) is consumption during period \( t \). Constraints (2a) and (2c) are the budget and credit constraints respectively (\( \kappa > 0 \) is an exogenous parameter.) Constraint (2b) is the equation used by the household to determine the level of aggregate investment, which in turn pins down the future prices of capital and labor which is information relevant for the calculation of the optimal consumption plan. In the following I will refer to the free parameters of equation (2b) \( (a_0, a_1, c_0, \) and \( c_1 ) \) as the \textit{expectational parameters} which, at an equilibrium, are determined to deliver the best forecast of aggregate capital in a mean-squared error sense, consistent with a log-linear AR(1) econometric specification. The algorithm used to solve the model involves these steps: (1) guess initial values for the expectational parameters, (2) determine the policy function through a value
3 Expectational Sensitivity Analysis

\[ \beta = 0.99, \quad \delta = 0.025 \]
\[ \eta = 1, \quad \alpha = 0.36 \]
\[ z_g = 1.01, \quad z_b = 0.99 \]
\[ l_g = 0.96, \quad l_b = 0.90 \]

Tab. 1: Parameter values. The parameterization follows Krusell and Smith (1998).

The fact that the forecasting equation (2b) occupies such a central position in the households decision-making process and the existence of an endogenous non-trivial distribution of wealth makes this model an ideal framework to discuss the impact of small changes in expectations on wealth inequality. The remainder of the paper therefore will be concerned with the effects of changes of the expectational parameters from their equilibrium values.

3 Expectational Sensitivity Analysis

In this section I show that the dynamic equilibrium of this model is very sensitive to small changes of the expectational parameters. The parameterization – reported in Tables 1 and 2 – follows Krusell and Smith (1998). When a household is employed it inelastically supplies \( \frac{1}{3} \) units of labor, corresponding to a work day of eight hours.\(^4\) Table 3 reports the descriptive statistics of the stochastic equilibrium selected by the algorithm described in section 2. The average level of capital during expansions is slightly higher than during recessions, and wealth is tightly distributed around the mean in all of the phases of the business cycle. The values of the expectational parameters reported in Table 3 are self-confirming: when households adopt the log-linear forecasting model,

\(^3\) This strategy relies on the assumption that equation (2b) delivers appropriate forecasts of aggregate capital at an equilibrium. Krusell and Smith (1998) perform several robustness checks and argue that the log-linear AR(1) is indeed a satisfactory specification for forecasting purposes.

\(^4\) This parameterization is consistent with the values originally used in Krusell and Smith (1998).
### 3 Expectational Sensitivity Analysis

\[
\begin{bmatrix}
\pi_{gg00} & \pi_{gg01} & \pi_{gb00} & \pi_{gb01} \\
\pi_{gg10} & \pi_{gg11} & \pi_{gb10} & \pi_{gb11} \\
\pi_{bg00} & \pi_{bg01} & \pi_{bb00} & \pi_{bb01} \\
\pi_{bg10} & \pi_{bg11} & \pi_{bb10} & \pi_{bb11}
\end{bmatrix}
= 
\begin{bmatrix}
0.2927 & 0.5834 & 0.0938 & 0.0313 \\
0.0243 & 0.8507 & 0.0091 & 0.1159 \\
0.0313 & 0.0938 & 0.5250 & 0.3500 \\
0.0021 & 0.1229 & 0.0389 & 0.8361
\end{bmatrix}
\]

Tab. 2: Transition probabilities. The subscripts of each entry indicate the aggregate transition (first two characters) and the individual transition (the following two digits). For example \(\pi_{gb01}\) is the probability that a currently unemployed household finds a job when the economy transitions from high productivity to low productivity.

their aggregate behavior produces data that, when used to estimate the best fitting log-linear AR(1) model, confirms the households' initial beliefs.\(^5\)

To assess the impact of permanent changes to expectations on the equilibrium I change each of the expectational parameters by amounts in the order of magnitude of \(10^{-4}\). Such changes are small in the sense that the resulting bias in the unconditional expectation of capital is at most 2%, and the bias in interest and wage rates change is at most 0.045% and 0.75%, respectively. I simulate the model with a new expectational parameter at a time, and then I calculate the time-averages of the mean and standard deviation of the cross sectional distribution of wealth, discarding an initial portion of the data to insure ergodicity. Figure 1 shows the behavior of the first two moments of the ergodic distribution of wealth as the expectational parameters change by the amount reported on the horizontal axis, relative to those reported in Table 3. Notably, for the range of changes I consider, there is a negative relation between the values of the expectational parameters and the mean of the distribution of wealth, and a positive relation between the expectational parameters and the standard deviation of the distribution of wealth. This shows that the equilibrium reported in Table 3 is non-trivially affected by small changes in expectations, and furthermore this opens the possibility that transitory expectational shocks may affect the distribution of capital as well as...

\(^5\) Defining the equilibrium reported in Table 3 as a rational expectations equilibrium may be misleading. In fact, the behavior of this model could be different when the consumption function is obtained through different representations of the state space. For this reason, the classification that best suits this equilibrium is that of a Restricted Perception Equilibrium proposed by Evans and Honkapohja (2001) and Sargent (2001). An RPE requires that the economic agents choose the best parameterization possible, within a limited class of available expectational models and that the parameter values are self-confirming.
Fig. 1: Time-averages of the average ($\mu$) and standard deviation ($\sigma$) of the cross-sectional distribution of capital in correspondence with small deviation in the expectational parameters. Each panel refers to one parameter of the forecasting equation used by the households.
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<table>
<thead>
<tr>
<th>Variable</th>
<th>average</th>
<th>st.dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>11.60</td>
<td>0.2692</td>
</tr>
<tr>
<td>$\mu_g$</td>
<td>11.67</td>
<td>0.2584</td>
</tr>
<tr>
<td>$\mu_b$</td>
<td>11.52</td>
<td>0.2591</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>7.361</td>
<td>0.2244</td>
</tr>
<tr>
<td>$\sigma_g$</td>
<td>7.375</td>
<td>0.2206</td>
</tr>
<tr>
<td>$\sigma_b$</td>
<td>7.348</td>
<td>0.2273</td>
</tr>
<tr>
<td>$a^*_0$</td>
<td>0.0924</td>
<td>-</td>
</tr>
<tr>
<td>$a^*_1$</td>
<td>0.9633</td>
<td>-</td>
</tr>
<tr>
<td>$c^*_0$</td>
<td>0.0826</td>
<td>-</td>
</tr>
<tr>
<td>$c^*_1$</td>
<td>0.9652</td>
<td>-</td>
</tr>
</tbody>
</table>

Tab. 3: Summary statistics for the economy at the stochastic equilibrium; $\mu$ and $\sigma$ denote the mean and standard deviation of the cross sectional distribution of capital averaged over 10,000 periods; the subscripts $b$ and $g$ denote recessions and expansions respectively. The standard errors of the expectational parameters are unreported since the precision of these estimates can be made arbitrarily small by increasing the length of the simulation.

its average. Before further elaborating on this idea, I propose – and test – in the following section an intuitive explanation the patterns displayed in Figure 1.

3.1 An Intuitive Explanation

Expectations about aggregate capital change systematically with the expectational parameters: by increasing (decreasing) each expectational parameter, one introduces a positive (negative) bias in the expected level of aggregate capital. For this reason, I will refer to expectations that imply higher levels of capital as “optimism,” while those that lead to lower capital as “pessimism.” In Figure 1 the regions on the right of the dashed lines are characterized by optimism, and those to the left by pessimism.

To fix ideas, consider the case of optimistic households, and denote with $E^o$ their expectational operator. From equations (1) it is immediate that, under $E^o$, the expected wage rate is higher, and the return rate to capital lower than those that materialize under the equilibrium expectational operator. As usual, the implications of such changes of expected returns to productive factors for the consumption-saving decisions of households can be decomposed into substitution and income

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6 This statement holds for both conditional and unconditional expectations. Since this distinction is redundant, I will omit it in the following.
effects. First, the higher wages have a positive wealth effect on current consumption. Second, the expected decrease in the interest rates further increases consumption through the substitution effect. Third, the decrease of interest rates produces an income effect that may reinforce or mitigate the increase in consumption depending on each household’s net worth: a household with positive net assets faces a negative income effect, while one with negative net assets faces a positive income effect. All three effects work in the same direction for households with negative net worth and therefore their current consumption increases when $E^o$ is used instead of equilibrium expectations.

The net effect of $E^o$ on the consumption of households with positive net worth is ambiguous and in order to sign it, I use the relation between permanent income and consumption implied by the Euler equations. Intuitively, the effect of $E^o$ on future expected income is twofold since higher wages balance lower interest rates. As the share of total income earned from capital increases with the wealth of the household, it is intuitive that there exists a critical threshold of capital ownership above (below) which $E^o$ has a negative (positive) net effect on the household’s perceived permanent income. The following Proposition 1 proves that this threshold exists, is positive and is finite. The proof, as given below, is considerably simpler when $E^o$ is a point-expectational operator, and the proof in the general case is given in the appendix.

**Proposition 1.** A household with point expectational operator $E^o$ will anticipate higher permanent income if and only if

$$k_{i+1}^i < \psi E^o \mu_{t+1}$$

where $\psi = E_t \left[ \frac{z_{t+1}^i \epsilon_{t+1}^i}{\mu_{t+1}} \right] / E_t \left[ \frac{z_{t+1}^i}{\mu_{t+1}} \right]$, and $E_t$ is the operator consistent with the probabilities reported in Table 2.

**Proof:** Household $i$’s income at $t+1$ is given by

$$y_{t+1}^i \equiv z_{t+1}^i \alpha \left[ \frac{\mu_{t+1}}{\ell_{t+1}} \right]^{\alpha-1} k_{t+1}^i + z_{t+1}^i (1-\alpha) \left[ \frac{\mu_{t+1}}{\ell_{t+1}} \right] \epsilon_{t+1}^i$$

Logarithmic preferences and market completeness imply that the income and substitution effects offset each other exactly for every household. However, under incomplete markets, Aiyagari (1994) and Huggett (1993) show that the interest rate is lower than the rate of time preference.

A point expectational operator is relative to a degenerate probability distribution that places a unit mass of probability on a single event. Point expectations correspond to the situation in which households perceive their beliefs to be very precise. This is an accurate approximation for the purpose of applying this proposition to the simulations summarized in Figure 1 since the standard errors of the regressions run by the households are extremely small.
using the assumption of point expectations, we have that \( E^o[\mu_{t+1}] = [E^o\mu_{t+1}]^\alpha \). Therefore, the 
time-\( t \) expectation for \( y^i_{t+1} \), conditional on the current aggregate and individual states, can be 
written as
\[
E_t E^o[y^i_{t+1}] = \sum_{s \in \{b, g\}} P_{z_t z_s} \left( z_s \alpha \left[ \frac{E^o\mu_{t+1}}{l_s} \right]^{\alpha-1} k^i_{t+1} + z_s (1 - \alpha) \left[ \frac{E^o\mu_{t+1}}{l_s} \right]^\alpha P_{z_{t+1}} \right)
\]
where \( P_{z_t z_g} \) and \( P_{z_t z_b} \) are the probabilities that the aggregate state transitions from \( z_t \) to \( z_g \) and 
\( z_b \) respectively, and \( P_{z_t z_g}^{i-1} \) and \( P_{z_t z_b}^{i+1} \) are the probabilities that household \( i \) is employed at \( t + 1 \) 
conditional on \( \epsilon^i_t \) and the aggregate state transition. These probabilities can be calculated directly 
from Table 2. Now, calculate the effect of a change expected level of average capital at \( t + 1 \) on 
expected income
\[
\frac{\partial E_t E^o[y^i_{t+1}]}{\partial E^o\mu_{t+1}} = \alpha (1 - \alpha) E^o \mu_{t+1}^{\alpha-2} \left\{ -k^i_{t+1} E_t \left[ \frac{z_{t+1}}{l_{t+1}} \right] + E^o \mu_{t+1} E_t \left[ \frac{z_{t+1} \epsilon^i_{t+1}}{l_{t+1}} \right] \right\}
\]
This expression shows that the necessary and sufficient condition for a marginal increase in the 
expected aggregate capital level \( \langle \mu_{t+1} \rangle \) to increase expected personal income \( (\partial E_t E^o[y^i_{t+1}]/\partial E^o\mu_{t+1} > 0) \) is \( k^i_{t+1} < \psi E^o\mu_{t+1} \).

Furthermore, by reversing the inequality sign in condition (3), Proposition 1 applies to pessimistic 
households. Accordingly, optimism affects capital-poor and capital-rich households (relative 
to the threshold) asymmetrically. But why does the capital stock decrease under optimistic 
parameterizations in Figure 1? The reason is that the equilibrium ergodic distribution of wealth 
does not feature very rich households: at the equilibrium the wealthiest percentile owns about 3 
times the average capital, no household receives as much income from interest payments as from 
wages, and the top 0.1 percentile earns five times more from labor than from capital. These facts 
are consistent with the standard deviations reported in Table 3. Accordingly, the vast majority of 
households are below the threshold identified in Proposition 1. This implies that optimism has a 
negative effect on the savings of most households, which explains the decreasing relation between 
the expectational parameters and the mean of the distribution of wealth.

To explain the positive relation between the standard deviation of the distribution of capital 
and the expectational parameters it is possible to use a similar argument. When optimism induces
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the majority of households to consume more (and thereby save less), *realized* returns to capital increase. This, jointly with the expected negative permanent income shock, induces a small number of households – the ones that happen to be wealthiest at the time of the shock – to save more.⁹ Consistently with this explanation, the high standard deviations in Figure 1 under optimism are driven by few households that become significantly wealthier than average. For example, in the case in which \( a_1 \) increases by 0.0003, the top percentile of the ergodic distribution of wealth owns 58% of total capital, and the top 0.3 percentile owns 57% of total capital.

### 3.2 Two Experiments

In this section I conduct two experiments that show that the spectacular increases in the standard deviation of capital shown in Figure 1 are caused by the changes in expectations in a manner that is consistent with the explanation provided in the previous section. The first experiment is based on a simulation of the model at the equilibrium to obtain a time series of factor prices and aggregate shocks. Then, I simulate and track the capital accumulation patterns chosen by three experimental households—an optimist, a household with equilibrium expectations, and a pessimist— that experience a common sequence of idiosyncratic shocks and have no relevance for the aggregate economy.⁰ Figures 2 and 3 show two replications of this experiment. In both cases the optimistic household accumulates less capital than the equilibrium one, and the pessimist accumulates *significantly* higher amounts of capital. Additionally, the intensity of these effects matches the size of the expectational shocks and the decreasing effects on the pessimist are quicker than the effects on the optimist. This is consistent with the average capital patterns shown in Figure 1.¹¹

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⁹ Recall that the labels “optimism” and “pessimism” refer to aggregate investment rather than to personal income. As a matter of fact, in the case of optimism considered here, wealthy households forecast lower personal income. Also, the presence of idiosyncratic shocks implies that the households that are wealthiest when the expectational shock takes place are more likely—but not certain—to become very rich.

⁰ In the simulations I account for this by simply not aggregating the experimental households’ capital with the rest of the economy: in this way it is possible to ignore the endogenous responses of interest rates and wages that result when the expectational biases affect the whole mass of households, and therefore the results have a cleaner intuitive explanation.

¹¹ A Monte-Carlo-type experiment is unnecessary to guarantee that these results are not due to chance. The policy function of the equilibrium household is sandwiched between the optimist’s policy function from below and the pessimist’s from above. The reported experiment is nevertheless more informative than this simple comparison as it provides a quantitative assessment of the differences that expectational shocks make on the individual household’s capital accumulation patterns.
Fig. 2: Asset holdings of three households facing an identical environment. The economy-wide expectations are at the equilibrium values and the experimental households are endowed with an initial wealth equal to the cross-sectional average. The top panel shows the asset holdings of a household that has optimistic expectations (achieved by setting the parameter $a_1$ equal to $3 \times 10^{-4}$ units higher than the rest of the economy). The bottom panel shows the asset holdings of a pessimistic household (parameter $a_1$ is set $3 \times 10^{-4}$ units lower than the rest of the economy). The middle panel shows an household with equilibrium expectations.
Fig. 3: This figure shows the same type of experiment shown in Figure 2. The expectational shocks here are bigger by one order of magnitude (the parameter \( a_1 \) is set at \( \pm 3 \times 10^{-3} \).)
Fig. 4: Capital accumulation paths displayed by the experimental households in an optimistic economy. A household starting from a capital level of 2410 has a higher chance of being richer in the future than if the same household started from a wealth level of 2140. The horizontal axis measures thousands of time periods.

The second test to confirm the intuition proposed in section 3 consists in the identification of the critical threshold of wealth above which an optimistic household decides to accumulate more capital. This threshold cannot be identified analytically because condition (3) implicitly involves the optimal policy function. It is nevertheless possible to solve for it numerically by simulating an economy with optimistic households, and then tracking the capital accumulation paths of a set of households that differ only in their initial capital endowments. The results of a typical simulation of this type is displayed in Figure 4, which shows that the hypothesized critical level is somewhere around the dashed line. Table 4 reports the results of a Monte Carlo experiment involving 100 simulations of the kind shown in Figure 4. In this table I report the percentage of simulations in which a household endowed with the amount reported in each row ends with a level of wealth that is below the threshold corresponding to each column. For example, a household that is initially endowed with 1600 units of capital ends with a wealth level that does not exceed 7 units in 77 cases out of 100. When instead the household is endowed with 1870 units, this percentage drops to 13%. Between 2140 and 2410 it is possible to see a clear break in the pattern of the data which suggests that the critical level of capital lies between 2140 and 2410. This supports the proposed intuition that the polarization of the distribution of wealth is a consequence of the expectational shocks.

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\[12\] The experimental households share the optimistic expectations of the rest of the economy and, as before, they have no relevance for the aggregate economy.
Tab. 4: Results from a capital accumulation experiment, generated for economies with positive expectational shocks. Fifteen households are exposed to identical sequences of choices and their capital accumulation is tracked over an extended number of time periods. Each entry of the table shows the percentage of simulations in which the households starting with the wealth level specified in the rows end with total wealth below the levels specified by each column. Somewhere between 2140 and 2410 there lies the critical level of wealth of Proposition 1.
3.3 Self-Insurance and Precautionary Savings

In this model, the households are exposed to uninsurable shocks and therefore they accumulate extra capital (precautionary savings) to serve as self-insurance against prolonged spells of unemployment. However, the demand for precautionary savings is unlikely to drive the increased dispersion of the distribution of wealth. There are two reasons for not considering precautionary savings in this discussion. First, Aiyagari (1994) and Diaz, Pijoan-Mas, and Rios-Rull (2003) argue that precautionary savings account only for a small proportion of total wealth in this type of models, and therefore consumption smoothing and intertemporal substitution motives dominate the dynamics of individual savings decisions. Second, expectational shocks change precautionary savings in the same direction for all wealth levels. When households are optimistic, the stochastic environment appears more favorable because the positive payoff (the wage in case of employment) is expected to increase, and the probabilities do not change. Furthermore, as interest payments decrease, capital accumulation becomes a less effective form of self-insurance, and therefore the net effect of positive expectational shocks on precautionary savings is unambiguously negative, independent of the household’s individual level of wealth. Opposite and symmetric considerations apply to the case of negative expectational shocks.

4 Learning

The analysis conducted this far concerns a permanent one-time change in expectations. In this section I focus instead on the effect of transitory expectational shocks and the dynamics that result under least-squares learning. The main question is whether transitory shocks can produce significant effects in the macroeconomy and particularly whether higher levels of inequality arise as a consequence.

The fact that the numerical procedure used to find the equilibrium is itself based on adaptive-learning, does not imply that it is trivial to anticipate how households’ learning will impact the model’s dynamics. In fact, under learning there exists an endogenous interaction between the current estimates of the expectational parameters and the process that generates the data used to update those estimates. This raises the issue of learnability of the equilibrium which can be addressed through the E-stability principle of Evans and Honkapohja (2001), i.e. a direct test for the stability of an equilibrium under least-squares learning. After having established expectational
stability, I simulate the transition dynamics of a model economy hit by a positive but small expectational shock. The main reason to favor small shocks is that E-stability is a local result, and consequently issues related to multiplicity of equilibria are less likely to arise. The shortcoming is the full effect of small shocks on the model’s dynamics take a large number of periods to unfold.

4.1 E-Stability

I assume that the households believe, as they did before, that the time-series of aggregate capital follows a state-contingent log-linear AR(1) process. Unlike before, they are now assumed to not know the equilibrium values of this process (i.e. $a_0^*$, $a_1^*$, $c_0^*$, and $c_1^*$ from Table 3) but rather they use the OLS estimator on the data available to them. The specification of the regression used by the households, together with the current estimates for its parameters, are known in the learning literature as the households’ Perceived Law of Motion (PLM). Each household saving-consumption plan obviously depends on their PLM, and the resulting choices, once aggregated, determine the economy’s Actual Law of Motion (ALM). The learning process can be conceptualized as a cycle between the PLM and the ALM, in which the time-$t$ PLM determines the ALM, which, in turn produces the data that yields the PLM at time $t+1$. Letting $\Phi_t \equiv [\hat{a}_{0,t}, \hat{a}_{1,t}, \hat{c}_{0,t}, \hat{c}_{1,t}]'$ be the vector of the time-$t$ estimates of the expectational parameters, the learning process can be summarized with the recursion $\Phi_{t+1} = T(\Phi_t)$, where $T$ is an unknown map that takes the current parameter estimates and maps them into their updated values. This map depends both on the behavior of the aggregate economy as well as on the learning algorithm, which is specified as follows

$$\Phi_t = \Phi_{t-1} + \gamma_t R_t^{-1} x_{t-1} [\ln \mu_t - x_{t-1} \Phi_{t-1}] \quad (4a)$$

$$R_t = R_{t-1} + \gamma_t (x_{t-1}' x_{t-1} - R_{t-1}) \quad (4b)$$

where $x_t = [I_g \ I_g \ I_b \ I_b \ln \mu_{t-1} \ I_b \ln \mu_{t-1}]$, $I_g$ is a dummy with value 1 if $z_t = z_g$ ($I_b = 1 - I_g$), and $\{\gamma_t\}_{t=0}^{\infty}$ is a positive non-increasing sequence of real numbers satisfying $\sum \gamma_t = \infty$ and $\sum \gamma_t^2 < \infty$.\(^\text{13}\)

Using the notation introduced in this section, the equilibrium expectational parameters satisfy the condition $\Phi^* = T(\Phi^*)$. E-stability requires that the matrix of first derivatives of $T$ evaluated at

\(^{13}\) I use dummies in the regression to keep the notation at a minimum. The matrix $R$ is block diagonal by construction, and allowing this type of heteroskedasticity in the regressions allows the estimation of a specification of the form (2b) with one regression only.
Φ∗ (DT|Φ∗) has eigenvalues with real parts less than one, in which case the equilibrium is (locally) stable under learning.

In the heterogeneous-agents economy with aggregate shocks the T-map cannot be pinned down analytically and consequently the evaluation of its matrix of first derivatives will be conducted numerically. This is done by initially setting the PLM at the equilibrium values and first increasing and then decreasing each parameter by a small quantity. The resulting pair of policy functions is used to perform two long Monte-Carlo simulations (again, an initial chunk of data is discarded to ensure ergodicity). The resulting ALMs are estimated from the data so generated. The ratios of the changes in the estimates relative to the small initial changes in the PLM provide a numerical estimate of one of the rows of DT|Φ∗. A detailed step-by-step description of the algorithm used to differentiate the map T is given in the appendix. Application of this algorithm yields the following numerical estimates

$$DT|Φ∗ = \begin{bmatrix}
2.6 & -1.7 & 3.4 & -1.9 \\
13.7 & -6.7 & 18.1 & -8.3 \\
2.9 & -1.7 & 3.1 & -1.8 \\
14.2 & -6.7 & 17.7 & -8.2
\end{bmatrix}.$$ 

The biggest eigenvalue of this matrix is equal to -0.12, so the equilibrium Φ∗ is locally stable under learning.

### 4.2 Transitional Dynamics Under Learning

To simulate the learning dynamics, in each period the set of expectational parameters is used to derive the consumption function of the households. Individual choices are then aggregated and a new data point is released to the households who use it in equations (4) thus obtaining new estimates to be used in the next period. The full details of the learning algorithm are given in the Appendix.

The model is first simulated at the equilibrium for an extended time to ensure that the expectational shock hit in the ergodic distribution of capital. Figure 5 shows the time-paths of several quantities that result after a positive expectational shock obtained by increasing the parameter a1 by 0.001. The size of this shock is small, as it implies an initial bias in the unconditional expectations for interest rates of -0.15% and of +2.5% for wages. Using small shocks decreases the chances that the transition dynamics will not converge back to the equilibrium. The learning gain
Fig. 5: Time paths of the estimated parameters values and of the the first two centered moments of the distribution of wealth. The initial distribution of wealth is the ergodic distribution obtained by a long simulation of the model at the equilibrium parameter values. The initial parameter values are such that the households are initially optimistic.
parameter progresses according to

\[ \gamma_1 = 5 \times 10^{-4}, \quad \gamma_{t+1} = \frac{\gamma_t}{\gamma_t + 1}. \]

which is a fairly common schedule for the gain parameter.

The initial shock to the value of \( a_1 \) affects all the expectational parameters and the saving behavior of the households changes in a way that is consistent with the intuition presented above. The majority of the households reacts to the expectational shock by decreasing savings and increasing consumption immediately. The drop of capital (bottom left panel of Figure 5) causes an increase in the interest rate, and a small minority of households starts slowly accumulating more capital. Since the positive effect on aggregate savings by the wealthier minority is slower to build up than the negative effect of the poorer households (see Figures 2 and 3), aggregate capital displays an abrupt decline followed by a slow and sustained increase back to higher values. In the mean time, the learning dynamics converge back to the equilibrium dynamics, and once the expectational parameters are close enough to the equilibrium values, the wealthier (poorer) households revert towards more (less) consumption, and the wealth distribution converges back to the pre-shock process.

The main implication conveyed by this quantitative exercise is that small expectational shocks are capable of increasing significantly wealth inequality: expectational shocks concerning the level of aggregate capital of the order of magnitude of few percentage points can more than double the standard deviation of the cross sectional distribution of capital. Furthermore, the second moment of the distribution of wealth increase steeply and steadily immediately after the expectational shock, but the return to the equilibrium levels is much slower: the expanding phase of the hump in the bottom right panel lasts six times less than the decreasing phase. Furthermore, high levels of inequality persist long after the average of the distribution of wealth returns to values close to the equilibrium ones. This indicates that the inequality induced by expectational shocks is particular persistent in this framework.

5 Conclusion

This paper illustrates that expectational shocks have interesting effects on the dynamics of heterogeneous-agents models. One consequence of this is that one must take into careful
consideration the determination of expectations in this class of models. Furthermore, while the centrality of expectations has been recognized for a long time in macroeconomics, this paper appears to be the first quantitative analysis of their effect on the distribution of wealth. The simulation in Figure 5 indicates that in this model a positive expectational shock produces a recession. This is the issue studied by Beaudry and Portier (2004) for representative-agent models, and the heterogeneous-agent economy with aggregate shocks is evidently no exception to their finding that one needs a richer description of the productive side of the economy to reverse this model-based correlation. The logical next step in this line of research involves the inclusion of Pigou cycles, and the quantitative assessment of the linkages between macroeconomic aggregates and wealth and income inequality.

Appendices

A General Proof of Proposition 1

**Proposition 2.** A household with expectational operator $E^o$ will anticipate higher personal income in the future if and only if

$$k_{t+1}^i < -\frac{(1 - \alpha)E_t \left[ \frac{z_{t+1}l_{t+1}^{\alpha}}{\mu_{t+1}} \right] Cov^*[\mu_{t+1}, \mu_{t+1}^{\alpha}]}{\alpha E_t \left[ \frac{z_{t+1}l_{t+1}^{\alpha}}{\mu_{t+1}} \right] Cov^*[\mu_{t+1}, \mu_{t+1}^{\alpha-1}]} \quad (5)$$

**Proof:** Household $i$’s expected income is

$$E_tE^o[y_{t+1}^i] = P_1 z_g \alpha E^o \left[ \frac{\mu_{t+1}}{l_g^{\alpha-1}} \right] k_{t+1}^i + P_1 z_g (1 - \alpha) E^o \left[ \frac{\mu_{t+1}}{l_g^{\alpha-1}} \right] P_1^1$$

$$+ P_2 z_b \alpha E^o \left[ \frac{\mu_{t+1}}{l_b^{\alpha}} \right] k_{t+1}^i + P_2 z_b (1 - \alpha) E^o \left[ \frac{\mu_{t+1}}{l_b^{\alpha}} \right] P_2^1$$

where $P_1 = P_{z_t z_g}$, $P_1^1$ is prob. $i$ is employed conditional on the aggregate transition being $z_t \rightarrow z_g$. $P_2 = 1 - P_1 = P_{z_t z_b}$.

$$E_tE^o[y_{t+1}^i] = P_1 z_g \alpha l_g^{1-\alpha} E^o \left[ \frac{\mu_{t+1}^{\alpha-1}}{l_g^{\alpha}} \right] + P_1 z_g (1 - \alpha) l_g^{-\alpha} P_1^1 E^o[\mu_{t+1}^{\alpha}]$$
\[
+ P_2 z_b \alpha b^{l_{t+1}} k_{t+1} E^o \left[ \mu_{t+1} \right] + P_2 z_b (1 - \alpha) l_{t+1}^{l_{t+1}} k_{t+1} E^o \left[ \mu_{t+1} \right] \\
= \alpha k_{t+1} E^o \left[ \mu_{t+1} \right] E_t \left[ \frac{z_{t+1}^{l_{t+1}}}{l_{t+1}} \right] + (1 - \alpha) E^o \left[ \mu_{t+1} \right] E_t \left[ \frac{z_{t+1}^{l_{t+1}}}{l_{t+1}} \right] \]
\]

A change in the household’s regression parameters will change the mean of the normal probability distribution function used by the households to perform forecasts. Conditional on time \( t \) information I will denote the conditional mean with \( \mu^* \). That is
\[
\mu^* = \begin{cases} 
\mu_1^{\alpha_1} & \text{if } z_t = z_g \\
\mu_1^{\alpha_1} & \text{if } z_t = z_b 
\end{cases}
\]

The effect of optimistic expectations is singled out by calculating
\[
\frac{\partial E_t E^o[y_{t+1}]}{\partial \mu^*} = \alpha k_{t+1} X_1 \frac{\partial E^o[\mu_{t+1}^{\alpha_1}]}{\partial \mu^*} + (1 - \alpha) X_2 \frac{\partial E^o[\mu_{t+1}^{\alpha_1}]}{\partial \mu^*} 
\]

So one needs to calculate \( \frac{\partial E^o[\mu_{t+1}^{\alpha_1}]}{\partial \mu^*} \)
\[
\frac{\partial E^o[\mu_{t+1}^{\alpha_1}]}{\partial \mu^*} = \frac{\partial}{\partial \mu^*} \int x^\alpha \frac{1}{\sqrt{2\pi\sigma}} \exp \left[ -\frac{(x - \mu^*)^2}{2\sigma^2} \right] dx 
= \int x^\alpha (x - \mu^*) \frac{1}{\sqrt{2\pi\sigma}} \exp \left[ -\frac{(x - \mu^*)^2}{2\sigma^2} \right] dx 
= \frac{1}{\sigma^2} \left[ \int x^\alpha \phi(x) dx - \mu^* \int x^\alpha \phi(x) dx \right] 
= \frac{1}{\sigma^2} \left( E^o[\mu_{t+1}^{\alpha_1}] - \mu^* E^o[\mu_{t+1}^{\alpha_1}] \right) 
\]

And similarly
\[
\frac{\partial E^o[\mu_{t+1}^{\alpha_1} - \mu^*]}{\partial \mu^*} = \frac{1}{\sigma^2} \left( E^o[\mu_{t+1}^{\alpha_1}] - \mu^* E^o[\mu_{t+1}^{\alpha_1}] \right) 
\]

Now consider the two random variables \( X = \mu_{t+1} \) and \( Y = X^{\alpha_1} \). Obviously \( X \) and \( Y \) are not independent and exploiting the fact that \( \text{Cov}[X, Y] = E[XY] - E[X]E[Y] \) one sees that \( \frac{\partial E^o[\mu_{t+1}^{\alpha_1}]}{\partial \mu^*} = \text{Cov}^*[\mu_{t+1}, \mu_{t+1}^{\alpha_1}] / \sigma^2 \). With a similar line of reasoning it is possible to show that setting \( Y = X^{\alpha} \)

\[
\frac{\partial E^o[\mu^o_{t+1}]}{\partial \mu'^*} = \frac{\text{Cov}^*[\mu^o_{t+1}, \mu^a_{t+1}]}{\sigma^2}
\]

Finally the condition \( \frac{\partial E_t}{\partial \mu^*} > 0 \) requires

\[
\alpha k^i_{t+1} X_1 \text{Cov}^*[\mu^o_{t+1}, \mu^a_{t+1}] > -(1 - \alpha) X_2 \text{Cov}^*[\mu^o_{t+1}, \mu^a_{t+1}]
\]

\[
k^i_{t+1} < -\frac{(1 - \alpha) X_2 \text{Cov}^*[\mu^o_{t+1}, \mu^a_{t+1}]}{\alpha X_1 \text{Cov}^*[\mu^o_{t+1}, \mu^a_{t+1}]}
\]

where the reversal of the direction of the inequality sign is necessary because \( \text{Cov}^*[\mu^o_{t+1}, \mu^a_{t+1}] < 0 \).

This is because \( \mu^o_{t+1} \) is a decreasing function of \( \mu^a_{t+1} \). Finally, since \( \mu^a_{t+1} \) is an increasing function of \( \mu^o_{t+1} \), the quantity of the RHS is positive. This proves that optimism about the capital accumulation process implies a higher personal income for household \( i \) if and only if \( i \) is not “too wealthy.”

\[\square\]

**An Algorithm to Calculate \( DT \)**

The matrix of first derivatives of the \( T \) map is derived numerically, by changing the PLM and deriving the implied ALM. The following steps describe the algorithm to derive the matrix of first derivatives of the \( T \) map at the equilibrium.

1. Set \( \iota = 1 \) and pick a small constant \( \vartheta \).

2. Set \( \Phi = [a^*_0 \ a^*_1 \ c^*_0 \ c^*_1]' \), and let \( \Phi[i] \) denote the \( i \)-th element of \( \Phi \). Set \( \Phi[i] = \Phi[i] + \vartheta \).

3. Using a value function iteration algorithm derive the policy function \( \hat{f}^+_i \).

4. Using 5000 households, run an 210,000-periods long Monte Carlo experiment using \( \hat{f}^+_i \) to simulate the economy.

5. Drop the first 200,000 periods of the simulation and use OLS on the remaining sample to obtain an estimate of the ALM: \( (a_0)^+_i, \ (a_1)^+_i, \ (c_0)^+_i \), and \( (c_1)^+_i \)

6. Set \( \Phi = [a^*_0 \ a^*_1 \ c^*_0 \ c^*_1]' \) and \( \Phi[i] = \Phi[i] - \vartheta \). Repeat Steps 3, 4, and 5 but this time denote the implied ALM as \( (a_0)^-_i, \ (a_1)^-_i, \ (c_0)^-_i, \) and \( (c_1)^-_i \)
7. Populate the \( i \)-th row of \( DT \) as follows

\[
DT_i = \left[ \frac{(a_0)^+ - (a_0)^-}{2\vartheta} \quad \frac{(a_1)^+ - (a_1)^-}{2\vartheta} \quad \frac{(c_0)^+ - (c_0)^-}{2\vartheta} \quad \frac{(c_1)^+ - (c_1)^-}{2\vartheta} \right]
\]

8. If \( i < 4 \) set \( i = i + 1 \) and repeat from Step 2.

The matrix reported in the main text is derived by using \( \vartheta = 0.0001 \), which is appropriate when numbers are have a double precision representation in the computer.

**Least Squares Learning Algorithm**

The simulation of the model under least-squares learning is implemented in these steps.

1. Choose an initial point in the parameter space \( \Phi_0 \) and initial distribution of wealth \( \Gamma_0 \), a simulation length \( T \), and the number of households \( N \).

2. Choose a constant \( \gamma_0 \) and calculate the gain \( \gamma_t = \frac{\gamma_{t-1} - \gamma_{t-1}}{\gamma_{t-1} + \gamma_{t-1}} \) for \( t = 1, 2, \ldots, T \).

3. Set \( t = 0 \)

4. Solve the dynamic program (2).

5. Draw an aggregate shock and \( N \) idiosyncratic shocks. Use the policy obtained in step 4 to determine the individual saving decisions, and aggregate them to obtain \( \mu_{t+1} \).

6. Use equations (4) to update \( \Phi_t \).

7. Set \( t = t + 1 \) and repeat from step 4 until \( t = T \).

The initial distribution of wealth is obtained as the final distribution of a long simulation at the equilibrium, and the initial values for the parameters in \( \Phi \) are consistent with a positive expectational shock. Also, this algorithm requires the initialization of the matrix \( R_0 \) for the application of the recursive least squares algorithm, as it can be seen from equations (4). I use the variance-covariance matrix of a regression specified as in (2b) run data from a simulation at the equilibrium.
References


