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# A Simple Characterization of Dynamic Completeness in Continuous Time

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## Abstract

This paper examines if (and how) continuous-time trading renders dynamically-complete a financial market in which the underlying risk process is a Brownian motion and the securities pay dividends that are proportional to geometric Brownian motions. A sufficient condition, that the instantaneous dispersion matrix of the relative dividends is non-degenerate, has been established recently in the literature for the case in which the financial market in question is part of a single-commodity, pure-exchange economy with many heterogeneous agents, where all intermediate flows of utilities and endowments are analytic functions. The present paper shows that the condition is indeed sufficient, as well as necessary in some important cases, by means of a very different (and more intuitive) mathematical argument that assumes neither analyticity nor a particular economic environment. It requires only that the pricing kernels are continuous and satisfy a standard growth condition. In this sense, dynamic completeness obtains irrespectively of preferences, endowments, and other structural elements (such as whether or not the budget constraints include only pure exchange, whether or not the time horizon is finite with lump-sum dividends available on the terminal date, etc.).

**Keywords:** Dynamically-Complete Markets, Geometric Brownian Motion, Asset Pricing.

**JEL Classification Numbers:** G10, G12.

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# 1 Introduction

Whether or not a given asset market is dynamically complete is of fundamental importance in financial economics. If the pricing process of the underlying securities is dynamically complete, then options and other derivatives can be uniquely priced by arbitrage arguments and replicated by trading the underlying securities. In the absence of dynamic completeness, however, this is no longer the case; no-arbitrage restrictions do not suffice to guarantee unique option prices while replication may not be possible.

It is crucial therefore to be able to associate dynamic completeness with the economic primitives of a given financial environment - in a manner that remains unambiguously verifiable and holds at least generically across the space of these primitives. The present paper does so for the case in which the underlying risk process is a Brownian motion and the securities pay dividends that are proportional to geometric Brownian motions. We establish that a sufficient, as well as necessary in some important cases, condition for this financial market to be dynamically complete is that the matrix of factor loadings of the relative dividends is nonsingular.<sup>1</sup>

Of course, the sufficiency part follows also from the seminal papers by Anderson and Raimondo [1] and Hugonnier et al. [34] - as well as the subsequent treatments by Riedel and Herzberg [61] or Kramkov [41]. Yet, the respective arguments refer to the case in which the financial market in question is part of a single-commodity, pure-exchange economy with many heterogeneous agents, and where all intermediate flows of utilities and endowments are analytic functions. In sharp contrast, the current exposition is based upon a mathematical approach that assumes neither analyticity nor a particular underlying economic environment. It requires only that the pricing kernels are continuous and satisfy a standard growth condition.

Specifying the securities' dividends as geometric Brownian motions has been an important benchmark for the theoretical- as well as the applied-finance literature.<sup>2</sup> Recently, moreover, it has started featuring prominently also in applied macro- and micro-economic studies.<sup>3</sup> Its popularity rests upon the facilitation of quite realistic financial modeling, under which asset prices can be derived in closed form or as solutions to well-known stochastic differential equations. And, as established in the sequel, this lends itself also to theoretical justification. For it allows the property of dynamic completeness to be mapped exclusively to a property of the securities' dividends.

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<sup>1</sup>The typical relative dividend is the dividend of the typical security divided by the dividend of the particular security which has been designated as the numeraire. When the latter security is a money-market account or when its dividend is deterministic, the condition refers to the matrix of factor loadings of the risky dividends.

<sup>2</sup>To name but a few theoretical papers, see Bick [5], Cochrane et al. [12], Constandinides and Zarithpoulou [13], Oksendal and Sulem [52], Raimondo [59], or Anderson and Raimondo [2]. Applied studies include, for instance, Martens and van Dijk [44], Wong [64], Instefjord [37], Gerber and Shiu [28]-[29], Gatheral and Schied [27], Browne [8], or Bigger and Hull [6].

<sup>3</sup>See, for example, Postali and Picchetti [55], Farhi and Panageas [23], Epaulard and Pommeret [21], Hadjiladis [30], Hull [36], He [32], Candenillas and Zapatero [9], Capozza and Kazarian [10], Ericsson [22], Mella-Baral and Perraudin [46], Oren [53], Pennings [54], Promislow and Young [56], Maratha and Ryan [58], Schmidli [62], Milevsky [48], Fleten et al. [25], Deng et al. [16], or Carey and Zilberman [11].

The relevance of this result becomes evident when viewed in the context of general equilibrium. The typical approach in the literature for obtaining financial equilibria in continuous time has been to compute an Arrow-Debreu equilibrium and use the associated consumption process as pricing kernel in order to construct equilibrium prices for the traded securities.<sup>4</sup> To ensure, however, that the starting Arrow-Debreu allocation is implementable by trading the given set of securities, their market needs to be dynamically complete. Yet, the equilibrium pricing processes are determined endogenously (via fixed-point arguments) from the model's primitives (the utility functions of the agents, their endowments, and the dividend processes of the securities) and are expressed as expectations of properly discounted future payoffs. As a result, especially in economies with many heterogeneous agents (increasingly the focus of the asset-pricing literature), and apart from the extremely special cases where one can obtain sufficiently straightforward closed form solutions, verifying from the primitives that the equilibrium pricing process is indeed dynamically complete is a highly non-trivial problem, known as “endogenous completeness.”

Essential progress in this problem was achieved only recently by the four papers cited above. The fundamental insight is that the non-degeneracy of the instantaneous dispersion matrix of the relative dividends can be shown to suffice for dynamic completeness - and, thus, permit the construction of the equilibrium pricing process via a representative agent. The crucial assumption is that, with respect to flows during the trading horizon, the securities' dividends as well as the agents' utilities and endowments must all be real analytic functions. The present paper complements the four studies by showing that, when the underlying risk process is a Brownian motion and the risky dividends are proportional to geometric Brownian motions, the non-degeneracy condition in question characterizes dynamic completeness via a more straightforward and intuitive line of proof. The present analysis is based upon completely standard mathematical techniques that do not require analyticity on the models' primitives. More importantly, it indicates that the relation between dynamic completeness and the non-degeneracy condition in question extends in directions that are important for applications.

Under the risk and dividend specifications in the present paper, the relation remains valid for any pricing kernels, as long as they are continuous and satisfy a standard in the literature growth condition. This allows the non-degeneracy condition to characterize dynamic completeness in a very general sense regarding the underlying structure for economic activity or the agents' preferences and endowments. In addition, when there are only dividend flows during the trading horizon (a very common setup in applied finance models), the relation implies that the instantaneous dispersion matrix of the relative securities' prices is non-degenerate everywhere, not almost everywhere. This has significant implications when it comes to solving Merton's portfolio selection problem via numerical methods, a field of investigation for which the recent literature has registered a strong

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<sup>4</sup>See the introductory section in Anderson and Raimondo [1] for an extensive review and discussion of the seminal studies. The existence of the Arrow-Debreu equilibrium itself is due to some assumptions whose form varies in the literature.

revival of interest as advances in computing power render numerical methods increasingly more efficient and accurate.. For example, in a single-commodity, pure-exchange economy with many heterogenous agents, under mild additional assumptions, numerical convergence ought to obtain universally, not generically, for the optimal portfolio positions of every agent are everywhere, not almost everywhere, locally bounded.

The balance of the paper is organized as follows. The next section introduces the theoretical structure under study, and presents the main analysis. Section 3 interprets our results in the context of the pertinent literature. Section 4 concludes while the Appendix contains the proofs and the supporting technical material.

## 2 Setup and Analysis

Consider a financial market where trading occurs over a time-interval  $\mathcal{T} \subseteq \mathbb{R}_+$  while the informational structure is given by a  $K$ -dimensional ( $K \in \mathbb{N} \setminus \{0\}$ ) standard Brownian process, defined on a complete probability space  $(\Omega, \mathcal{F}, \pi)$  and depicted as  $\beta : \Omega \times \mathcal{T} \mapsto \mathbb{R}^K$  or  $\beta_k : \Omega \times \mathcal{T} \mapsto \mathbb{R}$  with  $k \in \mathcal{K} \equiv \{1, \dots, K\}$  for the typical dimension. As usual, the process is meant to fully describe the exogenous financial risk in the sense that the collection of the sample paths  $\{\beta(\omega, t) : t \in \mathcal{T}\}_{\omega \in \Omega}$  specifies all the distinguishable events.

The underlying risk process being a Brownian motion, a necessary condition for any securities market to be dynamically complete is that the number of securities exceeds that of independent Brownian motions by at least one (i.e that the market is potentially complete).<sup>5</sup> In what follows, the trading structure will consist of  $K + 1$  dividend-paying securities, indexed by  $j \in \mathcal{K} \cup \{0\}$  and traded continuously over  $\mathcal{T}$ . The securities' dividends will be assumed to take one or both of two different forms, a flow and a lump sum. More precisely, letting  $\mathcal{I} : \Omega \times \mathcal{T} \mapsto \mathcal{T} \times \mathbb{R}^K$  depict the process  $\{t, \beta(\omega, t)\}_{(\omega, t) \in \Omega \times \mathcal{T}}$ , the typical security will be paying the dividend flow  $g_j(\mathcal{I}(\omega, \cdot))$  along the Brownian path  $\{\beta(\omega, t)\}_{t \in \mathcal{T}}$  or the lump sum  $G_j(\mathcal{I}(\omega, T))$  on the terminal date if the time-horizon is finite ( $\mathcal{T} = [0, T]$  for some  $T > 0$ ), where

$$G_j(\mathcal{I}(\omega, t)) := \mu_j(t) e^{\sigma_j^\top \beta(\omega, t)} \quad g_j(\mathcal{I}(\omega, t)) := \tilde{\mu}_j(t) e^{\tilde{\sigma}_j^\top \beta(\omega, t)} \quad j \in \mathcal{K} \cup \{0\} \quad (1)$$

for some deterministic continuous functions  $\mu_j, \tilde{\mu}_j : \mathcal{T} \mapsto \mathbb{R}_+$  (with  $\tilde{\mu}_j(t) > 0$  a.e. on  $\mathcal{T}$  and  $\mu_j(T) > 0$ ), and constant factor loadings (instantaneous dispersion) vectors  $\sigma_j, \tilde{\sigma}_j \in \mathbb{R}^K$ .

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<sup>5</sup>To fix ideas about the underlying concepts, let  $\mathcal{T} = [0, T]$  for some  $T > 0$ . The financial market is said to be *complete* if at any  $t \in [0, T)$  it is possible to instantaneously enter into a portfolio position that will replicate any admissible contingent claim - i.e., any process  $\{Y(\omega, s) : (\omega, s) \in \Omega \times (t, T]\}$  such that  $Y(\omega, T)$  has finite variance (and therefore finite expectation) under  $\pi$ . By contrast, the market is said to be *dynamically complete* if any given admissible contingent claim can be replicated instead by a self-financing trading strategy (a notion to be defined in footnote 6). Either definition refers of course to the replication of contingent claims. Yet, a complete market allows for the necessary bets to be placed all at once, whereas a dynamically-complete market may require that subsequent trades are executed after the initial investment. Needless to say, any complete market is also dynamically complete.

The typical terminal dividend is proportional to a  $K$ -dimensional geometric Brownian motion as long as  $\sigma_j \neq \mathbf{0}^K$  and  $\mu_j(t) = \rho(t) e^{\hat{\mu}_j t}$  for some constant  $\hat{\mu}_j \in \mathbb{R} \setminus \{0\}$  and some deterministic supply function  $\rho : \mathcal{T} \mapsto \mathbb{R}_{++}$  (with  $\int_{\mathcal{T}} \rho(s) ds < \infty$ ). By contrast,  $\sigma_j = \mathbf{0}^K$  renders the dividend riskless and the corresponding security a bond, whose coupon could be, for instance,  $\mu_j(t) = 0$  on  $[0, T)$  and  $\mu_j(T) = e^{\int_0^T \hat{\mu}_j(s) ds}$  for some deterministic function  $\hat{\mu}_j : \mathcal{T} \mapsto \mathbb{R}_{++}$ , (with  $\int_{\mathcal{T}} \hat{\mu}_j(s) ds < \infty$ ). Similarly, the typical intermediate dividend is proportional to a  $K$ -dimensional geometric Brownian motion if  $\tilde{\sigma}_j \neq \mathbf{0}^K$ . When  $\sigma_j = \mathbf{0}^K$ , on the other hand, letting  $\tilde{\mu}_j(t) = e^{\int_0^t \hat{\mu}_j(s) ds}$  on  $[0, T)$  renders the security an annuity with coupon rate  $dc_j(t)/c_j(t) = \hat{\mu}_j dt$ .

Given that trading occurs over a time-interval while the informational structure is driven by Brownian motions, well-known no-arbitrage conditions ensure that the securities' prices are the current expectations of their future dividends valued at some pricing kernel, a strictly-positive one-dimensional Itô process.<sup>6</sup> Along the Brownian path  $\{\beta(\omega, t)\}_{t \in \mathcal{T}}$ , this is given by the functions  $m(\mathcal{I}(\omega, \cdot))$  and  $M(\mathcal{I}(\omega, \cdot))$  so that the price of the typical security can be written as<sup>7</sup>

$$P_j(t, \beta_t) = \mathbb{E}_\pi \left[ \frac{(MG_j)(\mathcal{I}(\omega, T))}{M(t, \beta_t)} + \int_{\mathcal{T} \setminus [0, t)} \frac{(mg_j)(\mathcal{I}(\omega, s))}{m(t, \beta_t)} ds | \mathcal{F}_t \right] \quad j \in \mathcal{K} \cup \{0\} \quad (2)$$

All prices above being strictly positive everywhere on  $\Omega \times \mathcal{T}$ , any one can be used as deflator. When studying, therefore, dynamic completeness it is without loss of generality to deflate all prices by that of the zeroth security and examine instead the relative pricing process  $p_j(\cdot) := P_n(\cdot)/P_0(\cdot)$  for  $j \in \mathcal{K} \cup \{0\}$ . Of course, re-normalizing in this way means that the price of the zeroth security is 1 everywhere on  $\Omega \times \mathcal{T}$ , rendering the security instantaneously risk-free and the strategy of buying-

<sup>6</sup>To fix ideas again (borrowing heavily from the exposition in Chapters 5A-C, 6I, and 6K in Duffie [17]), let  $\mathcal{T} = [0, T]$  for some  $T > 0$ . Take also  $\mathbf{P}_t = (P_{0t}, \dots, P_{Kt})$  to be a  $K+1$ -dimensional Itô process. This is to say that, for any  $j \in \mathcal{K} \cup \{0\}$  and  $t \in [0, T)$  we have  $P_{jt} = P_{j0} + \int_0^t a_{js} ds + \int_0^t \mathbf{b}_j^T d\beta_s$  for some real number  $P_{j0}$ , and  $\mathbb{R}$ -valued and  $\mathbb{R}^K$ -valued respectively adapted processes  $a_j$  and  $\mathbf{b}_j$  such that  $\int_0^T |a_{js}| ds < +\infty$  almost surely w.r.t.  $\pi$  ( $\pi$ -a.s.) and  $\int_0^T |\mathbf{b}_{js}|^2 ds < +\infty$   $\pi$ -a.s. A trading strategy  $\theta$  is an  $\mathbb{R}^{K+1}$ -valued adapted process with  $\theta_j(\omega, t)$  specifying the number of units of the security  $j \in \mathcal{K} \cup \{0\}$  to hold at the node  $(\omega, t) \in \Omega \times [0, T]$ , and such that, letting  $\mathbf{g}_t = (g_{0t}, \dots, g_{Kt})$ , the generated gains process  $\int \theta^T d(\mathbf{P} + \mathbf{g})$  is well-defined as a stochastic integral. We require in other words that  $\int_0^T |\sum_{j \in \mathcal{K} \cup \{0\}} \theta_{js} (a_{js} + (\tilde{\mu}_{js}/\tilde{\mu}_{js}) g_{js})| ds < +\infty$   $\pi$ -a.s., as well as  $\int_0^T |\sum_{j \in \mathcal{K} \cup \{0\}} \theta_{js} (\mathbf{b}_{js} + g_{js} \tilde{\sigma}_j)|^2 ds < +\infty$   $\pi$ -a.s. where  $\tilde{\mu}_j = d\tilde{\mu}_j/dt$ . A trading strategy  $\theta$  is *self-financing* if  $\theta_t^T \mathbf{P}_t = \theta_0^T \mathbf{P}_0 + \int_0^t \theta^T d(\mathbf{P}_s + \mathbf{g}_s)$  everywhere on  $[0, T]$ . A self-financing trading strategy  $\theta$  is an *arbitrage* if  $\theta_0^T \mathbf{P}_0 < 0$  and  $\theta_T^T \mathbf{P}_T \geq 0$  or  $\theta_0^T \mathbf{P}_0 \leq 0$  and  $\theta_T^T \mathbf{P}_T > 0$  (where “ $\theta_T^T \mathbf{P}_T \geq 0$ ” means that  $\theta_0^T \mathbf{P}_0$  is non-negative  $\pi$ -a.s. whereas “ $\theta_T^T \mathbf{P}_T > 0$ ” means that  $\theta_0^T \mathbf{P}_0$  is non-negative  $\pi$ -a.s. and non-zero with positive probability.) Let now  $\Theta(\mathbf{P})$  denote the space of self-financing trading strategies  $\theta$  that satisfy  $\mathbb{E}_\pi \left[ \left( \int_0^T \sum_{j \in \mathcal{K} \cup \{0\}} \theta_{js} (a_{js} + (\tilde{\mu}_{js}/\tilde{\mu}_{js}) g_{js}) ds \right)^2 | \mathcal{F}_0 \right] < +\infty$  and  $\mathbb{E}_\pi \left[ \int_0^T \sum_{j \in \mathcal{K} \cup \{0\}} \theta_{js} (\mathbf{b}_{js} + g_{js} \tilde{\sigma}_j) ds | \mathcal{F}_0 \right] < +\infty$ . Let also  $M(\mathbf{P}) \subseteq \mathbb{R}$  be the *marketed* space:  $W \in M(\mathbf{P})$  iff  $W = \theta_T^T \mathbf{P}_T$  for some  $\theta \in \Theta(\mathbf{P})$ . Under no-arbitrage, there is a function  $\psi : M(\mathbf{P}) \mapsto \mathbb{R}$  with  $\psi(W) = \theta_0^T \mathbf{P}_0$  being the unique initial investment required to obtain the payoff  $W \in M(\mathbf{P})$ . The function  $\psi$  can be extended to a real-valued functional  $\Psi$  on the space of admissible contingent claims (recall footnote 5). And there is a unique strictly positive process  $\{M(\omega, s) : (\omega, s) \in \Omega \times (t, T)\}$  such that  $M(\omega, T)$  has finite variance (and therefore finite expectation) under  $\pi$  while  $\Psi(W) = \mathbb{E}_\pi [M_T W | \mathcal{F}_0]$ .

<sup>7</sup>There has emerged recently some literature on rational asset-pricing bubbles via the martingale method (see, for example, Hugonnier [35] or Jarrow et al. [38]) or the so-called relative arbitrage method (see, for instance, Heston et al. [33] or Fernholz and Karatzas [24]). And either approach is resting on the very premise that (2) fails.

and-holding it a trivial money-market account. Which ensures in turn that dynamic completeness is equivalent to the matrix of instantaneous dispersions of the remaining relative prices being almost everywhere (a.e.) non-singular (see, for example, Sections 4.1-4.4 and Theorem 5.6 in Nielsen [50]). More precisely, letting  $D_k := \partial/\partial\beta_k$  depict the instantaneous dispersion operator, the financial market under study here will be dynamically complete if and only if the dispersion (Jacobian) matrix

$$J_p(t, \beta_t) = [D_k p_n(t, \beta_t)]_{(n,k) \in \mathcal{K} \times \mathcal{K}}$$

has full rank a.e. on  $\mathcal{T} \times \mathbb{R}^K$ . To examine whether the latter condition obtains, we will investigate the typical element of this matrix which, by Itô's lemma, is given by

$$D_k p_n(t, \beta_t) = \frac{D_k P_n(t, \beta_t) - p_n(t, \beta_t) D_k P_0(t, \beta_t)}{P_0(t, \beta_t)}, \quad (n, k) \in \mathcal{K} \times \mathcal{K} \quad (3)$$

## 2.1 Lump-sum Dividends

It will be instructive to begin our analysis with the setting in which the time-horizon is finite ( $\mathcal{T} = [0, T]$  for some  $T > 0$ ) and the securities pay only lump-sum dividends on the terminal date. In this case, the second term inside the expectation in (2) is identically zero. And since the increment  $\beta(\omega, T) - \beta(\omega, t)$  is independent of  $\mathcal{F}_t$  and distributed  $\mathcal{N}(\mathbf{0}^K, (T-t)\mathbf{I}_K)$  over  $\Omega$ , the equation reads<sup>8</sup>

$$P_j(t, \beta_t) = \int_{\mathbb{R}^K} \frac{M_j(T, \beta_t + \sqrt{T-t}\mathbf{x})}{M(t, \beta_t)} \phi(\mathbf{x}) d\mathbf{x} \quad t \in [0, T) \quad j \in \mathcal{K} \cup \{0\} \quad (4)$$

where  $\phi(\cdot)$  denotes the  $K$ -dimensional standard-normal pdf while

$$M_j(\cdot) := M(\cdot) G_j(\cdot) \quad j \in \mathcal{K} \cup \{0\}$$

To ensure that the dispersion of this price with respect to the typical Brownian dimension is well-defined, letting  $|\cdot|$  denote the Euclidean norm, it suffices to assume that

**A 1**  $M(T, \cdot) : \mathbb{R}^K \mapsto \mathbb{R}_{++}$  is continuous and s.t.

$$\forall j \in \mathcal{K} \cup \{0\} \quad \exists (r_j, C_j) \in \left(0, \frac{1}{2T}\right) \times \mathbb{R}_{++} : \quad |M_j(T, \mathbf{x})| \leq C_j e^{r_j |\mathbf{x}|^2} \quad \forall \mathbf{x} \in \mathbb{R}^K$$

Indeed, since  $r_j < \frac{1}{2T} \leq \frac{1}{2(T-t)}$  for any  $t \in [0, T)$ , Lemma A.7 ensures that the  $\mathcal{T} \times \mathbb{R}^K \mapsto \mathbb{R}_{++}$

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<sup>8</sup>To facilitate the exposition, in what follows we will not display the future nodes  $(\omega, s)$  as arguments in the corresponding functions. And even though the expectations will be shown w.r.t. to the corresponding normally-distributed vectors, it should be kept in mind that they are meant to be over  $\Omega$ , w.r.t  $\pi$ , and conditional on  $\mathcal{F}_t$  (unless otherwise stated). Moreover, all variables inside the expectation operator are meant to be fixed at their current values. Notice also that “a.e.” and “a.s.” denote, respectively, “almost everywhere on  $\Omega$ ” and “almost surely w.r.t.  $\pi$ .”

function<sup>9</sup>

$$(MP_j)(t, \beta_t) = \int_{\mathbb{R}^K} \frac{M_j(T, \mathbf{x})}{\sqrt{T-t}} \phi\left(\frac{\mathbf{x} - \beta_t}{\sqrt{T-t}}\right) d\mathbf{x} \quad t \in [0, T], j \in \mathcal{K} \cup \{0\} \quad (5)$$

(which results from (4) via a trivial change in the variables of integration) is well-defined, and that so is also its instantaneous dispersion

$$\begin{aligned} D_k(MP_j)(t, \beta_t) &= \int_{\mathbb{R}^K} \frac{M_j(T, \mathbf{x})(x_k - \beta_{kt})}{\sqrt{(T-t)^3}} \phi\left(\frac{\mathbf{x} - \beta_t}{\sqrt{T-t}}\right) d\mathbf{x} \quad (j, k) \in \mathcal{K} \cup \{0\} \times \mathcal{K} \\ &= \mathbb{E}_{\mathbf{x}} \left[ \frac{x_k}{\sqrt{T-t}} M_j\left(T, \beta_t + \sqrt{T-t}\mathbf{x}\right) \right] \end{aligned} \quad (6)$$

at any  $(t, \beta_t) \in [0, T] \times \mathbb{R}^K$ , and where  $\mathbf{x} \sim \mathcal{N}(\mathbf{0}^K, \mathbf{I}_K)$  (while the second equality above follows again from a change in the variables of integration). Given this, and since  $P_n(\cdot)/P_0(\cdot) = (MP_n)(\cdot)/(MP_0)(\cdot)$ , for the typical relative price we get

$$\begin{aligned} &(MP_0)(t, \beta_t)^2 D_k p_n(t, \beta_t) \\ &= (MP_0)(t, \beta_t) D_k(MP_n)(t, \beta_t) - (MP_n)(t, \beta_t) D_k(MP_0)(t, \beta_t) \end{aligned} \quad (7)$$

$$= \mathbb{E}_{(\mathbf{x}, \mathbf{y})} \left[ \frac{x_k - y_k}{\sqrt{T-t}} M_n\left(T, \beta_t + \sqrt{T-t}\mathbf{x}\right) M_0\left(T, \beta_t + \sqrt{T-t}\mathbf{y}\right) \right] \quad (8)$$

for any  $t \in [0, T]$ , any  $(n, k) \in \mathcal{K} \times \mathcal{K}$ , and where  $\mathbf{x}, \mathbf{y} \sim \text{i.i.d. } \mathcal{N}(\mathbf{0}^K, (T-t)\mathbf{I}_K)$ . Yet, under the terminal-dividend specification in (1), it cannot but be  $G_0(T, \cdot) > 0$  everywhere on  $\mathbb{R}^K$ . For any  $n \in \mathcal{K}$ , therefore, we may define the function  $G_{n/0}(T, \cdot) := G_n(T, \cdot)/G_0(T, \cdot)$  and write

$$(MP_0)(t, \beta_t)^2 D_k p_n(t, \beta_t) = \mathbb{E}_{(\mathbf{x}, \mathbf{y})} \left[ \begin{aligned} &M_0\left(T, \beta_t + \sqrt{T-t}\mathbf{x}\right) M_0\left(T, \beta_t + \sqrt{T-t}\mathbf{y}\right) \\ &\times \frac{x_k - y_k}{\sqrt{T-t}} G_{n/0}\left(T, \beta_t + \sqrt{T-t}\mathbf{x}\right) \end{aligned} \right] \quad (9)$$

This brings us in position to make an observation that will form the backbone of the subsequent analysis. By Itô's lemma, under the terminal-dividend specification in (1) the dispersion coefficient of the typical relative dividend is given by  $\nabla_{\beta} G_{n/0}(T, \beta_t) = G_{n/0}(T, \beta_t)(\sigma_n - \sigma_0)$ . By the mean value theorem then, we have

$$G_{n/0}(T, \beta_t + \mathbf{z}) = G_{n/0}(T, \beta_t) + G_{n/0}(T, \beta_t + \lambda_{n/0}(\mathbf{z})\mathbf{z})(\sigma_n - \sigma_0)^{\top} \mathbf{z}, \quad \mathbf{z} \in \mathbb{R}^K \quad (10)$$

for some function  $\lambda_{n/0} : \mathbb{R}^K \mapsto (0, 1)$  such that  $\lim_{\mathbf{z} \rightarrow \mathbf{0}^K} \lambda_{n/0}(\mathbf{z}) = 0$ . Given this, it is trivial to

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<sup>9</sup>As far as (5) is concerned, to apply Lemma A.7 we set  $(m_1, \dots, m_K) = \mathbf{0}^{\top}$  and adopt the standard convention that the zeroth derivative of a function denotes the function itself.



check that (9) reads

$$(MP_0)(t, \beta_t)^2 D_k p_n(t, \beta_t) = \mathbb{E}_{(\mathbf{x}, \mathbf{y})} \left[ \begin{array}{l} M_0(T, \beta_t + \sqrt{T-t}\mathbf{x}) M_0(T, \beta_t + \sqrt{T-t}\mathbf{y}) \\ \times \frac{\sqrt{T-t}x_k - \sqrt{T-t}y_k}{T-t} (\sigma_n - \sigma_0)^\top \sqrt{T-t}\mathbf{x} \\ \times G_{n/0}(T, \beta_t + \lambda_{n/0}(\sqrt{T-t}\mathbf{x}) \sqrt{T-t}\mathbf{x}) \end{array} \right] \quad (11)$$

Yet, the quantity of the second line inside the expectation above depicts a linear relation between Brownian increments. More precisely,  $\sqrt{T-t}\mathbf{x}$  and  $\sqrt{T-t}\mathbf{y}$  depict, respectively, Brownian increments  $\beta_T - \beta_t$  and  $\tilde{\beta}_T - \beta_t$  which are i.i.d.  $\mathcal{N}(\mathbf{0}^K, (T-t)\mathbf{I}_K)$ . Which means in turn that, as  $t \rightarrow T$ , the quantity in question depicts a linear relation between infinitesimal Brownian increments. It becomes  $\frac{\Delta\beta_k - \Delta\tilde{\beta}_k}{\Delta t} \sum_{k \in \mathcal{K}} (\sigma_{nk} - \sigma_{0k}) \Delta\beta_k$ , reducing almost surely to  $\sigma_{nk} - \sigma_{0k}$ . This observation, in conjunction with the fact that as  $t \rightarrow T$  the quantity defined in (5) approaches  $M_j(T, \beta_T)$  (Lemma B.2) while the continuity in  $t$  commutes inside the expectation operator above, gives rise to the following result.

**Proposition 2.1** *Let  $\mathcal{T} = [0, T]$  for some  $T > 0$ . Suppose also that the terminal-dividend and price processes are given, respectively, by (1) and (4) while A1 holds. Then*

$$\lim_{t \rightarrow T} D_k p_n(\mathcal{I}(\omega, t)) = (\sigma_{nk} - \sigma_{0k}) G_{n/0}(\mathcal{I}(\omega, T)) \quad (n, k) \in \mathcal{K} \times \mathcal{K}, \omega \in \Omega$$

**Proof.** See Appendix B. ■

As  $t \rightarrow T$ , therefore, for any given  $\omega \in \Omega$  the dispersion matrix of the relative prices approaches the dispersion matrix of the relative terminal dividends

$$\begin{aligned} J_G(\mathcal{I}(\omega, T)) &:= [D_k G_{n/0}(\mathcal{I}(\omega, T))]_{(n,k) \in \mathcal{K} \times \mathcal{K}} \\ &= \begin{bmatrix} G_{1/0}(\mathcal{I}(\omega, T)) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & G_{K/0}(\mathcal{I}(\omega, T)) \end{bmatrix} \times \Sigma_0 \end{aligned}$$

where  $\Sigma_0 = [\sigma_{nk} - \sigma_{0k}]_{(n,k) \in \mathcal{K} \times \mathcal{K}}$ . Which leads in turn to a sufficient condition for the financial market to be dynamically complete.

**Theorem 2.1** *Let the conditions of Proposition 2.1 be satisfied. If  $\Sigma_0$  is non-singular,  $J_p(\cdot)$  has full rank a.e. on  $[0, T) \times \mathbb{R}^K$  so that the financial market is dynamically complete.*

**Proof.** See Appendix B. ■

Intuitively, the supporting argument proceeds as follows. Even though we have not required the pricing kernel to be analytic in any of its arguments, it turns out that the growth condition in A1 suffices for  $(MP_0)(\cdot)^2 D_k p_n(\cdot)$  to be analytic on  $(0, T) \times \mathbb{R}^K$  for any  $(n, k) \in \mathcal{K} \times \mathcal{K}$ . The

determinant of a matrix (which will be denoted by  $|\cdot|$ ) involving but sums and products of its entries,  $\left| (MP_0)(\cdot)^2 J_p(\cdot) \right|$  is also analytic on  $(0, T) \times \mathbb{R}^K$ . Fix now any  $\omega_0 \in \Omega$ . If  $\Sigma_0$  is non-singular, so must be  $J_G(\mathcal{I}(\omega_0, T))$ . And given that  $\lim_{t \rightarrow T} (MP_0)(\mathcal{I}(\omega_0, t)) = M_0(\mathcal{I}(\omega_0, T))$ , Proposition 2.1 implies in fact that, if  $\Sigma_0$  is non-singular,  $\left| (MP_0)(\mathcal{I}(\omega_0, t))^2 J_p(\mathcal{I}(\omega_0, t)) \right|$  approaches a non-zero limit as  $t \rightarrow T$ . Yet, the latter viewed as a real analytic function on  $(0, T) \times \mathbb{R}^K$ , this cannot be unless  $\left| (MP_0)(\cdot)^2 J_p(\cdot) \right| \neq 0$  a.e. on this domain.<sup>10</sup> Which implies of course that  $|J_p(\cdot)|$  is non-zero a.e. on  $(0, T) \times \mathbb{R}^K$ .

**Remark I** Under the terminal-dividend specification in (1), for the growth condition in A1 to be met it suffices that

$$\exists (r, C) \in \left(0, \frac{1}{2T}\right) \times \mathbb{R}_{++} : \quad |M(T, \mathbf{x})| \leq C e^{r|\mathbf{x}|^2} \quad \forall \mathbf{x} \in \mathbb{R}^K$$

Indeed, since  $|G_j(T, \mathbf{x})| \leq \mu_j(T) e^{\sigma_j^\top \mathbf{x}}$  everywhere on  $\mathbb{R}^K$ , Lemmas A.1-A.2 allow us to find  $\bar{r}_j \in (0, \frac{1}{2T} - r)$  and  $\bar{C}_j > 0$  s.t.  $|G_j(T, \mathbf{x})| \leq \bar{C}_j \mu_j(T) e^{\bar{r}_j |\mathbf{x}|^2}$  everywhere on  $\mathbb{R}^K$ . It suffices then to set  $r_j = r + \bar{r}_j$  and  $C_j = \max\{C, \bar{C}_j \mu_j(T)\}$ .

## 2.2 Dividend Flows

Next, we will examine the setting in which the securities pay only dividend flows during  $\mathcal{T}$  so that only the second term inside the expectation in (2) is relevant. In this case, the typical price is given by

$$P_j(t, \beta_t) = \mathbb{E}_\pi \left[ \int_t^T \frac{m_j(\mathcal{I}(\omega, s))}{m(t, \beta_t)} ds \mid \mathcal{F}_t \right] \quad t \in [0, T) \quad j \in \mathcal{K} \cup \{0\} \quad (12)$$

where

$$m_j(\cdot) := m(\cdot) g_j(\cdot) \quad j \in \mathcal{K} \cup \{0\}$$

Restricting first attention to the case in which the time interval is finite, recall that the time paths of a Brownian motion are almost surely continuous. Hence, given a continuous function  $f : \mathcal{T} \times \mathbb{R}^K \mapsto \mathbb{R}$  the paths  $\{f(\mathcal{I}(\omega, s))\}_{s \in \mathcal{T}}$  will be continuous almost everywhere on  $\Omega$ . Which means of course that, for almost all such paths and for any  $t \in [0, T)$ , the time-integral can be

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<sup>10</sup>A real analytic function defined on an open and convex subset of  $\mathbb{R}^n$  ( $n \in \mathbb{N} \setminus \{0\}$ ) is either zero everywhere on its domain or non-zero a.e. (see Theorem B.3 in Anderson and Raimondo [1]).

approximated as a Riemann-Stieltjes sum

$$\begin{aligned}
\int_t^T f(\mathcal{I}(\omega, s)) ds &= \lim_{\Pi_\tau \rightarrow 0} \sum_{i=1}^{\tau} f(\mathcal{I}(\omega, s_{i-1})) \Delta_i \\
&= \lim_{\Pi_\tau \rightarrow 0} \sum_{i=1}^{\tau} f(s_{i-1}, \beta(\omega, s_{i-1})) \Delta_i \\
&= \lim_{\Pi_\tau \rightarrow 0} \sum_{i=1}^{\tau} f\left(s_{i-1}, \beta(\omega, t) + \sum_{j=0}^{i-1} \beta(\omega, s_{j+1}) - \beta(\omega, s_j)\right) \Delta_i
\end{aligned}$$

where  $\Pi_\tau = \max_{i=1, \dots, \tau} \{\Delta_i = s_i - s_{i-1}\}$  denotes the mesh of the typical partition  $t = s_0 < s_1 < \dots < s_{\tau-1} < s_\tau = T$  for some  $\tau \in \mathbb{N}^*$  in the approximating sequence. Yet, for any  $s_j, s_{j+1} \in [t, T]$ , the increments  $\beta(\omega, s_{j+1}) - \beta(\omega, s_j)$  are independent of  $\mathcal{F}_t$  and i.i.d  $\mathcal{N}(\mathbf{0}^K, (s_{j+1} - s_j) \mathbf{I}_K)$  over  $\Omega$ . Hence, conditional on  $\beta_t \in \mathbb{R}^K$  and for each partition in the approximating sequence, the paths are distributed

$$\sum_{i=1}^{\tau} f(s_{i-1}, \beta_{s_{i-1}}) \Delta_i \sim \sum_{i=1}^{\tau} f\left(s_{i-1}, \beta_t + \sum_{j=0}^{i-1} \mathbf{x}_j\right) \Delta_i$$

where  $\mathbf{x}_j \sim \mathcal{N}(\mathbf{0}^K, \Delta_{j+1} \mathbf{I}_K)$  and, thus,  $\sum_{j=0}^{i-1} \mathbf{x}_j \sim \mathcal{N}(\mathbf{0}^K, (s_i - t) \mathbf{I}_K)$ . As  $\Pi_\tau \rightarrow 0$ , therefore, conditionally on  $\beta_t = \beta$  we have

$$\begin{aligned}
\int_t^T f(s, \beta_s) ds &\sim |_{\beta_t = \beta} \int_t^T f(s, \beta + \sqrt{s-t} \mathbf{x}) ds \\
\mathbb{E} \left[ \int_t^T f(s, \beta_s) ds \mid \beta_t = \beta \right] &= \mathbb{E}_{\mathbf{x}} \left[ \int_t^T f(s, \beta + \sqrt{s-t} \mathbf{x}) ds \right] \quad (t, \beta) \in [0, T] \times \mathbb{R}^K (13)
\end{aligned}$$

where  $\mathbf{x} \sim \mathcal{N}(\mathbf{0}^K, \mathbf{I}_K)$ . Given this, our preceding analysis can be applied here in a rather straightforward way if we assume the following.

**A 2**  $m(s, \mathbf{x}) : \mathcal{T} \times \mathbb{R}^K \mapsto \mathbb{R}_{++}$  is continuous, continuously-differentiable w.r.t.  $\mathbf{x}$ , and s.t. for any  $j \in \mathcal{K} \cup \{0\}$  we have

$$\begin{aligned}
\exists (\tilde{r}_j, \tilde{C}_j) \in \left(0, \frac{1}{2T}\right) \times \mathbb{R}_{++} & : |m_j(s, \mathbf{x})| \leq \tilde{C}_j e^{\tilde{r}_j |\mathbf{x}|^2} \\
|D_k m_j(s, \mathbf{x})| &\leq \tilde{C}_j e^{\tilde{r}_j |\mathbf{x}|^2} \quad \forall (s, \mathbf{x}) \in [0, T] \times \mathbb{R}^K \quad \forall k \in \mathcal{K}
\end{aligned}$$

For since  $\tilde{r}_j < \frac{1}{2T} < \frac{1}{2(T-t)} < \frac{1}{2(s-t)} \quad \forall s \in (t, T)$ , by Lemma A.7,  $\mathbb{E}_{\mathbf{x}} [m_j(s, \beta_t + \sqrt{s-t} \mathbf{x})]$  is well-defined for any  $(s, \beta_t) \in [t, T] \times \mathbb{R}^K$ . In addition, Corollary B.1 ensures that the expectation is

integrable on  $(t, T)$  so that, given (13), the pricing equation in (12) can be written as

$$\begin{aligned} (mP_j)(t, \beta_t) &= \mathbb{E}_{\mathbf{x}} \left[ \int_t^T m_j(s, \beta_t + \sqrt{s-t}\mathbf{x}) ds \right] \\ &= \int_t^T \mathbb{E}_{\mathbf{x}} [m_j(s, \beta_t + \sqrt{s-t}\mathbf{x})] ds \quad j \in \mathcal{K} \cup \{0\} \end{aligned} \quad (14)$$

the last equality by Fubini's theorem (see, for instance, Corollary 13.9 in Schilling [?]). Moreover, again by Lemma A.7, for any  $s \in (t, T]$  the integrand in (14) has a well-defined partial derivative w.r.t. the typical Brownian dimension given by

$$D_k \mathbb{E}_{\mathbf{x}} [m_j(s, \beta_t + \sqrt{s-t}\mathbf{x})] = \mathbb{E}_{\mathbf{x}} \left[ \frac{x_k}{\sqrt{s-t}} m_j(s, \beta_t + \sqrt{s-t}\mathbf{x}) \right] \quad (j, k) \in \mathcal{K} \cup \{0\} \times \mathcal{K}$$

Furthermore, the absolute value of this derivative is uniformly bounded on  $(t, T)$  by a continuous function of  $\beta_t$  (Lemma B.4). Hence, it is bounded in a neighborhood of  $(t, T) \times \{\beta_t\}$  in  $[0, T] \times \mathbb{R}^K$  and Lemma A.6 means that

$$\begin{aligned} D_k(mP_j)(t, \beta_t) &= \int_t^T D_k \mathbb{E}_{\mathbf{x}} [m_j(s, \beta_t + \sqrt{s-t}\mathbf{x})] ds \\ &= \int_t^T \mathbb{E}_{\mathbf{x}} \left[ \frac{x_k}{\sqrt{s-t}} m_j(s, \beta_t + \sqrt{s-t}\mathbf{x}) \right] ds \quad (j, k) \in \mathcal{K} \cup \{0\} \times \mathcal{K} \end{aligned} \quad (15)$$

This brings us in place to present our next fundamental result. To this end, observe first that, for any  $(n, k) \in \mathcal{K} \times \mathcal{K}$  and at any  $(t, \beta_t) \in \mathcal{T} \times \mathbb{R}^K$ , we have

$$\begin{aligned} (mP_0)(t, \beta_t)^2 D_k p_n(t, \beta_t) &= (mP_0)(t, \beta_t) D_k(mP_n)(t, \beta_t) \\ &\quad - (mP_n)(t, \beta_t) D_k(mP_0)(t, \beta_t) \end{aligned} \quad (16)$$

Taking then an arbitrary  $h \in (t, T)$ , it must be

$$\begin{aligned} &(mP_0)(t+h, \beta_{t+h})^2 D_k p_n(t+h, \beta_{t+h}) - (mP_0)(t, \beta_t)^2 D_k p_n(t, \beta_t) \\ &= (mP_0)(t+h, \beta_{t+h}) D_k(mP_n)(t+h, \beta_{t+h}) - (mP_0)(t, \beta_t) D_k(mP_n)(t, \beta_t) \\ &\quad + (mP_n)(t, \beta_t) D_k(mP_0)(t, \beta_t) - (mP_n)(t+h, \beta_{t+h}) D_k(mP_0)(t+h, \beta_{t+h}) \end{aligned}$$

Yet, as shown in (14)-(15), all terms above are time-integrals. Letting, therefore,  $\mathcal{D}_t$  denote the Dynkin operator, Lemma B.5 means that as  $h \rightarrow 0$  we have

$$\begin{aligned} &\mathcal{D}_t(mP_0)(t, \beta_t)^2 D_k p_n(t, \beta_t) \\ &= h^{-2} \int_t^{t+h} \int_t^{t+h} \mathbb{E}_{(\mathbf{x}, \mathbf{y})} \left[ m_0(s, \beta_t + \sqrt{s-t}\mathbf{x}) m_0(\tau, \beta_t + \sqrt{\tau-t}\mathbf{y}) \right. \\ &\quad \left. \times \left( \frac{x_k}{\sqrt{s-t}} - \frac{y_k}{\sqrt{\tau-t}} \right) g_{n/0}(s, \beta_t + \sqrt{s-t}\mathbf{x}) \right] ds d\tau \end{aligned} \quad (17)$$

Applying at this point our argument from the first part of the proof of Proposition 2.1 is rather

straightforward, albeit for one caveat. Here we are integrating also with respect to time, and the time-dependence of the relative dividend leaves the mean-value expansion - recall (10) - with a term  $g_{n/0}(s, \beta_t)$  that no longer comes out of the integrals. To tackle this issue, we will call upon an additional assumption.

**A 3** *None of the following sets*

$$\mathcal{K}_n = \{k \in \mathcal{K} : \tilde{\sigma}_{nk} \neq \tilde{\sigma}_{0k}\} \quad n \in \mathcal{K}$$

*is empty.*

As it turns out, this suffices for our previous argument to establish the following result.

**Proposition 2.2** *Let  $\mathcal{T} = [0, T]$  for some  $T > 0$ . Suppose also that the flow-dividends and price processes are given, respectively, by (1) and (12) while A2 and A3 are both satisfied. Then, for any  $(n, k) \in \mathcal{K} \times \mathcal{K}$  and any  $(\tau, \beta_\tau) \in [t, T) \times \mathbb{R}^K$ , we have*

$$D_t \mathbb{E} \left[ (mP_0)(\tau, \beta_\tau)^2 D_k p_n(\tau, \beta_\tau) | \mathcal{F}_t \right] = (\tilde{\sigma}_{nk} - \tilde{\sigma}_{0k}) \mathbb{E} [(m_n m_0)(\tau, \beta_\tau) | \mathcal{F}_t]$$

and, thus,

$$(mP_0)(t, \beta_t)^2 D_k p_n(t, \beta_t) = -(\tilde{\sigma}_{nk} - \tilde{\sigma}_{0k}) \int_t^T \mathbb{E} [(m_n m_0)(s, \beta_s) | \mathcal{F}_t] ds$$

for any  $(t, \beta_t) \in [0, T) \times \mathbb{R}^K$ .

**Proof.** See Appendix B. ■

Defining then the diagonal matrix

$$\tilde{\mathbf{g}}(t, \beta_t) = \begin{bmatrix} \int_t^T \mathbb{E} [(m_1 m_0)(s, \beta_s) | \mathcal{F}_t] ds & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \int_t^T \mathbb{E} [(m_K m_0)(s, \beta_s) | \mathcal{F}_t] ds \end{bmatrix}$$

the Jacobian of the relative prices gives

$$(mP_0)(t, \beta_t)^2 J_p(t, \beta_t) = -\tilde{\mathbf{g}}(t, \beta_t) \times \tilde{\Sigma}_0$$

so that its determinant must satisfy

$$\begin{aligned} (mP_0)(t, \beta_t)^2 |J_p(t, \beta_t)| &= \left| (mP_0)(t, \beta_t)^2 J_p(t, \beta_t) \right| \\ &= \left| \tilde{\Sigma}_0 \right| \prod_{k \in \mathcal{K}} \int_t^T \mathbb{E} [(m_k m_0)(s, \beta_s) | \mathcal{F}_t] ds \end{aligned}$$

where  $\tilde{\Sigma}_0 = [\tilde{\sigma}_{nk} - \tilde{\sigma}_{0k}]_{(n,k) \in \mathcal{K} \times \mathcal{K}}$ . Clearly, if  $\tilde{\Sigma}_0$  is non-singular [resp. singular],  $J_p(t, \beta_t)$  is also non-singular [resp. singular] everywhere on  $[0, T) \times \mathbb{R}^K$  - a conclusion that can be stated also as follows.

**Theorem 2.2** *Let the conditions of Proposition 2.2 be satisfied. Then the following statements are equivalent*

- (i)  $\tilde{\Sigma}_0$  is non-singular.
- (ii)  $J_p(t, \beta_t)$  has full rank everywhere on  $[0, T) \times \mathbb{R}^K$ .
- (iii) The financial market is dynamically complete.

**Remark II** Under the flow-dividend specification in (1), for the growth condition in A2 to be met it suffices that

$$\begin{aligned} \exists (\tilde{r}, \tilde{C}) \in \left(0, \frac{1}{2T}\right) \times \mathbb{R}_{++} \quad : \quad |m(s, \mathbf{x})| \leq \tilde{C}e^{\tilde{r}|\mathbf{x}|^2} \\ |D_k m(s, \mathbf{x})| \leq \tilde{C}e^{\tilde{r}|\mathbf{x}|^2} \quad \forall (s, \mathbf{x}) \in [0, T) \times \mathbb{R}^K \quad \forall k \in \mathcal{K} \end{aligned}$$

Indeed, by the same argument as in Remark I, given any  $j \in \mathcal{K} \cup \{0\}$  we can find  $\bar{r}_j \in (0, \frac{1}{2T} - \tilde{r})$  and  $\bar{C}_j > 0$  s.t.  $|g_j(s, \mathbf{x})| \leq \bar{\mu}_j \bar{C}_j e^{\bar{r}_j |\mathbf{x}|^2}$  and  $|D_k g_j(s, \mathbf{x})| \leq |\tilde{\sigma}_{jk}| \bar{\mu}_j \bar{C}_j e^{\bar{r}_j |\mathbf{x}|^2}$  everywhere on  $[0, T) \times \mathbb{R}^K$ , where  $\bar{\mu}_j$  is the maximum of  $\tilde{\mu}_j(\cdot)$  on  $[0, T]$ . It suffices, thus, to set  $\tilde{r}_j = \tilde{r} + \bar{r}_j$  and  $\tilde{C}_j = \max\{\tilde{C}, A_j \bar{\mu}_j \bar{C}_j\}$  where  $A_j = \max_{k \in \mathcal{K}} \{|\tilde{\sigma}_{jk}|\}$ .

**Remark III** If assumption A3 is not satisfied,  $\tilde{\Sigma}_0$  must be singular.

Turning next to the case in which  $\mathcal{T} = \mathbb{R}_+$ , the corresponding results ought to follow from the preceding analysis as  $T \rightarrow +\infty$ . Of course, this requires that the respective expectations on the right-hand sides of (14)-(15) are well-defined at all  $T \in \mathbb{R}_{++}$ . It presupposes in other words a strengthening of assumption A2 so that Lemma A.7 applies accordingly. To this end, we will assume the following.

**A 4**  $m(s, \mathbf{x}) : \mathbb{R}_+ \times \mathbb{R}^K \mapsto \mathbb{R}_{++}$  is continuous, continuously-differentiable w.r.t.  $\mathbf{x}$ , and s.t. for any  $j \in \mathcal{K} \cup \{0\}$  we have

$$\begin{aligned} \forall \tilde{r}_j > 0 \quad \exists \tilde{C}_j > 0 \quad : \quad |m_j(s, \mathbf{x})| \leq \tilde{C}_j e^{\tilde{r}_j |\mathbf{x}|^2} \\ |D_k m_j(s, \mathbf{x})| \leq \tilde{C}_j e^{\tilde{r}_j |\mathbf{x}|^2} \quad \forall (s, \mathbf{x}) \in \mathbb{R}_+ \times \mathbb{R}^K \quad \forall k \in \mathcal{K} \end{aligned}$$

Give this, the respective limiting versions of (14)-(15) obtain as

$$\begin{aligned} (mP_j)(t, \beta_t) &= \int_t^{+\infty} \mathbb{E}_{\mathbf{x}} [(mg_j)(s, \beta_t + \sqrt{s-t}\mathbf{x})] ds \quad j \in \mathcal{K} \cup \{0\} \\ &= \lim_{T \rightarrow +\infty} (mP_j)(t, \beta_t; T) \end{aligned}$$

and

$$\begin{aligned} D_k(mP_j)(t, \beta_t) &= \int_t^{+\infty} D_k \mathbb{E}_{\mathbf{x}} [(mg_j)(s, \beta_t + \sqrt{s-t}\mathbf{x})] ds \quad (j, k) \in \mathcal{K} \cup \{0\} \times \mathcal{K} \\ &= \lim_{T \rightarrow +\infty} D_k(mP_j)(t, \beta_t; T) \end{aligned}$$

where, abusing notation slightly,  $(mP_j)(t, \beta_t; T)$  and  $D_k(mP_j)(t, \beta_t; T)$  are meant to denote the right-hand sides of (14) and (15), respectively. And since Proposition 2.2 remains valid for any  $T \in \mathbb{R}_{++}$  under assumption A4, we have

$$\begin{aligned} (mP_0)(t, \beta_t)^2 D_k p_n(t, \beta_t) &= \lim_{T \rightarrow +\infty} \left[ \begin{aligned} &(mP_0)(t, \beta_t; T) D_k(mP_n)(t, \beta_t; T) \\ &- (mP_n)(t, \beta_t; T) D_k(mP_0)(t, \beta_t; T) \end{aligned} \right] \\ &= \lim_{T \rightarrow +\infty} (mP_0)(t, \beta_t; T)^2 D_k p_n(t, \beta_t; T) \end{aligned}$$

From which it follows that

$$(mP_0)(t, \beta_t)^2 |J(t, \beta_t)| = \lim_{T \rightarrow +\infty} (mP_0)(t, \beta_t; T)^2 |J(t, \beta_t; T)|$$

where  $J_p(t, \beta_t; T) = [D_k p_n(t, \beta_t; T)]_{(n,k) \in \mathcal{K} \times \mathcal{K}}$ . Recall now that for any  $j \in \mathcal{K} \cup \{0\}$  we have  $m_j(\cdot) > 0$  a.e. on  $\mathbb{R}_+ \times \mathbb{R}^K$ . For any  $k \in \mathcal{K}$  and  $T^* > 0$  therefore we get

$$\begin{aligned} \lim_{T \rightarrow +\infty} \int_t^T \mathbb{E}[(m_k m_0)(s, \beta_s) | \mathcal{F}_t] ds &= \int_t^\infty \mathbb{E}[(m_k m_0)(s, \beta_s) | \mathcal{F}_t] ds \\ &\geq \int_t^{T^*} \mathbb{E}[(m_k m_0)(s, \beta_s) | \mathcal{F}_t] ds \end{aligned}$$

And as

$$(mP_0)(t, \beta_t; T^*)^2 |J(t, \beta_t; T^*)| = \left| \tilde{\Sigma}_0 \right| \prod_{k \in \mathcal{K}} \int_t^{T^*} \mathbb{E}[(m_k m_0)(s, \beta_s) | \mathcal{F}_t] ds$$

it cannot but be

$$\begin{aligned} (mP_0)(t, \beta_t)^2 ||J(t, \beta_t)|| &= \left| \tilde{\Sigma}_0 \right| \lim_{T \rightarrow +\infty} \prod_{k \in \mathcal{K}} \int_t^T \mathbb{E}[(m_k m_0)(s, \beta_s) | \mathcal{F}_t] ds \\ &\geq \left| \tilde{\Sigma}_0 \right| \prod_{k \in \mathcal{K}} \int_t^{T^*} \mathbb{E}[(m_k m_0)(s, \beta_s) | \mathcal{F}_t] ds \geq 0 \end{aligned}$$

where  $||\cdot||$  denotes the absolute value of the determinant, while either inequality is strict if and only if  $\tilde{\Sigma}_0$  is non-singular. Needless to say, these observations can be stated formally as follows.

**Corollary 2.1** *Let  $\mathcal{T} = \mathbb{R}_+$ . Theorem 2.2 remains valid as long as A2 is replaced by A4 in the statement.*

**Remark IV** Under the flow-dividend specification in (1), as long as the functions  $\tilde{\mu}_j$  are bounded on  $\mathbb{R}_+$ , for the growth condition in A4 to be met it suffices that

$$\begin{aligned} \exists \tilde{r}, \tilde{C} > 0 \quad : \quad |m(s, \mathbf{x})| &\leq \tilde{C} e^{\tilde{r}|\mathbf{x}|^2} \\ |D_k m(s, \mathbf{x})| &\leq \tilde{C} e^{\tilde{r}|\mathbf{x}|^2} \quad \forall (s, \mathbf{x}) \in \mathbb{R}_+ \times \mathbb{R}^K \quad \forall k \in \mathcal{K} \end{aligned}$$

### 2.3 Lump-sums and Flows

It remains to examine the case in which the securities pay both dividend flows during the time-interval as well as lump sums on the terminal date. This presupposes a finite time-horizon and means of course that both terms inside the expectation in (??) apply. In this case, it is trivial to check that letting

$$\begin{aligned} P_{j1}(t, \beta_t) &= \int_{\mathbb{R}^K} \frac{M_j(T, \beta_t + \sqrt{T-t}\mathbf{x})}{M(t, \beta_t)} \phi(\mathbf{x}) d\mathbf{x} \\ P_{j2}(t, \beta_t) &= \int_t^T \frac{\mathbb{E}_{\mathbf{x}}[(mg_j)(s, \beta_t + \sqrt{s-t}\mathbf{x})]}{m(t, \beta_t)} ds \end{aligned}$$

we can write the pricing equation as follows

$$P_j(t, \beta_t) = (P_{j1} + P_{j2})(t, \beta_t) \quad t \in [0, T] \quad j \in \mathcal{K} \cup \{0\} \quad (18)$$

Letting, thus,

$$\tilde{P}_n(t, \beta_t) = \frac{P_{01}(t, \beta_t) P_{02}(t, \beta_t)}{P_0(t, \beta_t)} [p_{n2}(t, \beta_t) - p_{n1}(t, \beta_t)]$$

and  $p_{ni}(\cdot) = P_{ni}(\cdot) / P_{0i}(\cdot)$  for  $i \in \{1, 2\}$ , we get

$$\begin{aligned} &P_0(t, \beta_t) D_k p_n(t, \beta_t) \\ &= P_{01}(t, \beta_t) D_k p_{n1}(t, \beta_t) + P_{02}(t, \beta_t) D_k p_{n2}(t, \beta_t) \\ &- \tilde{P}_n(t, \beta_t) \left( \frac{D_k P_{01}(t, \beta_t)}{P_{01}(t, \beta_t)} - \frac{D_k P_{02}(t, \beta_t)}{P_{02}(t, \beta_t)} \right) \quad (n, k) \in \mathcal{K} \times \mathcal{K} \end{aligned} \quad (19)$$

Obviously, the dynamics of the typical relative price are now complex to the extent that a direct application of the preceding analysis does not lead to any clear conclusions. Nonetheless, a sufficient condition for dynamic completeness can be identified if, in addition to the  $K+1$  underlying securities, standard European options are also available for trading. To see this observe first that, since  $\lim_{t \rightarrow T} M_j(\mathcal{I}(\omega, t)) = M_j(\mathcal{I}(\omega, T))$  everywhere on  $\Omega$  (Lemma B.2), the absence of arbitrage



requires the existence of some constant  $A > 0$  such that<sup>11</sup>

$$P_j(\mathcal{I}(\omega, T)) = AG_j(\mathcal{I}(\omega, T)) \quad \text{a.e. on } \Omega \quad j \in \mathcal{K} \cup \{0\}$$

As a result, given any  $(t, \beta_t) \in [0, T) \times \mathbb{R}^K$ , the absolute prices of a European call with maturity date  $T$  and strike price  $P_0(t, \beta_t)$  as well as that of an equivalent European put must satisfy, respectively,<sup>12</sup>

$$\begin{aligned} (MP_j^C)(t, \beta_t) &= A \int_{V_j(t, \beta_t)} (MG_j)\left(T, \beta_t + \sqrt{T - t}\mathbf{x}\right) d\Phi(\mathbf{x}) \\ &\quad - P_0(t, \beta_t) \int_{V_j(t, \beta_t)} M\left(T, \beta_t + \sqrt{T - t}\mathbf{x}\right) d\Phi(\mathbf{x}) \\ (MP_j^P)(t, \beta_t) &= P_0(t, \beta_t) \int_{\mathbb{R}^K \setminus V_j(t, \beta_t)} M\left(T, \beta_t + \sqrt{T - t}\mathbf{x}\right) d\Phi(\mathbf{x}) \\ &\quad - A \int_{\mathbb{R}^K \setminus V_j(t, \beta_t)} (MG_j)\left(T, \beta_t + \sqrt{T - t}\mathbf{x}\right) d\Phi(\mathbf{x}) \end{aligned}$$

where

$$V_j(t, \beta_t) = \left\{ \mathbf{x} \in \mathbb{R}^K : G_j\left(T, \beta_t + \sqrt{T - t}\mathbf{x}\right) \geq P_0(t, \beta_t) / A \right\}$$

Which results of course in the put-call parity

$$P_j^C(t, \beta_t) - P_j^P(t, \beta_t) = AP_{j1}(t, \beta_t) - B_0(t, \beta_t) P_0(t, \beta_t) \quad j \in \mathcal{K} \cup \{0\} \quad (20)$$

where

$$B_0(t, \beta_t) = \mathbb{E}_{\mathbf{x}} \left[ M\left(T, \beta_t + \sqrt{T - t}\mathbf{x}\right) \right] / M(t, \beta_t)$$

is the current price of a zero-coupon bond maturing at  $T$  (should this security be traded).

For each  $j \in \mathcal{K} \cup \{0\}$ , we can construct now the portfolio that consists of being long one unit of the corresponding European call, short one unit of the equivalent put, and long  $B_0(t, \beta_t)$  units of the zeroth security. By (20), the current value of the typical portfolio is  $AP_{j1}(t, \beta_t)$  and, in addition to the original securities, we have all together  $2(K + 1)$  assets. Treating then the zeroth portfolio as the numeraire security, the new Jacobian of the relative prices is now a  $(2K + 1) \times K$

<sup>11</sup>For instance, if  $P_j(\mathcal{I}(\omega, T)) / P_i(\mathcal{I}(\omega, T)) > \lim_{t \rightarrow T} (MP_j)(\mathcal{I}(\omega, T)) / (MP_i)(\mathcal{I}(\omega, t))$  on a non-zero-measure subset of  $\Omega$ , a trivial arbitrage could be implemented by buying the  $j$ th security and short-sell the same amount of the  $i$ th one just prior to the terminal date.

<sup>12</sup>As usual, European calls and puts are said to be equivalent if they are written on the same underlying and have identical maturity dates and exercise prices.

matrix, which can be written in the following block-form

$$\bar{J}_{p_1}(t, \beta_t) = \begin{bmatrix} A^{-1} [D_k (P_j/P_{01}) (t, \beta_t)]_{(j,k) \in \mathcal{K} \cup \{0\} \times \mathcal{K}} \\ [D_k (P_{n1}/P_{01}) (t, \beta_t)]_{(n,k) \in \mathcal{K} \times \mathcal{K}} \end{bmatrix}$$

Similarly, for each  $j \in \mathcal{K} \cup \{0\}$ , we can construct the portfolio that consists of being long one unit of the security, short one unit of the corresponding European call, long one unit of the equivalent put, and short  $B_0(t, \beta_t)$  units of the zeroth security. By 18 and (20), the current value of this portfolio is now  $AP_{j2}(t, \beta_t)$  and, treating the new zeroth portfolio as the numeraire security, the Jacobian of the relative prices is given by

$$\bar{J}_{p_2}(t, \beta_t) = \begin{bmatrix} A^{-1} [D_k (P_j/P_{02}) (t, \beta_t)]_{(j,k) \in \mathcal{K} \cup \{0\} \times \mathcal{K}} \\ [D_k (P_{n2}/P_{02}) (t, \beta_t)]_{(n,k) \in \mathcal{K} \times \mathcal{K}} \end{bmatrix}$$

Given these constructions, the following is a direct application of Theorems 2.1 and 2.2.<sup>13</sup>

**Corollary 2.2** *Let  $\mathcal{T} = [0, T]$  for some  $T > 0$ . Suppose also that, for each  $j \in \mathcal{K} \cup \{0\}$ , the dividends and price processes are given, respectively, by (1) and (18) with assumptions A1 and A2-A3 satisfied, while a European call with maturity date  $T$  and strike price  $P_0(\mathcal{I}(\omega, t))$  as well as an equivalent put are traded almost everywhere on  $\Omega \times (0, T)$ .*

*If  $\Sigma_0$  [resp.  $\tilde{\Sigma}_0$ ] is non-singular,  $\bar{J}_{p_1}(t, \beta_t)$  [resp.  $\bar{J}_{p_2}(t, \beta_t)$ ] has rank  $K$  a.e. on  $[0, T) \times \mathbb{R}^K$  so that the financial market is dynamically complete.*

*In fact, if the requisite options are traded everywhere on  $\Omega \times (0, T)$  and  $\tilde{\Sigma}_0$  is non-singular,  $\bar{J}_{p_2}(t, \beta_t)$  has rank  $K$  everywhere on  $[0, T) \times \mathbb{R}^K$ .*

## 2.4 Money-market account

The preceding analysis extends readily to the case in which the zeroth security is a money market account - subject to two (purely-technical in nature) additional restrictions. Namely, that  $\mathcal{F}$  is in fact the Brownian filtration  $\mathcal{F}^\beta$ , and a strengthening of the no-arbitrage condition. To see this, let  $\mathcal{T} = [0, T]$  for some  $T > 0$ . Let also  $\{r_t : t \in \mathcal{T}\}$  be an (instantaneously) riskless-rate process such that  $\int_0^T |r_s| ds < +\infty$  a.s. and  $P_{0t} = P_0 \exp\left(\int_0^t r_s ds\right)$  for some initial value  $P_0 > 0$ . Suppose furthermore that the relative price process  $\mathbf{p}_t = (1, p_{1t}, \dots, p_{Kt})$  admits *no approximate arbitrage*.<sup>14</sup>

<sup>13</sup>Needless to say, the argument can be presented also by deploying instead the corresponding equivalent forward contracts. The position of being long on a standard European call and at the same time short on an equivalent put can be replicated by holding a forward contract to receive  $A$  units of the lump-sum terminal dividend at  $T$  at a price  $P_0(\mathcal{I}(\omega, t))$ .

<sup>14</sup>Recall footnote 6. An *approximate arbitrage* is a sequence  $\{W_n\}_{n \in \mathbb{N}}$  in  $M(\mathbf{P})$  with  $\psi(W_n) \leq 0$  for all  $n$ , s.t.  $\exists$  another sequence  $\{\tilde{W}_n\}_{n \in \mathbb{N}}$  with (i)  $\tilde{W}_n \leq W_n$  for all  $n$ , (ii)  $\mathbb{E} \left[ \left( \int_0^T \sum_{j \in \mathcal{K} \cup \{0\}} \theta'_{js} \left( a_{js} + \left( \tilde{\mu}_{js}/\tilde{\mu}_{js} \right) g_{js} \right) ds \right)^2 | \mathcal{F}_0 \right] < +\infty$  and  $\mathbb{E} \left[ \int_0^T \sum_{j \in \mathcal{K} \cup \{0\}} \theta'_{js} (\mathbf{b}_{js} + g_{js} \tilde{\sigma}_j) ds | \mathcal{F}_0 \right] < +\infty$ , and (iii)  $\mathbb{E} \left[ \left( \tilde{W}_n - Z \right)^2 | \mathcal{F}_0 \right] \rightarrow 0$  for some random variable  $Z$  which is everywhere strictly positive.

The latter condition guarantees the existence of an equivalent to  $\pi$  martingale measure  $\tilde{\pi}$  on  $(\Omega, \mathcal{F}_T^\beta)$  (see, for instance, Proposition 6.K in Duffie [17]).<sup>15</sup> And since we are working with the Brownian filtration, being equivalent means that  $\tilde{\pi}$  and  $\pi$  constitute a Girsanov pair (see the discussion that follows Theorem 5.7.1 in Bingham and Kiesel [7]). There will be that is a measurable, adapted,  $K$ -dimensional process  $\{\gamma_t : t \in [0, T]\}$  with  $\int_0^t |\gamma_s|^2 ds < +\infty$  a.s. such that the Radon-Nikodym derivative conditional on  $\mathcal{F}_t^\beta$ ,  $d\tilde{\pi}/d\pi = \exp\left(-\int_0^t \gamma_s^\top d\beta_s - \frac{1}{2} \int_0^t |\gamma_s|^2 ds\right)$ , is a continuous martingale; hence,  $\left\{\tilde{\beta}_t = \beta_t + \int_0^t \gamma_s ds : t \in [0, T]\right\}$  is a  $K$ -dimensional Brownian motion on  $(\Omega, \mathcal{F}^\beta, \tilde{\pi})$ .<sup>16</sup>

It follows then (see, for instance, Section 6.L in Duffie [17]) that the price process will be given now by

$$\begin{aligned} P_n(t, \tilde{\beta}_t) &= \mathbb{E}_{\tilde{\pi}} \left[ \tilde{G}_n(\tilde{\mathcal{I}}(\omega, T)) + \int_{T \setminus [0, t)} \tilde{g}_n(\tilde{\mathcal{I}}(\omega, s)) ds | \mathcal{F}_t \right], & n \in \mathcal{K} \\ P_0(t, \tilde{\beta}_t) &= P_0 \exp \left( \int_0^t r_s ds \right) \end{aligned}$$

while, for any  $n \in \mathcal{K}$  and at any  $(t, \beta_t) \in [0, T] \times \mathbb{R}^K$ , the dividend-processes evolve as

$$dG_n(t, \tilde{\beta}_t) = (\mu'_n(t) / \mu_n(t) - \sigma_n^\top \gamma_t) G_n(t, \tilde{\beta}_t) dt + G_n(t, \tilde{\beta}_t) \sigma_n^\top d\tilde{\beta}_t \quad (21)$$

$$dg_n(t, \tilde{\beta}_t) = (\tilde{\mu}'_n(t) / \tilde{\mu}_n(t) - \tilde{\sigma}_n^\top \gamma_t) g_n(t, \tilde{\beta}_t) dt + g_n(t, \tilde{\beta}_t) \tilde{\sigma}_n^\top d\tilde{\beta}_t \quad (22)$$

The instantaneous dispersion coefficients having the same form as before, the analysis in Sections 2.1-2.3 remains valid here with respect to the dynamics  $\tilde{D}_k = \partial p_{nt} / \partial \tilde{\beta}_{kt}$ . Indeed, letting

$$G_0(\tilde{\mathcal{I}}(\omega, t)) = \exp \left( - \int_0^t r_s ds \right) = \tilde{g}_0(\tilde{\mathcal{I}}(\omega, t)) \quad t \in [0, T]$$

the respective arguments proceed and the corresponding results can be stated exactly as before by setting  $M(\cdot) = 1 = m(\cdot)$  and  $\sigma_0 = \mathbf{0}^K = \tilde{\sigma}_0$ .

In fact, our analysis becomes more streamlined in two respects. On the one hand, given (21)-(22) and  $M(\cdot) = 1 = m(\cdot)$ , the growth conditions in A1-A2 are now satisfied trivially (recall Remarks I-II) whereas that in A4 amounts to requiring that  $\int_0^{+\infty} |r_s| ds < +\infty$  a.s. and  $\int_0^{+\infty} |\tilde{\mu}_n + \tilde{\sigma}_n^\top \gamma_s| ds < +\infty$  a.s. for all  $n \in \mathcal{K}$ . On the other hand, assumption A3 is no longer needed. This is because

<sup>15</sup>The no-arbitrage condition can be imposed instead on the absolute price process, as long as the riskless-rate process is bounded (i.e., there is a constant  $C > 0$  s.t.  $|r_s| \leq C$  everywhere on  $\Omega \times [0, T]$  - see Corollary 6.K in Duffie [17]).

<sup>16</sup>If  $\mathcal{F}$  is not the Brownian filtration, a well-known sufficient condition for  $\tilde{\beta}$  to be a Brownian motion is that  $\int_0^t |\gamma_s|^2 ds < +\infty$  a.s. in conjunction with Novikov's condition:  $\mathbb{E} \left[ \exp \left( \frac{1}{2} \int_0^t |\gamma_s|^2 ds \right) \right] < +\infty$  (see Theorem 5.7.1 in Bingham and Kiesel [7]).

(15) now gives

$$\begin{aligned}\tilde{D}_k P_0(t, \tilde{\beta}_t) &= 0 & k \in \mathcal{K} \\ \tilde{D}_k P_n(t, \tilde{\beta}_t) &= \int_t^T \mathbb{E}_{\mathbf{x}} \left[ \frac{x_k}{\sqrt{s-t}} g_n(s, \tilde{\beta}_t + \sqrt{s-t} \mathbf{x}) \right] ds & (n, k) \in \mathcal{K} \times \mathcal{K}\end{aligned}$$

while (17) reads

$$\begin{aligned}\mathcal{D}_t P_0(t, \tilde{\beta}_t)^2 \tilde{D}_k p_n(t, \tilde{\beta}_t) \\ = h^{-2} \int_t^{t+h} g_0(\tau) \int_t^{t+h} \mathbb{E}_{\mathbf{x}} \left[ \frac{x_k}{\sqrt{s-t}} g_n(s, \tilde{\beta}_t + \sqrt{s-t} \mathbf{x}) \right] ds d\tau\end{aligned}$$

And as  $\mathbb{E}_{\mathbf{x}} \left[ \frac{x_k}{\sqrt{s-t}} g_n(s, \tilde{\beta}_t) \right] = 0$  at any  $s \in (t, T]$ , the mean-value-expansion argument can be presented now exactly as in Section 2.1, regardless of the fact that we are integrating also with respect to time.

**Remark V** It is instructive to complete the exposition by recalling how an actual pricing-kernel and a riskless-rate processes are related. Given a pricing kernel  $M$ , the riskless-rate process can be constructed via the dynamics  $r_t dt = -\mathbb{E}_{\pi} [\mathcal{D}_t M(t) | \mathcal{F}_t] / M_t$ .<sup>17</sup> Conversely, given a riskless-rate process  $r$  and an equivalent to  $\pi$  martingale measure  $\tilde{\pi}$  with density process  $\xi_t = \mathbb{E}_{\pi} [d\tilde{\pi}/d\pi | \mathcal{F}_t]$ , a pricing kernel is given by  $M_t = \xi_t \exp \left( \int_0^t r_s ds \right)$ , as long as  $\text{Var}_{\pi} \left[ \xi_t \exp \left( \int_0^t r_s ds \right) | \mathcal{F}_0 \right] < +\infty$  for all  $t$ .<sup>18</sup> Of course, starting with one process, the other will not be necessarily uniquely-defined unless the financial market is dynamically complete.

### 3 Discussion and Related Literature

Using well-known mathematical tools, our analysis established that, as long as the underlying risk process is a Brownian motion while the securities' dividends and prices are given by (1) and (2) respectively, dynamic completeness can be characterized via the non-degeneracy of one of  $\Sigma_0$  or  $\tilde{\Sigma}_0$ , the matrices of factor loadings. As pointed out in the introductory section, this follows also from the recent studies by Anderson and Raimondo [1] and Hugonnier et al. [34] - as well as the subsequent treatments by Riedel and Herzberg [61] or Kramkov [41]. In Anderson and

<sup>17</sup>Given a pricing kernel  $M$ , an equivalent to  $\pi$  martingale measure  $\tilde{\pi}$  can be constructed via the density process  $\mathbb{E} [d\tilde{\pi}/d\pi | \mathcal{F}_t] \equiv \xi_t = \exp \left( \int_0^t r_s ds \right) M_t / M_0$  (see, for instance, Proposition 6.F in Duffie [17]). By Bayes' rule then,  $\mathbb{E}_{\pi} [\xi_t | \mathcal{F}_{\tau}] = \xi_{\tau}$  (see Chapter 6.F and Appendix C in Duffie [17]) and, thus,  $\mathbb{E}_{\pi} [d\xi_t | \mathcal{F}_{\tau}] = 0$  or  $\mathbb{E}_{\pi} \left[ \exp \left( \int_0^t r_s ds \right) (r_t dt M_t + dM_t) | \mathcal{F}_{\tau} \right] = 0$  at any  $t \in [\tau, T]$ . Which for  $\tau = t$  gives the riskless-rate dynamics in the text.

<sup>18</sup>The finite-variance condition is introduced in Proposition 6.F in Duffie [17] because equivalent martingale measures are defined in this book under the condition that the variance of the Radon-Nikodým derivative is finite, a technical convenience not uniformly adopted in the literature. In the current setting, we can ensure that the finite-variance condition is met by assuming that  $M(T, \cdot)$  (and, thus, also  $M^2(T, \cdot)$ ) satisfies the growth condition in A1 (recall Lemma A.7).

Raimondo [1] the underlying risk process is the same as here while the time-horizon is necessarily finite, on the terminal date of which the securities must pay lump-sum dividends.<sup>19</sup> The latter requirement is essential given that the non-degeneracy condition is imposed on the lump-sum dividends themselves.<sup>20</sup> By contrast, in Hugonnier et al. [34] the underlying risk process is allowed to follow a general diffusion and the time-horizon to be infinite. The approach in this paper is such that the non-degeneracy condition can be imposed instead on the flow-dividends.<sup>21</sup>

In either paper - as well as in Riedel and Herzberg [61] or Kramkov [41] - the argument for the sufficiency of the non-degeneracy condition is essentially the same, and hinges fundamentally upon the assumption that all flow primitive variables are real analytic functions. To facilitate comparison with the present analysis, it is easier to focus on Anderson and Raimondo [1] which examines the same setting as our Section 2.3 but for more general dividend specifications. In terms of our notation (recall in particular Section 2.1), the intuition as to why the equilibrium pricing process is dynamically complete can be described as follows (see Appendices B and D in Anderson and Raimondo [1]).

Let us assume the existence of an open set  $V \subseteq \mathbb{R}^K$  s.t. (i)  $G_0(T, \beta) > 0$  for all  $\beta \in V$ , and (ii)  $\exists \omega_0 \in \Omega$  and  $\beta(\omega_0, T) \in V$  s.t.  $|J_G(T, \beta(\omega_0, T))| \neq 0$ . Condition (i) guarantees of course that the relative terminal dividends are well-defined on  $\{T\} \times V$ . And so are the relative terminal prices and their dispersion matrix. For since there is no value left to a security on the terminal date other than its lump-sum dividend, it must be  $p_n(T, \cdot) = G_{n/0}(T, \cdot)$  and, thus,  $J_p(T, \cdot) = J_G(T, \cdot)$  on  $V$ .<sup>22</sup> Given, however, that all terminal and intermediate dividends, endowments, and utilities are

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<sup>19</sup>Anderson and Raimondo assume that the terminal lump-sum dividends and individual endowments are continuous almost everywhere in  $\{T\} \times \mathbb{R}^K$  while the agents' utilities over terminal lump-sums are analytic, strictly increasing, strictly concave, and satisfy the Inada conditions uniformly over this domain. The intermediate dividend and endowment flows are analytic on  $(0, T) \times \mathbb{R}^K$ , while the agents' utilities over flows are analytic, strictly increasing, strictly concave, and satisfy the Inada conditions uniformly over this domain. The agents' utilities are time- and state-dependent functions of the form  $u(c(t, \beta_t), t, \beta_t)$ .

<sup>20</sup>This requirement is essential also in Riedel and Herzberg [61] as well as in Kramkov [41]. The methodology of Anderson and Raimondo relies heavily on non-standard analysis. Riedel and Herzberg provide a proof that is based on standard analysis, albeit for a setting that is more general in some dimensions yet more restricted in others. Compared to Anderson and Raimondo [1], in Riedel and Herzberg [61] the underlying risk-process is allowed to be a time-homogenous diffusion (as long as the diffusion matrix satisfies a uniform ellipticity condition). However, attention is restricted to the case in which one of the securities is a zero-coupon bond maturing at the terminal date, the aggregate endowment is bounded and bounded away from zero, the agents' endowments are time- and state-independent while their utilities are state-independent and depend on time only via a common across all agents "impatience" rate. In Kramkov [41] the setting of Anderson and Raimondo is extended to allow for general diffusions (and weakly differentiable terminal lump-sum dividends and aggregate endowment). Yet, this is done at the expense of restricting the agents' absolute and relative risk aversion over terminal lump-sums to be bounded, as well as of either restricting the aggregate endowment flow and the agents' utilities over flows to be time-independent or imposing a stronger growth condition on the latter functions.

<sup>21</sup>Compared to Anderson and Raimondo [1], Hugonnier et al. restrict the individual utilities to be state-independent and to depend on time only via an "impatience" rate (which is common across all agents in the main formulation, although this is relaxed in Appendix 2). In the infinite-horizon case, moreover, the agents' relative risk aversion coefficients are bounded. These notwithstanding, Hugonnier et al. do not impose any growth condition.

<sup>22</sup>The argument here relies heavily on the absence of arbitrage. All four papers arrive at this argument after having established first the existence of an Arrow-Debreu equilibrium, which precludes of course arbitrage opportunities. For an explicit description of the absence-of-arbitrage requirement, albeit in terms of the relative jumps in prices as

taken to be, respectively, analytic on  $[0, T) \times \mathbb{R}^K$  and continuous on  $\{T\} \times \mathbb{R}^K$ , one can show that  $J_p$  is also analytic on  $[0, T) \times \mathbb{R}^K$  and continuous on  $\{T\} \times V$ . Analyticity implying continuity,  $J_p$  is then continuous on  $[0, T] \times V$  and condition (ii), which can re-stated as  $|J_p(T, \beta(\omega_0, T))| \neq 0$ , implies in fact that  $|J_p(\cdot)| \neq 0$  on  $(t_0, T) \times V_0$ , for some  $t_0$  arbitrarily close to  $T$  and  $V_0 \subseteq V$  an open neighborhood of  $\beta(\omega_0, T)$ . Yet, this can be only if  $|J_p(\cdot)| \neq 0$  a.e. on  $(0, T) \times \mathbb{R}^K$  (recall footnote 10).

Needless to say, the settings studied by Anderson and Raimondo [1] and the present paper intersect. Hence so do the respective sufficient conditions for dynamic completeness: under the dividend specifications in (1), condition (i) is satisfied trivially while (ii) is equivalent to  $|\Sigma_0| \neq 0$ . And the same can be said also with respect to the studies by Riedel and Herzberg [61], Kramkov [41], as well as Hugonnier et al. [34]. The former two papers deploy slightly different versions of conditions (i)-(ii) above.<sup>23</sup> The latter paper replaces  $J_G$  with the Jacobian of the flow dividends, and  $T$  by some intermediate date (taken sufficiently close to the end when the horizon is finite). Under the dividend specifications in (1), the new version of condition (ii) is equivalent to letting  $\sigma_0 = \mathbf{0}^K$  and  $|\tilde{\Sigma}_0| \neq 0$ .<sup>24</sup>

This notwithstanding, approaching the issue via a very different line of argument, the present study sheds new light onto some fundamental underpinnings of the four papers. On the one hand, even though the settings in Anderson and Raimondo [1], Riedel and Herzberg [61], as well as Kramkov [41] intersect with that in our Section 2.3, we did have to assume the availability of options in order to establish the same result. This attests to the mileage one gets out of assuming analyticity.<sup>25</sup> On the other hand, we did show the non-degeneracy condition to be both necessary and sufficient for dynamic completeness in Section 2.2, but only sufficient in Sections 2.1 and 2.3. This in line with the claim in Hugonnier et al. [34] that necessity may indeed not apply in some cases (see their Example 1), but indicates also that it may well do in others. Most importantly perhaps, our analysis in Sections 2.1-2.2 brings to the fore some of the subtle differences between the aforementioned four papers.

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we approach the terminal date, see Section B of the Supplement to Hugonnier et al. [34].

<sup>23</sup>Because neither paper assumes that the lump-sum dividends are everywhere continuous, either has to strengthen slightly condition (ii). This requires that  $J_G(T, \cdot)$  has full rank everywhere on  $V$  in Riedel and Herzberg [61] (see Assumption 7), and almost everywhere on  $\mathbb{R}^K$  in Kramkov [41] (see Assumption A2).

<sup>24</sup>As Hugonnier et al. [34] conduct their study in the presence of a money market account, the requirement for condition (i) is redundant. Of course, their analysis applies also when none of the securities is locally riskless (see their footnote 4). In this case though condition (i) comes back into play (with  $G_0$  replaced by  $g_0$ ) while condition (ii) must concern the Jacobian of the relative flow dividends.

<sup>25</sup>Of course, options are often claimed to complete markets dynamically when trading only the underlying securities is not enough, especially in situations of stochastic volatility. Even though the specification in (1) takes the *dividend*-volatility to be constant, the *price*-volatility implied by (6) and (15) will be stochastic, apart from very special cases. With respect to the former equation, for instance, Bick [5] considers a representative-agent, single-commodity, pure-exchange economy where  $K = 1$ , the zeroth security is a zero-coupon bond, and the agent has no non-financial endowment. It shows that the equilibrium relative price will be a geometric Brownian motion if and only if the agent's utility exhibits constant relative risk aversion. Raimondo [59] (see Remark 1 and Example 1) confirms this for  $K > 1$ .

In Anderson and Raimondo [1], restricting the underlying risk process to be a Brownian motion enables exploitation of the fact that the current expectations of terminal lump-sum variables are analytic, even though the variables themselves satisfy but a mild growth condition (see their Theorem B.4). Restricting the terminal-dividend specification, our Section 2.1 shows exactly how this translates into a dynamically-complete pricing system for terminal lump-sum payoffs, under a milder growth condition.<sup>26</sup> Of course, the analyticity result in question does not obtain when it comes to the current expectations of cumulative future flows, unless the flows are themselves analytic. Which is why in Section 2.2 we follow Hugonnier et al. [34] into investigating the rate of change of these expectations, rather than the expectations themselves.

Restricting again the dividend specification, we demonstrate how this approach delivers here a dynamically-complete pricing system for intermediate flow-payoffs, under continuity and mild growth conditions instead of analyticity. Yet, we come to observe also something else. Working with the non-degeneracy of the terminal dividends, Anderson and Raimondo allow their model to have time-dependent dividend flows. Working instead with the non-degeneracy of the intermediate dividend flows, Hugonnier et al. have to make them depend on time only indirectly, via the time-dependence of the underlying diffusion process.<sup>27</sup> Our exposition is indicative of what this restriction buys, as well as of its bite.<sup>28</sup>

Another insight born out of the relation the present study bears to the four papers above is the indication that the sufficiency of the non-degeneracy condition extends beyond pure-exchange economies. Of course, in the context of general equilibrium analysis the pricing kernel cannot be but a weighted average of the agents' equilibrium marginal utilities. Yet, the asset-pricing equation in (2) allows also for non-financial wealth (and, thus, production). In this sense, the essential premise is that the time- and state-dependency of the primitive variables (utilities, dividends, endowments, and other non-financial wealth) obtains as a function of  $(t, \beta_t)$ . As an approach towards equilibrium asset-pricing theory this has been the building block for much of the seminal literature.

The starting point has been to assume that the agents have identical preferences. This has been the launching pad of two related strands of the literature. The first restricts attention to what is essentially the continuous-time analogue of the static (one-period) model: the setting in which the time-horizon is finite and securities pay only lump-sum dividends on the terminal date. The setting

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<sup>26</sup>Given  $n \in \mathbb{N} \setminus \{0\}$ , the growth condition in Anderson and Raimondo [1] is met by a function  $f : [0, T] \times \mathbb{R}^n \mapsto \mathbb{R}$  if the latter satisfies  $|f(t, \mathbf{x})| \leq r + e^{r|\mathbf{x}|}$  for some constant  $r > 0$  and all  $(t, \mathbf{x}) \in [0, T] \times \mathbb{R}^n$ . This is stronger than the growth conditions imposed throughout Section 2 (recall Lemmas A.1-A.2). Anderson and Raimondo impose this condition on the dividends and the agents' marginal utilities and endowments (both in terms of flows as well as lump-sums) but also on the instantaneous dispersion of the dividend and endowment flows. In Kramkov [41], the condition is imposed on the agents' utilities and endowments (both in terms of flows as well as lump-sums) and the terminal lump-sum dividends (see Assumptions A2 and A4-9). In Riedel and Herzberg [61], the individual marginal utilities and endowments are in fact bounded (see Assumptions 3 and 6).

<sup>27</sup>The assumption that the intermediate dividend flows do not depend on time directly is made also in Riedel and Herzberg [61] (see Assumption 8.2).

<sup>28</sup>Recall that Assumption A3 was introduced in the proof of Proposition 2.2 to allow for a change in the drift of the Brownian motion so that the typical relative dividend flow no longer depends on time directly.

studied in Section 2.1 can be found, for example, in Bick [4]-[5], He and Leland [31], Raimondo [59], or Anderson and Raimondo [2].

The second approach has been to allow for securities that pay also dividend flows during the time-interval while the time-horizon may be infinite. Perhaps the most well-known paper in this strand is Cox et al. [14], the continuous-time analogue of the famous model in Lucas [43] enhanced to include production. The setting analyzed in Section 2.2 intersects with that in Cox et al. [14] while it encompasses the ones in Cochrane et al. [12], Martin [45], Farhi and Panageas [23], Merton [47] (see chapters 4-5), or Wang [63] (whose pricing formula derives from Example 3 in Duffie and Skiadas [18]).

The next step in the literature was to turn attention onto economies of agents with heterogeneous preferences. Even here though the pricing kernel remains a linear function of the equilibrium marginal utilities (the Negishi weights are constant) if the equilibrium allocation is Pareto-optimal.<sup>29</sup> And again, also in this case, the pricing formula retains the same basic form as in (2) - see (apart from the four papers mentioned at the beginning of this section) Basak and Cuoco [3], Duffie and Zame [19] (Theorem 1 and the subsequent discussion in Section 5), Dumas [20], Karatzas et al. [40] (Corollary 10.4), or Riedel [60] (see Theorem 2.1). Clearly, in the context of general equilibrium, as long as we restrict attention to single-commodity economies and maintain Pareto-optimality as a desideratum, the scope of the pricing process in (2) is quite large. And this is important in its own right, for it has become increasingly clear in the literature that the choice of the utility function can play a crucial role (see footnote 2 in Riedel and Herzberg [61] for relevant references).

Equally importantly perhaps, in this context the analysis of Section 2.2 in particular is directly relevant to the quantitative study of Merton's portfolio-selection theory (see Chapters 4-5 in Merton [47]), a field of investigation for which the recent literature has registered a strong revival of interest as advances in computing power render numerical methods increasingly more efficient and accurate. In Theorem 2.2 (or Corollary 2.1 in the infinite-horizon case) I establish the non-degeneracy of the instantaneous dispersion matrix of the relative flow dividends as a sufficient (as well as necessary) condition not simply for dynamic completeness, but for the instantaneous dispersion matrix of the relative asset prices to be non-degenerate everywhere, not almost everywhere, on the underlying domain. Aside from the mathematical import of the difference between generic non-degeneracy and non-degeneracy at every point, this can have significant implications for numerical approaches to Merton's consumption/portfolio problem.<sup>30</sup>

As I show in Appendix C, it means for instance that, in a single-commodity, pure-exchange economy with many heterogeneous agents, their portfolio positions are guaranteed to be locally-

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<sup>29</sup>When the equilibrium allocation is not Pareto-optimal, the representative agent's utility function will be state-dependent even if all individuals have state-independent preferences and homogenous beliefs (see, for example, Cuoco and He [15]). In fact, the Negishi weights in the construction of the representative agent may even play the role of endogenous state variables which cannot be recovered as functions of the exogenous ones.

<sup>30</sup>I am indebted to an anonymous referee for bringing this to my attention.



bounded everywhere, not almost everywhere, as long as mild growth conditions are imposed on the utilities and endowments.<sup>31</sup> Which implies in turn that imposing an approximating grid on the underlying time-state space does not by itself render a numerical approach to Merton’s problem susceptible to type-I error.<sup>32</sup> More importantly, once the approximating grid is sufficiently-fine, there are no real gains from refining it further - our computing efforts should focus instead on the other parameters of the numerical method at hand.<sup>33</sup>

## 4 Concluding Remarks

In an Arrow-Debreu economy, the agents may shift consumption or income across states and time by trading a complete set of contingent claims, once and for all at the beginning of time. When they are instead constrained to trade a given set of securities, the market is said to be dynamically complete if repeated trading of the securities can still deliver any allocation that would be feasible under a complete set of contingent claims. Under continuous-time trading, this may be possible by trading a finite set of securities rapidly enough, even though the information about the state of the world is revealed through a stochastic process. In particular, when the underlying uncertainty is driven by Brownian motions, this can happen if the securities’ market is potentially dynamically complete (i.e., the number of securities exceeds that of independent Brownian motions by at least one).<sup>34</sup>

Yet, potential dynamic completeness does not suffice by itself. Some form of independence amongst the securities’ payoffs must obtain in addition. In general, once the securities’ prices are appropriately deflated, this refers to the non-degeneracy of their instantaneous dispersion with respect to the underlying stochastic process. The present paper shows that, when the pricing process is given by (2) and the dividends are specified as in (1), a sufficient (and in some important settings also necessary) condition is the linear independence of the factor loadings of the relative dividends.

Within the realm of equation (2), our analysis refers to a general pricing kernel. It applies

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<sup>31</sup>Given  $n \in \mathbb{N} \setminus \{0\}$  and an open  $V \subseteq \mathbb{R}^n$ , a function  $f : V \mapsto \mathbb{R}$  is said to be *locally bounded* on  $V$  if for all  $\mathbf{x}_0 \in V$  there is a neighborhood  $\mathcal{B}_{\mathbf{x}_0} \subseteq V$  on which  $f$  is bounded.

<sup>32</sup>In sharp contrast, if the agents’ portfolios explode at some points on the underlying space, numerical estimations might reject a model (or an algorithm) simply because of failure to converge to a numerical solution. For nothing precludes a priori that the approximating grid includes points from the set of exploding points, nor guarantees that re-positioning the grid avoids such points. Even if we were to randomly re-position the grid, we cannot meet either requirement almost surely unless the randomization is generated under a probability measure that renders null the set of exploding points. Alas, it is very difficult to meet the latter condition in simulations without actually knowing this set, and impossible to do so with real data.

<sup>33</sup>Munk [49] compares the performance of different variations of the Markov chain approximation approach on Merton’s problem. As it turns out, under a sufficiently-fine common grid, the relative performance of the methods under examination depends indeed not on further reductions in the size of the grid-step but on other parameters (namely, the hypothesized values for the agent’s initial wealth, her “impatience” rate, and the grid-imposed upper-bound on the time-state space).

<sup>34</sup>When the underlying stochastic process is not Brownian, the required number of securities may be larger.

thus irrespectively of preferences, endowments, and other structural elements (such as whether the agents' budget constraints include only pure exchange, whether or not the time horizon is finite, whether or not lump-sum dividends are available on the terminal date etc.), as long as some standard in the literature growth conditions are met.<sup>35</sup> Equally importantly perhaps, it establishes that dynamic completeness obtains generically on the space of the primitive parameters, by means of a non-degeneracy condition that is verifiable universally on this space.<sup>36</sup>

Of course, our attention was restricted to a very specific functional form for the dividends, and to the exogenous risk process being a Brownian motion. And even though this combination has been used extensively in the literature, it does mean that the present results do not extend to a larger class of models (such as, for example, Ornstein-Uhlenbeck or more generally affine processes) that is becoming increasingly the forefront of the research in financial economics. Nevertheless, the setting of the present study has always been an important theoretical benchmark in the quest for fundamental equilibrium insight. In this sense, the importance of being able to determine explicitly if and when a pricing process is dynamically complete in this setting is obvious.

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<sup>35</sup>The growth condition in our assumption A2 is used for the study of the heat equation - see, for example, Appendix C in Nielsen [50].

<sup>36</sup>Either of  $\Sigma_0$  and  $\tilde{\Sigma}_0$  will be almost always non-singular. For within  $\mathbb{R}^{K^2}$ , the space of  $K \times K$  real matrices, the singular ones form a null subset. Notice also that, in most generic results on dynamic completeness in the literature, the corresponding sufficient condition is shown to hold except for a small set of the primitive parameters. It is nevertheless difficult, if not impossible in some cases, to establish whether it does so for particular values of these parameters. Notable exceptions of course are the results in Anderson and Raimondo [1], Riedel and Herzberg [61], Hugonnier et al. [34], and Kramkov [41].

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# Appendices

## A Preliminary Results

As usual,  $\mathcal{B}_\delta$  for  $\delta > 0$  denotes in what follows the open ball in  $\mathbb{R}^K$  of radius  $\delta$  that is centered at the origin.

**Lemma A.1** *Let  $f : \mathbb{R}^K \mapsto \mathbb{R}$  be such that*

$$\exists (\sigma, A) \in \mathbb{R}^K \times \mathbb{R}_{++} : |f(\mathbf{x})| \leq Ae^{\sigma^\top \mathbf{x}} \quad \forall \mathbf{x} \in \mathbb{R}^K$$

*Then*

$$\exists r_0, \delta_0 > 0 : |f(\mathbf{x})| < r_0 + e^{r_0|\mathbf{x}|} \quad \forall \mathbf{x} \in \mathbb{R}^K \setminus \mathcal{B}_{\delta_0}$$

**Proof.** Notice first that, for any  $\mathbf{x} \in \mathbb{R}^K$ , we have

$$|\sigma^\top \mathbf{x}| = |\sigma| \times |\mathbf{x}| \times |\cos \theta(\sigma, \mathbf{x})| \leq |\sigma| \times |\mathbf{x}|$$

where  $\theta(\sigma, \mathbf{x})$  denotes the angle between the vectors  $\vec{\sigma}$  and  $\vec{\mathbf{x}}$ . We ought to have, therefore,

$$Ae^{\sigma^\top \mathbf{x}} \leq Ae^{|\sigma| \times |\mathbf{x}|} = e^{\ln A + |\sigma| \times |\mathbf{x}|} \leq e^{2|\sigma| \times |\mathbf{x}|} \leq e^{r_0|\mathbf{x}|} < r_0 + e^{r_0|\mathbf{x}|}$$

for any  $\mathbf{x} \in \mathbb{R}^K : |\mathbf{x}| \geq \ln A/|\sigma|$  and any  $r_0 \geq 2|\sigma|$ . Letting, thus,  $\delta_0 = \ln A/|\sigma|$  gives the required result. ■

**Lemma A.2** *Let  $r > 0$ . Let also  $f : \mathbb{R}^K \mapsto \mathbb{R}$  be continuous and such that*

$$\exists \delta_0, r_0 > 0 : |f(\mathbf{x})| \leq r_0 + e^{r_0|\mathbf{x}|} \quad \forall \mathbf{x} \in \mathbb{R}^K \setminus \mathcal{B}_{\delta_0}$$

*Then*

$$\exists C > 0 : |f(\mathbf{x})| \leq Ce^{r|\mathbf{x}|^2} \quad \forall \mathbf{x} \in \mathbb{R}^K$$

**Proof.** By hypothesis, there exist  $r_0, \delta_0 > 0$  such that

$$|f(\mathbf{x})| \leq r_0 + e^{r_0|\mathbf{x}|}$$

for any  $\mathbf{x} \in \mathbb{R}^K : |\mathbf{x}| \geq \delta_0$ . Yet, for all  $\mathbf{x} \in \mathbb{R}^K : |\mathbf{x}| \geq r_0/r$ , we have

$$r_0 + e^{r_0|\mathbf{x}|} \leq r_0 + e^{r|\mathbf{x}|^2} \leq (1 + r_0) e^{r|\mathbf{x}|^2}$$

Hence, letting  $\delta = \max\{\delta_0, r_0/r\}$  gives

$$|f(\mathbf{x})| \leq (1 + r_0) e^{r|\mathbf{x}|^2} \quad \forall \mathbf{x} \in \mathbb{R}^K \setminus \mathcal{B}_\delta$$

To complete now the argument observe that, being continuous,  $f$  must be bounded on  $\mathcal{B}_\delta$ . There exists, therefore,  $B > 0$  such that  $|f(\mathbf{x})| \leq B \leq Be^{r|\mathbf{x}|^2}$  for any  $\mathbf{x} \in \mathcal{B}_\delta$ . The claim follows for  $C = \max\{1 + r_0, B\}$ . ■

**Corollary A.1** *Let the functions  $\lambda : \mathbb{R}^K \mapsto (0, 1)$  and  $f : \mathbb{R}^K \mapsto \mathbb{R}$  be continuous. Suppose also that the latter satisfies*

$$\exists (\sigma, A, \delta) \in \mathbb{R}^K \times \mathbb{R}_{++} : \quad |f(\mathbf{x})| \leq Ae^{\sigma^\top \mathbf{x}} \quad \forall \mathbf{x} \in \mathbb{R}^K \setminus \mathcal{B}_\delta$$

*Then*

$$\forall r > 0, \exists C > 0 : \quad |f(\lambda(\mathbf{x})\mathbf{x})| < Ce^{r|\mathbf{x}|^2} \quad \forall \mathbf{x} \in \mathbb{R}^K$$

**Proof.** By Lemma A.1, we can find  $\delta_0, r_0 > 0$  s.t.

$$|f(\lambda(\mathbf{x})\mathbf{x})| < r_0 + e^{r_0\lambda(\mathbf{x})|\mathbf{x}|} \quad \forall \mathbf{x} \in \mathbb{R}^K \setminus \{\mathbf{x} \in \mathbb{R}^K : \lambda(\mathbf{x})|\mathbf{x}| < \delta_0\}$$

As  $\lambda(\mathbf{x})|\mathbf{x}| < |\mathbf{x}|$ , however,  $\{\mathbf{x} \in \mathbb{R}^K : \lambda(\mathbf{x})|\mathbf{x}| < \delta_0\} \subsetneq \mathcal{B}_{\delta_0}$  so that it must be also

$$|f(\lambda(\mathbf{x})\mathbf{x})| < r_0 + e^{r_0\lambda(\mathbf{x})|\mathbf{x}|} < r_0 + e^{r_0|\mathbf{x}|} \quad \forall \mathbf{x} \in \mathbb{R}^K \setminus \mathcal{B}_{\delta_0}$$

The claim follows now from Lemma A.2. ■

**Lemma A.3** *Let  $(\lambda, \beta) \in \mathbb{R}_{++} \times \mathbb{R}^K$  be a parameter vector and suppose that  $f : \mathbb{R}^K \mapsto \mathbb{R}$  satisfies*

$$\exists (r_0, C_0) \in (0, \lambda/2) \times \mathbb{R}_{++} : \quad |f(\mathbf{x})| \leq e^{r_0|\mathbf{x}|^2} \quad \forall \mathbf{x} \in \mathbb{R}^K$$

*Then*

$$\exists r, \delta > 0 : \quad \left| f(\mathbf{x}) \phi\left(\sqrt{\lambda}(\mathbf{x} - \beta)\right) \right| \leq \frac{C_0 e^{-r|\mathbf{x}|^2}}{\sqrt{(2\pi)^K}} \quad \forall \mathbf{x} \in \mathbb{R}^K \setminus \mathcal{B}_\delta$$

**Proof.** By hypothesis, there exist  $(r_0, C_0) \in (0, \frac{\lambda}{2}) \times \mathbb{R}_{++}$  such that

$$|f(\mathbf{x})| \leq C_0 e^{r_0|\mathbf{x}|^2}$$

for any  $\mathbf{x} \in \mathbb{R}^K$ . Yet, given any  $\alpha > 0$  and for any  $\mathbf{x} \in \mathbb{R}^K$ :  $|\mathbf{x}| \geq |\beta|/\alpha$ , we have  $|\mathbf{x}^\top \beta| \leq |\mathbf{x}| \times |\beta| \leq \alpha |\mathbf{x}|^2$ . Hence,

$$\begin{aligned} r_0 |\mathbf{x}|^2 - \frac{\lambda}{2} |\mathbf{x} - \beta|^2 &= \left(r_0 - \frac{\lambda}{2}\right) |\mathbf{x}|^2 - \frac{\lambda}{2} |\beta|^2 + \lambda \mathbf{x}^\top \beta \\ &\leq \left(r_0 - \frac{\lambda}{2}\right) |\mathbf{x}|^2 + \lambda \mathbf{x}^\top \beta \leq \left(r_0 + \lambda \left(\alpha - \frac{1}{2}\right)\right) |\mathbf{x}|^2 \end{aligned}$$



Choosing, therefore,  $\alpha \in \left(0, \frac{\lambda - 2r_0}{2\lambda}\right)$  establishes that

$$\left| f(\mathbf{x}) \phi\left(\sqrt{\lambda}(\mathbf{x} - \beta)\right) \right| = \frac{1}{\sqrt{(2\pi)^K}} |f(\mathbf{x})| e^{-\frac{\lambda}{2}|\mathbf{x} - \beta|^2} \leq \frac{C_0}{\sqrt{(2\pi)^K}} e^{-r|\mathbf{x}|^2} \quad \forall \mathbf{x} \in \mathbb{R}^K \setminus \mathcal{B}_{|\beta|/\alpha}$$

where  $r = \left(\frac{1}{2} - \alpha\right)\lambda - r_0 > 0$ . ■

**Lemma A.4** *Let  $f : \mathbb{R}^K \mapsto \mathbb{R}$  be continuous. Suppose also that*

$$\exists r, C, \delta > 0 : |f(\mathbf{x})| \leq C e^{-r|\mathbf{x}|^2} \quad \forall \mathbf{x} \in \mathbb{R}^K \setminus \mathcal{B}_\delta$$

*Then  $f$  is integrable on  $\mathbb{R}^K$ .*

**Proof.** Being continuous,  $f$  is integrable on  $\mathcal{B}_\delta$ . That is,

$$\exists c > 0 : \int_{\mathcal{B}_\delta} |f(\mathbf{x})| d\mathbf{x} < c$$

Given this and the remaining hypothesis, we have

$$\begin{aligned} \int_{\mathbb{R}^K} |f(\mathbf{x})| d\mathbf{x} &= \int_{\mathcal{B}_\delta} |f(\mathbf{x})| d\mathbf{x} + \int_{\mathbb{R}^K \setminus \mathcal{B}_\delta} |f(\mathbf{x})| d\mathbf{x} \\ &< c + C \int_{\mathbb{R}^K \setminus \mathcal{B}_\delta} e^{-r|\mathbf{x}|^2} d\mathbf{x} \\ &< c + C \int_{\mathbb{R}^K} e^{-r|\mathbf{x}|^2} d\mathbf{x} = c + C \sqrt{\frac{(2\pi)^K}{2r}} \end{aligned}$$

as required. ■

The following two results are well-known in the literature as the continuity and differentiability lemmas for parameter-dependent integrals. The respective statements appear as Theorems 11.4-11.5 in Schilling [?]. They are meant to apply on some measure space  $(X, \mathcal{A}, \mu)$  and a non-empty, non-degenerate open interval  $(a, b) \subseteq \mathbb{R}$ .<sup>37</sup>

**Lemma A.5** *Let  $f : (a, b) \times X \mapsto \mathbb{R}$  be a function satisfying*

- (i)  $x \mapsto f(t, x)$  is  $\mu$ -integrable on  $X$  for every fixed  $t \in (a, b)$ ,
- (ii)  $t \mapsto f(t, x)$  is continuous for every fixed  $x \in X$ , and
- (iii) there exists a function  $h : X \mapsto \mathbb{R}_+$  such that  $\int_X h(x) d\mu(x) < +\infty$  and  $|f(t, x)| < h(x)$  for all  $(t, x) \in (a, b) \times X$ .

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<sup>37</sup>As usual, this means that  $\mathcal{A}$  is a  $\sigma$ -algebra and  $\mu$  a measure on a (non-empty) set  $X$ .

Then the function  $v : (a, b) \mapsto \mathbb{R}$  given by

$$t \mapsto v(t) := \int_X f(t, x) d\mu(x)$$

is continuous.

**Lemma A.6** Let  $f : (a, b) \times X \mapsto \mathbb{R}$  be a function satisfying

- (i)  $x \mapsto f(t, x)$  is  $\mu$ -integrable on  $X$  for every fixed  $t \in (a, b)$ ,
- (ii)  $t \mapsto f(t, x)$  is differentiable for every fixed  $x \in X$ , and
- (iii) there exists a function  $h : X \mapsto \mathbb{R}_+$  such that  $\int_X h(x) d\mu(x) < +\infty$  and  $|\partial f(t, x) / \partial t| < h(x)$  for all  $(t, x) \in (a, b) \times X$ .

Then the function  $v : (a, b) \mapsto \mathbb{R}$  given by

$$t \mapsto v(t) := \int_X f(t, x) d\mu(x)$$

is differentiable and its derivative is given by

$$v'(t) = \int_X \frac{\partial f(t, x)}{\partial t} d\mu(x)$$

**Lemma A.7** Let  $(\lambda, \beta) \in \mathbb{R}_{++} \times \mathbb{R}^K$  be a parameter vector. Let also  $f : \mathbb{R}^K \mapsto \mathbb{R}$  be continuous and such that

$$\exists (r, C) \in (0, \lambda/2) \times \mathbb{R}_{++} : |f(\mathbf{x})| \leq C e^{r|\mathbf{x}|^2} \quad \forall \mathbf{x} \in \mathbb{R}^K$$

Then, the  $\mathbb{R}^K \mapsto \mathbb{R}$  function

$$F^{(\mathbf{m})}(\beta) = \int_{\mathbb{R}^K} f(\mathbf{x}) \frac{\partial^{\sum_{k=1}^K m_k} \phi(\sqrt{\lambda}(\mathbf{x} - \beta))}{\prod_{k=1}^K \partial \beta_k^{m_k}} d\mathbf{x}$$

is well-defined and so are its partial derivatives

$$\frac{\partial F^{(\mathbf{m})}(\beta)}{\partial \beta_k} = \int_{\mathbb{R}^K} f(\mathbf{x}) \frac{\partial^{m_k+1+\sum_{l \in \mathcal{K} \setminus \{k\}} m_l} \phi(\sqrt{\lambda}(\mathbf{x} - \beta))}{\partial \beta_k^{m_k+1} \prod_{l \in \mathcal{K} \setminus \{k\}} \partial \beta_l^{m_l}} d\mathbf{x}$$

for any  $k \in \{1, \dots, K\}$  and any index multiple  $\mathbf{m} = (m_1, \dots, m_K) \in \mathbb{N}^K$ .

**Proof.** Fix an arbitrary  $\beta \in \mathbb{R}^K$  and an arbitrary dimension  $k \in \mathcal{K}$ . Our argument, which will focus upon the interval  $(\beta_k - \epsilon_k, \beta_k + \epsilon_k)$  for some  $\epsilon_k > 0$ , will be presented in steps.

Step 1(i). A well-known property of the pdf of a scalar standard normal variable is that its  $n$ th ( $n \in \mathbb{N}$ ) derivative is given by

$$\phi^{(n)}(x) = H_n(x) \phi(x) \quad x \in \mathbb{R}$$

$H_n(\cdot)$  being the Hermite polynomial of order  $n$ . Yet, for  $n \in \mathbb{N}$ , there exist constants  $c_0, \dots, c_n \in \mathbb{R}$  such that

$$H_n(x) = \sum_{i=0}^n c_i x^i$$

As a result, we must have

$$|H_n(x)| \leq \sum_{i=0}^n |c_i| |x|^i \leq C_n |x|^n \quad \forall x \in \mathbb{R} \setminus (-1, 1)$$

where  $C_n = \max\{|c_i| : i \in \{0, \dots, n\}\}$ . Observe also that

$$|x|^n \leq e^{n|x|} \leq e^{\alpha_n x^2/2} \quad \forall x \in \mathbb{R} : |x| \geq 2n/\alpha_n, \quad \forall \alpha_n > 0$$

Regarding the domain-conditions above (which qualify the second inequality), for any  $k \in \mathcal{K}$  and any  $\tilde{z}_k \in (-\epsilon_k, \epsilon_k)$ , we get

$$\begin{aligned} \sqrt{\lambda} |x_k - \beta_k - \tilde{z}_k| &\geq \sqrt{\lambda} (|x_k| - |\beta_k| - |\tilde{z}_k|) \\ &> \sqrt{\lambda} (|x_k| - |\beta_k| - \epsilon_k) \\ &\geq \sqrt{\lambda} (|x_k| - |\beta_k| - 1) \geq 2n/\alpha_n \end{aligned} \tag{23}$$

as long as  $\epsilon_k \leq 1$  and  $|x_k| \geq 1 + \frac{2n}{\alpha_n \sqrt{\lambda}} + |\beta_k|$ .

Step 1(ii). Let  $\mathbf{m} \neq \mathbf{0}^K$ . It follows from the preceding step that

$$\frac{\partial^{\sum_{k=1}^K m_k} \phi \left( \sqrt{\lambda} (\mathbf{x} - \beta) \right)}{\prod_{k=1}^K \partial \beta_k^{m_k}} = \prod_{k=1}^K H_{m_k} \left( \sqrt{\lambda} (x_k - \beta_k) \right) \phi \left( \sqrt{\lambda} (x_k - \beta_k) \right) \tag{24}$$

while

$$\begin{aligned} &\left| H_{m_k} \left( \sqrt{\lambda} (x_k - \beta_k - \tilde{z}_k) \right) \right| \phi \left( \sqrt{\lambda} (x_k - \beta_k - \tilde{z}_k) \right) \\ &\leq C_{m_k} e^{-\frac{\lambda(1-\alpha)(x_k - \beta_k - \tilde{z}_k)^2}{2}} = C_{m_k} \phi \left( \sqrt{\lambda(1-\alpha)} (x_k - \beta_k - \tilde{z}_k) \right) \end{aligned} \tag{25}$$

for any  $\alpha > 0$ , any  $\epsilon_k \in (0, 1)$ , all  $x_k \in \mathbb{R} : |x_k| \geq 1 + \frac{2m_k}{\alpha \sqrt{\lambda}} + |\beta_k|$ , and all  $\tilde{z}_k \in (-\epsilon_k, \epsilon_k)$ . Set then  $\tilde{z}_k = 0$  for all  $k \in \mathcal{K}$  and let  $\bar{\mathcal{K}} = \{k \in \mathcal{K} : m_k \geq 1\}$ . Let also  $m_0 = \max\{m_k : k \in \bar{\mathcal{K}}\}$  and

$\beta_0 = \max \{|\beta_k| : k \in \mathcal{K}\}$ . We must have

$$\begin{aligned} \left| \frac{\partial^{\sum_{k=1}^K m_k} \phi \left( \sqrt{\lambda} (\mathbf{x} - \beta) \right)}{\prod_{k=1}^K \partial \beta_k^{m_k}} \right| &\leq \prod_{k=1}^K C_{m_k} \phi \left( \sqrt{\lambda (1 - \alpha)} (x_k - \beta_k) \right) \\ &= \bar{C}_1 \phi \left( \sqrt{\lambda (1 - \alpha)} (\mathbf{x} - \beta) \right) \quad \forall \mathbf{x} \in \mathbb{R}^K \setminus \mathcal{B}_{\delta_0} \end{aligned} \quad (26)$$

where  $\delta_0 = 1 + \frac{2m_0}{\alpha\sqrt{\lambda}} + \beta_0$  and  $\bar{C}_1 = \prod_{k=1}^K C_{m_k}$ .

Step 1(iii). We are now in position to show that  $F^{(\mathbf{m})}(\cdot)$  is well-defined. If  $\mathbf{m} = \mathbf{0}^K$ , this follows immediately from Lemmas A.3-A.4.<sup>38</sup> Letting, on the other hand,  $\mathbf{m} \neq \mathbf{0}^K$ , we have

$$\left| f(\mathbf{x}) \frac{\partial^{\sum_{k=1}^K m_k} \phi \left( \sqrt{\lambda} (\mathbf{x} - \beta) \right)}{\prod_{k=1}^K \partial \beta_k^{m_k}} \right| \leq \bar{C}_1 |f(\mathbf{x})| \phi \left( \sqrt{\lambda (1 - \alpha)} (\mathbf{x} - \beta) \right)$$

by (26). Yet, there are  $r \in (0, \lambda/2)$  and  $C > 0$  s.t.  $|f(\mathbf{x})| \leq C e^{r|\mathbf{x}|^2}$  on  $\mathbb{R}^K$ . If we choose, therefore,  $\alpha \in (0, \frac{\lambda-2r}{\lambda})$  so that  $r < \frac{\lambda(1-\alpha)}{2}$ , Lemma A.3 ensures the existence of  $r_1, \delta > 0$  such that

$$\left| f(\mathbf{x}) \frac{\partial^{\sum_{k=1}^K m_k} \phi \left( \sqrt{\lambda} (\mathbf{x} - \beta) \right)}{\prod_{k=1}^K \partial \beta_k^{m_k}} \right| \leq \frac{C \bar{C}_1}{\sqrt{(2\pi)^K}} e^{-r_1 |\mathbf{x}|^2} \quad \forall \mathbf{x} \in \mathbb{R}^K \setminus \mathcal{B}_{\delta_1}$$

with  $\delta_1 = \max \{\delta_0, \delta\}$ . And as the functions  $f$ ,  $\phi$ , and  $H_{m_k}$  (for any  $k \in \mathcal{K}$ ) are all continuous with respect to  $\mathbf{x}$ , the claim now follows from Lemma A.4.

Step 2(i). To show next differentiability, recall (24) and (to economize on the length of exposition) let  $g_{m_k}(x) := (H_{m_k} \phi)(x)$  for  $x \in \mathbb{R}$  and  $k \in \mathcal{K}$ . Any  $z_k \in \mathbb{R} \setminus \{0\}$  gives

$$\begin{aligned} &\left| \frac{F^{(\mathbf{m})}(\beta_k + z_k, \beta_{-k}) - F^{(\mathbf{m})}(\beta)}{z_k} - \int_{\mathbb{R}^K} f(\mathbf{x}) \frac{\partial^{m_k+1+\sum_{l \in \mathcal{K} \setminus \{k\}} m_l} \phi \left( \sqrt{\lambda} (\mathbf{x} - \beta) \right)}{\partial \beta_k^{m_k+1} \prod_{l \in \mathcal{K} \setminus \{k\}} \partial \beta_l^{m_l}} d\mathbf{x} \right| \\ &= \left| \int_{\mathbb{R}^K} f(\mathbf{x}) \left[ \frac{\prod_{l \in \mathcal{K} \setminus \{k\}} g_{m_l} \left( \sqrt{\lambda} (x_l - \beta_l) \right)}{\times \left( \frac{g_{m_k}(\sqrt{\lambda}(x_k - \beta_k - z_k)) - g_{m_k}(\sqrt{\lambda}(x_k - \beta_k))}{z_k} - \frac{\partial g_{m_k}(\sqrt{\lambda}(x_k - \beta_k))}{\partial \beta_k} \right)} \right] d\mathbf{x} \right| \\ &\leq \int_{\mathbb{R}^K} \left| f(\mathbf{x}) \frac{\prod_{l \in \mathcal{K} \setminus \{k\}} g_{m_l} \left( \sqrt{\lambda} (x_l - \beta_l) \right)}{\times \left( \frac{g_{m_k}(\sqrt{\lambda}(x_k - \beta_k - z_k)) - g_{m_k}(\sqrt{\lambda}(x_k - \beta_k))}{z_k} - \frac{\partial g_{m_k}(\sqrt{\lambda}(x_k - \beta_k))}{\partial \beta_k} \right)} \right| d\mathbf{x} \\ &= \frac{1}{\sqrt{\lambda}} \int_{\mathbb{R}^K} \left| f \left( \frac{\mathbf{x} - \beta}{\sqrt{\lambda}} + \beta \right) \frac{\prod_{l \in \mathcal{K} \setminus \{k\}} g_{m_l} (x_l - \beta_l)}{\times \left( \frac{g_{m_k}(x_k - \beta_k - z_k) - g_{m_k}(x_k - \beta_k)}{z_k} - \frac{\partial g_{m_k}(x_k - \beta_k)}{\partial \beta_k} \right)} \right| d\mathbf{x} \end{aligned}$$

<sup>38</sup>Recall that the zeroth derivative of a function denotes the function itself. Notice also that, throughout the proof,  $\phi$  is taken to denote the standard normal pdf regardless of the dimensionality of the domain. The latter of course will always be obvious.

the last equality following from a change in the variables of integration (done purely to simplify the presentation). Yet, by the mean value theorem, there exist  $\gamma, \rho \in (0, 1)$  such that

$$\begin{aligned}
& \int_{\mathbb{R}^K} \left| f \left( \frac{\mathbf{x} - \beta}{\sqrt{\lambda}} + \beta \right) \prod_{l \in \mathcal{K} \setminus \{k\}} g_{m_l}(x_l - \beta_l) \right. \\
& \quad \left. \times \left( \frac{g_{m_k}(x_k - \beta_k - z_k) - g_{m_k}(x_k - \beta_k)}{z_k} - \frac{\partial g_{m_k}(x_k - \beta_k)}{\partial \beta_k} \right) \right| d\mathbf{x} \\
&= |z_k| \int_{\mathbb{R}^K} \left| f \left( \frac{\mathbf{x} - \beta}{\sqrt{\lambda}} + \beta \right) \prod_{l \in \mathcal{K} \setminus \{k\}} g_{m_l}(x_l - \beta_l) \right| \left| \frac{\partial g_{m_k}(x_k - \beta_k - \gamma z_k)}{\partial \beta_k} - \frac{\partial g_{m_k}(x_k - \beta_k)}{\partial \beta_k} \right| d\mathbf{x} \\
&= |\gamma z_k| \int_{\mathbb{R}^K} \left| f \left( \frac{\mathbf{x} - \beta}{\sqrt{\lambda}} + \beta \right) \prod_{l \in \mathcal{K} \setminus \{k\}} g_{m_l}(x_l - \beta_l) \frac{\partial^2 g_{m_k}(x_k - \beta_k - \gamma \rho z_k)}{\partial \beta_k^2} \right| d\mathbf{x} \\
&< |z_k| \int_{\mathbb{R}^K} \left| f \left( \frac{\mathbf{x} - \beta}{\sqrt{\lambda}} + \beta \right) \prod_{l \in \mathcal{K} \setminus \{k\}} g_{m_l}(x_l - \beta_l) \frac{\partial^2 g_{m_k}(x_k - \beta_k - \gamma \rho z_k)}{\partial \beta_k^2} \right| d\mathbf{x} \\
&= |z_k| \int_{\mathbb{R}^K} \left| f \left( \frac{\mathbf{x} - \beta}{\sqrt{\lambda}} + \beta \right) \prod_{l \in \mathcal{K} \setminus \{k\}} g_{m_l}(x_l - \beta_l) g_{m_k+2}(x_k - \beta_k - \gamma \rho z_k) \right| d\mathbf{x}
\end{aligned}$$

As a result, taking now  $z_k \in (-\epsilon_k, \epsilon_k) \setminus \{0\}$  we have

$$\begin{aligned}
& \left| \frac{F^{(\mathbf{m})}(\beta_k + z_k, \beta_{-k}) - F^{(\mathbf{m})}(\beta)}{z} - \int_{\mathbb{R}^K} f(\mathbf{x}) \frac{\partial^{m_k+1+\sum_{l \in \mathcal{K} \setminus \{k\}} m_l} \phi(\sqrt{\lambda}(\mathbf{x} - \beta))}{\partial \beta_k^{m_k+1} \prod_{l \in \mathcal{K} \setminus \{k\}} \partial \beta_l^{m_l}} d\mathbf{x} \right| \\
&< \frac{|z_k|}{\sqrt{\lambda}} \int_{\mathbb{R}^K} \left| f \left( \frac{\mathbf{x} - \beta}{\sqrt{\lambda}} + \beta \right) \prod_{l \in \mathcal{K} \setminus \{k\}} g_{m_l}(x_l - \beta_l) g_{m_k+2}(x_k - \beta_k - \gamma \rho z_k) \right| d\mathbf{x} \\
&= |z_k| \int_{\mathbb{R}^K} \left| f(\mathbf{x}) \prod_{l \in \mathcal{K} \setminus \{k\}} g_{m_l}(\sqrt{\lambda}(x_l - \beta_l)) g_{m_k+2}(\sqrt{\lambda}(x_k - \beta_k - \gamma \rho z_k)) \right| d\mathbf{x} \\
&< \epsilon_k \int_{\mathbb{R}^K} \left| f(\mathbf{x}) \prod_{l \in \mathcal{K} \setminus \{k\}} g_{m_l}(\sqrt{\lambda}(x_l - \beta_l)) g_{m_k+2}(\sqrt{\lambda}(x_k - \beta_k - \gamma \rho z_k)) \right| d\mathbf{x} \tag{27}
\end{aligned}$$

the equality again due to a change in the variables of integration.

Step 2(ii). Take now  $\epsilon_k \in (0, 1)$ . Of course, this means that  $|z_k| < 1$ . Therefore,

$$|x_k - \beta_k - \gamma \rho z_k| \leq |x_k - \beta_k| + |\gamma \rho z_k| < |x_k - \beta_k| + 1$$

and

$$\begin{aligned}
\left| g_{m_k+2} \left( \sqrt{\lambda} (x_k - \beta_k - \gamma \rho z_k) \right) \right| &= \phi \left( \sqrt{\lambda} (x_k - \beta_k - \gamma \rho z_k) \right) \left| H_{m_k+2} \left( \sqrt{\lambda} (x_k - \beta_k - \gamma \rho z_k) \right) \right| \\
&\leq \phi \left( \sqrt{\lambda} (x_k - \beta_k - \gamma \rho z_k) \right) \sum_{i=0}^{m_k+2} |c_i| \lambda^{i/2} |x_k - \beta_k - \gamma \rho z_k|^i \\
&< \phi \left( \sqrt{\lambda} (x_k - \beta_k - \gamma \rho z_k) \right) \sum_{i=0}^{m_k+2} |c_i| \lambda^{i/2} (|x_k - \beta_k| + 1)^i \\
&< \phi \left( \sqrt{\lambda} (x_k - \beta_k - \gamma \rho z_k) \right) \sum_{i=0}^{m_k+2} |c_i| \lambda^{i/2} e^{i(|x_k| + |\beta_k| + 1)} \\
&< \overline{C}_2 \phi \left( \sqrt{\lambda} (x_k - \beta_k - \gamma \rho z_k) \right) e^{(m_k+2)|x_k|} \tag{28}
\end{aligned}$$

where  $\overline{C}_2 = e^{(m_k+2)(|\beta_k|+1)} \sum_{i=0}^{m_k+2} |c_i| \lambda^{i/2}$ . Recall next (23). Letting  $\delta_2 = 1 + |\beta_k|$ , any  $x_k \in \mathbb{R} : |x_k| \geq \delta_4$  gives

$$|x_k - \beta_k - \gamma \rho z_k| \geq |x_k - \beta_k| - \gamma \rho |z_k| > |x_k - \beta_k| - 1 \geq |x_k| - |\beta_k| - 1 \geq 0$$

so that

$$\begin{aligned}
\phi \left( \sqrt{\lambda} (x_k - \beta_k - \gamma \rho z_k) \right) &< \phi \left( \sqrt{\lambda} (|x_k - \beta_k| - 1) \right) \\
&= e^{\lambda(|x_k - \beta_k| - \frac{1}{2})} \phi \left( \sqrt{\lambda} (x_k - \beta_k) \right) < e^{\lambda|x_k - \beta_k|} \phi \left( \sqrt{\lambda} (x_k - \beta_k) \right)
\end{aligned}$$

and, thus,

$$\begin{aligned}
\left| g_{m_k+2} \left( \sqrt{\lambda} (x_k - \beta_k - \gamma \rho z_k) \right) \right| &< \overline{C}_2 \phi \left( \sqrt{\lambda} (x_k - \beta_k) \right) e^{(m_k+2)|x_k| + \lambda|x_k - \beta_k|} \\
&\leq \overline{C}_2 \phi \left( \sqrt{\lambda} (x_k - \beta_k) \right) e^{(\lambda+m_k+2)|x_k| + \lambda|\beta_k|} \\
&= \overline{C}_3 e^{(\lambda+m_k+2)|x_k|} \quad \forall x_k \in \mathbb{R} : |x_k| \geq \delta_4
\end{aligned}$$

where  $\overline{C}_3 = \overline{C}_2 e^{\lambda|\beta_k|}$ . By Lemma A.2, however, for any  $r_k > 0$  we can find  $\overline{C}_4 > 0$  such that  $e^{(\lambda+m_k+2)|x_k|} < \overline{C}_4 e^{r_k x_k^2}$ . And as, by hypothesis, there exist  $r \in (0, \lambda/2)$  and  $C > 0$  s.t.  $|f(\mathbf{x})| \leq C e^{r|\mathbf{x}|^2}$  on  $\mathbb{R}^K$ , it must be

$$\begin{aligned}
|f(\mathbf{x})| e^{(\lambda+m_k+2)|x_k|} &< C \overline{C}_4 e^{r|\mathbf{x}|^2 + r_k x_k^2} = C \overline{C}_4 e^{(r+r_k)x_k^2 + r \sum_{l \in \mathcal{K} \setminus \{k\}} x_l^2} \\
&< C \overline{C}_4 e^{(r+r_k)|\mathbf{x}|^2}
\end{aligned}$$

Hence, choosing  $r_k \in (0, \frac{\lambda}{2} - r)$  means that, viewed as a function of  $z_k$ , the integrand in (27) is bounded above on  $\mathbb{R}^K \setminus \mathcal{B}_{\delta_2}$  by a function that is independent of  $z_k$  and integrable on  $\mathbb{R}^K$  (Lemma

A.3) - hence, also on  $\mathbb{R}^K \setminus \mathcal{B}_{\delta_2}$ . Regarding the interior of the left-out neighborhood, (28) gives

$$\left| g_{m_k+2} \left( \sqrt{\lambda} (x_k - \beta_k - \gamma \rho z_k) \right) \right| < \frac{\overline{C}_2}{\sqrt{2\pi}} e^{(m_k+2)|x_k|}$$

By continuity, therefore, the claim that, as a function of  $z_k$ , the integrand in (27) is bounded above by a function that is independent of  $z_k$  and integrable holds also on  $\mathcal{B}_{\delta_2}$ . Given this, and as  $g_{m_k+2} \left( \sqrt{\lambda} (x_k - \beta_k - \gamma \rho z_k) \right)$  is continuous in  $z_k$  for any  $x_k \in \mathbb{R}$ , Lemma A.5 ensures that the continuity in  $z_k$  on  $(\epsilon_k, \epsilon_k)$  is preserved under the integral sign in (27). Hence,

$$\begin{aligned} & \lim_{|z_k| \rightarrow 0} \int_{\mathbb{R}^K} \left| f(\mathbf{x}) \prod_{l \in \mathcal{K} \setminus \{k\}} g_{m_l} \left( \sqrt{\lambda} (x_l - \beta_l) \right) g_{m_k+2} \left( \sqrt{\lambda} (x_k - \beta_k - \gamma \rho z_k) \right) \right| d\mathbf{x} \\ &= \int_{\mathbb{R}^K} \left| f(\mathbf{x}) \prod_{l \in \mathcal{K} \setminus \{k\}} g_{m_l} \left( \sqrt{\lambda} (x_l - \beta_l) \right) g_{m_k+2} \left( \sqrt{\lambda} (x_k - \beta_k) \right) \right| d\mathbf{x} \end{aligned} \quad (29)$$

and to establish the required differentiability it suffices to show that the last integral above is well-defined.

Step 2(iii). To this end, recall (25). Letting  $\tilde{z}_k = 0$  we have

$$\left| \prod_{l \in \mathcal{K} \setminus \{k\}} g_{m_l} \left( \sqrt{\lambda} (x_l - \beta_l) \right) g_{m_k+2} \left( \sqrt{\lambda} (x_k - \beta_k) \right) \right| \leq \overline{C}_5 \phi \left( \sqrt{\lambda(1-\alpha)} (\mathbf{x} - \beta) \right)$$

for any  $\mathbf{x} \in \mathbb{R}^K \setminus \mathcal{B}_{\delta_0}$  with  $\overline{C}_5 = C_{m_k+2} \prod_{l \in \mathcal{K} \setminus \{k\}} C_{m_l}$ . And as  $\alpha$  in Step 1(iii) was chosen so that  $r < \lambda(1-\alpha)/2$ , Lemma A.3 ensures the existence of  $r_2, \delta_3 > 0$  such that

$$\left| f(\mathbf{x}) \phi \left( \sqrt{\lambda(1-\alpha)} (\mathbf{x} - \beta) \right) \right| \leq \frac{C}{\sqrt{(2\pi)^K}} e^{-r_2|\mathbf{x}|^2} \quad \forall \mathbf{x} \in \mathbb{R}^K \setminus \mathcal{B}_{\delta_3}$$

Letting, thus,  $\delta_4 = \max \{\delta_3, \delta_0\}$  we have

$$\left| f(\mathbf{x}) \prod_{l \in \mathcal{K} \setminus \{k\}} g_{m_l} \left( \sqrt{\lambda} (x_l - \beta_l) \right) g_{m_k+2} \left( \sqrt{\lambda} (x_k - \beta_k) \right) \right| \leq \overline{C}_6 e^{-r_2|\mathbf{x}|^2} \quad \forall \mathbf{x} \in \mathbb{R}^K \setminus \mathcal{B}_{\delta_4}$$

where  $\overline{C}_6 = C\overline{C}_5/\sqrt{(2\pi)^K}$ . The quantity in the brackets above being continuous on  $\mathbb{R}^K$ , that the integral on the right-hand side of (29) is well-defined follows now from Lemma A.4. To complete the proof, let  $\epsilon_k \rightarrow 0$ . ■

## B Main Results

**Lemma B.1** *Let  $\mathcal{T} = [0, T]$  for some  $T > 0$  and fix some  $(t, \beta) \in [0, T] \times \mathbb{R}^K$ . Suppose also that  $f : \mathcal{T} \times \mathbb{R}^K$  is such that*

$$\exists (r, C) \in \left(0, \frac{1}{2T}\right) \times \mathbb{R}_{++} : \quad |f(s, \mathbf{x})| \leq Ce^{r|\mathbf{x}|^2} \quad \forall (s, \mathbf{x}) \in \mathcal{T} \times \mathbb{R}^K$$

*There exist then  $\tilde{r}, \tilde{C} > 0$  s.t.*

$$|f(s, \beta + \sqrt{s-t}\mathbf{x})| \phi(\mathbf{x}) < \tilde{C}e^{-\frac{\tilde{r}|\mathbf{x}|^2}{2}} \quad \forall (s, \mathbf{x}) \in [t, T] \times \mathbb{R}^K$$

**Proof.** Under the given growth condition, we have

$$\begin{aligned} |f(s, \beta_t + \sqrt{s-t}\mathbf{x})| \phi(\mathbf{x}) &\leq C\phi(\mathbf{x}) e^{r|\beta_t + \sqrt{s-t}\mathbf{x}|^2} \\ &= C\phi(\mathbf{x}) e^{r(|\beta_t|^2 + (s-t)|\mathbf{x}|^2 + 2\sqrt{s-t}|\beta_t||\mathbf{x}|)} \\ &\leq C\phi(\mathbf{x}) e^{r(|\beta_t|^2 + (T-t)|\mathbf{x}|^2 + 2\sqrt{T-t}|\beta_t||\mathbf{x}|)} \\ &\leq C\phi(\mathbf{x}) e^{r(|\beta_t|^2 + T|\mathbf{x}|^2 + 2\sqrt{T}|\beta_t||\mathbf{x}|)} \\ &< C\phi(\mathbf{x}) e^{|\beta_t|\left(\frac{|\beta_t|}{2T} + \frac{|\mathbf{x}|}{\sqrt{T}}\right) + rT|\mathbf{x}|^2} \\ &= \frac{C}{\sqrt{2\pi}} e^{|\beta_t|\left(\frac{|\beta_t|}{2T} + \frac{|\mathbf{x}|}{\sqrt{T}}\right) - \left(\frac{1}{2} - rT\right)|\mathbf{x}|^2} \quad \forall (s, \mathbf{x}) \in [t, T] \times \mathbb{R}^K \end{aligned}$$

By Lemma A.2, however, we can find constants  $\hat{r} \in (0, \frac{1}{2} - rT)$  and  $\hat{C} > 0$  such that

$$e^{\frac{|\beta_t|}{\sqrt{T}}|\mathbf{x}|} < \hat{C}e^{\hat{r}|\mathbf{x}|^2}$$

Setting  $\tilde{r} = 1 - 2(rT + \hat{r})$  and  $\tilde{C} = \frac{C\hat{C}}{\sqrt{2\pi}}e^{\frac{|\beta_t|^2}{2T}}$  gives the result. ■

**Corollary B.1** *Let  $\mathcal{T} = [0, T]$  for some  $T > 0$  and fix some  $(t, \beta) \in [0, T] \times \mathbb{R}^K$ . Suppose also that  $f : \mathcal{T} \times \mathbb{R}^K$  is continuous and such that*

$$\exists (r, C) \in \left(0, \frac{1}{2T}\right) \times \mathbb{R}_{++} : \quad |f(s, \mathbf{x})| \leq Ce^{r|\mathbf{x}|^2} \quad \forall (s, \mathbf{x}) \in \mathcal{T} \times \mathbb{R}^K$$

*If  $\mathbf{x} \sim \mathcal{N}(\mathbf{0}^K, \mathbf{I}_K)$  then  $\mathbb{E}_{\mathbf{x}}[f(s, \beta + \sqrt{s-t}\mathbf{x})]$  is integrable on  $(t, T)$ .*

**Proof.** Since  $r < \frac{1}{2T} < \frac{1}{2(T-t)} < \frac{1}{2(s-t)} \quad \forall s \in (t, T)$ , that

$$\mathbb{E}_{\mathbf{x}}[f(s, \beta + \sqrt{s-t}\mathbf{x})] = \frac{1}{\sqrt{s-t}} \int_{\mathbb{R}^K} f(s, \mathbf{x}) \phi\left(\frac{\mathbf{x} - \beta}{\sqrt{s-t}}\right) d\mathbf{x}$$



is well-defined on  $(t, T)$  follows from Lemma A.7 (for  $\mathbf{m} = \mathbf{0}^K$ ). By the preceding lemma, moreover, there are  $\tilde{r}, \tilde{C} > 0$  s.t.

$$\int_t^T \mathbb{E}_{\mathbf{x}} [|f(s, \beta_t + \sqrt{s-t}\mathbf{x})|] ds < \tilde{C} \int_t^T \int_{\mathbb{R}^K} e^{-\frac{\tilde{r}|\mathbf{x}|^2}{2}} d\mathbf{x} ds = (T-t) \tilde{C} \sqrt{2\pi/\tilde{r}}$$

The claim follows. ■

**Lemma B.2** *Let  $\mathcal{T} = [0, T]$  for some  $T > 0$ . Suppose also that the terminal-dividend and price processes are given, respectively, by (1) and (4) while A1 holds. Then*

$$\lim_{t \rightarrow T} (MP_j)(\mathcal{I}(\omega, t)) = M_j(\mathcal{I}(\omega, T)) \quad j \in \mathcal{K} \cup \{0\}, \omega \in \Omega$$

**Proof.** Recall equation (4) in the main text. As we have already discussed there, under the terminal-dividend specification in (1) and assumption (A1), Lemma A.7 ensures that the integral  $\int_{\mathbb{R}^K} M_j(T, \beta_t + \sqrt{T-t}\mathbf{x}) \phi(\mathbf{x}) d\mathbf{x}$  is well-defined at any  $(t, \beta_t) \in [0, T) \times \mathbb{R}^K$ . Moreover, the function  $M_j(T, \cdot)$  is continuous and, thus, so is  $t \mapsto M_j(T, \beta_t + \sqrt{T-t}\mathbf{x})$  everywhere on  $[0, T]$  for any given  $(\beta_t, \mathbf{x}) \in \mathbb{R}^K \times \mathbb{R}^K$ . Furthermore, by Lemma B.1, there exist constants (independent not only of  $\mathbf{x}$  but also of  $t$ )  $\tilde{r}_j, \tilde{C}_j > 0$  such that

$$\left| M_j\left(T, \beta_t + \sqrt{T-t}\mathbf{x}\right) \right| \phi(\mathbf{x}) < \tilde{C}_j e^{-\frac{\tilde{r}_j|\mathbf{x}|^2}{2}} \quad \forall (t, \mathbf{x}) \in [0, T) \times \mathbb{R}^K$$

the upper bound above being obviously integrable on  $\mathbb{R}^K$ .

By Lemma A.5, these observations mean that the continuity in  $t$  commutes under the integral in (4) everywhere on  $(0, T)$ . Taking now any  $\epsilon \in \left(0, \frac{1}{2r_j} - T\right)$ , it is trivial to check that assumption (A1) and, thus, the entire analysis above remains valid if  $T$  is replaced by  $T + \epsilon$ . Which means of course that the mapping  $t \mapsto \int_{\mathbb{R}^K} M_j(T, \beta_t + \sqrt{T-t}\mathbf{x}) \phi(\mathbf{x}) d\mathbf{x}$  is continuous also on  $(0, T + \epsilon)$ . It is, therefore, left-continuous at  $T$  in the original problem and, thus,

$$\lim_{t \rightarrow T} \int_{\mathbb{R}^K} M_j\left(T, \beta_t + \sqrt{T-t}\mathbf{x}\right) \phi(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{R}^K} M_j(T, \beta_t) \phi(\mathbf{x}) d\mathbf{x} = M_j(T, \beta_t)$$

The claim now follows because  $\beta_t$  above is arbitrary. For any given  $\omega \in \Omega$ , therefore, we let  $\beta_t = \beta(\omega, t)$  and observe that it approaches  $\beta_T = \beta(\omega, T)$  as  $t \rightarrow T$ . ■

### Proof of Proposition 2.1

Recall equation (30) in the main text. Letting  $\tau = T - t$ , it reads

$$(MP_0)(t, \beta_t)^2 D_k p_n(t, \beta_t) = \frac{1}{\tau} \mathbb{E}_{(\mathbf{x}, \mathbf{y})} \left[ \begin{aligned} & M_0(T, \beta_t + \sqrt{\tau}\mathbf{x}) M_0(T, \beta_t + \sqrt{\tau}\mathbf{y}) \\ & \times \sqrt{\tau} (x_k - y_k) \sum_{k \in \mathcal{K}} (\sigma_{nk} - \sigma_{0k}) (\sqrt{\tau} x_k) \\ & \times G_{n/0}(T, \beta_t + \lambda_{n/0}(\sqrt{\tau}\mathbf{x}) \sqrt{\tau}\mathbf{x}) \end{aligned} \right]$$

For  $t \rightarrow T$ , we have  $\tau \rightarrow 0$  so that  $\sqrt{\tau}\mathbf{x}$  and  $\sqrt{\tau}\mathbf{y}$  depict, respectively, the infinitesimal increments  $\Delta\beta_t$  and  $\Delta\tilde{\beta}_t$  of two independent  $K$ -dimensional standard Brownian motions. And as well-known arguments (see, for instance, Theorem 4.2.1 in Øksendal [51]) establish that almost surely (a.s.)

$$\begin{aligned}\Delta\beta_{kt}\Delta\tilde{\beta}_{lt} &= 0 & \forall k, l \in \mathcal{K} \\ \Delta\beta_{kt}\Delta\beta_{lt} &= 0 = \Delta\tilde{\beta}_{kt}\Delta\tilde{\beta}_{lt} & \forall k, l \in \mathcal{K} : k \neq l \\ (\Delta\beta_{kt})^2 &= \tau & \forall k \in \mathcal{K}\end{aligned}$$

it cannot but be

$$\sqrt{\tau}(x_k - y_k) \sum_{k \in \mathcal{K}} (\sigma_{nk} - \sigma_{0k}) (\sqrt{\tau}x_k) = \left( \Delta\beta_k - \Delta\tilde{\beta}_k \right) \sum_{k \in \mathcal{K}} (\sigma_{nk} - \sigma_{0k}) \Delta\beta_k = \tau (\sigma_{nk} - \sigma_{0k}) \quad \text{a.s.}$$

As  $t \rightarrow T$ , therefore, it must be

$$\begin{aligned}(MP_0)(t, \beta_t)^2 D_k p_n(t, \beta_t) &= (\sigma_{nk} - \sigma_{0k}) \mathbb{E}_{(\mathbf{x}, \mathbf{y})} \left[ \begin{aligned} &M_0(T, \beta_t + \sqrt{T-t}\mathbf{x}) M_0(T, \beta_t + \sqrt{T-t}\mathbf{y}) \\ &\times G_{n/0}(T, \beta_t + \lambda_{n/0}(\sqrt{T-t}\mathbf{x}) \sqrt{T-t}\mathbf{x}) \end{aligned} \right] \\ &= (\sigma_{nk} - \sigma_{0k}) \mathbb{E}_{\mathbf{y}} \left[ M_0(T, \beta_t + \sqrt{T-t}\mathbf{y}) \right] \\ &\times \mathbb{E}_{\mathbf{x}} \left[ M_0(T, \beta_t + \sqrt{T-t}\mathbf{x}) G_{n/0}(T, \beta_t + \lambda_{n/0}(\sqrt{T-t}\mathbf{x}) \sqrt{T-t}\mathbf{x}) \right]\end{aligned}$$

which under (4) reduces to

$$(MP_0)(t, \beta_t) D_k p_n(t, \beta_t) = (\sigma_{nk} - \sigma_{0k}) \mathbb{E}_{\mathbf{x}} \left[ \begin{aligned} &M_0(T, \beta_t + \sqrt{T-t}\mathbf{x}) \\ &\times G_{n/0}(T, \beta_t + \lambda_{n/0}(\sqrt{T-t}\mathbf{x}) \sqrt{T-t}\mathbf{x}) \end{aligned} \right] \quad (30)$$

for any given  $\beta_t \in \mathbb{R}^K$ . It remains to show that continuity in  $t$  commutes under the last expectation above. To this end, notice first that Corollary A.1 ensures that for any  $r_{n/0} > 0$  we can find  $C_{n/0} > 0$  such that

$$\begin{aligned}\left| G_{n/0}(T, \beta_t + \lambda(\sqrt{T-t}\mathbf{x}) \sqrt{T-t}\mathbf{x}) \right| &< \bar{C}_{n/0} e^{r_{n/0}(T-t)|\mathbf{x}|^2} \\ &< \bar{C}_{n/0} e^{Tr_{n/0}|\mathbf{x}|^2} \quad \forall (t, \mathbf{x}) \in (0, T) \times \mathbb{R}^K\end{aligned}$$

where  $\bar{C}_{n/0} = C_{n/0} |\gamma_n(T)/\gamma_0(T)| e^{(\sigma_n - \sigma_0)^\top \beta_t}$ . Recall now (??). Choosing  $r_{n/0} \in (0, \tilde{r}_j)$  means that

$$\begin{aligned}&\left| M_0(T, \beta_t + \sqrt{T-t}\mathbf{x}) G_{n/0}(T, \beta_t + \lambda_{n/0}(\sqrt{T-t}\mathbf{x}) \sqrt{T-t}\mathbf{x}) \right| \phi(\mathbf{x}) \\ &< C_{n/0} \tilde{C}_j e^{-\frac{(\tilde{r}_j - r_{n/0})|\mathbf{x}|^2}{2}} \quad \forall (t, \mathbf{x}) \in (0, T) \times \mathbb{R}^K\end{aligned}$$

And as the mappings  $M_0, G_{n/0}$  are continuous in  $t$  for any given  $(\beta_t, \mathbf{x}) \in \mathbb{R}^K \times \mathbb{R}^K$ , it is trivial to check that the argument in the proof of Lemma B.2 establishes again here that the right-hand side

of (30) is left-continuous at  $T$ . Given then that

$$\begin{aligned}\lim_{t \rightarrow T} G_{n/0} \left( T, \beta_t + \lambda_{n/0} \left( \sqrt{T-t} \mathbf{x} \right) \sqrt{T-t} \mathbf{x} \right) &= G_{n/0} (T, \beta_t) \\ \lim_{t \rightarrow T} M_0 \left( T, \beta_t + \sqrt{T-t} \mathbf{x} \right) &= M_0 (T, \beta_t)\end{aligned}$$

under (30) we obtain in fact that

$$\begin{aligned}\lim_{t \rightarrow T} (MP_0) (t, \beta_t) D_k p_n (t, \beta_t) &= (\sigma_{nk} - \sigma_{0k}) \mathbb{E}_{\mathbf{x}} [M_0 (T, \beta_t) G_{n/0} (T, \beta_t)] \\ &= (\sigma_{nk} - \sigma_{0k}) M_n (T, \beta_t)\end{aligned}$$

To complete the argument, observe that the choice of  $\beta_t$  above was arbitrary. Letting, therefore,  $\beta_t = \beta(\omega, t)$  for any given  $\omega \in \Omega$  we get  $\lim_{t \rightarrow T} M_n (T, \beta(\omega, t)) = M_n (T, \beta(\omega, T))$ , by the continuity of the respective mappings, and  $\lim_{t \rightarrow T} (MP_0) (t, \beta(\omega, t)) = M_0 (T, \beta(\omega, T))$  by Lemma 2.1. ■

The following result derives almost immediately from the proof of Theorem B.4 in Anderson and Raimondo [1]. To facilitate comparison, we state it in as close a way as possible to the original theorem, but of course for our (weaker - recall Lemma A.2) growth condition.

**Lemma B.3** *Suppose  $f : \mathbb{R}^K \mapsto \mathbb{R}$  is measurable on  $\mathbb{R}^K$  and satisfies*

$$\exists C > 0, r \in \left( 0, \frac{1}{2T} \right) : |f(\mathbf{x})| \leq C e^{r|\mathbf{x}|^2} \quad \forall \mathbf{x} \in \mathbb{R}^K$$

*Let also  $\beta$  be a standard  $K$ -dimensional Brownian motion and*

$$F(t, \beta) = \mathbb{E}[f(\beta(T)) | \beta(t) = \beta]$$

*Then  $F(\cdot)$  is an analytic function of  $(t, \beta)$  on  $(0, T) \times \mathbb{R}^K$ .*

**Proof.** The proof proceeds in exactly the same way as in that of Theorem B.4 in Anderson and Raimondo [1], but for a trivial adaptation of the argument that supports equations (23)-(27) in that paper. Specifically, fixing  $t < T$ , equation (21) in Anderson and Raimondo [1] gives

$$\begin{aligned}F(t, \beta) &= (2\pi(T-t))^{-K/2} \int_{\mathbb{R}^K} f(\beta + \mathbf{x}) e^{-\frac{|\mathbf{x}|^2}{2(T-t)}} d\mathbf{x} \\ &= \frac{e^{-\frac{|\beta|^2}{2(T-t)}}}{(2\pi(T-t))^{K/2}} \int_{\mathbb{R}^K} \sum_{k=0}^{\infty} \frac{1}{(T-t)^k} \sum_{k_1 + \dots + k_K = k} \frac{(\beta_1 y_1)^{k_1} \dots (\beta_K y_K)^{k_K}}{k_1! \dots k_K!} f(\mathbf{y}) e^{-\frac{|\mathbf{y}|^2}{2(T-t)}} d\mathbf{y}\end{aligned}$$

Under the present growth condition, however, equations (22)-(27) in Anderson and Raimondo [1] can be replaced now by the following steps

$$\begin{aligned}
& \left| \frac{1}{(2\pi(T-t))^{K/2}} \int_{\mathbb{R}^K} \frac{y_1^{k_1} \cdots y_K^{k_K}}{k_1! \cdots k_K!} f(\mathbf{y}) e^{-\frac{|\mathbf{y}|^2}{2(T-t)}} d\mathbf{y} \right| \\
& \leq \frac{1}{(2\pi(T-t))^{K/2}} \int_{\mathbb{R}^K} \frac{|y_1|^{k_1} \cdots |y_K|^{k_K}}{k_1! \cdots k_K!} |f(\mathbf{y})| e^{-\frac{|\mathbf{y}|^2}{2(T-t)}} d\mathbf{y} \\
& \leq \frac{C}{(2\pi(T-t))^{K/2}} \int_{\mathbb{R}^K} \frac{|y_1|^{k_1} \cdots |y_K|^{k_K}}{k_1! \cdots k_K!} e^{-\frac{(1-2r(T-t))|\mathbf{y}|^2}{2(T-t)}} d\mathbf{y} \\
& \leq \frac{C}{\sqrt{k_1! \cdots k_K!}} \left( \frac{T-t}{1-2r(T-t)} \right)^{K/2} \leq C \frac{k_1! \cdots k_K!}{\sqrt{\frac{1}{T-t} - 2r}}
\end{aligned}$$

where the second inequality above follows from the growth condition itself and the third from the formula for the  $k_i$ th moment of the absolute value of a normal random variable. Clearly, the last inequality above provides an upper bound for the absolute value of the coefficient of  $\beta_1^{k_1} \cdots \beta_K^{k_K}$  in the power series of equation (27) in Anderson and Raimondo [1]. Hence, the power series in question converges also here absolutely within a positive radius of convergence (see Proposition 2.2.10 in Krantz and Parks [42]). The remainder of the proof is identical to that in Anderson and Raimondo [1]. ■

### Proof of Theorem 2.1

First, we will establish that, for any  $(j, k) \in \mathcal{K} \cup \{0\} \times \mathcal{K}$ , the functions  $(MP_j)(\cdot)$  and  $D_k(MP_j)(\cdot)$  are analytic on  $(0, T) \times \mathbb{R}^K$ . Given the growth condition (A1) and Lemma B.3, that the first function is analytic follows immediately by its definition in (4) in the main text. Observe next that, since  $|x_k| \leq |\mathbf{x}| < e^{|\mathbf{x}|} < 1 + e^{|\mathbf{x}|}$  everywhere on  $\mathbb{R}^K$ , by Lemma A.2, there exist  $C_{jk} > 0$  and  $r_{jk} \in (0, \frac{1}{2T} - r_j)$  such that  $|x_k| \leq C_{jk} e^{r_{jk}|\mathbf{x}|^2}$  for any  $\mathbf{x} \in \mathbb{R}^K$ . As a result, we have

$$|x_k M_j(T, \mathbf{x})| \leq \max\{C_j, C_{jk}\} e^{(r_{jk} + r_j)|\mathbf{x}|^2} \quad \forall \mathbf{x} \in \mathbb{R}^K$$

where  $r_{jk} + r_j \in (0, \frac{1}{2T})$ . Given (6) in the main text, this allows Lemma B.3 to establish that

$$\begin{aligned}
\beta_{kt}(MP_j)(t, \beta_t) + (T-t) D_k(MP_j)(t, \beta_t) &= \mathbb{E}_{\mathbf{x}} \left[ \left( \beta_k + \sqrt{T-t} x_k \right) M_j \left( T, \beta_t + \sqrt{T-t} \mathbf{x} \right) \right] \\
&= \mathbb{E} [\beta_{kT} M_j(T, \beta_T) | \mathcal{F}_t]
\end{aligned}$$

is an analytic function of  $(t, \beta_t)$  on  $(0, T) \times \mathbb{R}^K$ . That  $D_k(MP_j)(\cdot)$  itself is analytic on  $(0, T) \times \mathbb{R}^K$  follows then from the fact that  $(MP_j)(\cdot)$ ,  $\beta_{kt}$ , and  $T-t$  are all analytic on  $(0, T) \times \mathbb{R}^K$  while the sum, product, and ratio of two real analytic functions are also analytic (see Propositions 1.1.7 and 1.1.12 in Krantz and Parks [42]).

Under (7) in the main text, these observations imply that all entries of the matrix

$$(MP_0)(t, \beta_t)^2 J_p(t, \beta_t) = \left[ (MP_0)(t, \beta_t)^2 D_k p_n(t, \beta_t) \right]_{(n,k) \in \mathcal{K} \times \mathcal{K}}$$

are analytic on  $(0, T) \times \mathbb{R}^K$ . The determinant of a matrix involving nothing but the operations of product and sum on its entries, the same is true also for the function

$$\left| (MP_0)(t, \beta_t)^2 J_p(t, \beta_t) \right| = (MP_0)(t, \beta_t)^2 |J_p(t, \beta_t)|$$

Fix now any  $\omega_0 \in \Omega$  and let  $\beta_t = \beta(\omega_0, t)$ . By Proposition 2.1, it must be  $\lim_{t \rightarrow T} J_p(t, \beta(\omega_0, t)) = J_G(T, \beta(\omega_0, T))$ . Which, the determinant of a matrix being a continuous operator, implies in turn that

$$\lim_{t \rightarrow T} |J_p(t, \beta(\omega_0, t))| = |J_G(T, \beta(\omega_0, T))| \times |\Sigma| \prod_{n \in \mathcal{K}} G_{n/0}(T, \beta(\omega_0, T))$$

and, thus,

$$\begin{aligned} \lim_{t \rightarrow T} \left| (MP_0)(t, \beta(\omega_0, t))^2 J_p(t, \beta(\omega_0, t)) \right| &= \lim_{t \rightarrow T} (MP_0)(t, \beta(\omega_0, t))^2 \lim_{t \rightarrow T} |J_p(t, \beta(\omega_0, t))| \\ &= M_0(T, \beta(\omega_0, T))^2 |\Sigma_0| \prod_{n \in \mathcal{K}} G_{n/0}(T, \beta(\omega_0, T)) \end{aligned}$$

the last equality using also Lemma B.2. Now, if  $\Sigma_0$  is non-singular, the last quantity above is non-zero. Which requires of course the existence of some  $t_0 \in (0, T)$  and a neighborhood  $V_{\beta_T^0}$  of  $\beta_T^0 = \beta(\omega_0, T)$  in  $\mathbb{R}^K$  such that

$$\left| (MP_0)(s, \beta_s)^2 J_p(s, \beta_s) \right| \neq 0 \quad \forall (s, \beta_s) \in (t_0, T) \times V_{\beta_T^0}$$

The determinant in question being analytic, however, this can be only if (recall footnote 10 in the main text)

$$\left| (MP_0)(t, \beta_t)^2 J_p(t, \beta_t) \right| \neq 0 \quad \text{a.e. on } (0, T) \times \mathbb{R}^K$$

Equivalently, only if  $|J_p(t, \beta_t)| \neq 0$  a.e. on  $(0, T) \times \mathbb{R}^K$  as required. ■

**Lemma B.4** *Let  $\mathcal{T} = [0, T]$  for some  $T > 0$  and fix some  $(t, \beta) \in \mathcal{T} \times \mathbb{R}^K$ . Suppose also that  $f(s, \mathbf{x}) : \mathcal{T} \times \mathbb{R}^K$  is continuous, and continuously-differentiable w.r.t.  $\mathbf{x}$  such that*

$$\exists (r, C) \in \left(0, \frac{1}{2T}\right) \times \mathbb{R}_{++} : \quad |D_k f(s, \mathbf{x})| \leq C e^{r|\mathbf{x}|^2} \quad \forall (s, \mathbf{x}) \in \mathcal{T} \times \mathbb{R}^K \quad \forall k \in \mathcal{K}$$

If  $\mathbf{x} \sim \mathcal{N}(\mathbf{0}^K, \mathbf{I}_K)$  there exists a continuous function  $h : \mathbb{R}^K \mapsto \mathbb{R}_{++}$  such that

$$\forall k \in \mathcal{K} \quad \left| \mathbb{E}_{\mathbf{x}} \left[ \frac{x_k}{\sqrt{s-t}} f(s, \beta + \sqrt{s-t} \mathbf{x}) \right] \right| < h(\beta) \quad \forall s \in [t, T]$$

**Proof.** Take arbitrary  $(k, s) \in \mathcal{K} \times (t, T)$ . By the mean value theorem, we must have

$$\begin{aligned} f(s, \beta + \sqrt{s-t} \mathbf{z}) &= f(s, \beta) \\ &+ \sqrt{s-t} \sum_{m \in \mathcal{K}} z_m D_m f(s, \beta + \sqrt{s-t} \lambda(\mathbf{z}) \mathbf{z}), \quad \mathbf{z} \in \mathbb{R}^K \end{aligned}$$

for some function  $\lambda : \mathbb{R}^K \mapsto (0, 1)$  such that  $\lim_{\mathbf{z} \rightarrow \mathbf{0}^K} \lambda(\mathbf{z}) = 0$ . As a result,

$$\begin{aligned} \mathbb{E}_{\mathbf{x}} \left[ \frac{x_k}{\sqrt{s-t}} f(s, \beta + \sqrt{s-t} \mathbf{x}) \right] &= \mathbb{E}_{\mathbf{x}} \left[ \frac{x_k}{\sqrt{s-t}} f(s, \beta) \right] \\ &+ \mathbb{E}_{\mathbf{x}} \left[ x_k \sum_{m \in \mathcal{K}} x_m D_m f(s, \beta + \sqrt{s-t} \lambda(\mathbf{x}) \mathbf{x}) \right] \\ &= \sum_{l \in \mathcal{K}} \mathbb{E}_{\mathbf{x}} [x_k x_m D_m f(s, \beta + \sqrt{s-t} \lambda(\mathbf{x}) \mathbf{x})] \end{aligned}$$

For any  $m \in \mathcal{K}$ , however, we have

$$\begin{aligned} |x_k x_m D_m f(s, \beta + \sqrt{s-t} \lambda(\mathbf{x}) \mathbf{x})| &< C e^{|x_k| + |x_m|} e^{r|\beta + \sqrt{s-t} \lambda(\mathbf{x}) \mathbf{x}|^2} \\ &\leq C e^{2|\mathbf{x}| + r|\beta + \sqrt{s-t} \lambda(\mathbf{x}) \mathbf{x}|^2} \\ &= C e^{2(1 + \sqrt{s-t})|\lambda(\mathbf{x}) \mathbf{x}| + r(|\beta|^2 + (s-t)|\lambda(\mathbf{x}) \mathbf{x}|^2)} \\ &\leq C e^{2(1 + \sqrt{T-t})|\mathbf{x}| + r(|\beta|^2 + (T-t)|\mathbf{x}|^2)} \\ &\leq e^{2(1 + \sqrt{T})|\mathbf{x}| + r(|\beta|^2 + T|\mathbf{x}|^2)} \end{aligned}$$

and, by the same arguments as in the proofs of Lemma B.1 and Corollary B.1, we can find constants  $\tilde{r}, \tilde{C} > 0$  such that

$$|\mathbb{E}_{\mathbf{x}} [x_k x_m D_m f(s, \beta + \sqrt{s-t} \lambda(\mathbf{x}) \mathbf{x})]| < \tilde{C} e^{\frac{|\beta|^2}{2T}} / \sqrt{\tilde{r}}$$

The claim follows by letting  $h(\beta) := K e^{\frac{|\beta|^2}{2T}} \tilde{C} / \sqrt{\tilde{r}}$ . ■

**Lemma B.5** *Let the functions  $f, g : \mathbb{R} \mapsto \mathbb{R}$  be integrable on  $\mathcal{T} \subseteq \mathbb{R}$ . Then*

$$\lim_{h \rightarrow 0} \int_{a+h}^b \int_{a+h}^b f(x) g(y) dx dy = \lim_{h \rightarrow 0} \int_a^{a+h} \int_a^{a+h} f(x) g(y) dx dy \quad \forall (a, b) \subseteq \mathcal{T}$$

**Proof.** To economize on the length of exposition, we will use the notation  $I_f(a, b) := \int_a^b f(s) ds$  with  $I_g(a, b)$  defined similarly. Observe now that

$$\begin{aligned} I_f(a+h, b) I_g(a+h, b) - I_f(a, b) I_g(a, b) &= [I_f(a+h, b) - I_f(a, b)] [I_g(a+h, b) - I_g(a, b)] \\ &+ [I_f(a+h, b) - I_f(a, b)] I_g(a, b) \\ &+ [I_g(a+h, b) - I_g(a, b)] I_f(a, b) \end{aligned}$$

As  $h \rightarrow 0$ , however, the continuity of the integral function means that

$$\lim_{h \rightarrow 0} I_f(a+h, b) - I_f(a, b) = 0 = \lim_{h \rightarrow 0} I_g(a+h, b) - I_g(a, b)$$

The claim follows now since

$$I_f(a+h, b) - I_f(a, b) = - \int_a^{a+h} f(s) ds$$

(and similarly for  $g$ ). ■

## Proof of Proposition 2.2

Take an arbitrary  $n \in \mathcal{K}$ . Observe first that, (17) in the main text involving but integrals, without any loss of generality we may define the  $[0, T] \mapsto \mathbb{R}_{++}$  function  $\mu_j^*(\cdot) := e^{\tilde{\mu}_j(\cdot)}$  and re-write the flow-dividend specification in (1) as

$$g_j(t, \beta_t) = e^{\ln \mu_j^*(t) + \tilde{\sigma}_j^\top \beta_t} \quad j \in \mathcal{K} \cup \{0\}$$

Recall next assumption 3. Since  $\mathcal{K}_n \neq \emptyset$ , we can define the vector-valued function  $\hat{\mu}_n : [0, T] \mapsto \mathbb{R}^K$  as follows

$$\hat{\mu}_{nk}(t) = \begin{cases} \frac{\ln \mu_n^*(t) - \ln \mu_0^*(t)}{|\mathcal{K}_n|(\tilde{\sigma}_{nk} - \tilde{\sigma}_{0k})} & \text{if } \tilde{\sigma}_{nk} \neq \tilde{\sigma}_{0k} \\ 0 & \text{otherwise} \end{cases}$$

Observe now that (17) in the main text can be written also as

$$\begin{aligned} &\mathcal{D}_t(mP_0)(\mathcal{I}(\omega, t))^2 D_k p_n(\mathcal{I}(\omega, t)) \\ &= h^{-2} \int_t^{t+h} \int_t^{t+h} \mathbb{E}_{(\mathbf{x}, \mathbf{y})} \left[ m_0(s, \beta_t + \sqrt{s-t}\mathbf{x}) m_0(\tau, \beta_t + \sqrt{\tau-t}\mathbf{y}) \right. \\ &\quad \left. \times \left( \frac{\hat{\mu}_{nk}(t)(s-t) + \sqrt{s-t}x_k}{s-t} - \frac{\hat{\mu}_{nk}(t)(\tau-t) + \sqrt{\tau-t}y_k}{\tau-t} \right) g_{n/0}(s, \beta_t + \sqrt{s-t}\mathbf{x}) \right] ds d\tau \end{aligned}$$

By construction, however, we have  $\ln \mu_n^*(t) - \ln \mu_0^*(t) = (\tilde{\sigma}_n - \tilde{\sigma}_0)^\top \hat{\mu}_n(t)$ . Defining then the function  $\tilde{g}_{n/0} : \mathbb{R}^K \mapsto \mathbb{R}$  by  $\tilde{g}_{n/0}(\mathbf{x}) = e^{(\tilde{\sigma}_n - \tilde{\sigma}_0)^\top \mathbf{x}}$ , and letting  $\hat{\mathbf{x}} = \hat{\mu}(s) - \hat{\mu}(t) + \sqrt{s-t}\mathbf{x}$  gives

$$\tilde{g}_{n/0}(\hat{\mu}_n(t) + \beta_t + \hat{\mathbf{x}}) = g_{n/0}(s, \beta_t + \sqrt{s-t}\mathbf{x}) \quad (s, \mathbf{x}) \in [t, T] \times \mathbb{R}^K$$

Moreover, letting also  $\hat{\mathbf{y}} = \hat{\mu}(s) - \hat{\mu}(t) + \sqrt{s-t}\mathbf{y}$  we may write

$$\begin{aligned} & \mathcal{D}_t(mP_0)(\mathcal{I}(\omega, t))^2 D_k p_n(\mathcal{I}(\omega, t)) \\ &= h^{-2} \int_t^{t+h} \int_t^{t+h} \mathbb{E}_{(\mathbf{x}, \mathbf{y})} \left[ m_0(s, \beta_t + \sqrt{s-t}\mathbf{x}) m_0(\tau, \beta_t + \sqrt{\tau-t}\mathbf{y}) \right. \\ & \quad \left. \times \left( \frac{\hat{x}_k}{s-t} - \frac{\hat{y}_k}{\tau-t} \right) \tilde{g}_{n/0}(\hat{\mu}_n(t) + \beta_t + \hat{\mathbf{x}}) \right] ds d\tau \end{aligned}$$

Fix now an arbitrary  $(s, \tau, \mathbf{y}) \in \mathcal{T} \times \mathcal{T} \times \mathbb{R}^K$ . Given the flow-dividend specification in (1), the dispersion coefficient of  $\tilde{g}_{n/0}(\cdot)$  is given by  $\nabla_{\beta} \tilde{g}_{n/0}(\cdot) = \tilde{g}_{n/0}(\cdot)(\tilde{\sigma}_n - \tilde{\sigma}_0)$ . By the mean value theorem then, we have

$$\begin{aligned} \tilde{g}_{n/0}(\hat{\mu}_n(t) + \beta_t + \mathbf{z}) &= \tilde{g}_{n/0}(\hat{\mu}_n(t) + \beta_t) \\ &+ \tilde{g}_{n/0}\left(\hat{\mu}_n(t) + \beta_t + \tilde{\lambda}_{n/0}(\mathbf{z})\mathbf{z}\right) \sum_{k \in \mathcal{K}} (\tilde{\sigma}_{nk} - \tilde{\sigma}_{0k}) z_k, \quad \mathbf{z} \in \mathbb{R}^K \end{aligned}$$

for some function  $\tilde{\lambda}_{n/0} : \mathbb{R}^K \mapsto (0, 1)$  such that  $\lim_{\mathbf{z} \rightarrow \mathbf{0}^K} \tilde{\lambda}_{n/0}(\mathbf{z}) = 0$ . Notice also that letting  $s \wedge \tau \rightarrow t$  means that  $\hat{\mathbf{x}}$  and  $\hat{\mathbf{y}}$  depict, respectively, the infinitesimal increments  $\Delta\beta_t$  and  $\Delta\tilde{\beta}_t$  of two independent  $K$ -dimensional Brownian motions with instantaneous drift  $\hat{\mu}'_n(t)$ . Hence, by the same standard arguments as in the proof of Proposition 2.1 (see Theorem 4.2.1 in Øksendal [51]), it must be

$$\begin{aligned} \lim_{s \wedge \tau \rightarrow t} \left( \frac{\hat{x}_k}{s-t} - \frac{\hat{y}_k}{\tau-t} \right) \sum_{k \in \mathcal{K}} (\tilde{\sigma}_{nk} - \tilde{\sigma}_{0k}) \hat{x}_k &= \lim_{\Delta t \rightarrow 0} \left( \frac{\Delta\beta_k}{\Delta t} - \frac{\Delta\tilde{\beta}_k}{\Delta t} \right) \sum_{k \in \mathcal{K}} (\tilde{\sigma}_{nk} - \tilde{\sigma}_{0k}) \Delta\beta_k \\ &\stackrel{\text{a.s.}}{=} \tilde{\sigma}_{nk} - \tilde{\sigma}_{0k} \end{aligned}$$

As a result, for  $h \rightarrow 0$ , it cannot but be

$$\begin{aligned} & \mathcal{D}_t(mP_0)(\mathcal{I}(\omega, t))^2 D_k p_n(\mathcal{I}(\omega, t)) \\ &= \frac{\tilde{\sigma}_{nk} - \tilde{\sigma}_{0k}}{h^2} \int_t^{t+h} \int_t^{t+h} \mathbb{E}_{(\mathbf{x}, \mathbf{y})} \left[ m_0(s, \beta_t + \sqrt{s-t}\mathbf{x}) m_0(\tau, \beta_t + \sqrt{\tau-t}\mathbf{y}) \right. \\ & \quad \left. \times \tilde{g}_{n/0}(\hat{\mu}_n(t) + \beta_t + \tilde{\lambda}_{n/0}(\hat{\mathbf{x}})\hat{\mathbf{x}}) \right] ds d\tau \end{aligned} \tag{31}$$

Recall now Lemma B.1. Under assumption A2, there exist constants  $\bar{r}_0, \bar{C}_0 > 0$  such that

$$|m_0(s, \beta_t + \sqrt{s-t}\mathbf{x})| \phi(\mathbf{x}) < \bar{C}_0 e^{-\frac{\bar{r}_0|\mathbf{x}|^2}{2}} \quad \forall (s, \mathbf{x}) \in (t, T) \times \mathbb{R}^K \tag{32}$$

By Corollary A.1, however, choosing  $\bar{r}_{n/0} \in (0, \frac{\bar{r}_0}{4T})$  we can find  $\bar{C}_{n/0} > 0$  s.t.

$$\tilde{g}_{n/0}(\hat{\mu}_n(t) + \beta_t + \mathbf{z}) < \bar{A}_{n/0} \bar{C}_{n/0} e^{\bar{r}_{n/0}|\mathbf{z}|^2} \quad \forall \mathbf{z} \in \mathbb{R}^K$$



where  $\bar{A}_{n/0} = e^{(\tilde{\sigma}_n - \tilde{\sigma}_0)^T (\hat{\mu}_n t + \beta_t)}$ . As a result, we must have

$$\begin{aligned}
\tilde{g}_{n/0} \left( \hat{\mu}_n(t) + \beta_t + \tilde{\lambda}_{n/0}(\hat{\mathbf{x}}) \hat{\mathbf{x}} \right) &< \bar{A}_{n/0} \bar{C}_{n/0} e^{\bar{r}_{n/0} |\tilde{\lambda}_{n/0}(\hat{\mathbf{x}}) \hat{\mathbf{x}}|^2} \\
&< \bar{A}_{n/0} \bar{C}_{n/0} e^{\bar{r}_{n/0} |\hat{\mathbf{x}}|^2} \\
&= \bar{A}_{n/0} \bar{C}_{n/0} e^{\bar{r}_{n/0} |\hat{\mu}_n(s-t) + \sqrt{s-t} \mathbf{x}|^2} \\
&< \bar{A}_{n/0} \bar{C}_{n/0} e^{\bar{r}_{n/0} ((s-t) |\hat{\mu}_n| + \sqrt{s-t} |\mathbf{x}|)^2} \\
&= \bar{A}_{n/0} \bar{C}_{n/0} e^{\bar{r}_{n/0} ((s-t)^2 |\hat{\mu}_n|^2 + (s-t) |\mathbf{x}|^2 + 2\sqrt{s-t} (s-t) |\hat{\mu}_n| |\mathbf{x}|)} \\
&< \bar{A}_{n/0} \bar{C}_{n/0} e^{T \bar{r}_{n/0} (T |\hat{\mu}_n|^2 + 2\sqrt{T} |\hat{\mu}_n| |\mathbf{x}| + |\mathbf{x}|^2)} \\
&= \hat{A}_{n/0} e^{T \bar{r}_{n/0} (2\sqrt{T} |\hat{\mu}_n| |\mathbf{x}| + |\mathbf{x}|^2)} \quad \forall \mathbf{x} \in \mathbb{R}^K
\end{aligned}$$

where  $\hat{A}_{n/0} = \bar{A}_{n/0} \bar{C}_{n/0} e^{\bar{r}_{n/0} T^2 |\hat{\mu}_n|^2}$ . Yet, by Lemma A.2, we may also choose  $\hat{r}_{n/0} \in (0, 3T \bar{r}_{n/0})$  and find  $\hat{C}_{n/0} > 0$  such that

$$\tilde{g}_{n/0} \left( \hat{\mu}_n(t) + \beta_t + \tilde{\lambda}_{n/0}(\hat{\mathbf{x}}) \hat{\mathbf{x}} \right) < \hat{A}_{n/0} \hat{C}_{n/0} e^{(\hat{r}_{n/0} + T \bar{r}_{n/0}) |\mathbf{x}|^2} < \tilde{C}_{n/0} e^{4T \bar{r}_{n/0} |\mathbf{x}|^2}$$

where  $\tilde{C}_{n/0} = \hat{A}_{n/0} \hat{C}_{n/0}$ . For all  $(s, \mathbf{x}) \in (t, T) \times \mathbb{R}^K$ , therefore, we have

$$\left| m_0(s, \beta_t + \sqrt{s-t} \mathbf{x}) \tilde{g}_{n/0} \left( \hat{\mu}_n(t) + \beta_t + \tilde{\lambda}_{n/0}(\hat{\mathbf{x}}) \hat{\mathbf{x}} \right) \right| \phi(\mathbf{x}) < \tilde{C}_{n/0} \bar{C}_0 e^{-\frac{(\bar{r}_0 - 4T \bar{r}_{n/0})}{2} |\mathbf{x}|^2} \quad (33)$$

And as the upper bounds in (32)-(33) above are clearly integrable on  $\mathbb{R}^K$ , Lemma A.5 ensures that the continuity in  $s$  and  $\tau$  commutes under the expectation operator in (31) to give

$$\lim_{s, \tau \rightarrow t} \mathbb{E}_{(\mathbf{x}, \mathbf{y})} \left[ \begin{aligned} &(mg_0)(s, \beta_t + \sqrt{s-t} \mathbf{x}) (mg_0)(\tau, \beta_t + \sqrt{\tau-t} \mathbf{y}) \\ &\times \tilde{g}_{n/0} \left( \hat{\mu}_n(t) + \beta_t + \tilde{\lambda}_{n/0}(\hat{\mathbf{x}}) \hat{\mathbf{x}} \right) \end{aligned} \right] = (mg_n)(t, \beta_t) (mg_0)(t, \beta_t)$$

As  $h \rightarrow 0$ , therefore, we have

$$\begin{aligned}
&D_t(mP_0)(\mathcal{I}(\omega, t))^2 D_k p_n(\mathcal{I}(\omega, t)) \\
&= (\tilde{\sigma}_{nk} - \tilde{\sigma}_{0k}) \lim_{h \rightarrow 0} \frac{1}{h^2} \int_t^{t+h} m_0(t, \beta_t) ds \int_t^{t+h} m_0(t, \beta_t) ds \\
&= (\tilde{\sigma}_{nk} - \tilde{\sigma}_{0k}) (m_n m_0)(t, \beta_t)
\end{aligned}$$

To complete the argument for the first relation in the claim, recall that the choices of  $t$  and  $\beta_t$  above were arbitrary. In addition, the distribution of the Brownian increments process being stationary, the law of iterated expectations allows us to replace the current date-realization pair  $(t, \beta_t)$  above by any future one  $(t', \beta_{t'})$  where  $t' \in (t, T)$ . Moreover, we can do so also with respect to the argument that established the validity of equations (14)-(15) in the main text.

For the second relation of the claim, observe that, the quantities in (14)-(15) being time-integrals,

we obviously have

$$\lim_{t \rightarrow T} (mP_j) (\mathcal{I} (\omega, t)) = 0 = \lim_{t \rightarrow T} D_k (mP_j) (\mathcal{I} (\omega, t)) \quad (j, k) \in \mathcal{K} \cup \{0\} \times \mathcal{K}, \quad \omega \in \Omega$$

Given (16), therefore, it must be

$$\lim_{t \rightarrow T} (mP_0) (\mathcal{I} (\omega, t))^2 D_k (mP_n) (\mathcal{I} (\omega, t)) = 0 \quad (n, k) \in \mathcal{K} \times \mathcal{K}, \quad \omega \in \Omega$$

and, thus,

$$\begin{aligned} (mP_0) (\mathcal{I} (\omega, t))^2 D_k p_n (\mathcal{I} (\omega, t)) &= - \int_t^T \mathcal{D}_t \mathbb{E} \left[ (mP_0) (s, \beta_s)^2 D_k p_n (s, \beta_s) | \mathcal{F}_t \right] ds \\ &= - (\tilde{\sigma}_{nk} - \tilde{\sigma}_{0k}) \int_t^T \mathbb{E} [(m_n m_0) (s, \beta_s) | \mathcal{F}_t] ds \end{aligned}$$

as required. ■

## C Locally-bounded portfolios

**Lemma C.1** *Let  $M \in \mathbb{N} \setminus \{0\}$  and  $(\theta_1, \dots, \theta_M) \in \mathbb{R}^M \setminus \{\mathbf{0}^M\}$ . Suppose also that, for each  $m \in \{1, \dots, M\}$ , the function  $f_m : \mathbb{R}^K \mapsto \mathbb{R}$  satisfies*

$$\exists r_m, \delta_m > 0 : |f_m(\mathbf{x})| < r_m + e^{r_m |\mathbf{x}|} \quad \forall \mathbf{x} \in \mathbb{R}^K \setminus \mathcal{B}_{\delta_m}$$

*Then letting  $r = \max_{m \in \{1, \dots, M\}} \{r_m\}$*

$$\exists C, \delta > 0 : \left| \sum_{m=1}^M \theta_m f_m(\mathbf{x}) \right| < C \left( r + e^{r |\mathbf{x}|} \right) \quad \forall \mathbf{x} \in \mathbb{R}^K \setminus \mathcal{B}_\delta$$

**Proof.** Let  $\delta = \max_{m \in \{1, \dots, M\}} \{\delta_m\}$ . For any  $\mathbf{x} \in \mathbb{R}^K \setminus \mathcal{B}_\delta$  we have

$$\left| \sum_{m=1}^M \theta_m f_m(\mathbf{x}) \right| \leq \sum_{m=1}^M |\theta_m| |f_m(\mathbf{x})| \leq \sum_{m=1}^M |\theta_m| \left( r_m + e^{r_m |\mathbf{x}|} \right) \leq M \theta \left( r + e^{r |\mathbf{x}|} \right)$$

where  $\theta = \max_{m \in \{1, \dots, M\}} \{|\theta_m|\}$  and  $r = \max_{m \in \{1, \dots, M\}} \{r_m\}$ . ■

A trivially similar argument establishes also the following.

**Lemma C.2** *Let  $M \in \mathbb{N} \setminus \{0\}$  and  $(\theta_1, \dots, \theta_M) \in \mathbb{R}^M \setminus \{\mathbf{0}^M\}$ . Suppose also that, for each*

$m \in \{1, \dots, M\}$ , the function  $f_m : \mathbb{R}^K \mapsto \mathbb{R}$  satisfies

$$\exists r_m, C_m, \delta_m > 0 : |f_m(\mathbf{x})| < C_m e^{r_m |\mathbf{x}|^2} \quad \forall \mathbf{x} \in \mathbb{R}^K \setminus \mathcal{B}_{\delta_m}$$

Then letting  $r = \max_{m \in \{1, \dots, M\}} \{r_m\}$

$$\exists C, \delta > 0 : \left| \sum_{m=1}^M \theta_m f_m(\mathbf{x}) \right| < C e^{r |\mathbf{x}|^2} \quad \forall \mathbf{x} \in \mathbb{R}^K \setminus \mathcal{B}_{\delta}$$

**Lemma C.3** Let  $r > 0$ . Suppose also that  $f : \mathbb{R} \mapsto \mathbb{R}$  and  $h : \mathbb{R}^K \mapsto \mathbb{R}$  satisfy

$$\begin{aligned} \exists r_0, \delta_0 > 0 : |h(\mathbf{x})| &< r_0 + e^{r_0 |\mathbf{x}|} & \forall \mathbf{x} \in \mathbb{R}^K \setminus \mathcal{B}_{\delta_0} \\ \exists \gamma_1, \gamma_2, \delta_1 > 0 : |f(z)| &< C_1 + e^{r_1 |z|} & \forall z \in \mathbb{R} \setminus (-\delta_1, \delta_1) \end{aligned}$$

The composition  $f \circ h : \mathbb{R}^K \mapsto \mathbb{R}$  then satisfies

$$\exists C, \delta > 0 : |f(h(\mathbf{x}))| \leq C e^{r |\mathbf{x}|^2} \quad \forall \mathbf{x} \in \mathbb{R}^K \setminus \mathcal{B}_{\delta}$$

**Proof.** By Lemma A.2, letting  $\tilde{r} = r/\gamma_2$  and  $\tilde{\delta} = \max\{\delta_0, r_0/\tilde{r}\}$ , we have

$$|h(\mathbf{x})| \leq (1 + r_0) e^{\tilde{r} |\mathbf{x}|^2} \quad \mathbf{x} \in \mathbb{R}^K \setminus \mathcal{B}_{\tilde{\delta}}$$

It follows, therefore, that

$$\begin{aligned} |f(h(\mathbf{x}))| &\leq \gamma_1 (1 + |h(\mathbf{x})|^{\gamma_2}) \\ &= \gamma_1 \left( 1 + (1 + r_0)^{\gamma_2} e^{r |\mathbf{x}|^2} \right) \leq \gamma_1 [1 + (1 + r_0)^{\gamma_2}] e^{r |\mathbf{x}|^2} \quad \mathbf{x} \in \mathbb{R}^K \setminus \mathcal{B}_{\delta} \end{aligned}$$

where  $\delta = \max\{\tilde{\delta}, \delta_1\}$ . ■

## An Example

In what follows, we will restrict attention to the financial market in Section 2.2, allowing also for the case in which the zeroth security is a money-market account. We will consider this market as part of a single-commodity, pure-exchange economy with  $M \in \mathbb{N} \setminus \{0\}$  agents. The typical agent has preferences over consumption flows represented by the objective

$$U_m(c) = \mathbb{E} \left[ \int_{\mathcal{T} \setminus [0, t)} u_m(s, c_s) ds | \mathcal{F}_t \right] \quad c = \{c_\tau : \tau \in [0, T)\} \quad t \in [0, T) \quad (34)$$

where  $T \in \mathbb{R}_{++} \cup \{+\infty\}$  while  $u_m : \mathcal{T} \times \mathbb{R}_+ \mapsto \mathbb{R}$  is an increasing, strictly concave, twice-differentiable Bernoulli utility function that satisfies the growth condition

$$\exists \gamma_1, \gamma_2 > 0 : |u_m(t, x)| \leq \gamma_1 (1 + |x|^{\gamma_2}) \quad \forall (t, x) \in [0, T) \times \mathbb{R} \quad (35)$$

The typical agent is also endowed with some (possibly stochastic) intermediate flow  $\tilde{\rho}_m : [0, T) \times \mathbb{R}^K \mapsto \mathbb{R}_+$  that is a continuous function satisfying the Anderson-Raimondo (AR) growth condition of footnote 26 in the main text. Namely,

$$\exists r_m > 0 : |\tilde{\rho}_m(t, \mathbf{x})| \leq r_m + e^{r_m |\mathbf{x}|} \quad \forall (t, \mathbf{x}) \in [0, T) \times \mathbb{R}^K \quad (36)$$

We will suppose also that the equilibrium pricing process in (12) is an Itô process evolving as

$$d\mathbf{P}_t = \alpha(t, \mathbf{P}_t) dt + \xi(t, \mathbf{P}_t) d\beta_t \quad t \in [0, T) \quad (37)$$

where  $\mathbf{P}_t = (P_{0t}, \dots, P_{Kt})^\top$  while  $\alpha : [0, T) \times \mathbb{R}^K \mapsto \mathbb{R}$  and  $\mathbf{B} : [0, T) \times \mathbb{R}^K \mapsto \mathbb{R}^{K+1}$  are adapted processes satisfying the integrability conditions in footnote 6, as well as the following growth and Lipschitz conditions. Namely, there exist constants  $A, C > 0$  such that<sup>39</sup>

$$\begin{aligned} |\alpha(t, \mathbf{x})| + |\mathbf{B}(t, \mathbf{x})| &\leq A(1 + |\mathbf{x}|) & \forall (t, \mathbf{x}) \in [0, T) \times \mathbb{R}^K \\ |\alpha(t, \mathbf{x}) - \alpha(t, \mathbf{y})| + |\mathbf{B}(t, \mathbf{x}) - \mathbf{B}(t, \mathbf{y})| &\leq C|\mathbf{x} - \mathbf{y}| & \forall (t, \mathbf{x}, \mathbf{y}) \in [0, T) \times \mathbb{R}^K \times \mathbb{R}^K \end{aligned}$$

Let then  $\theta_{mj}$  be the typical agent's portfolio position on the security  $j \in \mathcal{K} \cup \{0\}$ . The typical price in (12) being ex-dividend, the self-financing condition requires that her financial wealth

$$W_{mt} = \sum_{j \in \mathcal{K} \cup \{0\}} \theta_{mj} P_{jt} \quad t \in [0, T) \quad (38)$$

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<sup>39</sup>The notation is meant as  $|\mathbf{B}(\cdot)|^2 = \sum_{(j,k) \in \mathcal{K} \cup \{0\} \times \mathcal{K}} \mathbf{B}_{jk}(\cdot)^2$ . The two conditions ensure that, given an initial value  $\mathbf{P}_0 = Z$  - where  $Z \in \mathbb{R}_{++}^{K+1}$  is a random vector on  $\Omega$  independent of the Brownian filtration  $\mathcal{F}^\beta$  with  $\mathbb{E}_\pi[|Z|^2] < +\infty$ , - (37) has a unique  $t$ -continuous solution (see, for instance, Theorem 5.2.1 in Øksendal [51]).

evolves according to the process<sup>40</sup>

$$\begin{aligned}
dW_{mt} &= (\tilde{\rho}_{mt} - c_{mt}) dt + \sum_{j \in \mathcal{K} \cup \{0\}} \theta_{mjt} (\tilde{g}_{jt} + \alpha_{jt}) dt + \sum_{k \in \mathcal{K}} \sum_{j \in \mathcal{K} \cup \{0\}} \theta_{mjt} \mathbf{B}_{jkt} d\beta_k \\
&\stackrel{(38)}{=} (\tilde{\rho}_{mt} - c_{mt}) dt + \frac{W_{mt}}{P_{0t}} \left( (\tilde{g}_{0t} + \alpha_{0t}) dt + \sum_{k \in \mathcal{K}} \mathbf{B}_{0kt} d\beta_k \right) \\
&+ \sum_{n \in \mathcal{K}} \theta_{mnt} (\tilde{g}_{nt} + \alpha_{nt} - (\tilde{g}_{0t} + \alpha_{0t}) p_{nt}) dt \\
&+ P_{0t} \sum_{n \in \mathcal{K}} \sum_{k \in \mathcal{K}} \theta_{njt} \tilde{\mathbf{B}}_{nkt} d\beta_k \quad t \in [0, T)
\end{aligned} \tag{39}$$

where  $\tilde{\mathbf{B}}_{nk} := (P_0 \mathbf{B}_{nk} - P_n \mathbf{B}_{0k}) / P_0^2$  for  $(n, k) \in \mathcal{K} \times \mathcal{K}$ . Next, we will assume that maximizing (34) subject to the wealth dynamics above admits a feedback control (that is measurable on  $[0, T) \times \mathbb{R}_+^{K+1}$  and satisfies the growth and Lipschitz conditions in footnote 14) such that the value function

$$V_m(t, W_t, \mathbf{P}_t) = \sup_{c, \theta_{m0}, \theta_m} U_m(c)$$

- where  $\theta_m = (\theta_{m1}, \dots, \theta_{mK})^\top$  - satisfies the dynamic programming equation

$$V_m(t, W_t, \mathbf{P}_t) = \sup_{c, \theta_{m0}, \theta_m} \mathbb{E} \left[ \int_t^\tau u(s, c_s) ds + V_m(\tau, W_\tau, \mathbf{P}_\tau) | \mathcal{F}_t \right]$$

at any  $\tau \in (t, T]$ . We will suppose furthermore that  $\partial V_m(t, W_t, \mathbf{P}_t) / \partial t$ ,  $\partial V_m(t, W_t, \mathbf{P}_t) / \partial W$ ,  $\partial^2 V_m(t, W_t, \mathbf{P}_t) / \partial W^2$ , as well as  $\partial^2 V_m(t, W_t, \mathbf{P}_t) / \partial W \partial P_j$  and  $\partial^2 V_m(t, W_t, \mathbf{P}_t) / \partial P_i \partial P_j$  for  $j \in \mathcal{K} \cup \{0\}$  are all well-defined and continuous everywhere on  $[0, T) \times \mathbb{R}_+^{K+1}$ .<sup>41</sup> The dynamic programming equation corresponds of course to the Hamilton-Jacobi-Bellman one

$$\sup_{c, \theta_{m0}, \theta_m} \{u_m(c) + \mathcal{D}V_m(t, W_t, \mathbf{P}_t)\} = 0$$

<sup>40</sup>The wealth dynamics here are essentially those in Chapter 5 of Merton [47] - see also Riedel and Herzberg [61].

<sup>41</sup>In general, explicit solutions to this kind of stochastic optimal control problem are extremely rare. Imposing therefore these assumptions is the modus operandi of theoretical investigation. They are made implicitly, for instance, in Chapter 5 of Merton [47], and explicitly as assumptions A9-A10 in Cox et al. [?] (see also sections 9.A-D in Duffie [17]). With respect to the current setting, where only intermediate dividend flows are present, the analytical verification of the assumptions in question has been accomplished in a few special cases. Namely, when  $u_m(s, c_s) = f(s) \tilde{u}_m(c_s)$  where the function  $f : [0, T) \mapsto \mathbb{R}$  is everywhere continuously differentiable while  $\tilde{u}_m : \mathbb{R}_+ \mapsto \mathbb{R}$  is a member of the HARA family (see, for example, sections 5.6 and 5.9 in Merton [47] or 9.C-D in Duffie [17]). It is trivial to show of course that this class of functions  $u_m$  satisfies the growth condition in (35).

where the Dynkin operator and the terminal-value condition are given, respectively, by

$$\begin{aligned}
\mathcal{D}V_m(t, W_t, \mathbf{P}_t) &= \frac{\partial V_m(t, W_t, \mathbf{P}_t)}{\partial t} \\
&+ \left( \tilde{\rho}_{mt} - c_{mt} + \frac{W_t}{P_{0t}} (\tilde{g}_{0t} + \alpha_{0t}) \right. \\
&\quad \left. + \sum_{n \in \mathcal{K}} \theta_{mnt} (\tilde{g}_{nt} + \alpha_{nt} - (\tilde{g}_{0t} + \alpha_{0t}) p_{nt}) \right) \frac{\partial V_m(t, W_t, \mathbf{P}_t)}{\partial W} \\
&+ \frac{W_t}{2} \sum_{n \in \mathcal{K}} \sum_{k \in \mathcal{K}} \theta_{mnt} \tilde{\mathbf{B}}_{nkt} \mathbf{B}_{0kt} \frac{\partial^2 V_m(t, W_t, \mathbf{P}_t)}{\partial W^2} \\
&+ \frac{P_{0t}^2}{2} \sum_{k \in \mathcal{K}} \sum_{n \in \mathcal{K}} \sum_{j \in \mathcal{K}} \theta_{mnt} \theta_{mjt} \tilde{\mathbf{B}}_{nkt} \tilde{\mathbf{B}}_{jkt} \frac{\partial^2 V_m(t, W_t, \mathbf{P}_t)}{\partial W^2} \\
&+ \frac{W_t}{2P_{0t}} \sum_{k \in \mathcal{K}} \mathbf{B}_{0kt} \frac{\partial^2 V_m(t, W_t, \mathbf{P}_t)}{\partial W^2} + \sum_{k \in \mathcal{K}} \alpha_{kt} \frac{\partial V_m(t, W_t, \mathbf{P}_t)}{\partial P_k} \\
&+ \frac{1}{2} \sum_{i \in \mathcal{K} \cup \{0\}} \sum_{j \in \mathcal{K} \cup \{0\}} \sum_{k \in \mathcal{K}} \mathbf{B}_{ikt} \mathbf{B}_{jkt} \frac{\partial^2 V_m(t, W_t, \mathbf{P}_t)}{\partial P_i \partial P_j} \\
&+ \frac{W_t}{P_{0t}} \sum_{j \in \mathcal{K} \cup \{0\}} \sum_{k \in \mathcal{K}} \mathbf{B}_{0kt} \mathbf{B}_{jkt} \frac{\partial^2 V_m(t, W_t, \mathbf{P}_t)}{\partial P_j \partial W} \\
&+ P_{0t} \sum_{j \in \mathcal{K} \cup \{0\}} \sum_{k \in \mathcal{K}} \sum_{n \in \mathcal{K}} \theta_{mnt} \tilde{\mathbf{B}}_{nkt} \mathbf{B}_{jkt} \frac{\partial^2 V_m(t, W_t, \mathbf{P}_t)}{\partial P_j \partial W}
\end{aligned}$$

and  $V_m(T, W_T, \mathbf{P}_T) = 0$ .<sup>42</sup> Hence, the first-order conditions are given by (38) and

$$\partial u_m(t, c_t) / \partial t = \frac{\partial V_m(t, W_t, \mathbf{P}_t)}{\partial W} \quad (40)$$

$$\begin{aligned}
0 &= \frac{P_{0t}^2}{2} \sum_{j \in \mathcal{K}} \theta_{mjt} \sum_{k \in \mathcal{K}} \tilde{\mathbf{B}}_{nkt} \tilde{\mathbf{B}}_{jkt} \frac{\partial^2 V_m(t, W_t, \mathbf{P}_t)}{\partial W^2} \\
&+ (\tilde{g}_{nt} + \alpha_{nt} - (\tilde{g}_{0t} + \alpha_{0t}) p_{nt}) \frac{\partial V_m(t, W_t, \mathbf{P}_t)}{\partial W} \\
&+ \frac{W_t}{2} \sum_{k \in \mathcal{K}} \mathbf{B}_{0kt} \tilde{\mathbf{B}}_{nkt} \frac{\partial^2 V_m(t, W_t, \mathbf{P}_t)}{\partial W^2} \\
&+ P_{0t} \sum_{j \in \mathcal{K} \cup \{0\}} \sum_{k \in \mathcal{K}} \tilde{\mathbf{B}}_{nkt} \mathbf{B}_{jkt} \frac{\partial^2 V_m(t, W_t, \mathbf{P}_t)}{\partial P_j \partial W}
\end{aligned} \quad (41)$$

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<sup>42</sup>When  $T = +\infty$ , the condition reads  $\lim_{t \rightarrow \infty} V_m(t, W_t, \mathbf{P}_t) = 0$ .

Let now

$$\begin{aligned}
v_n(t, W_t, \mathbf{P}_t) &:= (\tilde{g}_{nt} + \alpha_{nt} - (\tilde{g}_{0t} + \alpha_{0t}) p_{nt}) \frac{\partial V_m(t, W_t, \mathbf{P}_t)}{\partial W} \\
&+ \frac{W_t}{2} \sum_{k \in \mathcal{K}} \mathbf{B}_{0kt} \tilde{\mathbf{B}}_{nkt} \frac{\partial^2 V_m(t, W_t, \mathbf{P}_t)}{\partial W^2} \\
&+ P_{0t} \sum_{j \in \mathcal{K} \cup \{0\}} \sum_{k \in \mathcal{K}} \tilde{\mathbf{B}}_{nkt} \mathbf{B}_{jkt} \frac{\partial^2 V_m(t, W_t, \mathbf{P}_t)}{\partial P_j \partial W}, \quad n \in \mathcal{K}
\end{aligned}$$

and  $\mathbf{v} = (v_1, \dots, v_K)^\top$ . Note also that under the notation in the main text

$$\mathbf{B}_t \equiv [D_k P_{jt}]_{(j,k) \in \mathcal{K} \cup \{0\} \times \mathcal{K}} \quad \tilde{\mathbf{B}}_t \equiv [D_k p_{nt}]_{(n,k) \in \mathcal{K} \times \mathcal{K}}$$

while under (35) we have  $\partial^2 V_m(t, W_t, \mathbf{P}_t) / \partial W^2 < 0$  everywhere on  $[0, T] \times \mathbb{R}_+^{K+1}$  (see Lemma 2 in Cox et al. [?]). It is trivial to check therefore that (41) can be written also as

$$J_{pt} J_{pt}^\top \theta_{mt} = - \frac{2\mathbf{v}(t, W_t, \mathbf{P}_t)}{P_{0t}^2 \partial^2 V_m(t, W_t, \mathbf{P}_t) / \partial W^2} \quad (42)$$

Observe finally that the equilibrium aggregate consumption in this exchange economy must equal the aggregate endowment. That is, in equilibrium we must have

$$\begin{aligned}
\sum_{m=1}^M c_{mt} &= \sum_{m=1}^M \left( \tilde{\rho}_m(t, \beta_t) + \sum_{j \in \mathcal{K} \cup \{0\}} \theta_{mjt} g_j(t, \beta_t) \right) \\
\sum_{m=1}^M \theta_{mjt} &= \eta_j \quad j \in \mathcal{K} \cup \{0\} \quad t \in [0, T]
\end{aligned} \quad (43)$$

$\eta_j$  for  $j \in \mathcal{K} \cup \{0\}$  being the net aggregate supply of the  $j$ th security. Yet, by Lemmas A.1 and C.1 the dividend specifications in (1) ensure that the summation inside the brackets on the right-hand side of (43) satisfies the AR growth condition (recall footnote 26). And since (36) requires that so does the first term inside the brackets, by Lemma C.1 again, the entire bracketed term on the right-hand side of (43) satisfies the AR growth condition. Clearly, so does the overall summation itself. That is, in equilibrium, the aggregate and, thus, also individual consumption flows do satisfy the AR growth condition.

Define then the pricing kernel  $m_t := \sum_{m=1}^M \tilde{\lambda}_m \partial u_m(t, c_{mt}) / \partial c$  for some fixed weights  $\tilde{\lambda}_m \in (0, 1)$  with  $\sum_{m=1}^M \tilde{\lambda}_m = 1$ . It is trivial to check that (35) implies that the function  $u_m$  satisfies the AR growth condition. Since the individual equilibrium consumption flows satisfy this condition as well, Lemmas C.3 and C.2 together ensure that the pricing kernel above satisfies the growth condition in remark IV of the main text. Which, given the flow dividend specification in (1), means of course that assumption A4 (and thus also its weaker version, A2) is satisfied.

For the case where  $T \in \mathbb{R}_{++}$  [resp.  $T = +\infty$ ] recall now Theorem 2.2 [resp. Corollary 2.1]. If  $\tilde{\Sigma}_0$  is non-singular, the financial market will be dynamically complete. Hence, the equilibrium pricing kernel will indeed be of the form surmised above. Furthermore, the matrix pre-multiplying  $\theta_{mt}$  on the left-hand side of (42) will be invertible and, thus, the typical  $\theta_{mt}$  well-defined everywhere on  $\Omega \times \mathcal{T}$ .

In fact, the typical portfolio will be locally bounded (recall footnote 31 in the main text) everywhere on  $[0, T) \times \mathbb{R}^{K+1}$ . This is because so is every other term in (42). By assumption, the first and second partial derivatives of the value function are continuous everywhere on  $[0, T) \times \mathbb{R}_+^{K+1}$ , while (being Lipschitz continuous) so are also the functions  $\alpha_j$  and  $\mathbf{B}_{jk}$  for any  $(j, k) \in \mathcal{K} \cup \{0\} \times \mathcal{K}$ . Moreover, the typical price  $P_j$  is everywhere continuous (with respect to time it is an integral of a continuous function, while with respect to the Brownian vector it is differentiable - recall Lemma A.7).<sup>43</sup> And as a result, so are the functions  $\tilde{\mathbf{B}}_{jk}$  again for any  $(j, k) \in \mathcal{K} \cup \{0\} \times \mathcal{K}$ . That is, the quantities defining the typical  $v_n$  are all everywhere continuous - apart possibly from  $W_t$ . Yet, it is obvious from the wealth dynamics in (39) that the latter quantity is locally bounded everywhere on  $[0, T) \times \mathbb{R}_+^{K+1}$ . Equally obviously, if the positions  $\theta_{mt}$  on all other securities are locally bounded, so must be by (38) the position on the zeroth one.

**Remark C.I** Strictly speaking, being the solution to the SDE (37),  $\{\mathbf{P}(\mathcal{I}(\cdot, t)) : t \in [0, T)\}$  (and, therefore, each of the endogenous variables above) is defined almost everywhere, not everywhere, on  $\Omega$ . In this sense, our result seems to be that the agents' portfolios are locally bounded a.e. on  $\Omega \times [0, T)$  (i.e., they are essentially locally bounded).<sup>44</sup> Yet, this overlooks the no-arbitrage assumption of Section 2 in the main text. Given the latter, even though there are many versions of the pricing process that agree with each other only a.e. on  $\Omega$ , there will be one that is defined everywhere. It is this version that the present paper investigates throughout.

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<sup>43</sup>The same argument establishes also that the typical derivative  $\mathbf{B}_{jk}$  is everywhere continuous. In this sense, the requirement for it to be Lipschitz continuous refers exclusively to the SDE that generates the equilibrium pricing process being “well-behaved.”

<sup>44</sup>Recall the definition in footnote 31 of the main text. The function  $f$  is said to be *essentially locally bounded* on  $V$  if the neighborhood  $\mathcal{B}_{\mathbf{x}_0}$  exists for almost all  $\mathbf{x}_0 \in V$ .