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Aggregation and Optimization with State-Dependent Pricing: A Comment [*]

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ABSTRACT

A key argument in Caplin and Leahy (1997) states that the correlation between monetary shocks and output is falling in the variance of the money supply. We demonstrate that this conclusion depends on solving for the correlation in the non-stationary state of the model. In the stationary state, that correlation is initially rising.

1. Introduction

In an important and influential paper, Caplin and Leahy (1997) develop a model of state dependent pricing within a monopolistically competitive price-setting framework. To characterize an equilibrium of their model, they require that aggregate

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outcomes be consistent with individual optimizing outcomes. They demonstrate one way to do that with a few additional assumptions, especially as regards the distribution of relative prices. One of their key conclusions, Proposition 3 in their paper, is that the correlation between money and output is falling in the variance of the money supply, the only driving process in their model. In this note, we want to offer some clarification on that last point. We show that it is crucial to distinguish between the stationary and non-stationary state of the model. First, we briefly set out the key relations of the Caplin-Leahy model that we need to recall their analysis and develop ours. We then demonstrate that in the stationary state the correlation between money and output is initially rising in the variance of money. We then reconcile our result with their Proposition 3. Finally we offer a brief intuition for our result and indicate why it may be important.

2. A Brief Review of the Caplin-Leahy (1997) Model

If firms face a cost of changing prices, then when shocks to the profit maximizing price occur they have to weigh the benefits of changing prices against that cost. In general, there will be a zone of inaction where firms prefer optimally to change output rather than prices. Following an earlier contribution[1], Caplin and Leahy (1997) set up their model so that, in equilibrium, firms adjust their nominal prices at only two levels of real balances, $\pm\overline{Y}$. Whenever the relative price of a particular firm hits the boundary, price adjustment takes place moving the firm's relative price to the opposite end of the price distribution. As a result, they demonstrate that the distribution of relative prices remains uniform over time on some interval, $[-\overline{S}, \overline{S})$, while aggregate output, driven by a two-sided nominal shock, follows a regulated Brownian motion between the barriers $\pm\overline{Y}$. The boundaries themselves are the result of firms having solved a dynamic programming problem. Each period they minimize the distance between the price they set and the optimal price knowing that there is a fixed real resource cost to meet, C, if they decide to change prices. This cost aside, the loss function for the ith firm is proportional to the squared deviation of p_i from the price which maximizes instantaneous profit for each firm, P^*, which is a linear combination of the price index P and real aggregate demand Y,

$$L(x_i, Y) \equiv \gamma(p_i - P^*)^2 = \gamma(p_i - P - \alpha Y)^2 = \gamma(x_i - \alpha Y)^2,$$

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[1](Caplin and Leahy 1991)

2

where $\gamma > 0$, $\alpha > 0$. After solving the corresponding Bellman equation the following set of equations for optimal boundaries can be obtained (see e.g., Stokey, 2002),

$$\tanh(\beta\overline{Y}) + \alpha(\beta\overline{Y} - \tanh(\beta\overline{Y})) = \beta S \coth(\beta S) \tanh(\beta\overline{Y}); \qquad (2.1)$$

$$S(\beta\overline{Y} - \tanh(\beta\overline{Y})) = \delta C, \qquad (2.2)$$

where $\delta = r\beta/(4\alpha\gamma)$ and $\beta = \sqrt{2r}/\sigma$. r is the instantaneous discount rate, and σ is the standard deviation of the money supply process which follows a driftless Brownian motion. For given C these two relations provide the optimal values of Y and S. A notable contribution of Caplin and Leahy (1997) is that they can reconcile the outcome of this individual stochastic control problem, with the aggregate outcomes for the macroeconomy in an internally consistent way.

One of the key claims that Caplin and Leahy make is in their Proposition 3 which states that the correlation between money and output is diminishing for all $t' > t$ (where t denotes time). In this note we want to emphasize that this conclusion is correct for the nonstationary state of the model. In the stationary state, however, the conclusion is somewhat different; initially the correlation between money and output is rising, before falling.

3. The Correlation between Money and Output: The Stationary State

In this section we show that the behavior of this correlation in stationary state, when output is uniformly distributed on the interval $[-\overline{Y}, \overline{Y}]$, differs dramatically for small σ. Our claim is that before falling the correlation rises on some interval $[0, \sigma^*]$. In order to show that the correlation is an increasing function on that interval, it suffices to show that in stationary state $\rho(\sigma = 0) = 0$. We turn now to that proof and we also show that equations (1) and (2) can be solved for the the optimal boundaries in the case of 'static' policies where the money supply is, and is expected to be, constant i.e., $\sigma = 0$. This is important as we have to demonstrate that there are no discontinuities as $\sigma \to 0$. Before this, however, we find it useful to state Lemma 1[2]:

Lemma 3.1. *In the limit $\sigma \to 0$, $E(y\Delta M) \approx \sigma^2$.*

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[2]In what follows we suppress time dependence normalizing it to unity.

Here we provide an heuristic proof of this Lemma. Assume for simplicity that both processes for output and money start from the origin i.e., $y(0) = \Delta M(0) = 0$. Then, approximating the money supply by a random walk in the usual way, we may write: $y(n) = \sum_{i=1}^{n} \Delta y_i$ where $\Delta y_i = y_i - y_{i-1}$ and $\Delta y_1 = y_1$. Similarly $\Delta M(n) = \sum_{i=1}^{n} \Delta M_i$ where $\Delta M_i = M_i - M_{i-1}$ and $\Delta M_1 = M_1$. Further,

$$E(y\Delta M) = E(\sum_{i=1}^{n} \Delta y_i \sum_{j=1}^{n} \Delta M_j) = \sum_{i=1}^{n} E(\Delta y_i \Delta M_i) + \sum_{i<j}^{n} \Delta y_i \Delta M_j + \sum_{i>j}^{n} \Delta y_i \Delta M_j.$$

The second sum is zero simply because today's change in Y does not depend on future changes in the money supply. The third term can be neglected because changes in Y today are correlated with changes in money in the past only via boundary effects. Over any finite interval of time and for sufficiently small variance, it can safely be neglected. In other words, during a finite period of time the number of events when $\Delta y = \Delta M$ is much bigger than the number of events when $\Delta y = 0$; so on the average, $\Delta y = \Delta M$. The same reasoning applies when Y is uniformly distributed over the discrete interval $[-\overline{Y}, \overline{Y}]$: $E(y\Delta M) = P(Y \in (-\overline{Y}, \overline{Y}))(\Delta M)^2 + P(Y = \overline{Y})(0 + \frac{(\Delta M)^2}{2}) + P(Y = -\overline{Y})(0 + \frac{(\Delta M)^2}{2}) = (\frac{N-2}{N} + \frac{1}{N})(\Delta M)^2 = (\Delta M)^2$ for sufficiently large number of discrete points N in the interval $[-\overline{Y}, \overline{Y}]$. Now, since $E(y\Delta M) = \sum_{i=1}^{n} E(\Delta y_i \Delta M_i) = n(\Delta M)^2 = \sigma^2$, the statement of the Lemma follows immediately. We can now turn to our key proposition.

Proposition 1 *In the stationary state of the model, and in the limit as $\sigma \to 0$, the correlation function $\rho \to 0$.*

Proof. As the money supply follows a zero mean Brownian motion $E(\Delta M) = 0$, the correlation is given by:

$$\rho(Y, \Delta M) = \frac{E(Y\Delta M)}{\sqrt{Var(Y)}\sqrt{Var(\Delta M)}}; \qquad (3.1)$$

$$= \sqrt{3}\frac{E(Y\Delta M)}{\overline{Y}(\sigma)\sigma}, \qquad (3.2)$$

where $Var(Y) = \overline{Y}(\sigma)^2/3$ is the variance of a uniformly distributed random variable on the interval $[-\overline{Y}, \overline{Y}]$. When $\sigma = 0$ we are in a regime of 'static' policies,

4

where the change in the money supply will always be zero. Hence, the correlation is zero as well. If the correlation were a decreasing function over the entire interval of σ it would mean that it has a discontinuity at $\sigma = +0$. To demonstrate that this is not the case we have to approach $\sigma = 0$ from above i.e., to start from the set of equations for the optimal boundaries and see how their solution behaves as the variance approaches zero. From Lemma 1 we have:

$$E(Y\Delta M) \approx \sigma^2. \tag{3.3}$$

Combining (4) and (3.3) we have that when $\sigma \to 0$

$$\rho(Y, \Delta M) \sim \sigma/\overline{Y}(\sigma). \tag{3.4}$$

We need to see how $\overline{Y}(\sigma)$ behaves as $\sigma \to 0$. We start from the set of equations for optimal boundaries (2.1) and (2.2). Rewriting them in a more convenient form:

$$\frac{\tanh(\beta\overline{Y})}{\beta\overline{Y}} + \alpha(1 - \frac{\tanh(\beta\overline{Y})}{\beta\overline{Y}}) = \frac{\beta S}{\beta Y}\coth(\beta S)\tanh(\beta\overline{Y}); \tag{3.5}$$

$$(\beta S)(\beta\overline{Y})(1 - \frac{\tanh(\beta\overline{Y})}{\beta\overline{Y}}) = \overline{c}\beta^2, \tag{3.6}$$

where $\overline{c} = Cr/(4\alpha\gamma)$. Note that $\beta\overline{Y} \sim \overline{Y}/\sigma$ and both $\overline{Y}$ and S are monotonic and increasing in σ. In the following Lemma we show that $\beta\overline{Y} \to \infty$ when $\sigma \to 0$.

Lemma 3.2.
$$\lim_{\sigma \to 0} \beta\overline{Y} = \infty.$$

Proof. Assume that this is not true. Then there exists a positive number K so that we can find a sequence of numbers $\{\sigma_n\}$ such that

$$\sigma_n < \sqrt{2r}/n,$$

and[3]

$$\beta_n\overline{Y_n} < K. \tag{3.7}$$

Caplin and Leahy (1997) proved that

$$\alpha\overline{Y} < S < \overline{Y}, \tag{3.8}$$

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[3]We have suppressed the dependence on σ in our notation. So, when we write $\beta_n\overline{Y_n}$ we really mean $\beta(\sigma_n)\overline{Y}(\sigma_n)$.

and that proof holds good in the stationary state. Therefore, it follows that $\beta_n S_n$ satisfies the inequality

$$\beta_n S_n < \beta_n \overline{Y_n} < K. \tag{3.9}$$

Furthermore we note that

$$\beta_n = \sqrt{2r}/\sigma_n > n. \tag{3.10}$$

Now using (3.10), (3.6), the fact that $\tanh(x)/x \leq 1$ and (3.9) we write

$$\overline{c}n^2 < \overline{c}\beta^2 = (\beta S)(\beta \overline{Y})(1 - \frac{\tanh(\beta \overline{Y})}{\beta \overline{Y}}) < K^2.$$

However, we have obtained a contradiction, because no mater how large K is we can always choose an N such that $N > n = K^2/\sqrt{\overline{c}}$ so that inequality (3.7) does not hold. Therefore $\lim_{\sigma \to 0} \beta \overline{Y} = \infty.\blacksquare$

Inequality (3.8) and Lemma (2) imply that $\beta S \to \infty$ as well. Therefore $\tanh(\beta \overline{Y})/\beta \overline{Y} \to 0$ and $\coth(\beta S)\tanh(\beta \overline{Y}) \to 1$, and from equation (3.5) in the limit $\sigma \to 0$ we recover the case of static policies when the price deviation is such that the expected value of lost profits is just equal to the costs of price adjustment, C:

$$\frac{S}{\overline{Y}} = \alpha. \tag{3.11}$$

From equation (3.6) we have another relation:

$$S\overline{Y} = \overline{c}. \tag{3.12}$$

So, in the limit $\sigma \to 0$, S and $\overline{Y}$ are given by the solution to the simple system of equations:

$$\overline{Y}(0) = (\overline{c}/\alpha)^{1/2};$$

$$S(0) = (\overline{c}\alpha)^{1/2}.$$

From the above equations we see that $\alpha < 1$ is equivalent to $\overline{Y}(0) > S(0)$[4]. Finally we have that $\rho \sim \sigma/\overline{Y}(\sigma) \sim \sigma/\overline{Y}(0) \sim \sigma \to 0$ when $\sigma \to 0$ which completes the proof of Proposition 1.$\blacksquare$

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[4]$\alpha < 1$ implies that goods are strategic complements. All our arguments go through also if goods are strategic substitutes, $\alpha > 1$. For very high degrees of substitutability the interval $[0, \sigma^*]$ is somewhat reduced. Calculations demonstrating this are available on request.

4. The Correlation between Money and Output: The Non-Stationary State

We now contrast this with the non-stationary case. First, we provide a brief intuitive discussion for Caplin and Leahy's (1997) conclusion.

We already know that $\overline{Y}(0)$ is different from zero at the origin. Once the money supply process starts with infinitesimally small variance the barrier $\overline{Y}(\sigma)$ is practically inaccessible. In other words, there exists a positive probability that Y will never hit the boundary during some time interval. The expected time to hitting either the lower $(-\overline{Y})$ or upper $(+\overline{Y})$ barrier is given by $E(T) = \overline{Y}^2/\sigma^2$ (Karlin and Taylor, 1975). Because $\overline{Y}$ is finite for small values of σ, the expected hitting time may be some way off. Consequently, output follows money one-to-one. The correlation starts from one decreasing as σ increases because then the incidence of output reaching the boundaries increases. And so, as the time horizon extends, $\Delta t_1 < \Delta t_2 < ...$ we shall have a set of correlation functions $1 \geq \rho(\Delta t_1, \sigma) > \rho(\Delta t_2, \sigma) > ...$ decreasing in both σ and time.

In the stationary state this reasoning is no longer valid. Output is distributed uniformly over the whole interval $[-\overline{Y}, \overline{Y}]$ and, by Proposition 1, the correlation starts from zero. As the variances increase away from zero, the correlation can rise only on account of widening barriers. And this, in fact, is what happens; as the variance of the money supply increases from zero, $\overline{Y}$ rises more than proportionally with σ, $\Delta\overline{Y}(\sigma)/\Delta\sigma > 1$. This situation is depicted in Figure 1, where the line $y = \sigma$ is given for comparison purposes; what is key is the ratio $\overline{Y}/\Delta\sigma$. Starting from $\sigma = 0$, the correlation coefficient rises up to some small but finite value, σ^*, which is given by the solution of equations $\overline{Y}(\sigma^*) = \sigma^*$ and $\partial\rho/\partial\sigma > 0$. Damjanovic and Nolan (2005) demonstrate that the range of values over which this correlation is rising becomes larger as we add heterogenous sectors distinguished by differing costs of price adjustment, C. For $\sigma > \sigma^*$ the correlation coefficient falls and $\partial\rho/\partial\sigma < 0$ as Caplin and Leahy (1997, Figure 2) demonstrate in the nonstationary state.
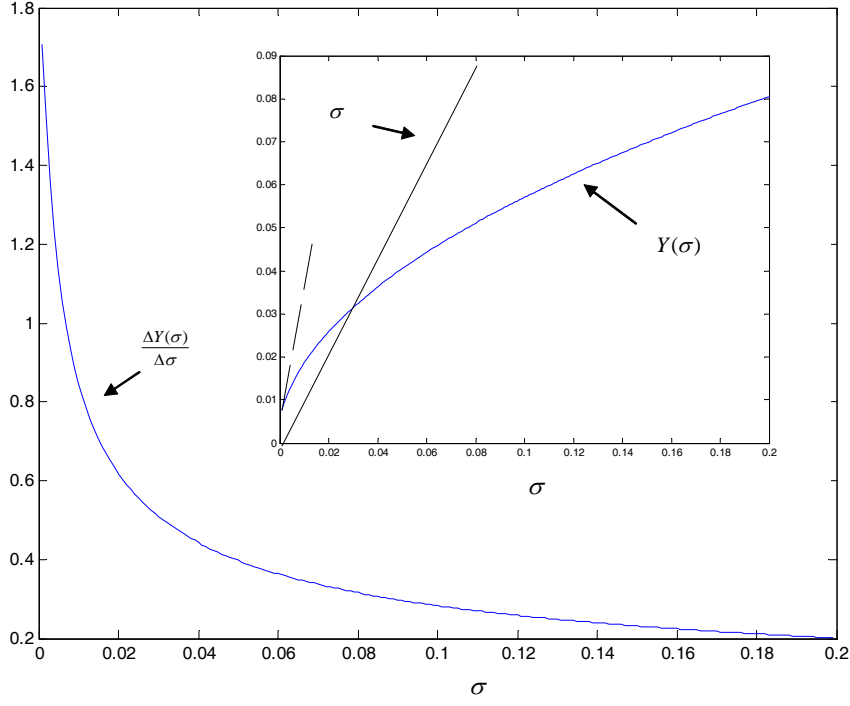
7

Figure 1. Functions $Y(\sigma)$ and $\Delta Y(\sigma)/\Delta\sigma$. The line $y = \sigma$ is given to facilitate comparison between the two slopes. $r = 0.05$, $\gamma = 0.5$ and $c = 0.001$, as in Caplin and Leahy (1997). Strategic complementarity is determined by setting $\alpha = 0.8$.
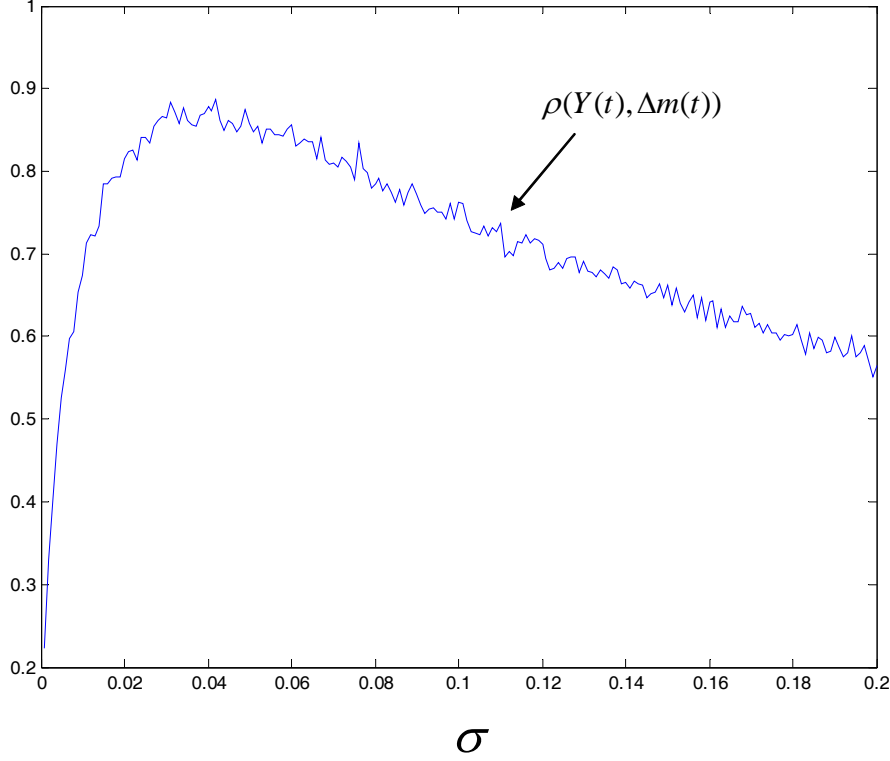
8

Figure 2. For each value of σ, with increments $\Delta\sigma = 0.001$, we compute time series $Y(i)$, $\Delta M(i)$ with initial conditions $Y(0) = \Delta M(0) = 0$. We do this via a random walk approximation on the discretized space with $\Delta Y = \sigma\sqrt{\Delta t}$ where $\Delta t = 1/N_1$. Time $t' - t$ is normalized to one as $t' - t = N_1 \Delta t = 1$. Then every $N_1 th$ pair of $(Y(i), \Delta M(i))$ where $\Delta M(i) = M(iN_1) - M((i-1)N_1)$ and $i = 1, ..N_2$ is taken out of $N_2 N_1$ pairs to make up our time series. In the simulations reported, $N_1 = 100$, $N_2 = 10000$.

5. Conclusion

It is interesting to note that Caplin and Leahy in proving their Proposition 3 use the fact that $\overline{Y}$ rises less than proportionally with σ. The fact that there exists a range of values for σ where that is not the case does not invalidate their proof; in

9

the nonstationary state the correlation, as a function of the variance of the money supply, *always* starts from its maximal value.

The fact that the correlation coefficient is initially rising with the variance of the money supply in the stationary state seems important. Lucas (1973) showed that, in an economy consisting of agents who are unable to distinguish between real and nominal shocks, an increase in the variance of the money supply reduces the effect of a given nominal shock on output. Caplin and Leahy (1997) showed that the same effect arises in an (s, S) model of costly price adjustment *i.e.,* that the correlation between money and output is diminishing in the variance of the money supply. These results are intuitively appealing to macroeconomists in the sense that with increasing variance of the money supply more and more firms adjust their prices more frequently. On the other hand, Lucas (and others) also mentioned that expansionary monetary policies in stable price countries may have relatively large effects on real output in contrast to countries with more volatile nominal policies. We find that the dynamic (S, s) menu cost model of Caplin and Leahy (1997) is capable of capturing such effects as well. The framework developed by Caplin and Leahy (1997) has proven most fruitful, and is consistent with a number of other important macroeconomic stylized facts. These are discussed more fully in Damjanovic and Nolan (2005).

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