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Partial Knowledge Restrictions on the Two-Stage Threshold Model of Choice

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Abstract

In the context of the two-stage threshold model of decision making, with the agent's choices determined by the interaction of three "structural variables," we study the restrictions on behavior that arise when one or more variables are exogenously known. Our results supply necessary and sufficient conditions for consistency with the model for all possible states of partial knowledge, and for both single- and multi-valued choice functions.

1 Introduction

Recent work in the theory of individual choice behavior has modified the classical preference maximization hypothesis in various ways. One approach has been to weaken the consistency properties that preferences are ordinarily assumed to possess.[1] Another has been to study relationships between preference and choice other than straightforward maximization.[2] And a third has been to permit additional, non-preference-related factors—as well as multiple preferences—to influence decision making in some way.[3]

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[1]For example, Eliaz and Ok [9], Mandler [14], Nishimura and Ok [23], and others allow preferences to be incomplete (following in the tradition of Aumann [2] and Bewley [5]).

[2]Models of this sort have been axiomatized by Baigent and Gaertner [3], Eliaz et al. [10], Mariotti [19], and Tyson [30], among others.

[3]In addition to the contributions cited below, we have for example the work of Bossert and Sprumont [6] and Masatlioglu and Ok [22] on status-quo bias; Ambrus and Rozen [1] and Rubinstein and Salant [25] on multi-self and framing models; Caplin and Dean [7],

Under the last of the three approaches just listed, the revealed preference exercise required to characterize a given model can be quite complex, since multiple factors must be inferred simultaneously from behavior. Moreover, models with more than one component make possible a variant of the usual characterization problem: An outside observer can test a collection of choice data for consistency with the model *while treating one or more components as known*.

For example, suppose that we postulate a decision maker who maximizes a utility function over the alternatives that he or she notices, but pays attention only to those options with a sufficiently high level of salience (with regard to the visual or another sensory system). If salience is directly measurable, then the relevant question is whether these measurements and the choice data together can be reconciled with our behavioral hypothesis.[4] And this means, of course, finding suitable assignments of the unobserved components—namely, the utility function and the salience thresholds.

As another example, imagine a choice among lotteries by a satisficing agent who decides between the options deemed satisfactory by following a social-norm ordering. On the one hand, the social norm might be known to the theorist, in which case it and the choice data must be jointly reconciled with the model by specifying the utilities and satisficing thresholds. Alternatively, perhaps the norm is unknown but we wish to introduce a maintained assumption of risk neutrality. In the latter case our search will be for satisficing thresholds (relative to expected value) plus a social norm that together generate the observed behavior.

Evidently, questions of this sort can be posed for any multiple-component model of choice, with any subset of the components taken to be known. In an electoral setting we might plausibly know the economic interests of a voter but not his or her ideology, while in a managerial setting we might assume profit maximization subject to an unobserved market-share constraint. Note that a model component could be designated as "known" due to an assumption, a physical observation, econometric estimates from a separate data set, or background knowledge of the agent's environment, among other reasons.

In this paper we explore the issue of testing model consistency under partial knowledge—one that appears to be largely unexamined in the context of axiomatic choice theory. To give this enterprise some concreteness, we shall commit to a particular model of how choices are determined by the

Cherepanov et al. [8], and Masatlioglu and Nakajima [20] on search and consideration sets; and Mandler et al. [15], Manzini and Mariotti [17], and Bajraj and Ülkü [4] on procedural models.

[4]The observer might be able to determine salience levels, say, using knowledge of the physiology of vision and the spatial arrangement of the choice alternatives.

interaction of various factors. We adopt a framework that is deliberately very general, and can accommodate the examples mentioned above. For each menu A of alternatives, the "two-stage threshold" (TST) model of choice specifies that the agent will select an option that solves

$$\max_{x \in A} g(x) \text{ subject to } f(x) \geq \theta(A). \tag{1}$$

Here the model components, which we shall call "structural variables," are real-valued functions f and g defined on the space of alternatives, plus a real-valued function θ defined on the space of menus.

The TST framework has no fixed interpretation. Indeed, the model overlaps with several existing theories based on very different hypotheses about the process of decision making. One possibility is to interpret f as a measure of consideration or attention priority, θ as a cognition-threshold map, and g as a utility function; as in the contributions of Lleras et al. [13] and Masatlioglu et al. [21].[5] Another possibility is to interpret f as the utility function, θ as a utility-threshold map, and g as a salience measure; as in Tyson [31]. Under these two interpretations the first stage of the model captures, respectively, the "consideration set" (a concept from the marketing literature) and Simon's [27] notion of satisficing.[6]

In its general form the TST model has been characterized by Manzini et al. [18], who demonstrate that Equation 1 can accommodate a wide range of behavior patterns. Indeed, when each set of acceptable choices is required to be a singleton, it is straightforward to show that *any* observed data set can be generated by the model (see Proposition 2.6). Moreover, even if we allow multiple acceptable choices, the constraints imposed by the framework itself remain weak (see Theorem 2.5). While the theories mentioned above reduce this freedom by imposing specialized restrictions on the structural variables, our approach will be to fix one or more variables completely and leave the others entirely unconstrained.[7] We then seek to identify the forms of behavior that remain consistent with the model.

Given a particular interpretation of the model, some structural variables will be more naturally assumed to be known than others. Since our intention is to avoid favoring any specific viewpoint, we provide a complete and hence interpretation-free collection of characterization results: For any strict subset of the three structural variables, we supply necessary and sufficient conditions for behavior to be compatible with the TST model when the variables in the

[5] Related models are studied by Eliaz and Spiegler [11] and Spears [28].

[6] For further details of these interpretations of the TST framework, see [18, pp. 879–881].

[7] These two approaches can also be combined. For instance, Theorem 4.7 below can be modified to incorporate the "expansiveness" restriction on $\langle f, \theta \rangle$ imposed by Tyson [31].

subset are known and all others are unrestricted.[8] This collection of results—together with posing the partial knowledge question for multiple-component choice models—makes up the contribution of the paper.

Broadly speaking, our analytical method is to use the choice data together with the known variables to infer as much information as we can about the unobserved variables. We then look for ways in which this information could be self-contradictory, and formulate axioms that rule them out. Such axioms will always be necessary for behavior to be compatible with the model. And if our search for contradictions is thorough enough, they will also be sufficient (though demonstrating this may require extended arguments).

For example, suppose that g is known while both f and θ are unobserved (cf. Theorem 4.7). If alternatives x and y are both on menu A, and if also $g(x) > g(y)$, then clearly x and y cannot both be chosen from A. This is the simplest illustration of how choice data and a known structural variable together can lead to a contradiction, which must be ruled out axiomatically.

Suppose now that f and g are both known, with only θ unobserved (cf. Theorem 4.8). Since g is known, the variety of contradiction seen in the preceding paragraph must still be avoided. Furthermore, if alternatives x and y are both on menu A, and if also $f(x) \geq f(y)$ and $g(x) \geq g(y)$, then we cannot have that y is chosen from A unless x too is chosen. These two types of contradictions turn out to exhaust the implications of the model when both f and g are known, which is to say that axioms ruling them out provide the desired characterization.

The remainder of the paper is structured as follows. Section 2 defines the TST framework and reviews the axiomatization of the unconstrained model given by Manzini et al. [18]. Our novel results are stated first in Section 3 for the special case of single-valued choice, and then in Section 4 for the multi-valued case. Section 5 contains a brief concluding discussion. Proofs of the general (i.e., multi-valued) versions of our results can be found in the Appendix.

[8] We assume that knowledge of one structural variable has no direct implications for the unknown variables, which can be chosen arbitrarily to generate the observed behavior. This assumption will not hold under interpretations of the model that motivate joint restrictions on the variables. For example, in [31] the functions f and θ are linked by the property of "expansiveness." It is even possible that knowledge of one variable could completely determine another, for instance if $\theta(A)$ equals the average $|A|^{-1} \sum_{x \in A} f(x)$ of the available f-values. Dependencies like these could certainly be taken into account in the characterization exercises we carry out, but we shall not impose any such link between structural variables as a blanket restriction on the model.

2 The two-stage threshold model

Let X be a nonempty, finite set, and let $\mathcal{D} \subseteq \mathcal{A} = 2^X \setminus \{\emptyset\}$. The elements of X are called *alternatives*, the elements of $\mathcal{D}$ are called *menus*, and any map $C : \mathcal{D} \to \mathcal{A}$ such that $\forall A \in \mathcal{D}$ we have $C(A) \subseteq A$ is called a *choice function*. The *choice set* $C(A)$ contains the alternatives that are chosen from menu A. A choice function is *single-valued* if it returns only singleton choice sets. Without loss of generality, we shall assume that $\forall x \in X$ we have $\{x\} \in \mathcal{D}$.

In the TST model, the choice set associated with menu A is constructed by maximizing $g(x)$ subject to $f(x) \geq \theta(A)$. Here $f : X \to \Re$ is the *primary criterion*, $g : X \to \Re$ the *secondary criterion*, and $\theta : \mathcal{D} \to \Re$ the *threshold map*. These three components of the model are termed *structural variables*, any triple $\langle f, \theta, g\rangle$ is a *profile*, and any pair $\langle f, \theta\rangle$ is a *primary profile*.

Given a primary profile $\langle f, \theta\rangle$ and an $A \in \mathcal{D}$, write $\Gamma(A|f, \theta) = \{x \in A : f(x) \geq \theta(A)\}$ for the subset of available alternatives whose primary criterion values are above the relevant threshold. The TST model can now be defined formally as follows.

2.1 Definition. A *two-stage threshold representation* of C is a profile $\langle f, \theta, g\rangle$ such that $\forall A \in \mathcal{D}$ we have $C(A) = \operatorname{argmax}_{x\in\Gamma(A|f,\theta)} g(x)$.

In order to axiomatize this model, Manzini et al. [18] use several binary relations that are revealed by the agent's choices. The *separation relation* encodes situations where one alternative is chosen and a second (available) alternative is rejected.

2.2 Definition. Let xSy if $\exists A \in \mathcal{D}$ such that $x \in C(A)$ and $y \in A \setminus C(A)$.

The *togetherness relation* encodes situations where two alternatives both are chosen, and its transitive closure is the *extended togetherness relation*.[9]

2.3 Definition. Let xTy if $\exists A \in \mathcal{D}$ such that $x, y \in C(A)$, and let xEy if $\exists z_1, z_2, \ldots, z_n \in X$ such that $x = z_1Tz_2T \cdots Tz_n = y$.

Finally, the *first-stage separation relation* encodes separations that must be attributed to the primary criterion, since an extended togetherness relationship guarantees equal values of the secondary criterion.[10]

2.4 Definition. Let xFy if xEy and xSy.

[9] Recall that a relation R is transitive if $xRyRz \Rightarrow xRz$, and that the transitive closure of R is the smallest transitive relation containing it.

[10] More explicitly, if xEy then $x = z_1Tz_2T \cdots Tz_n = y$, which under the TST model implies that $g(x) = g(z_1) = g(z_2) = \cdots = g(z_n) = g(y)$ for some $z_1, z_2, \ldots, z_n \in X$. If also xSy, then under the model we must have $f(x) \geq \theta(A) > f(y)$ for some $A \in \mathcal{D}$.

Manzini et al. [18, pp. 876–879] prove that acyclicity of this last relation is the one and only condition needed to characterize the TST model in the absence of known structural variables.[11]

2.5 Theorem. *A choice function has a two-stage threshold representation if and only if the relation F is acyclic.*

Moreover, since the relations T, E, and F are all empty in the single-valued case, here the acyclicity condition holds trivially and so consistency with the model is assured.

2.6 Proposition. *Any single-valued choice function has a two-stage threshold representation.*

3 Single-valued choice

3.1 One known variable

We study partial knowledge restrictions on the TST model first under the simplifying assumption of single-valued choice, and considering situations where just one of the three structural variables is known. As we shall see, even this minimal type of partial knowledge restriction gives empirical content to a framework that, as shown in Proposition 2.6, is otherwise vacuous: In each situation with one known variable the model can be characterized by an acyclicity condition like that employed in Theorem 2.5.

Known primary criterion. Suppose that the primary criterion f is known, while the threshold map θ and secondary criterion g are not, and let $x, y \in A$. In this case knowledge that $f(y) \geq f(x)$ implies that y survives the first stage of choice from A if x does so; i.e., that $x \in \Gamma(A|f, \theta) \Rightarrow y \in \Gamma(A|f, \theta)$. But then from the observation xSy we can deduce that $g(x) > g(y)$, since x and y must have been separated at the second stage. In other words, second-stage superiority of one alternative over another is revealed by the relation defined as follows in terms of the known f and observed C.

3.1 Definition. Let xH_fy if $f(y) \geq f(x)$ and xSy.

Since H_f implies *strict* second-stage superiority between alternatives, this relation must be acyclic for the model to hold. That is to say, acyclicity of H_f is necessary for C to admit a TST representation consistent with the partial profile $\langle f, \cdot, \cdot \rangle$.

[11] Recall that a relation R is acyclic if $x_1Rx_2R\cdots Rx_n \Rightarrow x_1 \neq x_n$.

3.2 Example. Let $f(x) = f(y) = 1$, $C(xy) = x$, and $C(xyz) = y$. If $\langle f, \theta, g \rangle$ were a TST representation of C, then $f(y) \geq f(x)$ and $C(xy) = x$ together would imply $g(x) > g(y)$, but at the same time $f(x) \geq f(y)$ and $C(xyz) = y$ would imply $g(y) > g(x)$, a contradiction.

Our first partial-knowledge result for the TST model states that acyclicity of H_f is sufficient as well as necessary, and thus characterizes the situation in question.[12]

3.3 Proposition. *A single-valued choice function has a two-stage threshold representation consistent with* $\langle f, \cdot, \cdot \rangle$ *if and only if* H_f *is acyclic.*

Observe that once we have acyclicity of H_f, which is needed to construct a suitable revealed secondary criterion, no further axiom is required to ensure the existence of an appropriate threshold map—this comes "for free."

Known secondary criterion. Now suppose that it is the secondary criterion g that is known, so that our task is to construct the primary profile. Here the first-stage separation relation used in Theorem 2.5 is unhelpful (since it is always empty in the single-valued case), but we can use our knowledge of g to define a new revealed relation that performs the same role.

3.4 Definition. Let $xH_g y$ if both $g(y) \geq g(x)$ and xSy.

3.5 Proposition. *A single-valued choice function has a two-stage threshold representation consistent with* $\langle \cdot, \cdot, g \rangle$ *if and only if* H_g *is acyclic.*

Somewhat surprisingly, both the definitions of H_f and H_g and the associated characterization results are exactly parallel, even though the two stages of the TST model are not in any sense interchangeable.

Known threshold map. Next we consider the possibility of a known threshold map θ. In this situation if we can find $A, B \in \mathcal{D}$ and $y \in A \cap B$ such that $y \in C(B)$ and $\theta(B) \geq \theta(A)$, then clearly we can conclude that $f(y) \geq \theta(A)$. If moreover both $x \in C(A)$ and $y \notin C(A)$ (noting that under single-valuedness the latter is immediate for $y \neq x$), then y must have been eliminated from A at the second stage and hence we must have $g(x) > g(y)$.

To capture this method of deducing second-stage superiority from the known θ and observed C, we define the *critical threshold* for alternative y.

3.6 Definition. Let $M(y|\theta) = \max\{\theta(A) : A \in \mathcal{D} \wedge y \in C(A)\}$.

[12]Since Proposition 3.3 and all other results in Section 3 follow from the corresponding results in Section 4, they are not given separate proofs.

In other words, the critical threshold is the highest threshold of any menu to whose choice set the alternative belongs, with the obvious consequence that $y \in C(A) \Rightarrow M(y|\theta) \geq \theta(A)$.[13] The argument above is then expressed in the construction of the following revealed relation.

3.7 Definition. Let $xH_\theta y$ if $\exists A \in \mathcal{D}$ such that $M(y|\theta) \geq \theta(A)$, $x \in C(A)$, and $y \in A \setminus C(A)$.

Once again our characterization imposes acyclicity, and no additional axioms are needed.

3.8 Proposition. *A single-valued choice function has a two-stage threshold representation consistent with* $\langle \cdot, \theta, \cdot \rangle$ *if and only if* H_θ *is acyclic.*

3.2 Two known variables

To complete our analysis of single-valued choice functions, we proceed to axiomatize the TST model in situations where two of the structural variables are known.

Known secondary criterion plus one primary variable. When both criteria are known, the assertion $xH_f y$ continues to imply that $g(x) > g(y)$. But instead of merely being checked for cycles, as in Proposition 3.3, inequalities of this sort can now be tested directly against the observed g.

3.9 Example. Let $f(x) = 1$, $f(y) = 2$, $g(x) = g(y) = 0$, and $C(xy) = x$. If $\langle f, \theta, g \rangle$ were a TST representation of C, then $f(y) \geq f(x)$ and $C(xy) = x$ together would imply $g(x) > g(y)$, which is known to be false.

Our result for this situation states that consistency of deduced and observed second-stage superiority is precisely what is needed for the desired characterization.

3.10 Proposition. *A single-valued choice function has a two-stage threshold representation consistent with* $\langle f, \cdot, g \rangle$ *if and only if* $xH_f y \Rightarrow g(x) > g(y)$.

In a similar way, our axiom for the case of known θ can be adapted to characterize the case of known θ and g. Instead of checking the revealed relation H_θ for cycles, we simply test it against the observed secondary criterion.

3.11 Proposition. *A single-valued choice function has a two-stage threshold representation consistent with* $\langle \cdot, \theta, g \rangle$ *if and only if* $xH_\theta y \Rightarrow g(x) > g(y)$.

[13] Note that since for each $y \in X$ we have $\{y\} \in \mathcal{D}$ and $y \in C(\{y\})$, and since $\mathcal{D}$ is a finite set, the critical threshold is always well defined.

Known primary profile. Finally, consider a situation in which the full primary profile $\langle f, \theta \rangle$ is known. Since f is known, the condition $M(y|\theta) \geq \theta(A)$ in the definition of H_θ (which guarantees that y survives the first stage of choice from menu A) can be replaced with a direct assumption that $f(y) \geq \theta(A)$. This modification leads to the following revealed relation, which continues to encode second-stage superiority.

3.12 Definition. Let $xH_{f\theta}y$ if $\exists A \in \mathcal{D}$ such that $f(y) \geq \theta(A)$, $x \in C(A)$, and $y \in A \setminus C(A)$.

As might be expected, our new relation must be acyclic if C is to admit a TST representation. This acyclicity is not sufficient, however, because it is also possible for the known $\langle f, \theta \rangle$ and the observed C to contradict each other outright—in a way that has nothing to do with inferences about the second stage.

3.13 Example. Let $f(x) = 1$, $f(y) = 3$, $\theta(xyz) = 2$, and $C(xyz) = x$. If $\langle f, \theta, g \rangle$ were a TST representation of C, then $x \in C(xyz)$ would imply that $f(x) \geq \theta(xyz)$, which is known to be false.

The additional axiom required says simply that alternatives in the choice set assigned to A can have primary criterion values no smaller than $\theta(A)$.[14] Our characterization result for the present case therefore appears as follows.

3.14 Proposition. *A single-valued choice function has a two-stage threshold representation consistent with* $\langle f, \theta, \cdot \rangle$ *if and only if* $H_{f\theta}$ *is acyclic and* $f[C(A)] \geq \theta(A)$.

Our axiomatizations of the TST model with partial knowledge and single-valued choice are summarized in Figure 1, which also shows the logical relationships among these results. For example, starting from Proposition 3.10, we can discard our knowledge of g and thereby arrive at Proposition 3.3. Hence any choice function covered by the former result must be covered by the latter (as well as by Proposition 3.5, since we can discard our knowledge of f).[15] While our axioms will need to be generalized appropriately to deal with the multi-valued case, these logical relationships will remain intact.

[14] In other words, $\forall x \in C(A)$ we have $f(x) \geq \theta(A)$; an assertion that we shall abbreviate as $f[C(A)] \geq \theta(A)$.

[15] In terms of the relevant axioms, $xH_fy \Rightarrow g(x) > g(y)$ can hold only if H_f is acyclic.

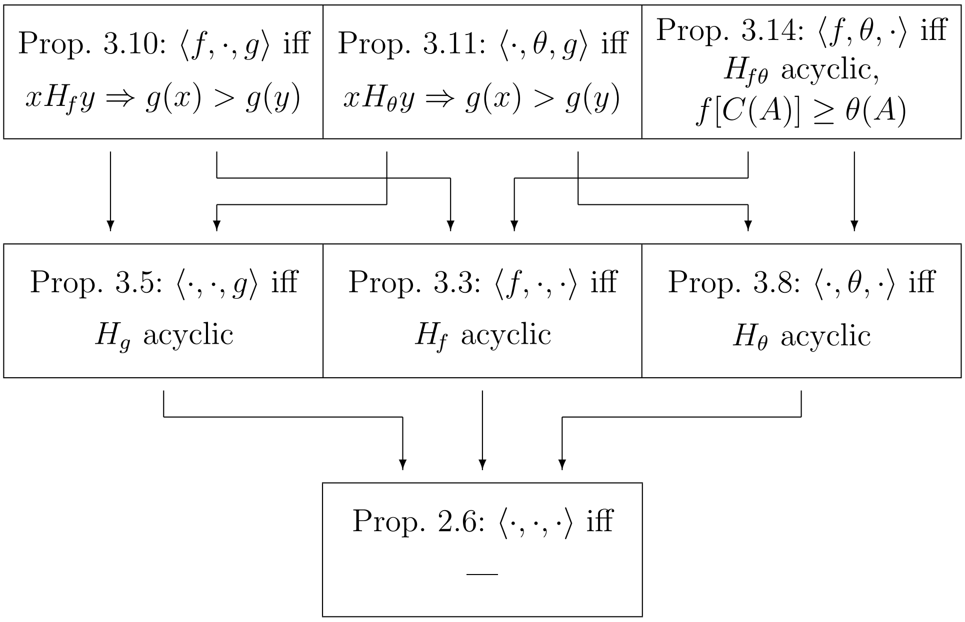

Figure 1: Summary of characterization results for single-valued choice.

4 Multi-valued choice

4.1 Preliminary comments

While useful and intuitive, the results in Section 3 are insufficient for a complete understanding of TST representations with known structural variables. This is because the maintained assumption of single-valued choice substantially limits the scope of the theory and—as we shall argue below—has no fully satisfactory justification in the present context. On the contrary, there are good reasons here to prefer results derived in a multi-valued setting.

One reason is that the standard theory of utility maximization requires multi-valued choice in order to allow for indifference. The supposedly more flexible TST framework should be a true generalization of the standard theory, and for this an extension to the multi-valued environment is needed. A related point is that under the consideration-set interpretation of the TST model (see Section 1), it is desirable to avoid imposing a no-indifference assumption on the agent's utility function g.

A second reason to prefer results derived in the more general setting is that single-valuedness is antithetical to the concept of multi-stage models of decision making, where final choices arise from two or more discrete phases during which the available alternatives are narrowed down. If any given phase always reduces the menu to a single alternative, then of course all later phases become irrelevant. Thus if we routinely study choice procedures under the assumption of single-valuedness, then we cannot meaningfully embed them in a multi-stage model.

4.2 One known variable

We proceed now to generalize Propositions 3.3, 3.5, and 3.8 to the multi-valued setting.

One known primary variable. When f is known, the relation H_f continues to reveal second-stage superiority. But in the multi-valued case alternatives can also be related by E, which is easily seen to reveal second-stage indifference. Choice data can thus be incompatible with the TST model even when there is no H_f-cycle, as seen in the following example.

4.1 Example. Let $f(x) = 1$, $f(y) = 2$, $f(z) = 0$, $C(xy) = x$, $C(xz) = z$, and $C(yz) = yz$. If $\langle f, \theta, g \rangle$ were a TST representation of C, then $f(y) \geq f(x)$ and $C(xy) = x$ together would imply $g(x) > g(y)$, and likewise $f(x) \geq f(z)$ and $C(xz) = z$ would imply $g(z) > g(x)$. But then $C(yz) = yz$ would imply $g(y) = g(z) > g(x) > g(y)$, a contradiction.

The choice function in this example has F empty (and therefore vacuously acyclic), so there is no difficulty in exhibiting a TST representation when all three structural variables are free. It is only in combination with the specified f that C conflicts with the model, due to the mixed cycle $yEzH_fxH_fy$.

With such cases in mind, we define formally the relation of being linked by a chain of alternatives connected sequentially by either H_f or E.

4.2 Definition. Let xW_fy if $\exists z_1, z_2, \ldots, z_n \in X$ such that $z_1 = x$, $z_n = y$, and for each $k \in \{1, 2, \ldots, n-1\}$ we have either $z_kH_fz_{k+1}$ or z_kEz_{k+1}.

This relation reveals weak second-stage superiority, and strict superiority if at least one link in the chain is via H_f. The condition needed for a characterization, analogous to Richter's [24, p. 637] Congruence axiom, is then that no alternative bear W_f to itself except via extended togetherness.

4.3 Theorem. *A choice function has a two-stage threshold representation consistent with $\langle f, \cdot, \cdot \rangle$ if and only if $xW_fy \Rightarrow \neg[yH_fx]$.*

The case of a known θ can be handled similarly. Here H_θ reveals strict second-stage superiority, and the following relation reveals a weak analog.

4.4 Definition. Let $xW_\theta y$ if $\exists z_1, z_2, \ldots, z_n \in X$ such that $z_1 = x$, $z_n = y$, and for each $k \in \{1, 2, \ldots, n-1\}$ we have either $z_k H_\theta z_{k+1}$ or $z_k E z_{k+1}$.

Our result then uses a Congruence-like condition to identify the data sets consistent with the model.

4.5 Theorem. *A choice function has a two-stage threshold representation consistent with* $\langle \cdot, \theta, \cdot \rangle$ *if and only if* $xW_\theta y \Rightarrow \neg[yH_\theta x]$.

Known secondary criterion. When g is the known variable, it remains true in the multi-valued case that H_g must be acyclic. But since this relation reveals first-stage superiority, combining it with extended togetherness is unhelpful. Instead, we need to check that *revealed* second-stage indifference agrees with the *observed* secondary criterion, which is to say that alternatives related by E have identical g-values.

4.6 Example. Let $g(x) = 1$, $g(y) = 2$, $C(xz) = xz$, and $C(xyz) = yz$. If $\langle f, \theta, g \rangle$ were a TST representation of C, then $C(xz) = xz$ would imply that $g(x) = g(z)$, while $C(xyz) = yz$ would imply that $g(z) = g(y)$. But then we would have $g(x) = g(y)$, which is known to be false.

4.7 Theorem. *A choice function has a two-stage threshold representation consistent with* $\langle \cdot, \cdot, g \rangle$ *if and only if* H_g *is acyclic and* $xEy \Rightarrow g(x) = g(y)$.

4.3 Two known variables

Finally, we develop multi-valued versions of Propositions 3.10, 3.11, and 3.14.

Known secondary criterion plus one primary variable. When the primary and secondary criteria are both known, we must again test revealed second-stage indifference for consistency with the observed g. This test may be conducted independently from the consistency check on H_f used in Proposition 3.10, resulting in the following characterization.

4.8 Theorem. *A choice function has a two-stage threshold representation consistent with* $\langle f, \cdot, g \rangle$ *if and only if* $xH_f y \Rightarrow g(x) > g(y)$ *and* $xEy \Rightarrow g(x) = g(y)$.

Conveniently, the same straightforward modification also succeeds in the case of known g together with θ.

4.9 Theorem. *A choice function has a two-stage threshold representation consistent with* $\langle \cdot, \theta, g \rangle$ *if and only if* $xH_\theta y \Rightarrow g(x) > g(y)$ *and* $xEy \Rightarrow g(x) = g(y)$.

Known primary profile. When the first-stage structural variables are both known, weak second-stage superiority is revealed by the following relation.

4.10 Definition. Let $xW_{f\theta}y$ if $\exists z_1, z_2, \ldots, z_n \in X$ such that $z_1 = x$, $z_n = y$, and for each $k \in \{1, 2, \ldots, n-1\}$ we have either $z_k H_{f\theta} z_{k+1}$ or $z_k E z_{k+1}$.

As in Theorems 4.3 and 4.5, we can use this relation to strengthen the requirement that $H_{f\theta}$ be acyclic. In combination with the condition that chosen alternatives survive the (fully observable) first stage, this strengthened axiom achieves our final characterization.[16]

4.11 Theorem. *A choice function has a two-stage threshold representation consistent with* $\langle f, \theta, \cdot \rangle$ *if and only if* $xW_{f\theta}y \Rightarrow \neg[yH_{f\theta}x]$ *and* $f[C(A)] \geq \theta(A)$.

Our axiomatizations with partial knowledge and multi-valued choice are summarized in Figure 2. As anticipated in the discussion of Figure 1 above, the conditions used when two structural variables are known together imply those used when each is known separately (which are in turn always stronger than the condition needed when all three variables are free). Moreover, it is not difficult to confirm that each single-valued characterization in Section 3 is a corollary of the corresponding multi-valued result.[17]

5 Discussion

The axiomatizations summarized in Figures 1 and 2 constitute a complete analysis of choice under partial knowledge in the context of the two-stage threshold model. Our results involve a number of acyclicity and Congruence-like conditions similar to those used in traditional choice theory, as well as other conditions with fewer precedents in the literature.

In addition to establishing these specific results, a secondary goal of the paper has been to introduce the issue of partial knowledge itself. Outside of the TST context, partial-knowledge characterizations can be developed for choice-theoretic models that include different "structural variables." If the

[16]Unlike Theorems 4.3 and 4.5, Theorem 4.11 can be viewed as a direct consequence of Richter's [24] classical axiomatization. This is because when the entire primary profile is observable, the subsets $\Gamma(A|f, \theta)$ of alternatives that survive the first stage are themselves observable. Provided $f[C(A)] \geq \theta(A)$, these survivor subsets can be treated as surrogate menus, and the TST characterization problem reduces to the classical exercise. (The same is true, mutatis mutandis, for Proposition 3.14.)

[17]For example, under single-valued choice we have that E is empty, W_f is the transitive closure of H_f, and the condition that $xW_fy \Rightarrow \neg[yH_fx]$ (used in Theorem 4.3) amounts to acyclicity of H_f (used in Proposition 3.3).

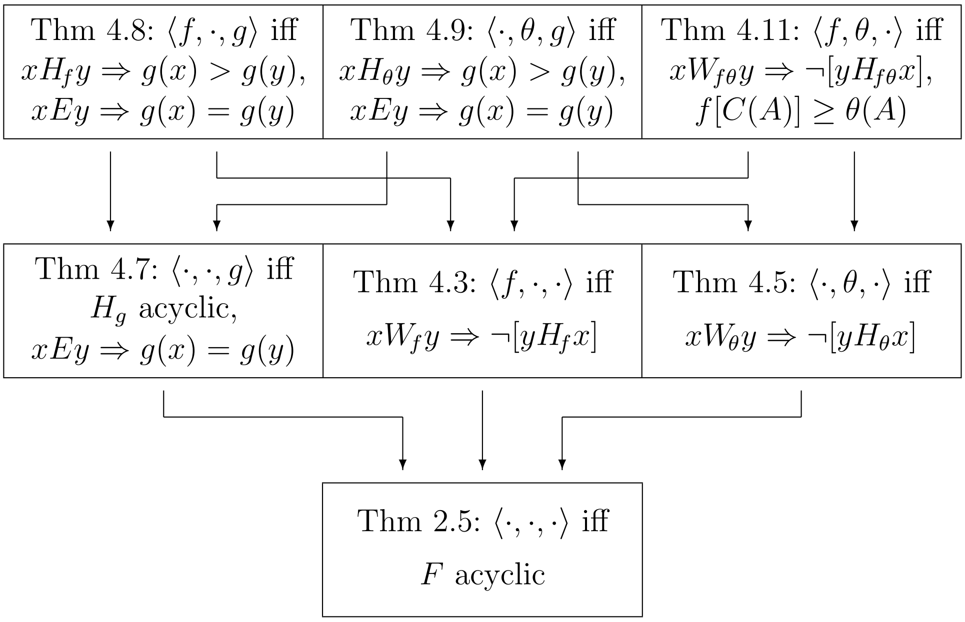

Figure 2: Summary of characterization results for multi-valued choice.

models have features in common with the present framework—such as multiple stages or threshold effects—then it may be hoped that our techniques will be transferable to some degree to these new settings.

For example, consider the following variant of the "rational shortlist method" (RSM) model proposed by Manzini and Mariotti [16]. In the first of two stages, the decision maker eliminates any alternative that is not maximal with respect to an asymmetric binary relation $\succ$.[18] Then, in the second stage, a criterion function g is optimized in the usual way. Since maximization over menu A of an asymmetric $\succ$ cannot in general be represented with a threshold structure $\Gamma(A|f, \theta)$, this model is not covered by the TST framework. Moreover, since optimization of a secondary criterion is stronger than the second-stage procedure specified by Manzini and Mariotti, the new model is a special case of an RSM.

The model described above would place restrictions on behavior even under single-valued choice.[19] Furthermore, we might wonder what *additional* restrictions are implied by knowledge of either $\succ$ or g. Assuming a known $\succ$ would lead to a situation very similar to that in Theorem 4.11, whose proof can be suitably modified (see Footnote 16). Alternatively, assuming a known

[18]Recall that a relation R is asymmetric if $xRy \Rightarrow \neg[yRx]$.

[19]This fact is implied by Manzini and Mariotti's [16] characterization of RSMs in general.

g would lead to a situation resembling that in Theorem 4.7, and one that poses more of a challenge. Here the objective would be to use the known g and the observed C together to infer information about the unknown $\succ$, and to assemble a set of axioms that rules out all possible contradictions.[20]

Another setting in which partial knowledge restrictions could be studied is that of Salant and Rubinstein's [26] "salient consideration functions." Here choice sets have the structure $C(A) = \bigcup_{i=1}^{n} \{x \in A : \forall y \in A \ \neg[yP_i x]\}$, where n is a natural number and each P_i is a relation. Apart from the constraints intrinsic to this model, we can ask what additional behavioral restrictions are implied by knowledge of n, or of one or more of the P_i relations. And similar questions can be posed in the setting of Kalai et al.'s [12] "rationalization by multiple rationales," another prominent multiple-factor model of choice.

A Appendix

As illustrated in Figures 1 and 2, results for TST representations with more known variables can be used to help prove results with less known variables. For example, to demonstrate that the conditions in Theorem 4.7 are sufficient for a representation consistent with $\langle \cdot, \cdot, g \rangle$, it is enough to define a primary criterion f such that the conditions in Theorem 4.8 hold.

We shall make good use of this proof strategy, and so in order to preserve a sequential logical progression we shall prove our characterization results non-consecutively. Specifically, we prove first Theorem 4.8, followed by Theorems 4.3 and 4.7. We then prove Theorem 4.11, followed by Theorem 4.5. And lastly we prove Theorem 4.9.

A few items of notation not employed in the main text will be used in the proofs: We write xR^*y if $\exists z_1, z_2, \ldots, z_n \in X$ such that $x = z_1 R z_2 R \cdots R z_n = y$ (thereby defining the transitive closure R^* of the relation R). Furthermore, we write $K(x)$ for the E-equivalence class of $x \in X$, and $\mathcal{K} = \{K(x) : x \in X\}$ for the associated partition of X.[21]

Proof of Theorem 4.8. Let $\langle f, \theta, g \rangle$ be a TST representation of C, whereupon the implication $xEy \Rightarrow g(x) = g(y)$ is immediate. For $x, y \in X$, if $xH_f y$ then $f(y) \geq f(x)$ and xSy. Hence $\exists A \in \mathcal{D}$ with $x \in C(A)$ and $y \in A \setminus C(A)$, so that $f(y) \geq f(x) \geq \theta(A)$ and $g(x) > g(y)$. Thus $xH_f y \Rightarrow g(x) > g(y)$.

[20] As a first step, observe that if $x \in C(A)$, $y \in A \setminus C(A)$, and $g(y) \geq g(x)$, then for any $B \supseteq A$ we cannot have $y \in C(B)$.

[21] Recall that a binary relation is an equivalence if it is reflexive, symmetric, and transitive. (Reflexivity of R means that xRx, and symmetry means that $xRy \Rightarrow yRx$.) Extended togetherness inherits the properties of reflexivity and symmetry from togetherness, and is transitive by construction.

Conversely, suppose that both $xH_fy \Rightarrow g(x) > g(y)$ and $xEy \Rightarrow g(x) = g(y)$. Given $A \in \mathcal{D}$, let $\theta(A) = \min_{x \in C(A)} f(x)$, so that for each $x \in C(A)$ we have $f(x) \geq \theta(A)$. Moreover, for any $y \in C(A)$ we have xTy, xEy, and $g(x) = g(y)$. Now let $w \in C(A)$ be such that $f(w) = \theta(A)$. If $\exists z \in A \setminus C(A)$ with $f(z) \geq \theta(A)$, then both wSz and $f(z) \geq \theta(A) = f(w)$. But then wH_fz and so $g(w) > g(z)$. It follows that $\langle f, \theta, g \rangle$ is a TST representation of C. □

Proof of Theorem 4.3. Let $\langle f, \theta, g \rangle$ be a TST representation of C. We then have $xH_fy \Rightarrow g(x) > g(y)$ and $xEy \Rightarrow g(x) = g(y)$ by Theorem 4.8, and it follows that $xW_fy \Rightarrow g(x) \geq g(y) \Rightarrow \neg[yH_fx]$.

Conversely, suppose that $xW_fy \Rightarrow \neg[yH_fx]$. For $K_1, K_2 \in \mathcal{K}$, let $K_1 \gg K_2$ if there exist $x_1 \in K_1$ and $x_2 \in K_2$ such that $x_1H_fx_2$.

A.1 Lemma. *$\gg$ is acyclic.*

Proof. Suppose instead that $\exists K_1, K_2, \ldots, K_n \in \mathcal{K}$ with $K_1 \gg K_2 \gg \cdots \gg K_n \gg K_1$. For each $k \in \{1, 2, \ldots, n\}$ there must exist $x_k, y_k \in K_k$ such that $x_1H_fy_2Ex_2H_fy_3E \cdots H_fy_nEx_nH_fy_1Ex_1$. But then both $y_2W_fx_1$ and $x_1H_fy_2$, contradicting $y_2W_fx_1 \Rightarrow \neg[x_1H_fy_2]$. □

Since $\gg$ is acyclic, $\gg^*$ is a strict partial order. By Szpilrajn's Theorem [29], it follows that there exists a linear order $\ggg$ such that $\forall K_1, K_2 \in \mathcal{K}$ we have $K_1 \gg K_2 \Rightarrow K_1 \ggg K_2$. Now let xQy if $K(x) \ggg K(y)$, so that Q is a weak order, and take any numerical representation g of Q.[22]

For $x, y \in X$ we now have xEy only if $K(x) = K(y)$, and so $g(x) = g(y)$. Moreover, xH_fy only if $K(x) \gg K(y)$, $K(x) \ggg K(y)$, and $g(x) > g(y)$. But then C has a TST representation consistent with $\langle f, \cdot, g \rangle$ by Theorem 4.8. □

Proof of Theorem 4.7. Let $\langle f, \theta, g \rangle$ be a TST representation of C, whereupon the implication $xEy \Rightarrow g(x) = g(y)$ is immediate. Moreover, for $x, y \in X$ we have $xH_fy \Rightarrow g(x) > g(y)$ by Theorem 4.8, which is logically equivalent to $xH_gy \Rightarrow f(x) > f(y)$. But then H_g is acyclic.

Conversely, suppose that both H_g is acyclic and $xEy \Rightarrow g(x) = g(y)$. Let xQy if $g(y) > g(x)$ or xH_gy, so that $\forall w, z \in X$ we have $wQ^*z \Rightarrow g(z) \geq g(w)$.

A.2 Lemma. *Q is acyclic.*

[22] Recall that a binary relation is a strict partial order if it is irreflexive and transitive; a weak order if it is a strict partial order that is negatively transitive; and a linear order if it is a weak order that is weakly complete. (Irreflexivity of R means that $\neg[xRx]$; negative transitivity means that $[\neg[xRy] \wedge \neg[yRz]] \Rightarrow \neg[xRz]$; and weak completeness means that $[\neg[xRy] \wedge \neg[yRx]] \Rightarrow x = y$.)

Proof. Suppose instead that $\exists x_1, x_2, \ldots, x_n \in X$ such that $x_1 Q x_2 Q \cdots Q x_n = x_1$. Since H_g is acyclic, there must exist a $k < n$ such that $g(x_{k+1}) > g(x_k)$. But since $x_{k+1} Q^* x_k$ we have also $g(x_k) \geq g(x_{k+1})$, a contradiction. □

Since Q is acyclic, Q^* is a strict partial order. By Szpilrajn's Theorem [29], it follows that there exists a weak order P such that $\forall x, y \in X$ we have $xQ^*y \Rightarrow xPy$. Let f be any numerical representation of P.

For $x, y \in X$ we now have $xH_f y$ only if xSy and $f(y) \geq f(x)$; and thus only if $\neg[xPy]$, $\neg[xQ^*y]$, $\neg[xQy]$, and $g(x) > g(y)$, using the definitions of Q and H_g. But then C has a TST representation consistent with $\langle f, \cdot, g \rangle$ by Theorem 4.8. □

Proof of Theorem 4.11. Let $\langle f, \theta, g \rangle$ be a TST representation of C, whereupon the implications $x \in C(A) \Rightarrow f(x) \geq \theta(A)$ and $xEy \Rightarrow g(x) = g(y)$ are both immediate. For $x, y \in X$, if $xH_{f\theta}y$ then $\exists A \in \mathcal{D}$ such that $f(y) \geq \theta(A)$, $x \in C(A)$, and $y \in A \setminus C(A)$, which implies $g(x) > g(y)$. And it follows that $xW_{f\theta}y \Rightarrow g(x) \geq g(y) \Rightarrow \neg[yH_{f\theta}x]$.

Conversely, suppose that both $xW_{f\theta}y \Rightarrow \neg[yH_{f\theta}x]$ and $f[C(A)] \geq \theta(A)$. For $K_1, K_2 \in \mathcal{K}$, let $K_1 \gg K_2$ if there exist $x_1 \in K_1$ and $x_2 \in K_2$ such that $x_1 H_{f\theta} x_2$.

A.3 Lemma. *$\gg$ is acyclic.*

Proof. Suppose instead that $\exists K_1, K_2, \ldots, K_n \in \mathcal{K}$ with $K_1 \gg K_2 \gg \cdots \gg K_n \gg K_1$. For each $k \in \{1, 2, \ldots, n\}$ there must exist $x_k, y_k \in K_k$ such that $x_1 H_{f\theta} y_2 E x_2 H_{f\theta} y_3 E \cdots H_{f\theta} y_n E x_n H_{f\theta} y_1 E x_1$. But then both $y_2 W_{f\theta} x_1$ and $x_1 H_{f\theta} y_2$, contradicting $y_2 W_{f\theta} x_1 \Rightarrow \neg[x_1 H_{f\theta} y_2]$. □

Since $\gg$ is acyclic, $\gg^*$ is a strict partial order. By Szpilrajn's Theorem [29], it follows that there exists a linear order $\ggg$ such that $\forall K_1, K_2 \in \mathcal{K}$ we have $K_1 \gg K_2 \Rightarrow K_1 \ggg K_2$. Now let xQy if $K(x) \ggg K(y)$, so that Q is a weak order, and take any numerical representation g of Q.

For $x, y \in X$ we now have xEy only if $K(x) = K(y)$, and so $g(x) = g(y)$. Moreover, we have $xH_{f\theta}y$ only if $K(x) \gg K(y)$, $K(x) \ggg K(y)$, and $g(x) > g(y)$.

Given $A \in \mathcal{D}$ and $x \in C(A)$, we have $f(x) \geq \theta(A)$. Moreover, for any $y \in C(A)$, we have xTy, xEy, and $g(x) = g(y)$. If there exists a $z \in A \setminus C(A)$ with $f(z) \geq \theta(A)$, then we have $xH_{f\theta}z$ and so $g(x) > g(z)$. It follows that $\langle f, \theta, g \rangle$ is a TST representation of C. □

Proof of Theorem 4.5. Let $\langle f, \theta, g \rangle$ be a TST representation of C. We then have $xW_{f\theta}y \Rightarrow \neg[yH_{f\theta}x]$ and $f[C(A)] \geq \theta(A)$ by Theorem 4.11. It follows that $\forall x \in X$ we have $f(x) \geq M(x|\theta)$. Moreover, for $x, y \in X$ we have $xH_\theta y$

only if $\exists A \in \mathcal{D}$ such that $f(y) \geq M(y|\theta) \geq \theta(A)$, $x \in C(A)$, and $y \in A \backslash C(A)$, which implies $xH_{f\theta}y$. Hence $xW_\theta y \Rightarrow xW_{f\theta}y \Rightarrow \neg[yH_{f\theta}x] \Rightarrow \neg[yH_\theta x]$.

Conversely, suppose that $xW_\theta y \Rightarrow \neg[yH_\theta x]$. For each $x \in X$, let $f(x) = M(x|\theta)$. Given $A \in \mathcal{D}$ and $x \in C(A)$, we then have $f(x) \geq \theta(A)$. Moreover, for each $x, y \in X$ we have $xH_\theta y \iff xH_{f\theta}y$ and hence $xW_\theta y \iff xW_{f\theta}y$. But then we can conclude that $xW_{f\theta}y \Rightarrow xW_\theta y \Rightarrow \neg[yH_\theta x] \Rightarrow \neg[yH_{f\theta}x]$, and so C has a TST representation consistent with $\langle f, \theta, \cdot\rangle$ by Theorem 4.11. □

Proof of Theorem 4.9. Let $\langle f, \theta, g\rangle$ be a TST representation of C, whereupon the implication $xEy \Rightarrow g(x) = g(y)$ is immediate. Moreover, for $x, y \in X$ we have $xH_\theta y$ only if $\exists A \in \mathcal{D}$ with $M(y|\theta) \geq \theta(A)$, $x \in C(A)$, and $y \in A \setminus C(A)$. Now let $B \in \mathcal{D}$ be such that $y \in C(B)$ and $M(y|\theta) = \theta(B)$. It follows that $f(y) \geq \theta(B) = M(y|\theta) \geq \theta(A)$, and so $g(x) > g(y)$. Thus $xH_\theta y \Rightarrow g(x) > g(y)$.

Conversely, suppose that both $xH_\theta y \Rightarrow g(x) > g(y)$ and $xEy \Rightarrow g(x) = g(y)$. For each $x \in X$, let $f(x) = M(x|\theta)$. Given $A \in \mathcal{D}$ and $x \in C(A)$, we then have $f(x) \geq \theta(A)$. Moreover, for any $y \in C(A)$ we have xTy, xEy, and $g(x) = g(y)$. If $\exists z \in A \setminus C(A)$ with $\theta(A) \leq f(z) = M(z|\theta)$, then $xH_\theta z$ and so $g(x) > g(z)$. It follows that $\langle f, \theta, g\rangle$ is a TST representation of C. □

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