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## $S,s$ Pricing in a General Equilibrium Model with Heterogeneous Sectors<sup>\*</sup>

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### ABSTRACT

We study the impact of two-sided nominal shocks in a simple dynamic, general equilibrium  $(S,s)$ -pricing macroeconomic model comprised of heterogeneous sectors. The simple model we develop has a number of appealing empirical implications; it captures why some sectors of the economy have systematically more flexible prices, the smooth dynamics of aggregate output following a monetary shock, and a degree of price asynchronization. Incorporating multiple sectors is central to arriving at these three results.

**JEL Classification:** E31, E32, E37, E58.

**Keywords:** Price rigidity,  $(S, s)$  pricing, macroeconomic dynamics.

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## 1. Introduction

In this paper we study the impact of two-sided nominal shocks in a dynamic, general equilibrium macroeconomic model comprised of heterogeneous sectors where changing prices is costly. In our model, heterogeneity arises from two principal sources. First, the degree of complementarity (or substitutability) may differ across sectors on account of differing price or cost elasticities. Alternatively, or in addition, the costs of changing prices may differ systematically across different sectors. Recent empirical and theoretical work seems to indicate that both of these sources of heterogeneity may be important. On the one hand, there is evidence of marked differences in the frequency of repricing across different categories of goods, even against the backdrop of more or less low and stable inflation; see [Bils and Klenow \(2004\)](#). On the other hand, the analysis of [Gertler and Leahy \(2006\)](#) suggests that, absent sufficient strategic complementarity, state-dependant pricing models may imply that nominal shocks have an implausibly low impact on aggregate output dynamics. We incorporate both sorts of heterogeneity and analyze the significance of nominal shocks. We demonstrate that, even in the case when strategic complementarities are equal across the sectors, the systematic differences in costs of price adjustment are sufficient for nominal shocks to have a sizeable impact on aggregate output and prices. We are able, despite the apparent complexity of our set-up, to exploit certain fundamental properties of Markov processes to obtain analytical expressions for the stationary distributions of aggregate output and prices for the case of two sectors, and this allows us to calculate the correlation between output and monetary (i.e., nominal) shocks.

In addition to demonstrating how to construct analytically the aggregate ergodic distributions just mentioned, we show that our simple model economy has some qualitative features that we may observe in actual economies: It implies different sectoral responses to aggregate shocks; we observe smooth dynamics and Phillips curve-type behavior at the aggregate level; and we can explain a sizeable effect on output from demand shocks.

### 1.1. $(S, s)$ Pricing and Macroeconomic Dynamics

The development of theoretical models based on  $(S, s)$  pricing constitutes an influential and important line of thought going back to [Barro \(1972\)](#) and [Sheshinski and Weiss \(1977, 1983\)](#), analyses that were in turn motivated by the seminal work of [Arrow, Harris and Marschak \(1951\)](#). This work has been developed recently by, amongst others, [Caplin and Spulber \(1987\)](#), [Caplin and Leahy \(1991, 1997\)](#),

and more recently still by Dotsey, King and Wollman (1999), Danziger (1999), Golosov and Lucas (2003) and Gertler and Leahy (2006).

The basic idea behind these  $(S, s)$  pricing models is straightforward enough; firms face a resource cost of adjusting prices when demand or cost conditions alter. As a consequence, observed prices of almost all goods will differ from what would be the optimal price in the absence of this cost.

Some analyses in the  $(S, s)$  pricing tradition imply that monetary shocks have little impact on output (Golosov and Lucas, 2003) or none at all (Caplin and Spulber, 1987). However, these conclusions may be driven by particular features of the models employed.<sup>1</sup> But, in general it seems that the  $(S, s)$ -pricing general equilibrium model is as consistent with significant short-run monetary nonneutrality as the more popular time-dependent approach to modelling price stickiness (at least in low-inflation environments); see, for example, Klenow and Kryvstov (2005), Dotsey, King and Wollman (1999) and Gertler and Leahy (2006).

But there may be other stylized facts that these models have difficulty confronting. In practice price changes may not be highly synchronized (i.e., prices may often move in different directions across different sectors), as Bils and Klenow (2004) show for the US and Dhyne et al. (2004) demonstrate for countries in the Euro area, whilst many  $(S, s)$  pricing models generally suggest a high level of synchronization. And notably, there is evidence that there are *systematic* differences across sectors in the economy in the frequency of price adjustment; again Bils and Klenow (2004) document this for the US and Dhyne et al. (2004) show that the same is true in the Euro area.<sup>2</sup>

This observed degree of price change asynchronization and the systematic sectoral asymmetries suggest that heterogeneity is an important issue that needs to be incorporated in any successful  $(S, s)$  model, as Golosov and Lucas (2003) emphasize.

In this paper we make a start at introducing sectoral heterogeneity into a popular  $(S, s)$  model. Motivated by the observation that we just mentioned, that there appears to be some systematic variation in the frequency of price adjustment across goods, we analyze the effects of heterogeneous costs of adjustment. We

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<sup>1</sup>In the analysis of Golosov and Lucas (2003), Gertler and Leahy (2006) suggest that the lack of complementarity across goods may be key to understanding their finding of a muted impact of monetary shocks. As regards Caplin and Spulber (1987), the key is the assumption that demand shocks are always and everywhere positive. In our model we analyse a two-sided driving process for money and incorporate complementarity.

<sup>2</sup>Dhyne et al. also document that the Euro area appears to have "stickier" prices than is the case in the US.

extend the influential work of Caplin and Leahy (1997) in a number of important directions. First, we introduce consumers into the model with heterogeneous preferences across goods in different sectors; this enables us to track expenditure flows across sectors following a monetary shock and incorporate heterogeneous degrees of complementarity across sectors (making some sectors inherently more flexible-price sectors, regardless of the actual resource costs incurred in changing prices). We also introduce multiple sectors into the model economy, where each of these sectors is indexed by a different cost of price adjustment (in addition to potentially differing degrees of complementarity).

We are able analytically to characterize the stationary distributions of output and prices, at the sectoral and aggregate level, for our model economy. We also show that the stationary distribution of aggregate output ceases to be uniform, as it would be if there were only one sector in the economy, and instead becomes dependent on the number of sectors. These calculations are necessary when we analyze the economy-wide impact of nominal shocks.

Our model with multiple sectors may also deliver more plausible aggregate dynamics than a single sector model. The class of  $(S, s)$  models that we employ here, based on homogeneity of costs and preferences, tends to imply somewhat rigid dynamics; a sequence of positive (negative) monetary shocks causes output to rise (fall), while entailing no nominal price response, until some boundary is reached; further shocks in that positive (negative) sequence affect only prices. With heterogeneous costs of price adjustment and preferences, the aggregate dynamics are more nuanced. For example, we demonstrate that, in the stationary state, the correlation coefficient between money shocks and output initially rises in the variance of the money stock before falling. In fact, as Damjanovic and Nolan (2006) demonstrate, that is also the case in the single-sector model, in contrast to the arguments of Caplin and Leahy (1997). We also show that as we add heterogeneous sectors the correlation between money shocks and output is lower in the multiple sector case (compared with the single sector case) for relatively low monetary variance, and higher for relatively high monetary variance. Adding heterogeneous sectors in the way we do appears to help us understand why nominal shocks may be quite important. Further, a natural implication of our set-up is a degree of price asynchronization since in some sectors average real prices may be rising or falling relative to their counterparts in other sectors. However as we emphasize, one has to distinguish between nonstationary (i.e., out of long-run equilibrium) and stationary behaviour. In equilibrium, independently of the type of heterogeneity, there exists a natural order of timing of price adjustments. This

timing or, more accurately, asynchronization of price adjustments is defined in a unique way by the sequence of maximal admissible sectoral outputs.

In Section 2 we set out our framework and characterize optimal behavior of consumers and firms. In Section 3 we define equilibrium, analyse the impact of the various types of heterogeneity and examine the dynamics of prices and output at the sectoral level. In Section 4 we characterize optimal and equilibrium pricing behavior. We show that at both sectoral and aggregate levels the dynamics of price adjustment are intimately related to the type of heterogeneity present in the economy and show how asynchronization of price adjustment emerges. We construct the joint density for sectoral outputs and we obtain an explicit solution for the stationary distribution of aggregate output in the case of two sectors. In Section 5 we draw on that analysis to examine the interaction at the macroeconomic level between money, output and prices and demonstrate that our model economy may be sensitive to monetary shocks. In Section 6 we summarize and conclude.

## 2. The Model

Our model is a  $K$ -sector model building on the basic framework pioneered by Blanchard and Kiyotaki (1987). In each sector, indexed by  $s = 1, \dots, K$ , output is denoted by  $\tilde{y}$ ,<sup>3</sup> and is produced by a continuum of monopolistic competitors of measure one, indexed by  $j$ . Given a total expenditure level  $\tilde{E}$ , and prices of differentiated goods across both sectors and firms denoted by  $\tilde{P}_{j,s}$ , the representative consumer allocates expenditure shares optimally across goods and sectors. We will assume that the utility of the representative consumer is logarithmically separable across sectoral aggregates.<sup>4</sup> The objective of the representative consumer is:

$$\max[\ln(\tilde{u}_1) + \ln(\tilde{u}_2)], \quad (2.1)$$

subject to

$$\int \tilde{y}_{j,1} \tilde{P}_{j,1} dj + \int \tilde{y}_{j,2} \tilde{P}_{j,2} dj \leq \tilde{E}, \quad (2.2)$$

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<sup>3</sup>With some abuse of notation unless otherwise stated for all variables  $\tilde{\star}$  the following labelling will be used  $\star \equiv \ln(\tilde{\star})$ .

<sup>4</sup>Here, for simplicity, we choose logarithmic preferences and restrict ourselves to the case of a two sector economy. It will be apparent that any separable utility of the form  $u_1 + u_2$  with  $u'_i > 0$  and  $u''_i < 0$ ,  $i = 1, 2$ , will produce qualitatively similar dynamics at the aggregate level.

with

$$\tilde{u}_s = \int \tilde{y}_{j,s}^{1-\theta_s} dj,$$

where  $0 < \theta_s < 1$  and  $-\frac{1}{\theta_s}$  denotes the price elasticity of demand in sector  $s$ .

Define aggregated sectoral output  $\tilde{Y}_s$  and sectoral price indices  $\tilde{P}_s$  as:

$$\tilde{Y}_s \equiv \left[ \int \tilde{y}_{j,s}^{1-\theta_s} dj \right]^{1/(1-\theta_s)} \quad (2.3)$$

$$\tilde{P}_s \equiv \left[ \int \tilde{P}_{j,s}^{(\theta_s-1)/\theta_s} dj \right]^{\theta_s/(\theta_s-1)}. \quad (2.4)$$

As a result of the optimization problem (2.1)–(2.2), it follows first that the demand  $\tilde{y}_{j,s}$  for good  $j$  in sector  $s = 1, 2$ <sup>5</sup> is proportional to the sectoral output  $\tilde{Y}_s$ ,

$$\tilde{y}_{i,s} = \left( \frac{\tilde{P}_{i,s}}{\tilde{P}_s} \right)^{-\frac{1}{\theta_s}} \tilde{Y}_s, \quad (2.5)$$

and that (2.2) holds with equality. That is, total expenditure,  $\tilde{E}$ , is divided into sectoral expenditure  $\tilde{E}_s$  as

$$\tilde{E}_1 \equiv \tilde{Y}_1 \tilde{P}_1 = \frac{1 - \theta_1}{2 - \theta_1 - \theta_2} \tilde{E}, \quad (2.6)$$

$$\tilde{E}_2 \equiv \tilde{Y}_2 \tilde{P}_2 = \frac{1 - \theta_2}{2 - \theta_1 - \theta_2} \tilde{E}. \quad (2.7)$$

We turn now to price setting. We shall make two further assumptions. First, the velocity of money is normalized to unity so that total expenditure  $\tilde{E}$  is equal to nominal money balances  $\tilde{M}$ . Second, firms in each sector face an isoelastic production cost function of the form  $\omega c_0 \tilde{y}_j^\gamma / \gamma$ ,  $c_0 > 0$ ,  $\gamma > 1$  where  $\omega$  is the (constant) real wage. Then, defining relative prices and real sectoral expenditures (which are equal to sectoral demands) as

$$\tilde{p}_{j,s} \equiv \frac{\tilde{P}_{j,s}}{\tilde{P}_s}, \text{ and } \tilde{e}_s \equiv \left( \frac{1 - \delta_{s,1}\theta_1 - \delta_{s,2}\theta_2}{2 - \theta_1 - \theta_2} \right) \frac{\tilde{M}}{\tilde{P}_s}, \quad (2.8)$$

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<sup>5</sup>The generalization to  $K$  - sectors case is straightforward.

(here  $\delta$  stands for the Kronecker symbol  $\delta_{s,s'} = 0$  for  $s \neq s'$ ) equation (2.5) may be written as

$$\tilde{y}_{j,s}(\tilde{p}_{j,s}, \tilde{e}_s) = \tilde{e}_s \tilde{p}_{j,s}^{-1/\theta_s},$$

and the real profit for firm  $j$  in sector  $s$  is given by:

$$\begin{aligned} \tilde{\pi}_{j,s}(\tilde{p}_{j,s}, \tilde{e}_s) &= \tilde{p}_{j,s} \tilde{y}_{j,s}(\tilde{p}_{j,s}, \tilde{e}_s) - \frac{wc_0}{\gamma} \tilde{y}_{j,s}^\gamma(\tilde{p}_{j,s}, \tilde{e}_s); \\ &= \tilde{e}_s \tilde{p}_{j,s}^{1-1/\theta_s} - \frac{wc_0}{\gamma_s} \tilde{e}_s^{\gamma_s} \tilde{p}_{j,s}^{-\gamma_s/\theta_s}. \end{aligned}$$

As a consequence the profit-maximizing relative price is given by

$$\tilde{p}_{j,s}^* = B_s \tilde{e}_s^{\alpha_s}$$

where  $B_s = \left(\frac{wc_0}{1-\theta_s}\right)^{\theta_s/(\theta_s+\gamma-1)}$  is a constant and  $\alpha_s = \frac{\theta_s(\gamma_s-1)}{(\theta_s+\gamma_s-1)}$ ;  $0 < \alpha_s < 1$  since  $\theta_s < 1$  and  $\gamma > 1$ .

Now in log terms it follows that the price  $P_{j,s}^*$ , all  $j$ , is a linear combination of the price index and aggregate demand, that is,

$$P_{j,s}^* = b_s + P_s + \alpha_s Y_s, \quad (2.9)$$

where  $b_s \equiv \ln B_s$  and  $\alpha_s$  is a constant interpreted as a measure of strategic complementarity across firms. Equation (2.9) is equivalent to  $P_{j,s}^* = b'_s + (1-\alpha_s)P_s + \alpha_s M$  so that as  $\alpha \rightarrow 0$  firms  $j$  tends to raise its nominal price more as  $P_s$  rises i.e., when others firms raise their prices. In a frictionless world all firms adjust prices continuously to keep their profits at the optimum and they all charge the same price equal to  $P_s$ . It then follows from (2.9) that the optimal equilibrium level of  $Y_s^*$ <sup>6</sup> is given by  $Y_s^* = -b_s/\alpha_s$  so that the optimal nominal price (2.9) of firm  $j$  in sector  $s$  in the absence of adjustment costs can be written as a linear combination of the (log of the) sectoral price index and the (log of) excess demand in that sector:

$$P_{j,s}^* = P_s + \alpha_s(Y_s - Y_s^*). \quad (2.10)$$

Menu costs imply that firms will not continuously adjust nominal prices. So, we approximate the instantaneous loss in real profits to firm  $j$  to be quadratic in the deviation from target,  $P_{j,s}^*$ . Hence,

$$L_{j,s}(t) = g(P_{j,s} - P_{j,s}^*)^2 = g[x_{j,s}(t) - \alpha_s Y_s(t)]^2, \quad (2.11)$$

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<sup>6</sup>Without confusion we may speak either of the equilibrium level of demand or real money balances as they differ only by constant factor.

where  $x_{j,s} = P_{j,s} - P_s$  is the firm's relative price,  $Y_s$  stands now for excess demand and  $g$  is a positive constant.<sup>7</sup> From now on, where we can safely do so, we suppress the index  $j$  and further distinguish between different sectors of the economy by the costs of price adjustment in each sector,  $C_s$ . We assume that these costs are 'sufficiently different' across sectors such that when one sector starts to change nominal prices, this does not immediately cause firms in 'nearby' sectors to change their prices.

The dynamic relationship between real money balances,  $M(t) - P_s(t)$ , and (demand-determined) sectoral output is governed by the equations,

$$dY_s(t) = dM(t) - dP_s(t). \quad (2.12)$$

The above equations easily follow from equations (2.6) and (2.7). We assume that the money supply evolves continuously following a driftless Brownian motion with infinitesimal variance  $\sigma^2$ ,

$$dM(t) = \sigma dW(t), \quad (2.13)$$

where  $W(t)$  is a Wiener process.

Since the optimal frictionless price in each sector depends only on the sectoral output and price index, firms in each sector face a control problem of the same sort; that is, they must choose a policy  $\pi$  of adjusting their nominal prices so as to minimize the expected present value of lost profit given the cost of price adjustment,  $C$ . Assuming that the discount factor  $r$  is constant, the value function of any firm  $j$  in sector  $s$  at moment  $t$  can be expressed in the form:

$$V_s = \min_{\pi} \left\{ E_t \int_t^{\infty} e^{-r\tau} g [x_s(\tau) - \alpha_s Y_s(\tau)]^2 d\tau + \sum_i e^{-rT_i} C_s \right\}, \quad (2.14)$$

where  $E_t$  stands for the expectation operator and the sequence  $\{T_i\}$  represents the time when the  $i^{th}$  adjustment takes place.

## 2.1. The optimal policy

The optimal policy  $\pi$  determines how firms divide the impact of changes in demand (the money supply) between their output and prices. Clearly, in deciding when and in what amount to change their nominal prices, firms have to keep track

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<sup>7</sup>From now on when we use term sectoral demand and/or sectoral output we mean deviations from their equilibrium values.



of both how their nominal price differs from the optimal frictionless price and aggregate output in their own sector. This implies the existence of a pair of admissible boundaries for both variables  $x$  and  $Y$ . Assume that in sector  $s$  at instant  $t$  the firm's relative price takes some lower admissible value,  $-S_s$ , and that the output in that sector is at its maximal possible level  $+\bar{Y}_s$ . Then from (2.11) it follows that the firm's instantaneous loss is at its maximal possible level  $g[S_s + \alpha_s \bar{Y}_s]^2$ . We assume that the firm then raises its relative price to  $S_s$ , that is changes it by  $2S_s$ , to be as far as possible from  $-S_s$ . Symmetrically when the firm's relative price takes the value  $+S_s$  and output is at its lowest level  $-\bar{Y}_s$ , the firm changes its relative price by  $-2S_s$ . For all other values of relative prices and aggregate sectoral output the firm does not change its nominal price. Assuming that initially demand  $Y_s$  is in between some boundaries  $[-\bar{Y}_s, \bar{Y}_s]$  and that the initial distribution of relative prices is uniform, then it can be shown (Caplin and Leahy (1997) and Stokey (2006)), that the distribution of relative prices remains uniform on  $[-S_s, +S_s]$  and that  $Y_s$  follows a Brownian motion regulated at  $\pm \bar{Y}_s$ . The optimal policy, then, is to find such values for  $\bar{Y}_s$  and  $S_s$  so that (2.14) holds. The optimality of such a policy in a one sector economy is established in Caplin and Leahy (1997) and Stokey (2006).

## 2.2. Value function and optimal boundaries

At any point in time, total expenditure is divided between the sectors in constant proportions, we use the expression for the value function given in Caplin and Leahy (1997) and write the solution of (2.14) for sector  $s$ :

$$\begin{aligned}
V_s(x, Y) = & \frac{g}{r}(x - \alpha_s Y)^2 + g \left[ \frac{\alpha_s \sigma}{r} \right]^2 + \frac{2g}{r} \frac{\alpha_s - 1}{(e^{2\beta \bar{Y}_s} - e^{-2\beta \bar{Y}_s})} \times \\
& \left\{ \left( \frac{x - \alpha_s \bar{Y}_s}{\beta} - \frac{1}{\beta^2} \right) e^{\beta \bar{Y}_s} e^{\beta Y} - \left( \frac{x + \alpha_s \bar{Y}_s}{\beta} - \frac{1}{\beta^2} \right) e^{-\beta \bar{Y}_s} e^{\beta Y} \right. \\
& \left. - \left( \frac{x + \alpha_s \bar{Y}_s}{\beta} + \frac{1}{\beta^2} \right) e^{\beta \bar{Y}_s} e^{-\beta Y} + \left( \frac{x - \alpha_s \bar{Y}_s}{\beta} + \frac{1}{\beta^2} \right) e^{-\beta \bar{Y}_s} e^{-\beta Y} \right\} \\
& - \kappa(e^{\beta x} e^{-\beta Y} + e^{-\beta x} e^{\beta Y}). \quad (2.15)
\end{aligned}$$

with  $Y \in [-\bar{Y}_s, \bar{Y}_s]$  and  $x \in [-S_s, S_s]$ . Optimal boundaries are uniquely determined by the following relations<sup>8</sup>,

$$\alpha_s \bar{Y}_s + \frac{(1 - \alpha_s)}{\beta} \tanh(\beta \bar{Y}_s) = S_s \coth(\beta S_s) \tanh(\beta \bar{Y}_s); \quad (2.16)$$

$$S_s(\beta \bar{Y}_s - \tanh(\beta \bar{Y}_s)) = \delta_s C_s, \quad (2.17)$$

where  $S_s$  is the upper bound of relative prices,  $\bar{Y}_s$  is the upper bound of total sector  $s$  output, and where  $\delta_s = r\beta/(4\alpha_s g)$ , and  $\beta = \sqrt{2r/\sigma^2}$  are constants.

### 3. Sectoral Dynamics

#### 3.1. Heterogeneities

Before we analyze the distribution of relative prices and output at the economy-wide level it is useful to examine more closely equations (2.16) and (2.17) as they contain information about various types of heterogeneity that are present in our model economy. It turns out that regardless of the type of heterogeneity the dynamics of aggregate output are essentially the same. What is more significant, the timing (that is, the degree of synchronization/asynchronization) of price adjustment is solely determined by the *sectoral* outputs. Moreover, again regardless of the nature of heterogeneity, the distribution of relative prices does not change through time, although it ceases to be uniform (as it is in the single sector set-up).

In the set-up we have presented, there are three differing scenarios reflecting heterogeneity. First, we may consider the case when the price-elasticity of demand is equal across sectors,  $\theta_1 = \theta_2 \equiv \theta$ , but where we allow  $C_s$  to differ,  $C_1 \neq C_2$ . We label this  $(C)$ -type heterogeneity. The second, which we label  $(\alpha)$ -type, is when costs of price adjustments are identical  $C_1 = C_2 \equiv$ , but where  $\theta_1 \neq \theta_2$  so that there are differences in strategic complementarity,  $\alpha_s$ , across sectors. The third case is  $(C, \alpha)$ -type heterogeneity when both costs and the degree of strategic complementarity differ across sectors. In what follows we shall index sectors either by  $C$ ,  $\alpha$  or  $(C, \alpha)$  indicating the type of heterogeneity. Also we restrict our analysis to the positive values of optimal boundaries. The symmetry of equations (2.16) and (2.17) in respect of  $S$  and  $\bar{Y}$  means that one simply reverses the sign to extend the analysis to the lower boundaries.

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<sup>8</sup>These equations are the same as equations (26) and (27) in Chapter 7 of Stokey (2006); we have rewritten them in a slightly different, and for our purposes more convenient, form.

### 3.1.1. $(C)$ -type

In this case in each sector  $s$  the behavior of firms is defined by their cost of price adjustment  $C$ . The next Lemma characterizes the optimal boundaries  $\bar{Y}(C)$  and  $S(C)$ .

**Lemma 3.1.** *The sequences of optimal boundaries  $\{S(C)\}$  and  $\{\bar{Y}(C)\}$  are strictly increasing functions of cost with  $S(0) = \bar{Y}(0) = 0$ .*

**Proof.** Rewrite equation (2.16) as:

$$\frac{\beta S \coth(\beta S) - 1}{\beta \bar{Y} \coth(\beta \bar{Y}) - 1} = \alpha. \quad (3.1)$$

Now, consider equation (2.17) and let  $C$  increase. It follows that the left hand side of (2.17) must also increase. Consequently, there are three possibilities: Both  $S$  and  $\bar{Y}$  increase;  $S$  increases, while  $\bar{Y}$  decreases or  $\bar{Y}$  increases, and  $S$  decreases. It turns out that only the first of these possibilities is consistent with condition (3.1). When  $C = 0$  (2.16) and (2.17) become  $\beta S = \tanh(\beta S)$  and  $\beta \bar{Y} = \tanh(\beta \bar{Y})$  with solutions  $S(0) = \bar{Y}(0) = 0$ . The solutions of (2.16) and (2.17) for various costs are depicted in Figure 3.1. ■

### 3.1.2. $(\alpha)$ -type

The next Lemma characterizes the behavior of optimal boundaries in terms of strategic complementarity.

**Lemma 3.2.** *Let  $\bar{Y}_s$  and  $S_s$  be the solutions to (2.16) and (2.17) and let  $C_s \equiv C$  be fixed and equal across sectors. Then, the following statements are true:*

- a)  $\bar{Y}_s$  is a decreasing, and  $S_s$  is an increasing, function of  $\alpha_s$ ;
- b) For  $\alpha_s = 0$ ,  $S_s(0)$  is finite and  $\bar{Y}_s(0) = \infty$ ;
- c) For all  $\alpha_s \in [0, 1)$  we have  $\bar{Y}_s(\alpha_s) > S_s(\alpha_s)$  with equality when  $\alpha_s = 1$ .<sup>9</sup>

**Proof.** From (3.1) due to the monotonicity of the function  $x \coth(x)$ , statement c) follows immediately. Combine (3.1) and (2.17) to get the equation  $e^{2\beta \bar{Y}} = (1 + \Psi(S))/(1 - \Psi(S))$  where  $\Psi(S) = \bar{g}/\beta S(\beta S \coth(\beta S) - 1)$  and  $\bar{g} = r\beta C/(4g)$ . For the solution to exist it must be the case that  $\Psi(S) \leq 1$  which is equivalent

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<sup>9</sup>In the case of strategic substitutability  $\alpha_s \in (1, +\infty)$ , and the inequality is reversed,  $\bar{Y}_s(\alpha_s) < S_s(\alpha_s)$ .

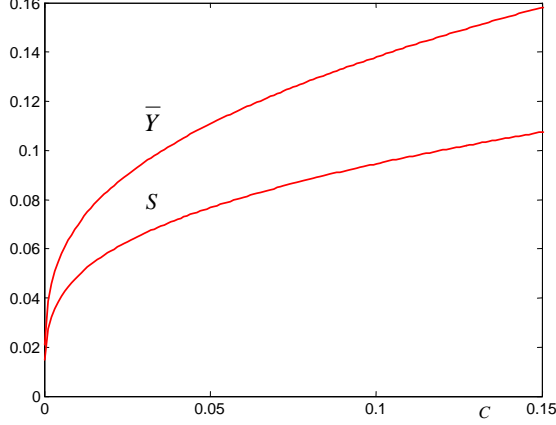


Figure 3.1: Optimal boundaries  $S$  and  $\bar{Y}$  as a function of the cost of price adjustments,  $C$ .

to  $\beta S(\beta S \coth(\beta S) - 1) \geq \bar{g}$ . The previous inequality is satisfied for all  $S \geq S(0)$ , where  $S(0)$  is a positive solution of the equation  $\Psi(S) = 1$ ; i.e.,  $\beta S(\beta S \coth(\beta S) - 1) = \bar{g}$ . The solution is unique and exists for all  $\bar{g}$  since the left side is a monotonic increasing function starting from zero. Clearly, when  $S = S(0)$ ,  $\bar{Y} = \infty$ . Now from (2.17) it follows that this pair of solutions corresponds to the case of  $\alpha_s = 0$ . This establishes b). To establish statement a) first note that solutions  $\bar{Y}_s$  and  $S_s$  are monotonic functions of  $\alpha_s$  which is easily seen from (3.1) and (2.17). Then it follows immediately that  $\bar{Y}_s$  is a decreasing function of  $\alpha_s$ . To show the second part of statement a) it suffices to show that there exists at least one  $\alpha_s > 0$  such that  $S_s(\alpha_s) > S_s(0)$ . Choose  $\alpha_s = 1$ . Then  $S(1)$  is a solution of  $\beta S(\beta S - \tanh(\beta S)) = \bar{g}$ . Now  $S(1) > S(0)$  follows from the fact that  $\beta S(\beta S - \tanh(\beta S)) < \beta S(\beta S \coth(\beta S) - 1)$  which is equivalent to  $\tanh(\beta S) < 1$ . ■

### 3.1.3. $(C, \alpha)$ – type

The previous analysis and results carry over to the more general  $(C, \alpha)$  – type of heterogeneity with the only difference that we now have increasing (decreasing) sequences  $\{\bar{Y}_s\}$  and  $\{S_s\}$ . That is, there exists  $C_s, \alpha_s, C_{s'}, \alpha_{s'}$  such that either  $S_s = S_{s'}$  or  $\bar{Y}_s = \bar{Y}_{s'}$ . To summarize, for all  $\alpha > 0$  and  $C > 0$  the sequence

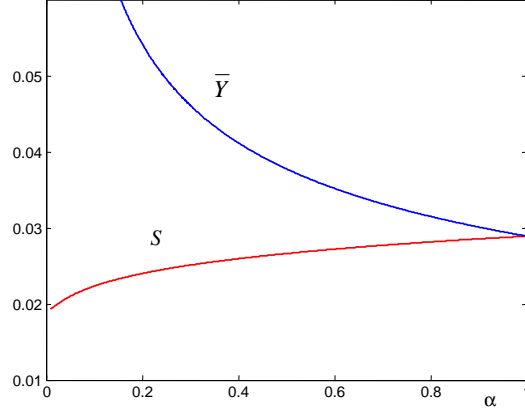


Figure 3.2: Optimal boundaries  $\bar{Y}$  and  $S$  as a function of strategic complementarity,  $\alpha$ .

$S(C, \alpha)$  is an increasing function of both  $C$  and  $\alpha$ , whereas  $\bar{Y}(C, \alpha)$  increases when  $C$  rises but falls when  $\alpha$  rises.

### 3.2. Equilibrium and Dynamics of Relative Prices and Outputs

Whilst the driving process impacts both sectors in a symmetric manner and hence, one might think, is a force for synchronization of decisions across sectors, we have seen that the firms' problems are essentially sectoral affairs; in solving its optimization problem a firm need only have regard to sectoral aggregates. In this section we examine how this tension is resolved and define equilibrium.

A firm's optimal strategy implies that the distribution of relative prices in each sector is uniform at any point in time. Since the firm adjusts its nominal price only when sectoral output is at its boundaries, it follows that the distribution of relative prices,  $\mu$ , depicted by the bold line in Figure (3.3) moves up and down in  $(S, \bar{Y})$  space in the direction of the money supply. In that case all changes in the money supply feed into changes in output. When both sectoral outputs differ from their extremes, no firm adjusts its price and its action is fully synchronized in the sense that all firms either increase or decrease output when money rises or falls. However this synchronization happens only at random time intervals when output reaches the boundary.

It is clear that, in general, neither price changes nor production decisions are synchronized across sectors. For simplicity, consider the two sector case with different costs of price adjustment and assume that at each consecutive time step  $\Delta t$  money falls by an amount,  $\Delta M$ . Then the "elevator" (that is, the distribution of relative prices) in the first sector (with the lower cost of adjustment) will hit the boundary  $-\bar{Y}_1$  first. Assume further that money continues to fall. Then with each consecutive time step,  $\Delta t$ , a constant fraction ( $\Delta M/2S_1$ ) of firms in sector 1 will adjust their nominal prices while the elevator ( $\mu_2$ ) in the second sector will continue to travel down towards boundary  $-\bar{Y}_2$  (see Figure 3.2). The situation where firms facing the higher cost of adjustment change nominal prices but not firms from the lower cost sector ( $\mu_2$  hits its own boundary first) is, of course, possible but only for particular paths defined on specific initial conditions; it can only happen a finite number of times. However once the aggregates  $Y_1$  and  $Y_2$  fall into the set of their stationary values that possibility is ruled out, as will be shown in the next section. In the stationary state, it cannot be the case that the higher cost sector adjusts but the lower cost sector does not. So in our model synchronization/asynchronization of relative prices is understood as follows: When some firms from the sector with cost  $C^*$  adjusts then in all sectors with costs  $C < C^*$  some fraction of firms also adjust. Identical reasoning applies to the case with sectors with equal costs of price adjustment but differing degrees of strategic complementarity, as well as to the general  $(C, \alpha)$  - type of heterogeneity.

It is important to realize that in all cases the synchronization/asynchronization of price adjustments is entirely guided by sectoral outputs. Also note that we can always order the sequence  $\{\bar{Y}_s(C_s, \alpha_s)\}$  in ascending order to get the new sequence  $\{\bar{Y}_{s'}\}$ . Then if some firms in sector  $s^*$  change their relative prices then at the same time in each sector  $s' < s^*$  some firms change their relative prices too.

Overall, then, our model predicts that the frequency of price changes is likely to differ systematically across sectors, that these changes will not be synchronized and that prices are as likely to be flexible downwards as upwards.<sup>10</sup>

We conclude this section with some observations regarding the fraction of

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<sup>10</sup>These findings appear to be broadly in line with Dhyne et al. (2004) where a high degree of heterogeneity in price setting behaviour, across both products and sectors, is clearly documented. The heterogeneity is reflected in the frequency of price setting and in the absence of price change synchronization across different sectors. Interestingly, Dhyne et al. also document that there is no evidence of strong downward price rigidity in the Euro area. On average 40% of price changes are downward movements. Our model with symmetric and nested boundaries is hence capable of generating such behaviour.

firms that adjust their nominal prices. In general our economy is characterized by periods of no price adjustment followed by periods when some fraction of firms adjust. Note that the fraction of firms which adjust is not constant but changes in time. In our example with two sectors at any point in time the fraction of firms which adjust can take one of three possible values; either zero when no one adjusts ( $Y_1 \neq \pm \bar{Y}_1$  and  $Y_2 \neq \pm \bar{Y}_2$ ) or  $|\Delta M|/2S_1$  when  $Y_1 = \pm \bar{Y}_1$  and  $Y_2 \neq \pm \bar{Y}_2$  or  $|\Delta M|(1/2S_1 + 1/2S_2)$  when  $Y_1 = \pm \bar{Y}_1$  and  $Y_2 = \pm \bar{Y}_2$ . Can we have a situation where at every point of time a constant fraction of firms change their relative price? The answer is positive and is related to strategic complementarity. Recall that a lower  $\alpha$  means that a firm will change its nominal price more when others do so. When  $\alpha = 0$  the optimal nominal price is equal to the price index so the only criterion for the firm is how far its nominal price is from its optimal value. In this case we know from Lemma (3.2) that  $\bar{Y}(0) = \infty$ , which in fact means the absence of any boundary for output. Then, because money changes continuously, at any point of time we have that a constant fraction of firms  $|\Delta M|(1/2S_1 + 1/2S_2)$  changes their nominal prices<sup>11</sup>.

We conclude this section by defining an equilibrium.

**Definition 3.1:** *An equilibrium is an increasing sequence of optimal boundaries  $\{\bar{Y}_s\}$ , a sequence of optimal boundaries  $\{S_s\}$ , an initial distribution of prices, and a set of pricing strategies such that:*

- (i) *in each sector  $s$  relative prices are distributed uniformly over the interval  $[-S_s, +S_s]$ ;*
- (ii) *in each sector  $s$  output is distributed uniformly over the interval  $[-\bar{Y}_s, +\bar{Y}_s]$ ;*
- (iii) *in each sector firms change their nominal prices only when sectoral output is at  $\pm \bar{Y}_s$  and their relative prices are at  $\mp S_s$ ;*
- (iv) *when the output of sector  $s$  is at  $\pm \bar{Y}_s$  then in all sectors  $s' < s$  sectoral outputs are at  $\pm \bar{Y}_{s'}$  and the aggregate price index changes by an amount of  $s\Delta M$ .*
- (v) *no firm deviates from the price adjustment strategy in (iii).*

## 4. Aggregate Dynamics

In a one sector version of the above model with two sided shocks, we know that relative prices will optimally remain uniformly distributed, if they are initially uniformly distributed. We also know that in each sector total output will be

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<sup>11</sup>There is a similarity here with the Calvo model where also a constant fraction of firms changes its nominal price each period.

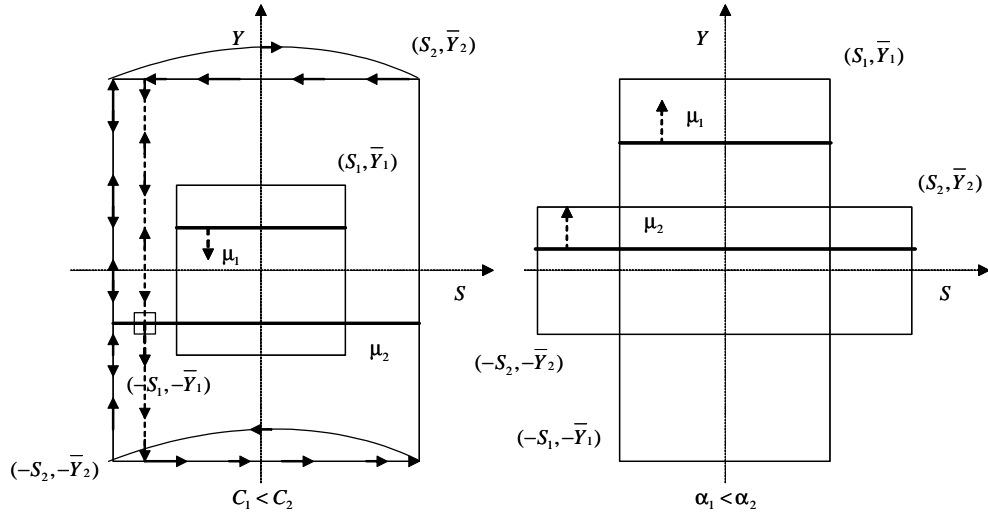


Figure 3.3: Optimal boundaries for the two sector model. The left-hand picture represents nested innaction regions for firms in both sectors having equal strategic complementarity. The arrowed line with jumps  $(-S_2, \bar{Y}_2) \rightarrow (S_2, \bar{Y}_2)$  and  $(S_2, -\bar{Y}_2) \rightarrow (-S_2, -\bar{Y}_2)$  represent one set of possible paths for the firm in sector 2 denoted by  $\square$  in the left-hand picture, returning to the initial state. The case of different complementarities is presented on the right-hand side picture. The only difference is that regions of innactions are not nested.



uniformly distributed in the stationary state.<sup>12</sup> Here we want to address two issues regarding the dynamics of relative prices and aggregate output. The first is to characterize the distributions of relative prices in the whole economy. The second is to characterize the distribution of aggregated sectoral outputs. We start with the first of these where the answer is more or less straightforward.

#### 4.1. The distribution of relative prices

From the fact that the measure of relative prices within each sector remains uniform it follows immediately that the distribution of relative prices within the sectors is simply obtained as a sum of sectoral measures. Consider again the case of two sectors each of measure one. Since in each sector the same rule of price adjustment applies firms of measure  $1/2$  are uniformly distributed over an interval of relative prices  $[-S_1, S_1]$  and so too are the firms (of measure  $1/2$ ) over the interval  $[-S_2, S_2]$ . We choose  $S_1 < S_2$ . The distribution of relative prices across the sectors is not uniform (as it would be in a single sector set-up) but it is time-invariant as depicted in Figure 3.3. It is not hard to see that the distributional shape is defined by the increasing sequence of optimal boundaries  $\{S_s\}$ . Recall that the timing of price adjustment is determined by the sequence  $\{\bar{Y}_s\}$ . The corresponding densities for the case of two and five sectors are depicted in Figure 3.3. By way of illustration, it is straightforward algebraically to describe the densities of relative prices for the two sector economy:

$$\mu_2(x) = \begin{cases} 1/2 (2S_2) & -S_2 \leq x < -S_1; \\ 1/2 (2S_1 + n_2/2S_2) & -S_1 \leq x \leq S_1; \\ 1/2 (2S_2) & S_1 < x \leq S_2. \end{cases} \quad (4.1)$$

The generalization of this analysis to  $K$  sectors is straightforward and we conclude this section by formulating two propositions for the cases of discrete and continuous costs of price adjustment <sup>13</sup>.

**Proposition 3.2.** *Let  $C$  take values from a discrete and strictly increasing set  $\{C_1 < C_2 < \dots < C_K\}$  Then for any relative price  $x_s \in [-S_s, -S_{s-1}) \cup (S_{s-1}, S_s]$ ,*

<sup>12</sup>Harrison (1985) proves that the stationary distribution of a regulated Brownian motion is uniformly distributed.

<sup>13</sup>Identical propositions apply for any type of heterogeneity as we will always have an increasing sequence of optimal boundaries  $\{S_s\}$ .

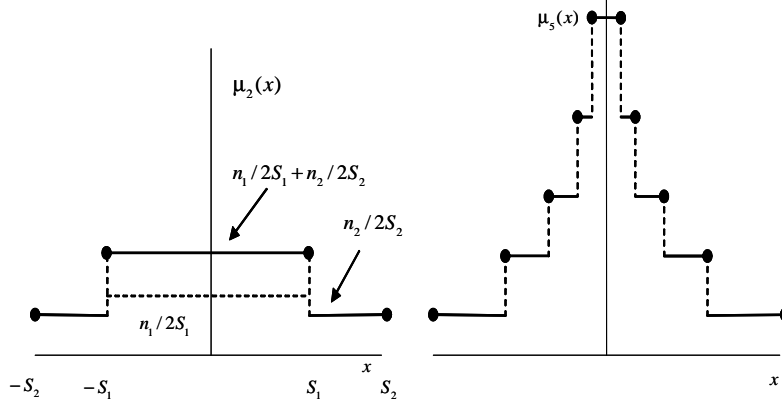


Figure 4.1: The left-hand picture represents the density of relative prices  $\mu_2(x)$  (bold line) in the case of two sectors defined by (4.1). The dashed horizontal line represents sectoral density  $1/2(2S_1)$  and  $1/2(2S_2)$  is a continuous line between  $-S_2$  and  $S_2$ ). The right-hand picture depicts the density  $\mu_5(x)$ .

$s = 1, 2, 3, \dots, K$  the density  $\mu(x)$  is given by:

$$\mu(x_s) = \frac{1}{K} \sum_{i=s}^K \frac{1}{2S_i}. \quad (4.2)$$

**Proposition 3.3.** *Let  $C$  take values from some bounded and continuous set i.e.,  $C \in [\underline{C}, \overline{C}]$ . Then for any relative price  $x \in [-S(\overline{C}), S(\overline{C})]$  the density is given by:*

$$\mu(x) = \int_x^{S(\overline{C})} \frac{d\zeta}{S(\zeta)}. \quad (4.3)$$

## 4.2. The distribution of aggregate output

Before we can analyze how sensitive our model economy is to nominal shocks we need to characterize the distribution of aggregate output. In this subsection we consider the dynamics of aggregate output  $Y = \sum Y_s$  where each variable  $Y_s$  is uniformly distributed over its own domain  $[-\overline{Y}_s, \overline{Y}_s]$  and the sequence  $\{\overline{Y}_s | s = 1, \dots, K\}$  is increasing. The result derived in this section holds for any type of heterogeneity but for concreteness consider  $(C)$ -type. In the one-sector economy

output is uniformly distributed. In the multiple sector case, this is no longer true. Intuitively, although firms in different sectors have different costs of price adjustment, there will still be some range over which their respective outputs rise and fall together. Eventually, however, demand rises sufficiently that some firms start adjusting nominal prices, whilst others continue to meet demand at their current posted prices. At this point, their supply responses diverge. It turns out that the stationary distribution of aggregate output can, therefore, be split into two parts. One part corresponds to the uniform distribution of absolutely correlated outputs. The other part of the distribution represents the sum of independent random variables. In terms of the aggregate dynamics of output following a monetary shock, this is an important result which, as we show in an appendix, is generalizable.

We calculate the stationary distribution of aggregate output in the following way. First, we employ a state-space discretization and define the space of joint stochastic processes for sectoral outputs  $(Y_1(t), Y_2(t))$ . Knowing transition probabilities between the different states, we then find stationary occupancy measures of states  $(Y_1, Y_2)$  by solving the corresponding vector Markov equation. Finally, taking the continuous limit we calculate the stationary distribution of aggregate output as the sum of two dependent random variables,  $Y_1$  and  $Y_2$ .

Let time take equidistant discrete values i.e.,  $t = 0, \Delta t, 2\Delta t, \dots, N\Delta t$ , and let all stochastic variables change by small discrete amounts. We first define output  $Y(t)$  as a regulated Brownian motion in terms of the underlying driving process,  $M(t)$ , which follows a simple random walk on a lattice<sup>14</sup>.

**Definition 4.1.** *We say that the process:*

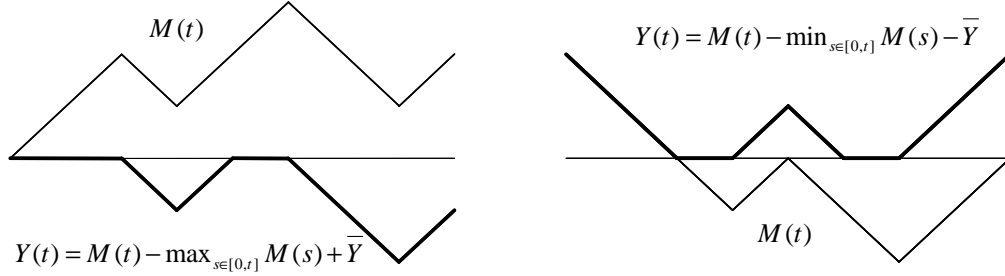
$$Y(t) = M(t) + L(t) - U(t),$$

*is a regulated Brownian motion if for all  $t \geq 0$  the following properties hold: i)  $Y(t) \in [-\bar{Y}, \bar{Y}]$ ; ii) regulators  $L(t)$  and  $U(t)$  are nondecreasing stochastic processes, with the following properties  $L(0) = U(0) = 0$  and iii)  $L(t)$  increases only when  $Y(t) = -\bar{Y}$ , and  $U(t)$  increases only when  $Y(t) = \bar{Y}$ .*

The role of regulators  $L, U$  is to keep output at the level  $Y = -\bar{Y}(+\bar{Y})$  when money further decreases (increases). Then, at the next instant of time, output

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<sup>14</sup>This means that  $\Delta M(t) := M(t) - M(t-1)$  takes value  $+\delta$  with probability  $1/2$  or  $-\delta$  with the same probability. To match driftless Brownian motion with infinitesimal variance  $\sigma^2$  in the continuous limit,  $\Delta t = T/N \rightarrow 0$ , we require that  $T\sigma^2 = N\delta^2$ . In addition we assume that  $M(0) = 0$ .



will stay at the boundary with positive probability. To see that more clearly we construct processes  $L, U$  in terms of the exogenous process  $M(t)$ . For what follows it suffices to define  $L$  and  $U$  in the presence of only an upper or lower boundary<sup>15</sup>. We formulate two auxiliary lemmas.

**Lemma 4.1.** *Let  $Y(0) \geq -\bar{Y}$  and define the stopping time  $T_{-\bar{Y}}$  as the first time when  $Y(t) = -\bar{Y}$ . Then the process  $Y(t) = M(t) + L(t)$ , where*

$$L(t) = \begin{cases} 0 & t \leq T_{-\bar{Y}} \\ -\bar{Y} - \min_{s \in [0,t]} M(s) & t > T_{-\bar{Y}} \end{cases}$$

*satisfies properties i) – iii) from Definition 4.1.*

**Lemma 4.2.** *Let  $Y(0) \leq \bar{Y}$  and define the stopping time  $T_{+\bar{Y}}$  as the first time when  $Y(t) = +\bar{Y}$ . Then the process  $Y(t) = M(t) - U(t)$ , where*

$$U(t) = \begin{cases} 0 & t \leq T_{+\bar{Y}} \\ \max_{s \in [0,t]} M(s) - \bar{Y} & t > T_{+\bar{Y}} \end{cases}$$

*satisfies properties i) – iii) from Definition 4.1.*

The proofs of these lemmas are straightforward; see Figure 4.1 where two cases with upper and lower barriers are depicted.

Consider now two sectors of the economy with different costs of price adjustment. As we already stated, in each sector firms face the same optimization problem and in each sector output follows a regulated Brownian motion but because the underlying money process is the same for both sectors we would anticipate that sectoral outputs are dependent random variables. The next proposition establishes the functional relation between sectoral outputs and also defines the state-space for joint process  $(Y_1(t), Y_2(t))$ .

<sup>15</sup>In the presence of two barriers the same constructions still apply but one has to keep track of consecutive sequences of stopping times between the barriers. To keep things simple we omit it. For more details see Stokey (2006).

**Proposition 4.1.** *Let*

$$Y_1(t) \in [-\bar{Y}_1, \bar{Y}_1] \text{ and } Y_2(t) \in [-\bar{Y}_2, \bar{Y}_2] \quad (4.4)$$

*all  $t$ , be two regulated Brownian motions with the same underlying driving process  $M(t)$  as in Definition 4.1 and  $\bar{Y}_1 < \bar{Y}_2$ . Then, for all  $t \geq 0$ , we have*

$$Y_2(t) = Y_1(t) + A, \quad (4.5)$$

*where  $A \in [-\bar{Y}_2 + \bar{Y}_1, \bar{Y}_2 - \bar{Y}_1]$  is a random variable changing its value with probability  $1/2$  to  $A + \Delta M$  or  $A - \Delta M$  when  $Y_1 = +\bar{Y}_1$  and money rises, or when  $Y_1 = -\bar{Y}_1$  and money further falls, respectively.*

*Proof:* Consider first the case with two upper boundaries  $\bar{Y}_1 < \bar{Y}_2$ . Without loss of generality assume that  $Y_1(0) = Y_2(0) = M(0) = 0$ . By Definition 4.1 and Lemma 4.2 we have:

$$Y_1(t) = \begin{cases} M(t) & t \leq T_1 \\ \bar{Y}_1 + M(t) - \max_{s \in [0, t]} M(s) & t > T_1 \end{cases} \quad (4.6)$$

and similarly

$$Y_2(t) = \begin{cases} M(t) & t \leq T_2 \\ \bar{Y}_2 + M(t) - \max_{s \in [0, t]} M(s) & t > T_2 \end{cases} \quad (4.7)$$

where  $T_1$  and  $T_2$  are stopping times for processes  $Y_1$  and  $Y_2$  respectively. Then, by (4.6) and (4.7), it follows that for all  $t \leq T_1$ ,  $Y_1(t) = Y_2(t) = M(t)$ . As money rises and falls the point  $(Y_1, Y_2)$  moves along 45% degree line starting from the origin with  $A = 0$ . At  $t = T_1$  we have  $Y_1(T_1) = Y_2(T_1) = \bar{Y}_1$ . In the next instant of time money can fall or rise with equal probability. If money falls then the movement is along the same 45% degree line; that is,  $(\bar{Y}_1, \bar{Y}_1) \rightarrow (\bar{Y}_1 - \Delta M, \bar{Y}_1 - \Delta M)$ . If money rises then the point  $(\bar{Y}_1, \bar{Y}_1)$  moves up to  $(\bar{Y}_1, \bar{Y}_1 + \Delta M)$  transferring movement of  $(Y_1, Y_2)$  onto the new 45% line and rising  $A$ . For  $T_1 < t < T_2$  we have  $Y_1(t) = \bar{Y}_1 + M(t) - \max_{s \in [0, t]} M(s)$  and  $Y_2(t) = M(t)$  or  $Y_2(t) = Y_1(t) - \bar{Y}_1 + \max_{s \in [0, t]} M(s)$ . As it is still the case that  $\max_{s \in [0, t]} M(s) < \bar{Y}_2$ , we have  $Y_2(t) < Y_1(t) + \bar{Y}_2 - \bar{Y}_1$ . But whenever we are initially below the line  $Y_2 = Y_1 + \bar{Y}_2 - \bar{Y}_1$ , eventually monetary growth will be such that, after some time, we shall arrive at  $\bar{Y}_2$ . Therefore for all  $t \geq T_2$  we have  $Y_2(t) = Y_1(t) + \bar{Y}_2 - \bar{Y}_1$ . Identical reasoning in the case of the pair of lower boundaries leads us to conclude that after  $Y_2$  hits the boundary  $-\bar{Y}_2$ , the point

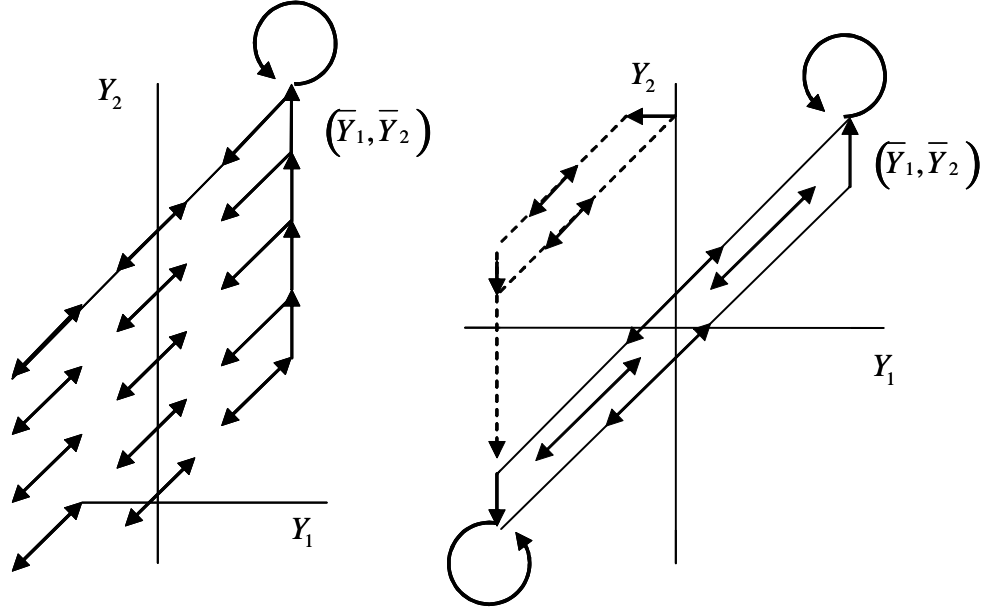


Figure 4.2: The left hand side represents how the point  $(Y_1, Y_2)$  moves in the state-space as time passes in the presence of two upper boundaries  $\bar{Y}_1$  and  $\bar{Y}_2$ . After  $t \geq T_2$  the point  $(Y_1, Y_2)$  never comes back to the interior staying forever on the line  $Y_2 = Y_1 + \bar{Y}_2 - \bar{Y}_1$ . It may remain at point  $(\bar{Y}_1, \bar{Y}_2)$  with a positive probability during the next instant of time,  $\Delta t$ , which is represented by the circular arrow. In the presence of two lower boundaries we have an absolutely symmetric situation. The right hand side picture represents the movement of point  $(Y_1, Y_2)$  in the case with both two upper and lower boundaries. The broken line represents one from a possible set of paths starting at  $(0, \bar{Y}_2)$  and terminating onto the ergodic set as indicated. During the transition it happens that a fraction of firms from sector 2 adjust their prices while firms from sector 1 do not. This situation is impossible once the state  $(Y_1, Y_2)$  reaches the ergodic set. This, of course, will happen with probability one in finite time.

$(Y_1, Y_2)$  will move along the line  $Y_2 = Y_1 - \bar{Y}_2 + \bar{Y}_1$  forever. Now it is clear that in the presence of two pairs of symmetric boundaries the state space of the joint stochastic process  $(Y_1(t), Y_2(t))$  can be represented by a parallelogram depicted in Figure 4.3.■

The fact that both  $Y_1$  and  $Y_2$  move along the 45% degree line (so that  $\Delta Y_1/\Delta Y_2 = 1$ ) except when at least one of them is at its extreme values also follows from (2.12). Recall that when  $Y_1 \neq \bar{Y}_1$  and  $Y_2 \neq \bar{Y}_2$  we are in the inaction region where no firm adjusts its nominal price. Here, all (log) changes in the money supply are reflected in changes of output and these changes are distributed equally between the sectors.

The money supply process generates an ergodic Markov chain<sup>16</sup> on the space  $(Y_1, Y_2)$  defined in Proposition 4.1. The next proposition defines limiting occupation probabilities  $\pi(Y_1, Y_2)$ <sup>17</sup> of states  $(Y_1, Y_2)$  depicted in Figure 4.3.

**Proposition 4.2.** *Let outputs  $Y_1$  and  $Y_2$  take only discrete and equidistant integer values in the intervals  $[-N_1, N_1]$  and  $[-N_2, N_2]$  respectively, as represented in Figure 4.3. The joint distribution  $\pi(Y_1, Y_2)$  in the stationary state is given by:*

$$\pi(Y_1, Y_2) = \begin{cases} \frac{\zeta}{n_1 n_2} & Y_2 = Y_1 + (\bar{Y}_2 - \bar{Y}_1) \\ \frac{n_1 + 1 - \zeta}{n_1 n_2} & Y_2 = Y_1 - (\bar{Y}_2 - \bar{Y}_1) \\ \frac{1}{n_1 n_2} & \text{everywhere else} \end{cases} \quad (4.8)$$

where  $\zeta = 0, 1, \dots, n_1$ ,  $n_1 = 2N_1 + 1$ , and  $n_2 = 2N_2 + 1$ .■

**Proof.** See Appendix 1.

Now we state our main result by formulating Theorem 4.1.

**Theorem 4.1.** *Let the economy consist of two different sectors with strictly increasing optimal boundaries  $\{\bar{Y}_i | i = 1, 2\}$ . Let output  $Y_i$  in each sector  $i$  follow a regulated Brownian motion. Then, in the stationary state, the density function of aggregate output, defined as  $Y = Y_1 + Y_2$ , is given by a weighted average of two densities:*

$$\mu(Y) = (1 - \omega)\mu_1 + \omega\mu_2(Y = 2z_1 + z_2), \quad (4.9)$$

<sup>16</sup>It is apparent from Figure 4.3 that all states communicate with each other, and hence that the chain is irreducible. Further, because by construction our chain is finite, it is necessarily positive-recurrent. In addition it is aperiodic so the existence of a stationary distribution is guaranteed.

<sup>17</sup>These probabilities are equal to the inverses of the mean recurrence times of the underlying Markov chain. They can also be interpreted as the proportion of time spent in the corresponding states. We interpret them as probabilities of the state  $(Y_1, Y_2)$  on the discretized state-space.

where weight  $\omega$  is given by:

$$\omega = \frac{1}{2} \frac{\bar{Y}_1(\bar{Y}_2 - \bar{Y}_1)}{\bar{Y}_1 \bar{Y}_2}.$$

$\mu_1$  denotes the density of a uniformly distributed random variable on the interval  $[-\bar{Y}_1 - \bar{Y}_2, \bar{Y}_1 + \bar{Y}_2]$  i.e.,

$$\mu_1 = \frac{1}{2(\bar{Y}_1 + \bar{Y}_2)},$$

and  $\mu_2$  is the density of the sum of two independently, non-identically and uniformly distributed random variables,  $2z_1$  and  $z_2$ , with corresponding densities:

$$\mu(z_1) = \frac{1}{2\bar{Y}_1};$$

$$\mu(z_2) = \frac{1}{2(\bar{Y}_2 - \bar{Y}_1)}.$$

**Proof.** See Appendix 2.

The density has two components, visualized in Figure 4.4. First, we recognize the influence of the uniformly distributed outputs from the two sectors which is reflected in the uniform density,  $\mu_1$ <sup>18</sup>. Second, there is the influence of the independent portions of the sectoral outputs, that is the density for the sum of two independent uniformly and nonidentically distributed random variables (see, for example, Rényi, 1970).

## 5. The Relationship Between Money, Output and Prices

In a single sector model, aggregate dynamics reflect sectoral dynamics implying that output and money are positively correlated, up to a certain point, after which further changes in the money stock would result merely in a rise in prices (Caplin and Leahy, 1997). In other words, a single sector state-dependent model is qualitatively consistent with Phillips-curve type behavior. However, these dynamics appear to be very ‘angular’: Inside the barriers, output rises with money one for

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<sup>18</sup>In the case of three and more sectors this "uniform" part of the density function still exists reflecting the simple fact that there is a positive probability of staying in state  $(\bar{Y}_1, \bar{Y}_2, \dots, \bar{Y}_K)$ . In the case of three sectors, for example, this part takes the form  $1/6\bar{Y}_K$ .



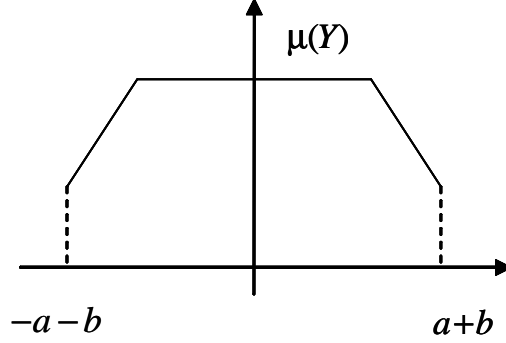


Figure 4.3: Density function  $\mu(Y)$ .

one; once the barriers are reached, all further monetary shocks (in the same direction as those that led output to hit the barrier) result merely in price changes<sup>19</sup>. In the case of  $K$  sectors these dynamic interactions are smoother. At the aggregate level, output will not in general change with money one-to-one, and the economy may be quite sensitive to demand shocks; these are the main results we are going to show in this section.

We may easily calculate the correlation between changes in the money supply and output for one and two sector economies. Since  $E(Y) = E(\Delta M) = 0$ , the correlation function is

$$\rho(Y, M(t') - M(t)) = E(Y \Delta M),$$

where we use normalized variables  $\Delta m/\sigma$  and  $Y/\sqrt{\text{Var}(Y)}$ . We also know, drawing on Theorem 4.1, that

$$\text{Var}(Y) = \int Y^2 \mu_{1(2)}(Y) dY, \quad (5.1)$$

where for the one sector case  $\mu_1(Y) = 1/2\bar{Y}$ , and for the two sector case  $\mu_2(Y)$  is given by (4.9). Using the definition of regulated Brownian motion we simulate paths of normalized  $Y$  and  $\Delta M$  for both one and two sectors for different values of the standard deviation of money. Optimal boundaries  $\bar{Y}(\sigma)$  are calculated from

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<sup>19</sup>It is worth emphasising that this qualitative feature of the model is not merely a function of having closed the model with a simple quantity-type equation. With richer nominal specification, the same basic features would be present.

equations (2.16) and (2.17) with parameters  $g = 0.5$ ,  $\alpha_1 = \alpha_2 = 0.8$ ,  $r = 0.05$ . We compare a one sector economy with a two sector economy. We choose for the single sector economy  $C_1 = 0.0015$  and for the two sector case economy  $C'_1 = 0.001$  and  $C'_2 = 0.002$ , so that average costs of changing prices are equal across our model economies. Then, time series of length of  $N = 10000$  for outputs  $Y_1(0.0015)$ , and  $Y = Y_1(0.001) + Y_2(0.002)$ , and  $\Delta M$  are generated. Normalizing time to  $T = 1$  we have that  $\Delta t = 10^{-5}$ . We found that  $t' - t = 100\Delta t$  was enough to achieve reasonable convergence of outputs to their limiting distributions. The results of these simulations are presented in Figure 5.1.

For both cases there exists a maximum of the correlation function. We can conjecture such a maximum with equations (2.16) and (2.17) from which it is easy to verify that  $\bar{Y}(\sigma) - \sigma$  changes sign from positive to negative after some value of the variance  $\sigma_c$ . For our chosen set of parameters,  $\sigma_c \approx 0.033$  in the one sector case. We show that in the region  $\sigma < \sigma_c$  the correlation is an increasing function of the variance of money. Before hitting the boundaries, the absolute value of output increases with increasing  $|\Delta M|$ . In the region  $\sigma > \sigma_c$  with rising variance,  $\bar{Y}$  rises allowing output to fluctuate more widely (output is still uniformly distributed) although at the same time fluctuations in money are much larger i.e.,  $\bar{Y} \ll \sigma$ . Again, this can be verified using (2.16) and (2.17).

If the standard deviation of the money supply process were to rise over any finite period of time, then it can be shown that the probability of output reaching its limits rises. As a result, the correlation coefficient characterizing the money-output relation necessarily falls. In the stationary state that reasoning is no longer valid. Output is distributed uniformly over the whole interval  $[-\bar{Y}, \bar{Y}]$ . Now as the variance of money increases away from zero,  $\bar{Y}$  rises more than proportionally with  $\sigma$ ,  $\Delta \bar{Y}(\sigma)/\Delta \sigma > 1$ . Eventually, however, that effect subsides and the correlation coefficient falls. Damjanovic and Nolan (2006) provides a fuller discussion of this point.

A striking feature of this model is that aggregate output is less responsive to changes in the money supply for small values of the variance of the money stock than is the case in the one sector economy. However, for larger variances, that effect is reversed and monetary shocks have a larger impact in the multi-sector economy.

The correlation coefficient between money and output in the two-sector case rises more slowly and peaks at a lower level (of the variance) than in the one sector case for the following reasons. As the variance starts away from zero the barriers are practically inaccessible in both the one and two sector cases and the correla-

tion coefficient rises. Recall, that we are here considering the ergodic distribution of aggregate output and so in the two sector case the correlation coefficient reflects the combination of sectors, one that will more often hit the boundary, and one that will less often hit the boundary, for a given level of infinitesimal variance. On the average therefore, the correlation function is lower for small variance. Eventually, the barriers rise less than one-for-one as variance rises, and in the two sector case that happens more gradually since one sector faces a larger cost of price-adjustment. This is one sense in which the aggregate dynamics are more nuanced than in the one sector case. Two implications follow from this. First, a given nominal shock in the multi-sector economy will, in general, result in price and output movements, in contrast to the single-sector economy. Second, the multi-sector economy may be somewhat more sensitive to nominal shocks since as the variance of the monetary process rises the impact on the correlation coefficient dies away at a slower rate than in the single-sector economy. This issue is worthy of more research as it is important to see how dependent these effects are on the relative sizes of the costs of price adjustment, the degree complementarity/substitutability and the variance of the driving process.

## 6. Conclusion

This paper has presented a macroeconomic model with multiple sectors. Incorporating multiple sectors in this framework permitted us to analyze extensively the impact of goods complementarity and differing costs of price adjustment on the behaviour of the sectors themselves and on the macroeconomy. We studied the impact of two-sided nominal shocks in this simple multi-sector dynamic, general equilibrium  $(S, s)$ -pricing model. The model has a number of appealing empirical implications; it captures why some sectors of the economy have systematically more flexible prices, the smooth dynamics of aggregate output following a monetary shock, and a degree of price asynchronization. We established that incorporating multiple sectors is central to arriving at these three results in the class of models under study.

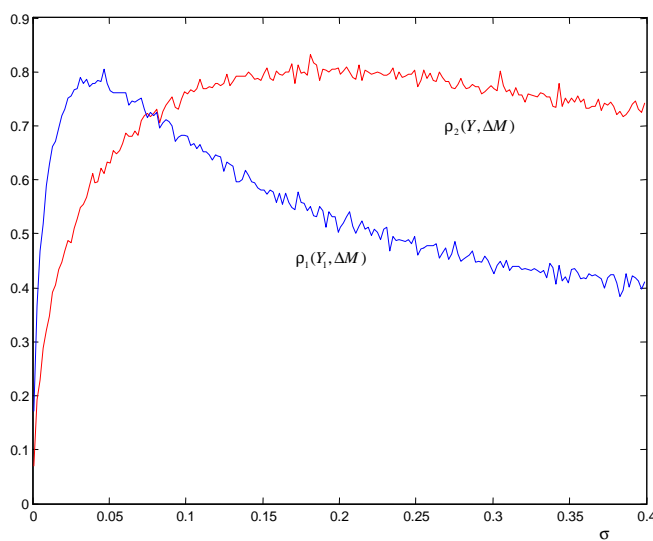


Figure 5.1: Correlations between output  $Y$  and changes in the money supply for one and two sectors respectively, as a function of the standard deviation of money,  $\sigma$ . For the one sector case  $\rho_1$  is simulated with cost  $C_1 = 0.0015$ . In the case of two sectors,  $\rho_2$  is simulated with costs 0.001 and 0.002 so that the average costs in both cases are equal.

## References

- [1] Arrow, K.J., Harris, T., and Marschak, J., (1951), Optimal Inventory Policy, *Econometrica*, 19, 205-272.
- [2] Bils, M and Klenow, P (2004), Some Evidence on the Importance of Sticky Prices, *Journal of Political Economy* 112, October, 947-985.
- [3] Blanchard, O.J., Kiyotaki, N., (1987), Monopolistic Competition and the Effects of Aggregate Demand, *The American Economic Review* 77, 647-666.
- [4] Barro, R. J., (1972), A Theory of Monopolistic Price Adjustment, *The Review of Economic Studies*, 39, 17-26.
- [5] Caplin, A. S., and Spulber D. F., (1987), Menu Costs and the Neutrality of Money, *Quarterly Journal of Economics*, 102: 703-726.
- [6] Caplin, A., Leahy, J., (1997), Aggregation and Optimization with State-Dependent Pricing, *Econometrica* 65, 601-625.
- [7] Damjanovic, V., Nolan, C., (2006), Aggregation and Optimization with State-Dependent Pricing: A Comment, *Econometrica* 74(2), 565-573.
- [8] Danziger, L., (1999), A Dynamic Economy with Costly Price Adjustments, *The American Economic Review*, Vol. 89, No. 4, September, pp.878-901.
- [9] Dhyne, E., Alvarez, L.J., Le Bihan, H., Veronese, G., Dias, D., Hoffman, J., Jonker, N., Lunermann, P., Rumler, F., and Vilmunen, J., (2004) Price Setting in the Euro Area: Some Stylised Facts from Individual Consumer Price Data mimeo, European Central Bank.
- [10] Dotsey, M., King R.G., Wolman, A. L., (1999), State-Dependent Pricing and the General Equilibrium Dynamics of Money and Output, *The Quarterly Journal of Economics*, Vol. 114, No. 2 (May, 1999), pp. 655-690.
- [11] Gertler, M. and Leahy J., (2006), A Phillips Curve with an SS Foundation, *NBER Working Paper 11971*.
- [12] Golosov, M., and Lucas E. R. Jr., (2003), Menu Costs and Phillips Curves, *NBER Working Paper* No.10187.

- [13] Harrison, J. M.,(1985), *Brownian Motion and Stochastic Flow System*, New York: Springer Verlag.
- [14] Klenow, P., Kryvstov, O., (2005), State-Dependent or Time-Dependent Pricing: Does it Matter for Recent US Inflation?, *NBER Working Paper* No. 11043.
- [15] Rényi, A., (1970), *Probability Theory* (North Holland).
- [16] Sheshinski, E., and Weiss Y., (1983), Optimum Pricing Policy under Stochastic Inflation, *Review of Economic Studies*, 60: 513-529.
- [17] Sheshinski, E., and Weiss Y., (1977), Inflation and Costs of Price Adjustment, *Review of Economic Studies*, 54: 287-303.
- [18] Stokey, N. L., (2002), *Brownian Models in Economics*. Manuscript.

### Appendix 1: Solving the Markov Equation

Proof of Proposition 4.1: We proceed directly to find the joint distribution in the stationary state, denoted by  $\pi(Y_1, Y_2)$ , by solving the eigenvalue problem for the Markov transition matrix  $P$ :

$$\pi P = \pi. \tag{6.1}$$

The number of different states, i.e., the number of points in the set  $ABCD$  (see Figure (6.1).), is  $n = n_1 \bar{n}_2$  where  $n_1 = 2N_1 + 1$  and  $\bar{n}_2 = 2(N_2 - N_1) + 1$ . Obviously,  $\dim(\pi) = n$  and  $P$  is a  $n \times n$  stochastic matrix with elements 0 and  $1/2$ . Instead of solving the system of equations (6.1) directly it is more straightforward to compute limiting probabilities  $\pi$  directly from the balance equations<sup>20</sup>.

Recall now Figure 4.3. If we consider any subset of admissible states with  $Y_2$  fixed (that is, points parallel to the  $Y_1$  axis), then for such a set the following equation must be satisfied:  $p(Y_2) = \sum_j \pi(Y_2, Y_j)$ , where  $p(Y_2)$  is the unconditional probability of  $Y_2$ , and  $\pi(Y_2, Y_i)$  are joint probabilities. However, it must be the case that  $p(Y_2) = 1/n_2$  where  $n_2 = 2N_2 + 1$  because in the stationary state outputs in each band are uniformly distributed. It follows, then, that the joint probabilities in corners  $A$  and  $C$  are  $\pi_A = \pi_C = 1/n_2$ . Now start from corner  $B$  and move to  $A$  along the edge  $BA$  and write

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<sup>20</sup>It has already been established in Section 4 that the underlying Markov chain is ergodic. Therefore in stationary state balance equations must be satisfied in each state.

down the corresponding balance equations. Now, starting from corner  $B$  and moving along to  $A$  along the edge  $\overline{BA}$  we may write the balance equations as follows,

$$\begin{aligned}\pi_B &= \pi(1); \\ \pi(1) &= \frac{1}{2}\pi(2); \\ \pi(2) &= \frac{1}{2}\pi(1) + \frac{1}{2}\pi(3); \\ &\dots\dots\dots\end{aligned}$$

From the above equations we receive the following recursion:

$$\begin{aligned}\pi(1) &= \pi_B; \\ \pi(2) &= 2\pi_B; \\ \pi(3) &= 3\pi_B; \\ &\dots\dots\dots \\ \pi(n_1) &= \pi_A = n_1\pi_B.\end{aligned}$$

Hence  $\pi_A = 1/n_2$  and from the last equation it follows that  $\pi_B = 1/n_1n_2$ . By symmetry  $\pi_D = \pi_B$ . So along the edge  $\overline{BA}$  joint probabilities are given by:

$$\pi_{BA}(x) = \frac{\zeta}{n_1n_2}$$

and along the edge  $CD$  by

$$\pi_{CD}(x) = \frac{n_1 + 1 - \zeta}{n_1n_2},$$

where  $\zeta = 1, 2, \dots, n_1$ . To see where this latter relation comes from, note that as we move along edge  $\overline{CD}$  the probabilities are falling, as opposed to rising (along  $\overline{BA}$ ). So, we have that

$$\begin{aligned}\pi_C &= \pi(1); \\ \pi(1) &= n_1\pi_D; \\ \pi(2) &= (n_1 - 1)\pi_D; \\ &\dots\dots\dots \\ \pi(n_1) &= (n_1 - 1 + 1)\pi_D.\end{aligned}$$

Proceeding in the same fashion we find that all remaining points in the joint probability equal  $\pi_B$ . Consequently, the balance equations are satisfied for all points. We can verify this by direct calculation. For any points on edge  $\overline{BA}$  we have that

$$\pi_{BA}(\zeta) = \frac{1}{2}\pi_{BA}(\zeta - 1) + \frac{1}{2}\pi_{BA}(\zeta + 1) = \frac{1}{2}\frac{\zeta - 1}{n_1 n_2} + \frac{1}{2}\frac{\zeta + 1}{n_1 n_2} = \frac{\zeta}{n_1 n_2},$$

while on edge  $\overline{CD}$  we have

$$\begin{aligned}\pi_{CD}(\zeta) &= \frac{1}{2}\pi_{CD}(\zeta - 1) + \frac{1}{2}\pi_{CD}(\zeta + 1); \\ &= \frac{1}{2}\frac{n_1 + 1 - \zeta + 1}{n_1 n_2} + \frac{1}{2}\frac{n_1 + 1 - \zeta - 1}{n_1 n_2}; \\ &= \frac{n_1 + 1 - \zeta}{n_1 n_2}.\end{aligned}$$

At corner  $A$  we see that

$$\pi_A = \frac{1}{2}\pi_A + \frac{1}{2}\frac{n_1 - 1}{n_1 n_2} + \frac{1}{2}\frac{1}{n_1 n_2} = \frac{1}{n_2},$$

and similarly at corner  $C$ . For any other point we have the obvious identity  $\frac{1}{n_1 n_2} = \frac{1}{2}\frac{1}{n_1 n_2} + \frac{1}{2}\frac{1}{n_1 n_2}$ . Finally, we require that the probabilities sum to unity:

$$\begin{aligned}&\sum_{\zeta}(\pi_{BA}(\zeta) + \pi_{CD}(\zeta)) + \frac{1}{n_1 n_2}(n_1 \bar{n}_2 - 2n_1) \\ &= \sum_x \frac{n_1 + 1}{n_1 n_2} + \frac{1}{n_1 n_2}n_1(\bar{n}_2 - 2) \\ &= \frac{n_1(n_1 + 1)}{n_1 n_2} + \frac{n_1(n_2 - n_1 - 1)}{n_1 n_2} \\ &= 1.\end{aligned}$$

This completes the solution of equation (6.1)<sup>21</sup>.

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<sup>21</sup>To further illustrate our solution we rely on the well known fact that  $P^k \rightarrow P^*$  in the limit as  $k \rightarrow \infty$  where  $P^*$  is a stationary Markov matrix whose rows are identical and equal to the stationary eigenvalue vector  $\pi^*$ . We chose  $N_1 = 2, N_2 = 4$



## Appendix 2: Deriving the Stationary Distribution of Aggregate Output

Proof of Proposition 4.2: Having calculated the stationary distribution of joint probabilities it is possible to find the stationary probability distribution of aggregate output. We start by calculating the probability function  $P(Y)$  defined in the usual way:

$$P(Y) = \sum_{\xi} P(\xi \leq Y). \quad (6.2)$$

On the other hand, the unconditional probabilities  $P(\xi)$  are given by:

$$P(\xi) = \sum \pi(Y_1, Y_2), \quad (6.3)$$

where the sum has to be taken over all values of  $Y_1$  and  $Y_2$  and where  $\xi = Y_1 + Y_2 \leq Y$ . In Figure 4.3 it corresponds to all points lying on the line  $Y_2 = \xi - Y_1$  for a certain  $\xi$ . Combining (6.2) and (6.3) we see that in order to find the probability function  $P(Y)$  one has to sum joint probabilities over all points lying on the lines  $Y_1 + Y_2 = \xi \leq Y$ .

We perform this summation noticing that lines from Figure 4.3 are divided into two groups; this sub-division provides us with our clue on how to take these discrete calculations to the continuous limit. The first group is represented by lines containing the points on the edges  $\overline{AB}$  and  $\overline{CD}$  where we have different joint probabilities. We label such points as elements in the ‘exterior’ group ( $\xi^E$  lines). The remaining probabilities are elements in the ‘interior’ group ( $\xi^I$  lines). These are depicted in Figure 4.1.

Consider first the exterior group of lines and calculate  $P(\xi)$  for an arbitrary line. To begin with, rewrite the joint probabilities of points lying on the exterior lines in a more convenient forms as:

$$\pi^{\pm}(\zeta) = (\pm\zeta + \overline{Y}_1 + 1)\pi_B$$

where  $\zeta = -\overline{Y}_1, -\overline{Y}_1 + 1, \dots, \overline{Y}_1 - 1, \overline{Y}_1$ , along the upper (+) and lower (−) edges respectively, and  $\pi_B$  otherwise.

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so that  $n_1 = 5$  and  $n_2 = 9$  and the dimension of the transition matrix is  $25 \times 25$  (see the Figure 4.3 in the main text). We enumerate states by index  $j$  as follows. The first state, represented by the point  $A$ , is the state with  $j = 1$  and the last state, represented by  $C$ , with  $j = 25$ . After a sufficiently large number,  $k$ , of iterations we obtain  $P^*$  with all rows equal and given by:  $\pi^* = (0.1111, 0.0889, 0.0667, 0.0444, 0.0222, 0.0222, \dots, 0.0222, 0.0222, 0.0444, 0.0667, 0.0889, 0.1111)$ , from which we read for example  $\pi_A = \pi_C = 1/n_2 = 1/9 = 0.1111$ ,  $\pi_B = 1/n_1 n_2 = 1/45 = 0.0222$  and so on.

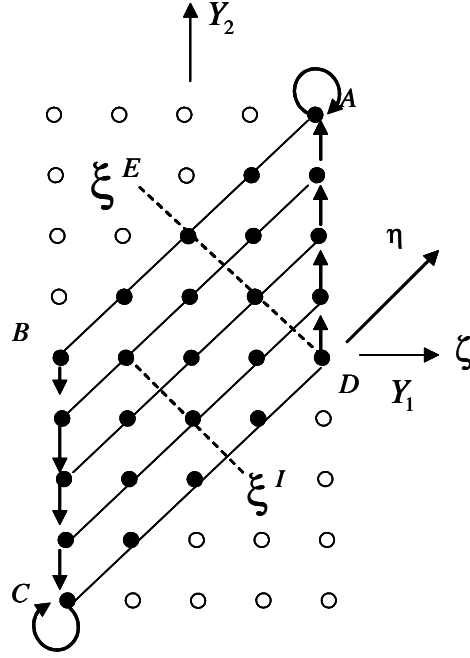


Figure 6.1: Joint state space  $(Y_1, Y_2)$  for outputs from two sectors is represented by bold dots for values  $\bar{Y}_1 = 2, \bar{Y}_2 = 4$  so that  $Y_1 = -2, -1, 0, 1, 2$  and  $Y_2 = -4, -3, \dots, 0, \dots, 3, 4$  ( $n_1 = 5, n_2 = 9$ ). Two way moving between the states is represented by bold lines while one way moving is represented by arrowed lines. Arrowed circles indicate the possibility of remaining at corners  $A$  and  $C$ .

Now calculate  $P(\xi)$  directly from (6.3) for some  $\xi$ . Let  $\Delta\bar{Y} \equiv \bar{Y}_2 - \bar{Y}_1$ . The line  $Y_2 = \xi - Y_1$  intersects the upper edge  $\overline{BA}$  in  $\zeta = (\xi - \Delta\bar{Y})/2$  and the lower edge  $\overline{CD}$  in  $\zeta = (\xi + \Delta\bar{Y})/2$ , as depicted in Figure 4.3. Then equation (6.3) reads:

$$P(\xi) = \pi^+\left(\frac{\xi - \Delta\bar{Y}}{2}\right) + \pi^-\left(\frac{\xi + \Delta\bar{Y}}{2}\right) + \frac{1}{2}(2\Delta\bar{Y} - 2)\pi_B,$$

where the last term is simply the number of points between intersections excluding the two points on the edges. By direct calculation, we then have that

$$\begin{aligned} &= \left(\frac{\xi - \Delta\bar{Y}}{2} + \bar{Y}_1 + 1\right)\pi_B + \left(-\frac{\xi + \Delta\bar{Y}}{2} + \bar{Y}_1 + 1\right)\pi_B + (\Delta\bar{Y} - 1)\pi_B; \\ &= -\Delta\bar{Y}\pi_B + 2(\bar{Y}_1 + 1)\pi_B + \Delta\bar{Y}\pi_B - \pi_B; \\ &= (2\bar{Y}_1 + 1)\pi_B; \\ &= 1/(2\bar{Y}_2 + 1) = 1/n_2. \end{aligned}$$

When the line  $\xi$  does not intersect both edges the above relations also hold. For example we can conclude immediately that  $P(\xi = N_1 + N_2) = P(\xi = -N_1 - N_2) = 1/n_2 = 1/(2\bar{Y}_2 + 1)$ . Moving one step further from, for example, corner  $C$  gives:  $\pi^-(-\bar{Y}_1 + 1) + \pi_B = (-(-\bar{Y}_1 + 1) + \bar{Y}_1 + 1)\pi_B + \pi_B = (2\bar{Y}_1 + 1)\pi_B = 1/(2\bar{Y}_2 + 1)$ . Continuing in a similar fashion we conclude that for any  $\xi$  representing an exterior line, the sum of joint probabilities is the same and equal to  $1/(2\bar{Y}_2 + 1)$ . For points belonging to interior lines the contribution is simply equal to  $\frac{1}{n_1 n_2} n(\xi)$ , where  $n(\xi)$  is the number of points on the interior line,  $\xi^I$ . It follows that (6.2) can be written as

$$P(Y) = \sum_{\xi^{Ee} \leq Y} \frac{1}{n_2} + \sum_{\xi^I \leq Y} \frac{1}{n_1 n_2} n(\xi) \quad . \quad (6.4)$$

Let  $Y = \bar{Y}_1 + \bar{Y}_2 = \frac{n_1 + n_2}{2} - 1$ . Then from (6.4) we have:

$$P\left(\frac{n_1 + n_2}{2} - 1\right) = \frac{n_1 + n_2}{2n_2} + \frac{1}{n_1 n_2} \frac{n_1(n_2 - n_1)}{2} = 1,$$

as must be the case.

It is easy to rewrite (6.4) in a more explicit form, but instead of that we now proceed to derive an expression for the probability function in the continuous case<sup>22</sup>. Introduce

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<sup>22</sup>All the results and analysis will apply in the continuous limit as  $\Delta Y \rightarrow 0$  ( $N \rightarrow \infty$ ), where  $\Delta Y = 2\bar{Y}/(2N + 1)$ . It is obvious that for arbitrarily large  $N$  system of equations

(6.1) have an unique set of solutions so that  $\lim_{N \rightarrow \infty} \sum_{k=1}^N |\pi_k| = 1$ .

the continuous variable  $\eta \in [0, 2(a+b)]$  to enumerate aggregate output from the interval  $[-a-b, a+b]$ . Then on the scale  $\eta$  point  $D$  in Figure 4.3 has coordinate  $4a$  and the measure of edge  $\overline{DA}$  is  $2(b-a)$ . To see that this is indeed the case recall that in the discrete case just half of the  $\xi$  lines (here we speak of  $\eta$  lines) terminate on edge  $\overline{CD}$ . The measure of edge  $\overline{DA}$  remains unchanged as all  $\xi$  lines terminate on this edge. In the continuous limit this must be preserved so that on the scale  $\eta$  properly applying (6.4) means changing  $\eta \rightarrow \eta/2$ . The situation is similar for  $n(\eta)$ , as it is just half of the interval lying on the  $\eta$  line. To illustrate this, we now provide an explicit calculation of  $P(\eta)$  for  $\eta \leq 2(b-a)$ .

Hence:

$$n(\xi) = \xi/2,$$

where we require that,

$$\sum_{\xi^I \leq x} \rightarrow \int_0^{\eta} d(\xi/2),$$

and

$$\sum_{\xi^{Ee} \leq x} \rightarrow \frac{\eta}{2}.$$

Therefore, it follows that

$$n_1 \rightarrow 2a, n_2 \rightarrow 2b.$$

And applying (6.4) for the continuous case we obtain:

$$P(\eta) = \frac{1}{2b} \frac{\eta}{2} + \frac{1}{4ab} \int_0^{\eta} \frac{\xi}{2} d\left(\frac{\xi}{2}\right) = \frac{1}{2b} \frac{\eta}{2} + \frac{1}{4ab} \frac{\eta^2}{8}.$$

Finally we write an expression for the probability function as a sum of two terms,  $P(\eta) = P^E(\eta) + P^I(\eta)$  where:

$$P^E(\eta) = \frac{1}{2b} \frac{\eta}{2}, \quad 0 \leq \eta \leq 2(b-a)$$

and

$$P^I(\eta) = \begin{cases} \frac{1}{4ab} \frac{\eta^2}{8} & 0 \leq \eta \leq 2(b-a) \\ \frac{1}{4ab} \frac{b-a}{2} (\eta - b + a) & 2(b-a) \leq \eta \leq 4a \\ \frac{1}{4ab} [2a(b-a) - \frac{(\eta-2a-2b)^2}{8}] & 4a \leq \eta \leq 2(a+b) \end{cases}$$

It follows immediately that the density function is given by:

$$\begin{aligned}\mu(\eta) &= \frac{dP(\eta)}{dx} \\ &= \begin{cases} \frac{1}{4b} + \frac{1}{4ab} \frac{\eta}{4} & 0 \leq \eta \leq 2(b-a) \\ \frac{1}{4b} + \frac{1}{4ab} \frac{b-a}{2} & 2(b-a) \leq \eta \leq 4a \\ \frac{1}{4b} + \frac{1}{4ab} \frac{2(a+b)-\eta}{4} & 4a \leq \eta \leq 2(a+b) \end{cases}\end{aligned}\tag{6.5}$$

Finally rewriting the first term in the above expression as

$$\frac{1}{4b} = \frac{2a(2a+2b)}{2(2a)(2b)} \frac{1}{2a+2b} = \left(1 - \frac{1}{2} \frac{2a(2b-2a)}{(2a)(2b)}\right) \frac{1}{2a+2b}$$

and in the same fashion the second terms as

$$\begin{aligned}\frac{1}{4ab} \frac{\eta}{4} &= \frac{1}{2} \frac{2a(2b-2a)}{(2a)(2b)} \frac{\eta}{4a(2b-2a)} \\ \frac{1}{4ab} \frac{b-a}{2} &= \frac{1}{2} \frac{2a(2b-2a)}{(2a)(2b)} \frac{1}{4a} \\ \frac{1}{4ab} \frac{2(a+b)-\eta}{4} &= \frac{1}{2} \frac{2a(2b-2a)}{(2a)(2b)} \frac{2(a+b)-\eta}{4a(2b-2a)}\end{aligned}$$

we recover expression (4.9) completing the proof.

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