CENTRE FOR DYNAMIC MACROECONOMIC ANALYSIS
WORKING PAPER SERIES

CDMA05/05

Aggregate Dynamics with Heterogeneous Agents and State-Dependent Pricing*

Vladislav Damjanovic†
University of St Andrews

Charles Nolan‡
University of St Andrews

JUNE 2005

ABSTRACT

This paper examines the consequences of (*S*, *s*) pricing rules in a dynamic economy with heterogeneous costs of price adjustment. We construct the stationary distributions for aggregate output and prices for our model economy. As a result of our assumption of heterogeneous costs we find that: (i) Some sectors change prices more regularly than others; (ii) Price changes are asynchronized (relative prices may be moving in opposite directions in different sectors); (iii) The economy may be more sensitive to demand shocks. There is broad empirical support for the predictions of the model.

JEL Classification: E31, E32, E37, E58.
Keywords: Price rigidity, (*S*, *s*) pricing, macroeconomic dynamics.

* Acknowledgments: Vladislav Damjanovic is grateful for a Royal Economic Society research fellowship. We are grateful for comments from Harald Uhlig on this, and related, work. We are also grateful to seminar participants at The University of St Andrews and The Scottish Economic Society Conference, Perth, 2005. The usual disclaimer applies.

† School of Economics and Finance, Castlecliffe, The Scores, St Andrews, Fife KY16 9AL, Scotland, UK. Tel: +44 (0) 1334 462445. E-mail: vd6@st-andrews.ac.uk. Web: www.st-andrews.ac.uk/economics/CDMA/pages/v.damjanovic.shtml.

‡ Corresponding author: School of Economics and Finance, Castlecliffe, The Scores, St Andrews, Fife, KY16 9AL, Scotland, UK. Tel: +44 1334 462449. E-mail: Charles.Nolan@st-andrews.ac.uk. Web: www.st-andrews.ac.uk/economics/staff/pages/c.nolan.shtml.

CASTLECLIFFE, SCHOOL OF ECONOMICS & FINANCE, UNIVERSITY OF ST ANDREWS, KY16 9AL
TEL: +44 (0)1334 462445 FAX: +44 (0)1334 462444 EMAIL: AWT2@ST-AND.AC.UK
WWW.ST-ANDREWS.AC.UK/ECONOMICS/CDMA/CDMA.SHTML

1. Introduction

In this paper we pursue further an influential line of thought regarding nominal price stickiness dating back to Barro (1972) and Sheshinski and Weiss (1977, 1983), analyses that were in turn motivated by the seminal work of Arrow, Harris and Marschak (1951). This work on (S, s) pricing policies has been developed recently by Caplin and Spulber (1987), Caplin and Leahy (1991), Caballero and Engel (1991) and Caplin and Leahy (1997), and more recently still by Golosov and Lucas (2003).

The basic idea behind all these (S, s) pricing models is straightforward enough; firms face a resource cost of adjusting prices when demand or cost conditions alter. As a consequence, observed prices of most, perhaps almost all, goods will differ from what would be the optimal price in the absence of this cost. The more recent analyses in this vein have probed the macroeconomic significance of these (S, s) pricing strategies. One of the basic insights that seems to emerge is that nominal shocks will generally result in fluctuations in real magnitudes, although that need not always be so, as Caplin and Spulber showed (1987). However, many macroeconomists believe such nonneutralities to be an important stylized fact that macroeconomic models must match.

But there are other aspects of the data that many of these models have difficulty matching. First, in these models the impact of nominal shocks on aggregate output is modest, short-lived and probably less than what we observe in the data. The best evidence we know of on this score is Christiano et al. (2001). Second, in practice price changes may not be highly synchronized (i.e., different prices may often move in different directions in different sectors), as Bils and Klenow (2004) show for the US and Dhyne et al. (2004) demonstrate for countries in the Euro area, whilst many (S, s) pricing models generally predict a high level of synchronization. Third, there is evidence that there are systematic differences across sectors in the economy in the frequency of price adjustment; again Bils and Klenow (2004) document this for the US and Dhyne et al. (2004) show that the same is true in the Euro area.[1]

This observed degree of price change asynchronization and the systematic sectoral asymmetries suggest that heterogeneity is an important issue that needs to be incorporated in any successful (S, s) model. In an important contribution, Golosov and Lucas (2003) introduce asymmetries in the form of idiosyncratic shocks and

[1] Dhyne et al. also document that the Euro area appears to have "stickier" prices than is the case in the US.

by simulating their model show that this helps rationalize price change asynchronization. However, nominal shocks (even when monetary policy lacks credibility) continue to explain only a minor amount of variation in output.

Introducing idiosyncratic shocks in this way is attractive, but it may be a less useful way of incorporating systematic variation across sectors in the degree of price flexibility. One could imagine a model with different sectors of the economy systematically being buffeted by different sized shocks. Such a model would appear capable of rationalizing price change asynchronization and systematic differences in the frequency of price changes across sectors, with sectors prone to big shocks (on a systematic basis) tending to change prices more frequently. However, as we are interested, as far as possible, in obtaining analytical insights to the questions we pose, such a model would pose formidable technical challenges.

In this paper we take a simpler approach which turns out to yield substantial analytical results. Motivated by the observation that we just mentioned, that there appears to be some systematic variation in the frequency of price adjustment across goods, we analyze the effects of heterogenous costs of price adjustment. We extend the important work of Caplin and Leahy (1997) by introducing multiple sectors into the model economy, where each of these sectors is indexed by a different cost of price adjustment. In each sector of the economy firms face the same generic optimization problem as in the single sector set-up of Caplin and Leahy (1997), but it turns out that their actions have starkly different implications for aggregate output and prices. Such a set-up is also consistent with a degree of price asynchronization. As we demonstrate, the distribution of relative prices ceases to be uniform as Caplin and Leahy (1997) found. This prediction of the model was, in any case, counterfactual. We are able analytically to characterize the stationary distributions of output and prices for our model economy. We also show that the stationary distribution of aggregate output ceases to be uniform, and becomes dependent on the number of sectors.

Our model with multiple sectors may also deliver more plausible aggregate dynamics than a single sector model. (S, s) models based on homogeneity of costs tend to imply somewhat rigid dynamics; a sequence of positive (negative) monetary shocks causes output to rise (fall), while entailing no nominal price response, until some boundary is reached; further shocks in that positive (negative) sequence affect only prices. With heterogeneous costs of price adjustment the aggregate dynamics are more nuanced. For example, we demonstrate that, in the stationary state, the correlation coefficient between money shocks and output initially rises in the variance of the money stock before falling, in contrast to the findings of Caplin and Leahy (1997). We reconcile our result with theirs. We also show that as we add heterogeneous sectors the correlation between money shocks and output is lower in the multiple sector case (compared with the single sector case) for relatively low monetary variance, and higher for relatively high monetary variance. Adding heterogenous sectors in the way we do appears to hold out the promise of increasing the importance of nominal shocks. Further, a natural implication of our set-up is a degree of price asynchronization as in some sectors average real prices may be rising or falling whilst in other sectors they are moving in the opposite direction.

The paper is organized as follows. In Section 2 we set out our framework by extending the model of Caplin and Leahy (1997). In Section 3 we define the optimal behavior of agents in different sectors and discuss the distributions of outputs and relative prices in each sector of the economy. We show that the correlation between outputs in different sectors is intimately related to the cost of price adjustment, and the sense in which price changes are asynchronized emerges here also. In Section 4 we obtain an explicit solution for the stationary distribution of aggregate output in the case of two sectors. Appendix 4 shows how to generalize that derivation, first to the case of three sectors and then to the $K-$sector case. In Section 5 we analyze the interaction at the macroeconomic level between money, output and prices and demonstrate that our model economy may be sensitive to monetary shocks. In Section 6 we summarize and conclude.

2. The Model

Our model is a $K-$sector model building on the basic framework pioneered by Blanchard and Kiyotaki (1987)[2]. In turn, each sector is comprised of a continuum of firms. Firms in each sector are monopolistic competitors. The marginal cost of production (absent fixed costs of price adjustment) is constant across firms and sectors although, in equilibrium, firms do not all charge the same price. At instant t the optimal frictionless price for any firm f in sector i is given by

$$p^*(t,i,f) = P(t) + \alpha Y(t). \tag{2.1}$$

That is, the 'optimal' price, in the absence of costs of adjustment, is a linear function of (the log of) the aggregate price level $P(t)$ and (the log of) real aggregate

[2] The underlying model is developed in more detail in the appendix, whilst in the main text we use a linear approximate version.

demand $Y(t)$. α is a measure of strategic interaction between firms. In particular, $\alpha < 1$ indicates strategic complementarity such that firms tends to raise their prices when other firms do the same. On the other hand, if $\alpha > 1$ firms exhibit strategic substitutability; firm i tends to decrease its nominal price when other firms increase them. From now on, where we can safely do so, we suppress the index i, assuming that α is the same across sectors. We distinguish between different sectors of the economy by the costs of price adjustment in each sector, $c \in C$. We assume that the costs of price adjustment are 'sufficiently different' across sectors such that when one sector starts to change nominal prices, this does not immediately cause firms in 'nearby' sectors to change prices.

The aggregate price-level and aggregate demand are defined in the following way,

$$
\begin{aligned}
P(t) &= \Phi\left[\sum_{c \in C}\left\{\int p(t,f,c)df\right\}\right]; \\
Y(t) &= \Theta\left[\sum_{c \in C}\left\{\int y(t,f,c)df\right\}\right].
\end{aligned}
$$

An appendix provides more details on the functions $\Phi[\cdot]$, $\Theta[\cdot]$. The aggregate price level is a function of sectoral prices, which are in turn themselves a basket of individual firms' prices. Similar considerations obtain with respect to output. There is a cost to changing prices and this cost differs *systematically* across sectors of the economy which are enumerated by the number of elements (K) in the set C[3]. In each sector we assume that the continuum of firms all face the same cost. In Caplin and Leahy (1997) C is a singleton. We assume that costs c are positive and bounded, $c \in (0, c_{\max}]$. Firms in this economy will therefore differ along two dimensions. First, there will be, in equilibrium, a non-degenerate distribution of relative prices within each sector. Second, the distribution of costs across sectors will mean that different sectors will produce different levels of output, before finally changing prices.

If velocity of money is constant, we may write the relationship between the log of aggregate real money balances, $M(t)$, and (demand-determined) output as

$$Y(t) = M(t) - P(t). \tag{2.2}$$

[3]We emphasise that costs of adjustment do not vary stochastically across firms. The result of randomly altering the costs of adjustment may have a limited impact on our analysis here, as in Caballero and Engel (1991b), where the distribution of relative prices remains unaltered.

We assume that money evolves continuously without drift,

$$dM(t) = \sigma dW(t), \tag{2.3}$$

where $W(t)$ is a Wiener process.

Each firm in each sector faces a control problem of the same sort. In what follows, to avoid notational clutter, we shall drop the t index. For each firm the instantaneous loss in real profits is a quadratic function of the deviation of its nominal price from the optimal price given by

$$\begin{aligned} L(p(i), Y, i) &= \gamma(p(i) - p^*(i))^2; & (2.4) \\ &= \gamma(p(i) - P - \alpha Y)^2; \\ &= \gamma(x(i) - \alpha Y)^2, & (2.5) \end{aligned}$$

where $x(i) \equiv p(i) - P$ is a relative price. In each sector agents face identical problems i.e., to minimize the expected present value of lost profit given the cost of price adjustment. Assuming that the discount factor r is constant, the value function of the firm at moment t can be expressed in the form:

$$V(p(i), Y) = \min \left\{ E_t \int_t^\infty e^{-rs} L\left[p(i), Y\right] ds + \sum_j e^{-rT_j} c \right\}, \tag{2.6}$$

where E_t stands for the expectations operator and the sequence $\{T_j\}$ represents the time when the j^{th} adjustment takes place.

As we show in the appendix, the value function is given by:

$$\begin{aligned} V(x, Y) = \frac{\gamma}{r}(x - \alpha Y)^2 + \gamma \left[\frac{\alpha\sigma}{r}\right]^2 + \frac{2\gamma}{r}\frac{\alpha - 1}{(e^{2\beta\overline{Y}} - e^{-2\beta\overline{Y}})} \times \\ \left\{ \begin{array}{c} \left(\frac{x-\alpha\overline{Y}}{\beta} - \frac{1}{\beta^2}\right) e^{\beta\overline{Y}} e^{\beta Y} - \left(\frac{x+\alpha\overline{Y}}{\beta} - \frac{1}{\beta^2}\right) e^{-\beta\overline{Y}} e^{\beta Y} \\ - \left(\frac{x+\alpha\overline{Y}}{\beta} + \frac{1}{\beta^2}\right) e^{\beta\overline{Y}} e^{-\beta Y} + \left(\frac{x-\alpha\overline{Y}}{\beta} + \frac{1}{\beta^2}\right) e^{-\beta\overline{Y}} e^{-\beta Y} \end{array} \right\} \\ - \kappa(e^{\beta x} e^{-\beta Y} + e^{-\beta x} e^{\beta Y}). \end{aligned}$$

There are several other optimality conditions related to an equilibrium of the model. First there is the smooth contact conditions,

$$\left.\frac{\partial V(x, Y)}{\partial x}\right|_{Y=\pm\overline{Y}} = - \left.\frac{\partial V(x, Y)}{\partial Y}\right|_{Y=\pm\overline{Y}}. \tag{2.7}$$

Then there is the value matching condition,

$$V(-S, \overline{Y}) = V(S, \overline{Y}) + c. \tag{2.8}$$

The optimal choice of target is given by

$$\left.\frac{\partial V(x, Y)}{\partial x}\right|_{x=-S} = -\left.\frac{\partial V(x, Y)}{\partial x}\right|_{x=S}, \tag{2.9}$$

and the smooth pasting condition is

$$\left.\frac{\partial V(x, Y)}{\partial x}\right|_{x=S, Y=\overline{Y}} = 0. \tag{2.10}$$

Appendix 2 derives the value function and evaluates the accompanying optimality conditions to find a value for κ, a variable defined in the appendix, and a pair of simultaneous equations necessary and sufficient to solve for $\overline{Y}$ and S. Here S denotes denotes the lower bound on the real relative price, and $\overline{Y}$ the bound on output.

3. The distribution of output and prices in different sectors

In this section we take a closer look at the distributions of relative prices and outputs in each sector of economy. In the one sector model with two sided shocks, we know that relative (i.e., real) prices will optimally remain uniformly distributed, if they are initially uniformly distributed. And the same is true for output; this will also be uniformly distributed in the stationary state (Harrison, 1985). The same applies in the case of a K-sector economy. In each sector the behavior of agents is defined by their costs of price adjustments. As already noted firms face the same optimization problem but with different costs of price adjustment. The next proposition will prove useful.

Proposition 3.1. *The sequences of optimal boundaries* $\{S(c)\}$ *and* $\{\overline{Y}(c)\}$ *are strictly increasing functions of cost.*

Proof: The set of equations for determining optimal boundaries are derived in Appendix 2 (A.17 and A.18) and reproduced here as (3.1) and (3.2):

$$\alpha\overline{Y} + \frac{(1-\alpha)}{\beta}\tanh(\beta\overline{Y}) = S\coth(\beta S)\tanh(\beta\overline{Y}); \tag{3.1}$$

$$S(\beta\overline{Y} - \tanh(\beta\overline{Y})) = \delta c, \tag{3.2}$$

where $\delta = r\beta/(4\alpha\gamma)$ and $\beta = \sqrt{2r/\sigma^2}$.

The solutions of equations (3.1) and (3.2) determine optimal S and $\overline{Y}$ for a particular sector c. Unfortunately, this system of equations does not yield an analytical solution, but it is still possible to obtain asymptotic solutions in the region of small and large costs of adjustments. We proceed by rewriting equation (3.1):

$$\frac{\beta S \coth(\beta S) - 1}{\beta\overline{Y}\coth(\beta\overline{Y}) - 1} = \alpha. \tag{3.3}$$

Consider equation (3.2) and let c increase. It follows that the left hand side of (3.2) must also increase. Consequently, there are three possibilities: Both S and $\overline{Y}$ increase; S increases, while $\overline{Y}$ decreases with $\lim \overline{Y} = \text{constant} > 0$; or $\overline{Y}$ increases, and S decreases with $\lim S = \text{constant} > 0$; the constant has to be positive to keep the left hand side of (3.2) positive. It turns out that only the first of these possibilities is a solution as the second and third violate condition (3.3). The solutions of (3.1) and (3.2) for various costs are given in Figure 3.1. ■

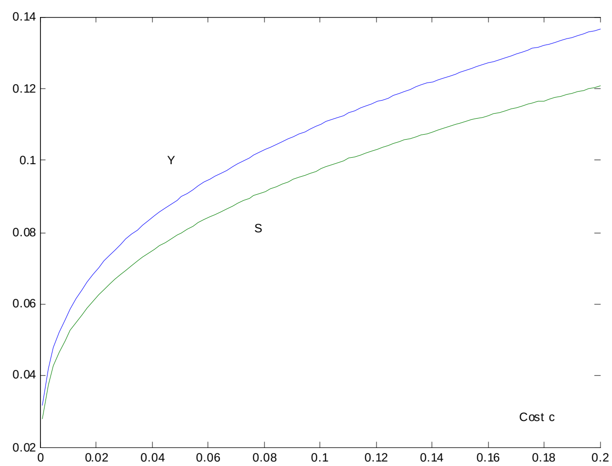

Figure 3.1: Optimal boundaries $S(c)$, $\overline{Y}(c)$ as a functions of cost of price adjustments.

A direct consequence of Proposition 3.1 is that, in the stationary state, the inaction regions $[-S(c), S(c)] \cup [-\overline{Y}(c), \overline{Y}(c)]$ are nested as depicted in Figure 3.2

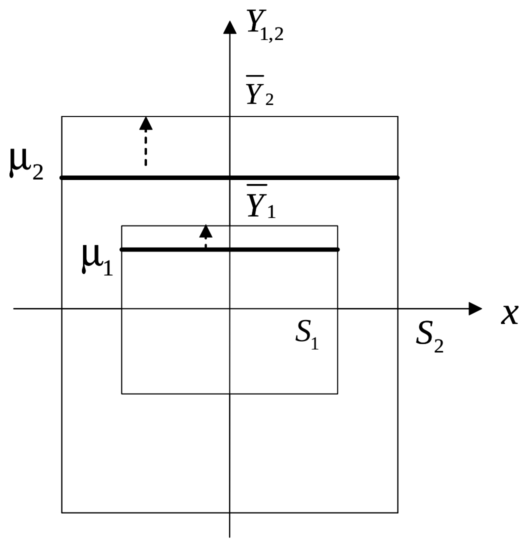

Figure 3.2: Nested regions of innaction for two sectors. The sectoral densities of relative prices ('elevators') in both sectors move up and down when money rises or falls.

for the case of two sectors. Hence we define a stationary equilibrium of our model as a straightforward generalization of the one sector model of Caplin and Leahy (1997):

Definition 3.2. *: An equilibrium is an initial distribution of prices and set of pricing strategies such that :*

(i) the aggregate price index only changes when at least one sectoral output level is at $\pm\overline{Y}(c)$ *and relative prices hit* $\mp S(c)$*;*

(ii) in each sector firms change their relative prices only when output is at $\pm\overline{Y}(c)$ *and relative prices are at* $\mp S(c)$*;*

(iii) in each sector relative prices are distributed uniformly over the interval $[-S(c), +S(c)]$*;*

(iv) no firm deviates from the price adjustment strategy in (ii).

We conclude this section by formulating and proving one important result about the distribution of relative prices: We show that the distribution of costs of price adjustment determines the distribution of relative prices in an equilibrium. For simplicity, consider the case of two sectors. Let total firms be of measure one. The measure of firms in sectors one and two are denoted by n_1 and n_2, respectively. We then have that $n_1 + n_2 = 1$. Since in each sector the same rule of price adjustment applies, the fraction n_1 of firms is uniformly distributed over an interval of relative prices $[-S_1, S_1]$ and so too is the fraction n_2 of firms over the interval $[-S_2, S_2]$. We choose $S_1 < S_2$ ($c_1 < c_2$). The distribution of aggregate relative prices is not uniform (as it would be in a single sector set-up) but it is invariant in time as depicted in Figure 3.3. It is not hard to see that any distributional shape can be achieved by choosing a specific distribution of costs, $n(c)$. The corresponding densities for the case of two sectors are depicted in Figure 3.3. By way of illustration, it is straightforward algebraically to describe the densities of relative prices for a two and three sector economy. In turn these are:

$$\mu_2(x) = \begin{cases} n_2/2S_2 & -S_2 \leq x < -S_1 \\ n_1/2S_1 + n_2/2S_2 & -S_1 \leq x \leq S_1 \\ n_2/2S_2 & S_1 < x \leq S_2 \end{cases} ; \qquad (3.4)$$

$$\mu_3(x) = \begin{cases} n_3/2S_3 & -S_3 \leq x < -S_2 \cup S_2 < x \leq S_3 \\ n_1/2S_1 + n_2/2S_2 + n_3/2S_3 & -S_1 \leq x \leq S_1 \\ n_2/2S_2 + n_3/2S_3 & -S_2 \leq x < -S_1 \cup S_1 < x \leq S_2 \end{cases} . \qquad (3.5)$$

It is easy to show that $\int \mu(x)dx = 1$. In particular, for the two dimensional case we have $\int \mu(x)dx = 2(S_2 - S_1)\frac{n_2}{2S_2} + 2S_1(\frac{n_1}{2S_1} + \frac{n_2}{2S_2}) = n_1 + n_2 = 1$.

Proceeding further in the same fashion we can construct any distributional shape by specifying $n(c)$. The generalization of this analysis to the continuous case is straightforward and we conclude this section by formulating it in the next proposition.

Proposition 3.3. *Let $n(c)$ be the density of firms in respect of their cost of relative price adjustment, i.e., $\int n(c)dc = 1$, where c takes its values from some bounded set. Then the density of relative prices, x, is given by:*

$$\mu(x) = \int_x^{c_{\max}} \frac{n(c)}{2S(c)} dc,$$

where $x \in [-S(c_{\max}), S(c_{\max})]$ and x and c are given by equations (3.1) and (3.2).

It is immediately clear that price changes are not synchronized across sectors. For simplicity, consider the two sector case and assume that money rises. Then

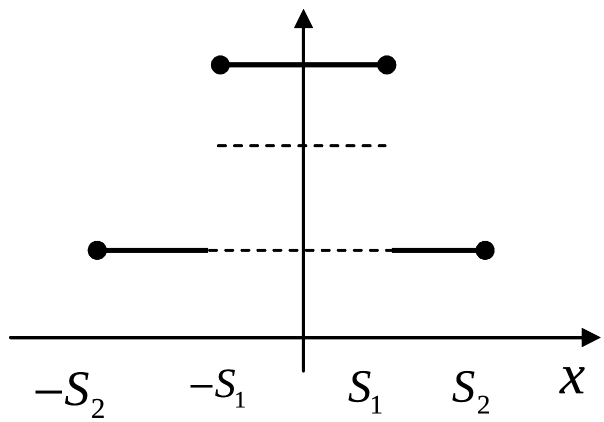

Figure 3.3: The density of relative prices $\mu_2(x)$ (bold line) in the case of two sectors defined by (3.4). Dashed lines represent sectoral densities: $1/2\mathrm{S}_2$ (lower line) and $1/2\mathrm{S}_1$(upper line).

the "elevator" (that is, the distribution of relative prices) in the first sector (with lower cost of adjustment) will hit the boundary $\overline{Y}_1$ first, causing nominal price adjustment in that sector while the elevator in the second sector will continue to travel up towards to boundary $\overline{Y}_2$ (see Figure 3.2). At that instant of time adjustment will take place in the first sector only. The situation where firms from the sector with higher cost of adjustment change nominal prices but not firms from the lower cost sector is, of course, possible but only for particular paths defined on specific initial conditions; it can only happen a finite number of times. In the stationary state it cannot be the case that the higher cost sector adjusts but the lower cost sector does not. So in this model synchronization/asynchronization of relative prices is understood as follows: When sector $c^* < c_{\max}$ adjusts then all sectors with costs $c < c^*$ also adjust, but those sectors with costs $c > c^*$, do not necessarily adjust prices.

Overall, then, our model predicts that the frequency of price changes is likely to differ systematically across sectors, and that these changes will not be synchronized. These findings appear to be broadly in line with Dhyne et al. (2004) where a high degree of heterogeneity in price setting behavior, across both products and sectors, is clearly documented. The heterogeneity is reflected in the frequency of price setting and in the absence of price change synchronization across different sectors. Interestingly, Dhyne et al. also document that there is no evidence of strong downward price rigidity in the Euro area. On average 40% of price changes are downward movements. Our model with symmetric and nested boundaries is also capable of generating similar behavior.

4. Stationary distribution of aggregate output in the case of two sectors

We now consider the dynamics of aggregate output for the case where different costs of price adjustment lead to optimal boundaries for output, derived from equations $(2.6)-(2.10)$, that are characterized by the strictly increasing finite set, $\left\{\overline{Y}_i | i=1,2...K\right\}$. Incorporating heterogeneity with respect to costs of price adjustment one would expect that the dynamics of aggregate output should change materially from the one sector case, where we know that output is uniformly distributed. Indeed this is the case. Intuitively, although firms in different sectors have different costs of price adjustment, there will still be some range over which their respective outputs rise and fall together. Eventually, however, demand rises sufficiently that some firms start adjusting nominal prices, whilst others continue to meet demand at their current posted prices. At this point, their supply responses diverge. It turns out that the stationary distribution of aggregate output can, therefore, be split into two parts. One part corresponds to the uniform distribution of absolutely correlated outputs. The other part of the distribution represents the sum of independent random variables. This is an important result which, as we show in an appendix, is generalizable. We formally state this in the next theorem for the two dimensional case, $K=2$.

Theorem 4.1. *Let the economy consists of K different sectors with strictly increasing optimal boundaries $\left\{\overline{Y}_i | i=1,2\right\}$. Let output Y_i in each sector i follow a regulated Brownian motion. Then, in the stationary state, the density function of aggregate output, defined as $Y=Y_1+Y_2$, is given by a weighted average of two densities:*

$$\mu(Y)=(1-\omega)\mu_1+\omega\mu_2(Y=2z_1+z_2), \tag{4.1}$$

where weight ω is given by:

$$\omega=\frac{1}{2}\frac{\overline{Y}_1(\overline{Y}_2-\overline{Y}_1)}{\overline{Y}_1\overline{Y}_2}.$$

μ_1 denotes the density of a uniformly distributed random variable on the interval $[-\overline{Y_1}-\overline{Y_2},\overline{Y_1}+\overline{Y_2}]$ i.e.,

$$\mu_1 = \frac{1}{2(\overline{Y_1} + \overline{Y_2})},$$

and μ_2 *is the density of the sum of two independently and non-identically distributed random variables,* $2z_1$ *and* z_2*, with corresponding densities:*

$$\mu(z_1) = \frac{1}{2\overline{Y_1}};$$

$$\mu(z_2) = \frac{1}{2(\overline{Y_2} - \overline{Y}_1)}.$$

PROOF: We are considering the two dimensional case $K = 2$ with boundaries $\overline{Y}_1 < \overline{Y}_2$. We may construct the state-space for pairs (Y_1, Y_2) shown in Figure 4.1. For simplicity, and only temporarily, suppose that outputs Y_1 and Y_2 take only discrete and equidistant integer values in the intervals $[-N_1, N_1]$ and $[-N_2, N_2]$ respectively, i.e., $Y_1 \in \{-N_1, -N_1+1, ., 0, .., N_1-1, N_1\}$ and $Y_2 \in \{-N_2, -N_2+1, ., 0, .., N_2-1, N_2\}$. The feasible set of states of the system are those represented by points of the parallelogram $ABCD$ including the points on its boundaries. In principle the system can start from any point outside of this set but once it reaches any point on its boundary (that will happen for sure in finite time) it remains inside that set forever. After we characterize this density for the discrete case, we generalize to the continuous case and all the results and analysis will apply in the limit as $\Delta Y \to 0$ ($N \to \infty$), where we demonstrate that $\Delta Y = 2\overline{Y}/(2N+1)$.

We proceed directly to find the joint distribution in the stationary state, denoted by $\pi(Y_1, Y_2)$, by solving the eigenvalue problem for the Markov transition matrix P:

$$\pi P = \pi. \tag{4.2}$$

The number of different states, i.e., the number of points in the set $ACBD$, is $n = n_1\overline{n}_2$ where $n_1 = 2N_1 + 1$ and $\overline{n}_2 = 2(N_2 - N_1) + 1$. Obviously, $\dim(\pi) = n$ and P is a square matrix with dimensions $n \times n$.

If we consider any subset of admissible states with Y_2 fixed (that is, points parallel to the Y_1 axis), then for such a set the following equation must be satisfied: $p(Y_2) = \sum_j \pi(Y_2, Y_j)$, where $p(Y_2)$ is the unconditional probability of Y_2, and $\pi(Y_2, Y_i)$ are joint probabilities. However, it must be the case that $p(Y_2) = 1/n_2$ where $n_2 = 2N_2 + 1$ because in the stationary state outputs in each band are uniformly distributed. It follows, then, that the joint probabilities in corners A and C are $\pi_A = \pi_C = 1/n_2$. Now start from corner B and move to A along the edge BA and write down the corresponding balance equations. As appendix 3 shows in detail, we find that along BA the joint probabilities are given by:

$$\pi_{BA}(\zeta) = \frac{\zeta}{n_1 n_2};$$

and along the edge CD by:

$$\pi_{CD}(\zeta) = \frac{n_1 + 1 - \zeta}{n_1 n_2},$$

where $\zeta = 1, 2, ...n_1$. At all other points $\pi = \pi_B = 1/n_1 n_2$.

Having calculated the stationary distribution of joint probabilities it is possible to find the stationary probability distribution of aggregate output. We start by calculating the probability function $P(Y)$ defined in the usual way:

$$P(Y) = \sum_{\xi} P(\xi \leq Y). \tag{4.3}$$

On the other hand, the unconditional probabilities $P(\xi)$ are given by:

$$P(\xi) = \sum \pi(Y_1, Y_2), \tag{4.4}$$

where the sum has to be taken over all values of Y_1 and Y_2 and where $\xi = Y_1 + Y_2 \leq Y$. In Figure 4.1 it corresponds to all points lying on the line $Y_2 = \xi - Y_1$ for a certain ξ. Combining (4.3) and (4.4) we see that in order to find the probability function $P(Y)$ one has to sum joint probabilities over all points lying on the lines

$$Y_1 + Y_2 = \xi \leq Y. \tag{4.5}$$

We perform this summation noticing that lines from (4.5) are divided into two groups. It is worth noting now, that this sub-division provides us with our clue on how to take these discrete calculations to the proper continuous limit. The first group is represented by lines containing the points on the edges $\overline{AB}$ and $\overline{CD}$ where we have different joint probabilities. We label such points as elements in the 'exterior' group (ξ^E lines). The remaining probabilities are elements in the 'interior' group (ξ^I lines). These are depicted in Figure 4.1.

Consider first the exterior group of lines and calculate $P(\xi)$ for an arbitrary line. Before we proceed, we rewrite the joint probabilities of points lying on the exterior lines in a more convenient forms as:

$$\pi^{\pm}(\zeta) = (\pm\zeta + \overline{Y}_1 + 1)\pi_B$$

where $\zeta = -\overline{Y}_1, -\overline{Y}_1 + 1, ..., \overline{Y}_1 - 1, \overline{Y}_1$, along the upper $(+)$ and lower $(-)$ edges respectively, and π_B otherwise.

Now calculate $P(\xi)$ directly from (4.4) for some ξ. Let $\Delta\overline{Y} \equiv \overline{Y}_2 - \overline{Y}_1$. The line $Y_2 = \xi - Y_1$ intersects the upper edge $\overline{BA}$ in $\zeta = (\xi - \Delta\overline{Y})/2$ and the lower edge $\overline{CD}$ in $\zeta = (\xi + \Delta\overline{Y})/2$, as depicted in Figure 4.1. Then equation (4.4) reads:

$$P(\xi) = \pi^{+}(\frac{\xi - \Delta\overline{Y}}{2}) + \pi^{-}(\frac{\xi + \Delta\overline{Y}}{2}) + \frac{1}{2}(2\Delta\overline{Y} - 2)\pi_B,$$

where the last term is simply the number of points between intersections excluding the two points on the edges. By direct calculation, we then have that

$$\begin{aligned}
&= (\frac{\xi - \Delta\overline{Y}}{2} + \overline{Y}_1 + 1)\pi_B + (-\frac{\xi + \Delta\overline{Y}}{2} + \overline{Y}_1 + 1)\pi_B + (\Delta\overline{Y} - 1)\pi_B; \\
&= -\Delta\overline{Y}\pi_B + 2(\overline{Y}_1 + 1)\pi_B + \Delta\overline{Y}\pi_B - \pi_B; \\
&= (2\overline{Y}_1 + 1)\pi_B; \\
&= 1/(2\overline{Y}_2 + 1) = 1/n_2.
\end{aligned}$$

When the line ξ does not intersect both edges the above relations also hold. For example we can conclude immediately that $P(\xi = N_1 + N_2) = P(\xi = -N_1 - N_2) = 1/n_2 = 1/(2\overline{Y}_2 + 1)$. Moving one step further from, for example, corner C gives: $\pi^{-}(-\overline{Y}_1 + 1) + \pi_B = (-(-\overline{Y}_1 + 1) + \overline{Y}_1 + 1)\pi_B + \pi_B = (2\overline{Y}_1 + 1)\pi_B = 1/(2\overline{Y}_2 + 1)$. Continuing in a similar fashion we conclude that for any ξ representing an exterior line, the sum of joint probabilities is the same and equal to $1/(2\overline{Y}_2 + 1)$. For points belonging to interior lines the contribution is simply equal to $\frac{1}{n_1 n_2} n(\xi)$, where $n(\xi)$ is the number of points on the interior line, ξ^I. It follows that (4.3) can be written as

$$P(Y) = \sum_{\xi^{Ee} \leq Y} \frac{1}{n_2} + \sum_{\xi^{I} \leq Y} \frac{1}{n_1 n_2} n(\xi) \quad . \tag{4.6}$$

Let $Y = \overline{Y}_1 + \overline{Y}_2 = \frac{n_1 + n_2}{2} - 1$. Then from (4.6) we have:

$$P(\frac{n_1 + n_2}{2} - 1) = \frac{n_1 + n_2}{2n_2} + \frac{1}{n_1 n_2} \frac{n_1(n_2 - n_1)}{2} = 1,$$

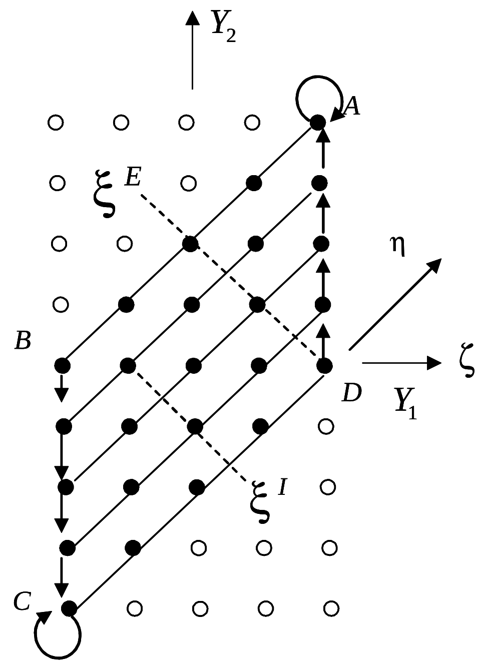

Figure 4.1: Joint state space (Y_1, Y_2) for outputs from two sectors is represented by bold dots for values $\overline{Y}_1 = 2, \overline{Y}_2 = 4$ so that $Y_1 = -2, -1, 0, 1, 2$ and $Y_2 = -4, -3, ..0, ..3, 4$ $(n_1 = 5, n_2 = 9)$. Two way moving between the states is represented by bold lines while one way moving is represented by arrowed lines. Arrowed circles represent possibilities of remaining at corners A and C.

as must be the case.

It is easy to rewrite (4.6) in a more explicit form, but instead of that we now proceed to derive an expression for the probability function in the continuous case. Introduce the continuous variable $\eta \in [0, 2(a+b)]$ to enumerate aggregate output from the interval $[-a-b, a+b]$. Then on the scale η point D in Figure 4.1 has coordinate $4a$ and the measure of edge $\overline{DA}$ is $2(b-a)$. To see that this is indeed the case recall that in the discrete case just half of the ξ lines (here we speak of η lines) terminate on edge $\overline{CD}$. The measure of edge $\overline{DA}$ remains unchanged as all ξ lines terminate on this edge. In the continuous limit this must be preserved so that on the scale η properly applying (4.6) means changing $\eta \to \eta/2$. The situation is similar for $n(\eta)$, as it is just half of the interval lying on the η line.

To illustrate this, we now provide an explicit calculation of $P(\eta)$ for $\eta \leq 2(b-a)$.

Hence:

$$n(\xi) = \xi/2,$$

where we require that,

$$\sum_{\xi^I \leq x} \rightarrow \int_0^\eta d(\xi/2),$$

and

$$\sum_{\xi^{Ee} \leq x} \rightarrow \frac{\eta}{2}.$$

Therefore, it follows that

$$n_1 \rightarrow 2a, n_2 \rightarrow 2b.$$

And applying (4.6) for the continuous case we obtain:

$$P(\eta) = \frac{1}{2b}\frac{\eta}{2} + \frac{1}{4ab}\int_0^\eta \frac{\xi}{2} d(\frac{\xi}{2}) = \frac{1}{2b}\frac{\eta}{2} + \frac{1}{4ab}\frac{\eta^2}{8}.$$

Finally we write an expression for the probability function as a sum of two terms, $P(\eta) = P^E(\eta) + P^I(\eta)$ where:

$$P^E(\eta) = \frac{1}{2b}\frac{\eta}{2}, \ 0 \leq \eta \leq 2(a+b)$$

and

$$P^I(\eta) = \begin{cases} \frac{1}{4ab}\frac{\eta^2}{8} & 0 \leq \eta \leq 2(b-a) \\ \frac{1}{4ab}\frac{b-a}{2}(\eta - b + a) & 2(b-a) \leq \eta \leq 4a \\ \frac{1}{4ab}[2a(b-a) - \frac{(\eta-2a-2b)^2}{8}] & 4a \leq \eta \leq 2(a+b) \end{cases}$$

It follows immediately that the density function is given by:

$$\begin{aligned} \mu(\eta) &= \frac{dP(\eta)}{dx} \\ &= \begin{cases} \frac{1}{4b} + \frac{1}{4ab}\frac{\eta}{4} & 0 \leq \eta \leq 2(b-a) \\ \frac{1}{4b} + \frac{1}{4ab}\frac{b-a}{2} & 2(b-a) \leq \eta \leq 4a \\ \frac{1}{4b} + \frac{1}{4ab}\frac{2(a+b)-\eta}{4} & 4a \leq \eta \leq 2(a+b) \end{cases} \end{aligned} \tag{4.7}$$

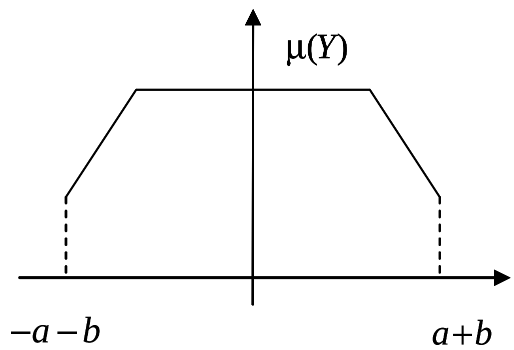

Figure 4.2: Figure 4.2: Density function $\mu(Y)$.

Finally rewriting the first term in the above expression as

$$\frac{1}{4b} = \frac{2a(2a+2b)}{2(2a)(2b)}\frac{1}{2a+2b} = (1 - \frac{1}{2}\frac{2a(2b-2a)}{(2a)(2b)})\frac{1}{2a+2b}$$

and in the same fashion the second terms as

$$\frac{1}{4ab}\frac{\eta}{4} = \frac{1}{2}\frac{2a(2b-2a)}{(2a)(2b)}\frac{\eta}{4a(2b-2a)}$$
$$\frac{1}{4ab}\frac{b-a}{2} = \frac{1}{2}\frac{2a(2b-2a)}{(2a)(2b)}\frac{1}{4a}$$
$$\frac{1}{4ab}\frac{2(a+b)-\eta}{4} = \frac{1}{2}\frac{2a(2b-2a)}{(2a)(2b)}\frac{2(a+b)-\eta}{4a(2b-2a)}$$

we recover expression (4.1) completing the proof.

As we conjectured above, we see that the density has two components. First, we recognize the influence of the uniformly distributed outputs from the two sectors that we intuited earlier. This is captured by the uniform density, $1/2(2b)$. In Appendix 4 it is shown that in the general case of K sectors the uniform part takes the form $1/K(2\overline{Y}_K)$. Second, there is the influence of the independent portions of the sectoral outputs, that is the density for the sum of two independent uniformly and nonidentically distributed random variables (see Rényi, 1970). In the above formulas for the probability function and the density function one should make the substitution $\eta \to Y + a + b$ where $-a-b \leq Y \leq a+b$ (see Figure 4.2).

5. The Relationship Between Money, Output and Prices

In the model of Caplin and Leahy (1997), the implied dynamic interactions between money, output and prices were an improvement on earlier (S, s) pricing models, but still somewhat rudimentary. The improvement was that output and money were positively correlated, up to a certain point, after which further changes in the money stock resulted merely in a rise in prices. In other words, state-dependent models were shown indeed to be consistent with Phillips-curve type behavior of a general sort that seems to characterize the data in many advanced industrial economies. On the downside, these dynamics seem somewhat 'angular'. Inside the barriers, output rises with money one for one; once the barriers are reached, all further monetary shocks (in the same direction as those that led output to hit the barrier) result merely in price rises[4]. In the case of K sectors these dynamic interactions are smoother. At the aggregate level, output will not in general change with money one-to-one, and the economy may be quite sensitive to demand shocks; these are the main results we are going to show in this section.

It will be useful to work with an alternative definition of regulated Brownian motion using a state-space discretization, in the spirit of the previous section. In the presence of two barriers $-\overline{Y}, \overline{Y}$ output Y follows regulated Brownian motion

$$dY(t) = dM(t) + dL(t) - dU(t)$$

where lower and upper regulators, $L(t)$ and $U(t)$ respectively, are continuous and non-decreasing stochastic processes, $Y(t) \in [-\overline{Y}, \overline{Y}]$ for all t and $L(t)$ increases only when $Y(t) = -\overline{Y}$ and $U(t)$ increases only when $Y(t) = \overline{Y}$. The role of regulators L, U is to keep output at the level $Y = -\overline{Y}(+\overline{Y})$ when money further decreases (increases). Then, at the next instant of time, output will stay at the boundary with positive probability. To see that more clearly we construct processes L, U in terms of the exogenous process $M(t)$. We do that by approximating the process $M(t)$ by a simple random walk on a lattice. Consider first the case with a single barrier $+\overline{Y}$ and define

$$U(t) = \begin{cases} 0 & t \leq T_{\overline{Y}} \\ \max_{s\in[0,t]} M(s) - \overline{Y} & t > T_{\overline{Y}} \end{cases}$$

where the stopping time, $T_{\overline{Y}}$, is defined as the first time when $M(t) = \overline{Y}$. Clearly $Y(t) \leq \overline{Y}$ for all t. In the presence of the lower barrier only, we have:

$$L(t) = \begin{cases} 0 & t \leq T_{-\overline{Y}} \\ -\overline{Y} - \min_{s\in[0,t]} M(s) & t > T_{-\overline{Y}} \end{cases}$$

[4] It is worth emphasising that this qualitative feature of the model is not merely a function of having closed the model with a simple quantity-type equation. With richer nominal specification, the same basic features would be present.

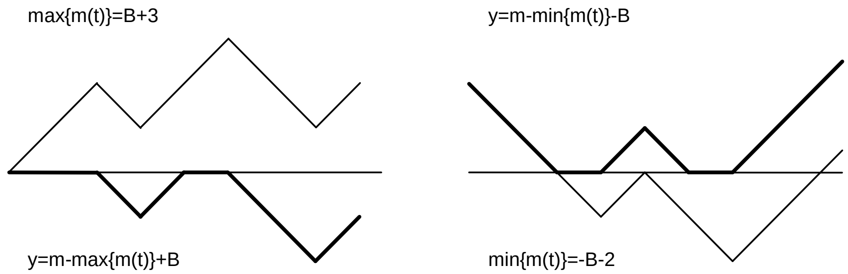

Figure 5.1: Regulated Brownian motion starting at $t = T_1$ and terminating at $t = T_1+8$ is illustrated. For example, on the upper boundary (left side) we have $Y(T_1+8) = M(T_1+8) - \max M + \overline{Y} = (\overline{Y}+2) - (\overline{Y}+2) + \overline{Y} = \overline{Y} - 1$; and $Y(T_1+5) = M(T_1+5) - 3 + \overline{Y} = \overline{Y} + 3 - 3 - \overline{Y} + \overline{Y} = \overline{Y}$ and on the lower boundary (right side) $Y(T_1+8) = M(T_1+8) - \overline{Y} - \min M = -\overline{Y} - \overline{Y} - (-\overline{Y} - 2) = -\overline{Y} + 2$.

where $T_{-\overline{Y}}$ is defined as the first time when $M(t) = -\overline{Y}$. The above construction ensures that $Y(t) \geq -\overline{Y}$ for all t. In Figure 5.1 two cases with upper and lower barriers are depicted. In the presence of two barriers the above constructions still apply but one has to keep track of consecutive sequences of stopping times $[T_{2i+1}, T_{2i}]$ or $[T_{2i}, T_{2i+1}]$ depending on wether the process $M(t)$ crosses first the upper or lower barrier; for more details see Stokey (2002). Instead of that, we provide an alternative definition for the regulated random walk.

Definition 5.1. *Let random variable X follow a simple random walk*[5]*:*

$$\Delta X(n) = X(n) - X(n-1) = \begin{cases} +\delta & \textit{with probability } 1/2 \\ -\delta & \textit{with probability } 1/2 \end{cases}$$

We say that random variable Z follows a random walk regulated by variable X and barriers $\pm\overline{Y}$ or follows a regulated random walk if

a) Z is bounded below and above, i.e.,$-\overline{Y} \leq Z(n) \leq +\overline{Y}$,($\overline{Y} > 0$) for all n and

b)

$$\Delta Z(n) = Z(n) - Z(n-1) = \begin{cases} 0 & if\ Z(n-1) = +\overline{Y} \textit{ and } \Delta X(n) = +\delta \\ 0 & if\ Z(n-1) = -\overline{Y} \textit{ and } \Delta X(n) = -\delta \\ \Delta X(n) & otherwise \end{cases}$$

[5] "simple" means that we exclude the possibility: $X(n) = X(n-1)$.

We can now justify our claim that the dynamic between money and output is indeed smoother in the heterogenous cost case. We do that by performing simulations of the correlation between changes in the money supply and output for one and two sector economies. As $E(Y) = E(\Delta M) = 0$ the correlation function is

$$\rho(Y, M(t') - M(t)) = E(Y \Delta M),$$

where we use normalized variables $\Delta m/\sigma$ and $Y/\sqrt{Var(Y)}$. On the other hand

$$Var(Y) = \int Y^2 \mu_{1(2)}(Y) dY, \qquad (5.1)$$

where for the one sector case $\mu_1(Y) = 1/2\overline{Y}$, and for the two sector case $\mu_2(Y)$ is given by (4.1). Using the definition of regulated Brownian motion we simulate paths of normalized Y and ΔM for both one and two sectors for different values of the standard deviation of money. Optimal boundaries $\overline{Y}(\sigma)$ are calculated from equations (3.1) and (3.2) with parameters $\gamma = 0.5$, $\alpha = 0.8$, $r = 0.05$ and $c_1 = 0.001$, $c_2 = 0.002$. Then time series of length $N = 10000$ for outputs Y_1, Y and ΔM are generated. Normalizing time to $T = 1$ we have that $\Delta t = 10^{-5}$. We found that $t' - t = 100\Delta t$ was enough to achieve reasonable convergence of outputs to their limiting distributions. The results of these simulations are presented in Figure 5.2.

For both cases there exists a maximum of the correlation function. We can conjecture such a maximum with equations (3.1) and (3.2) from which it is easy to verify that $\overline{Y}(\sigma) - \sigma$ changes sign from positive to negative after some value of the variance σ_c. For our chosen set of parameters, $\sigma_c = 0.033$ in the one sector case. We show that in the region $\sigma < \sigma_c$ the correlation is an increasing function of the variance of money. Before hitting the boundaries, the absolute value of output increases with increasing $|\Delta M|$ and, in addition, with rising variance more and more firms find themselves at their respective boundaries, and hence are compelled optimally to change their nominal prices. In the region $\sigma > \sigma_c$ with rising variance, $\overline{Y}$ rises allowing output to fluctuate more widely (output is still uniformly distributed) while at the same time fluctuations in money are much larger i.e., $\overline{Y} \ll \sigma$. Again, this can be verified using (3.1) and (3.2).

If the standard deviation of the money supply process were to rise over any finite period of time, then it can be shown that the probability of output reaching its limits rises. As a result, the correlation coefficient characterizing the money-output relation necessarily falls. That is the intuition behind Proposition 3 in Caplin in Leahy (1997).

In the stationary state that reasoning is no longer valid. Output is distributed uniformly over the whole interval $[-\overline{Y}, \overline{Y}]$. Now as the variance of money increases away from zero, $\overline{Y}$ rises more than proportionally with σ, $\Delta\overline{Y}(\sigma)/\ \Delta\sigma > 1$. Eventually, however, that effect subsides and the correlation coefficient falls. In Damjanovic and Nolan (2005) we analyze this issues further and show how it is possible to derive an analytical expression for the correlation function, drawing on the state-space discretization developed above.

Another striking features of this model is that aggregate output is less responsive to changes in the money supply for small values of the variance of the money stock than is the case in the one sector economy. However, for larger variances, that effect is reversed and monetary shocks may have a larger impact in the muti-sector economy. In the mutli-sector economy prices are less flexible and so monetary shocks can have a relatively large impact, particularly as the variance grows; for the high cost sector it requires a relatively large change in the variance of the money stock to counter the effect of widening barriers. Moreover, it appears from Figure 5.2 that the correlation between money and output may decline only rather gradually, even as the variance becomes quite large. This may well be a function of the fact that all goods in our economy are complements (perhaps goods in different sectors ought to be substitutes). We leave further investigation of that issue to future research. Suffice to say for the moment that the degree of complementarity that we have assumed does not seem wildly out of line with what other researchers have assumed.

6. Conclusion

This paper has presented a simple macroeconomic model with multiple sectors, which differ because firms in one sector face higher costs of price adjustment than firms in other sectors. Almost by construction, then, we build in an assumption that firms will tend to differ in the frequency of nominal price adjustment. That systematic difference was one of our motivating stylized facts. Based on that assumption, our model is then capable of replicating, qualitatively, the other two stylized facts; a potentially sizeable impact on output of a nominal shock, and a degree of price change asynchronization across sectors.

We derived these results by characterizing the stationary distribution of aggregate prices and output, given a stochastic driving process for money. We

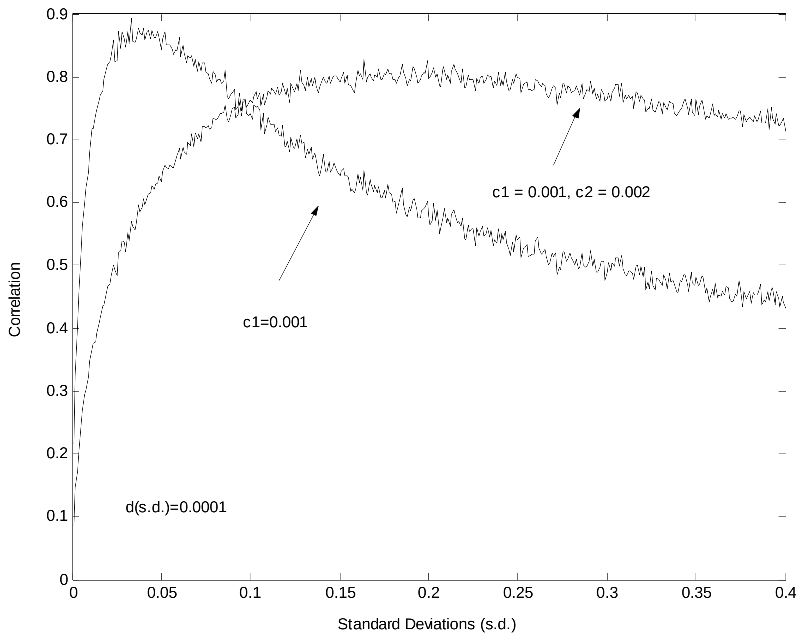

Figure 5.2: Correlation between output Y and changes in the money supply for one and two sectors respectively, as a function of the standard deviation of money, σ.

showed that introducing sectors with different costs of relative price adjustment may smooth the dynamics of aggregate output and money significantly in the sense that output does not follow the money supply one-to-one any more. Although sectoral outputs follow a regulated Brownian path with their own uniform distribution in stationary state, in characterizing output at the aggregate level we have to deal with the fact that sectoral outputs are also partially correlated. The source of the correlation is, of course, due to the underlying process for the money supply but the strength of the correlation depends how different sectors are (in terms of their costs of adjustment). The more similar are their costs of adjustment, the more correlated are sectoral outputs, while the further apart they are, the less correlated are outputs and the more asynchronized are sectoral price changes. We spelled out the behavior of a two sector economy in some detail, but by increasing the number of sectors the picture appears to remain qualitatively the same. For example, the density of aggregate output continues to consist of two parts; one component being simply the sum of independent random variables, while the other reflects the correlation across sectoral outputs. Regarding the distribution of relative prices, the picture is no less interesting. The density of relative prices is invariant in time and can in fact take any form. The particular form it takes depends not only of the size of the adjustment costs, but also on the distribution of firms in respect of those costs.

(S, s) pricing models based on heterogeneous costs of price adjustment appear to be a promising modelling route for macroeconomic models incorporating price rigidity. Our simple extensions resulted in a model with interesting aggregate dynamics, but also able to capture a number of important stylized facts recently documented in empirical analyses of price rigidity.

References

[1] Arrow, K.J., Harris, T., and Marschak, J., (1951), Optimal Inventory Policy, *Econometrica*, 19, 205-272.

[2] Bils, M and Klenow, P (2004), Some Evidence on the Importance of Sticky Prices, *Journal of Political Economy* 112, October, 947-985.

[3] Blanchard, O.J., Kiyotaki, N., (1987), Monopolistic Competition and the Effects of Aggregate Demand, *The American Economic Review* 77, 647-666.

[4] Barro, R. J., (1972), A Theory of Monopolistic Price Adjustment, *The Review of Economic Studies*, 39, 17-26.

[5] Caballero, R.J., Engel, E., (1993), Heterogeneity and Output fluctuations in a Dynamic Menu-Cost model, *The Review of Economics Studies* 60, 95-119.

[6] Caballero, R. and E. Engel, (1991), Dynamic (S, s) Economies, *Econometrica*, 59(6),1659-86.

[7] Canzoneri, M. B., Cumby, R. E., Diba, B. T., (2004), The Cost of Nominal Inertia in NNS Models, NBER Working Paper, 10889.

[8] Caplin, A. S., and Spulber D. F., (1987), Menu Costs and the Neutrality of Money, *Quarterly Journal of Economics*, 102: 703-726.

[9] Caplin, A., Leahy, J., (1997), Aggregation and Optimization with State-Dependent Pricing, *Econometrica* 65, 601-625.

[10] Chari, V.V., Kehoe, P.J., McGrattan, E.R., (2000), Sticky Price Models of the Business Cycle: Can the Contract Multiplier Solve the Persistence Problem?, *Econometrica*, vol. 68(5), pages 1151-1180.

[11] Christiano, L. J., Eichenbaum, M., and Evans, C., (2005), Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy, forthcoming in *The Journal of Political Economy*.

[12] Damjanovic, V., Nolan, C., (2005), On the Correlation between Money and Output in a Dynamic (S, s) Economy.

[13] Dhyne, E., Alvarez, L.J., Le Bihan, H., Veronese, G., Dias, D., Hoffman, J., Jonker, N., Lunermann, P., Rumler, F., and Vilmunen, J., (2004) Price Setting in the Euro Area: Some Stylised Facts from Individual Consumer Price Data mimeo, European Central Bank.

[14] Dotsey, M, King R.G., and Wolman A.L., (1999), State-Dependent Pricing and the General Equilibrium Dynamics of Money and Output, *Quarterly Journal of Economics,* 114: 655-690.

[15] Golosov, M., and Lucas E. R. Jr., (2003), Menu Costs and Phillips Curves, NBER, w10187 .

[16] Harrison, J. M.,(1985), Brownian Motion and Stochastic Flow System, New York: Springer Verlag.

[17] Lucas E. R. Jr., (1973), Some International Evidence on Output-Inflation Tradeoffs, *The American Economic Review* 63, 326-334.

[18] Rényi, A., (1970), Probability Theory (North Holland).

[19] Sheshinski, E., and Weiss Y., (1983), Optimum Pricing Policy under Stochastic Inflation, *Review of Economic Studies*, 60: 513-529.

[20] Sheshinski, E., and Weiss Y., (1977), Inflation and Costs of Price Adjustment, *Review of Economic Studies*, 54: 287-303.

[21] Stokey, N. L., (2002), Brownian Models in Economics. Manuscript.

Appendix 1: The Microfoundations of the K-Sector Model

The simple approximate linear model that we use in this paper may be motivated as a more or less straightforward generalization of the model of Blanchard and Kiyotaki (1987). Consider an economy with $k = 1, ..., K$ final composite good sectors. Following Chari, Kehoe and McGrattan (2001)[6] we employ the notion of a competitive bundler who composes the output of the various sectors in the economy for competitive resale. The aggregate for final output, $Q(t)$, is then given by,

$$Q(t) = \prod_{k=1}^{K} q(t,k)^{\frac{1}{K}},$$

where $q(t,k)$ is itself an aggregate, competitively bundled, of firms across the various sectors. We detail the sectoral aggregates presently. The corresponding price index for the economy can be shown to be,

$$X(t) = K \prod_{k=1}^{K} x(t,k)^{\frac{1}{K}},$$

where $x(t,k)$ is again a sectoral index of prices, specified below.

We follow the convention in the (sS) pricing literature and tie nominal expenditure to the money stock, $L(t)$, in a simple way:

$$Q(t) = L(t)/X(t).$$

The demand for the sectoral bundle is inversely related to its real price (measured with respect to the economy wide price-level, $X(t)$:

$$q^d(t,k) = K^{-1} [x(t,k)/X(t)]^{-1} Q(t).$$

It follows immediately that nominal expenditure on each sectoral composite is proportional to total nominal expenditure,

$$x(t,k) q^d(t,k) = K^{-1} X(t) Q(t).$$

[6]Our precise notation is, however, closer to Canzoneri et al. (2004).

We turn now to the sectoral aggregates. In each sector k there is a continuum of firms, denoted by h in what follows. The output of each firm is bundled as follows,

$$q(t,k) = \left[\int_k^{k+1} q(t,k,h)^{\frac{\theta-1}{\theta}} dh\right]^{\frac{\theta}{\theta-1}},$$

so that $q(t,k,h)$ denotes the output of firm h in sector k and time t. The corresponding sectoral price and demand functions are given by the following pair of expressions:

$$x(t,k) = \left[\int_k^{k+1} x(t,k,h)^{1-\theta} dh\right]^{\frac{1}{1-\theta}};$$

$$q^d(t,k,h) = \left(\frac{x(t,k,h)}{x(t,k)}\right)^{-\theta} q(t,k).$$

We now turn to calculate what the optimal price of a firm in sector k would be, were it free of any rigidity in the setting of price. Let us write total cost as being proportional to output, where d is an arbitrary constant.

$$TC = dq(t,k,h)^{\frac{1}{1-\alpha}}.$$

It follows that marginal cost is given by

$$MC = \frac{1}{1-\alpha} dq(t,k,h)^{\frac{\alpha}{1-\alpha}}.$$

The optimal (frictionless) real price is given

$$\frac{x^*(t,k,h)}{X(t)} = \frac{\theta d}{(\theta-1)(1-\alpha)} q(t,k,h)^{\frac{\alpha}{1-\alpha}}.$$

Since all firms would behave symmetrically in such a friction free scenario, we have that $q^d(t,k,h) = q^d(t,k)$, and so we may write

$$\frac{x^*(t,k,h)}{X(t)} = \left[\left(\left(\frac{\theta d}{(\theta-1)(1-\alpha)}\right)^{\frac{1-\alpha}{\alpha}} \frac{1}{K}\right)^{\alpha}\right] Q(t)^{\alpha}.$$

By an appropriate choice of d we may normalize the first term to unity to recover the origins of equation (2.1) in the main text, where $\log x^*(t,k,h) \equiv p^*(t,i)$, $\log X(t) \equiv P(t)$ and $\log Q(t) \equiv Y(t)$.

Appendix 2: Deriving the Value Function

Initially we work with the following two relationships.

$$rV(x,Y)dt = L(x,Y)dt + EdV(x,Y); \tag{A.1}$$

$$V_1(x,Y) = -V_2(x,Y), \quad \text{for } Y = \overline{Y} \text{ or } -\overline{Y}. \tag{A.2}$$

Ito's Lemma yields

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial Y}dY + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(dx)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial Y^2}(dY)^2 + \frac{\partial^2 V}{\partial x \partial Y}dxdY$$

We have that $E(dM) = 0$, and $E(dM)^2 = \sigma^2 dt$. We also know that

$$dx = \begin{cases} -dP, & \text{if the firm does not adjust at moment } t; \\ 0, & \text{if other firms do not adjust at moment } t. \end{cases}$$

$dY = dM - dP = dM$, if $dP = 0$, no-one adjusts. So, at interior points

$$dV = \frac{\partial V}{\partial Y}dM + \frac{1}{2}\frac{\partial^2 V}{\partial Y^2}(dM)^2$$

Taking expectations through this function and using the resulting expression in (A.1), yields

$$rV(x,Y)dt = \gamma(x - \alpha Y)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial Y^2}\sigma^2 dt. \tag{A.3}$$

As the loss function is quadratic, it is reasonable to assert $\partial^2 V/\partial Y^2 = 2\alpha^2\gamma/r$, so that

$$V(x,Y) = \frac{\gamma}{r}(x - \alpha Y)^2 + \gamma\left[\frac{\alpha\sigma}{r}\right]^2. \tag{A.4}$$

(A.3) also implies that $\beta = \pm\sqrt{2r/\sigma^2}$, so that the solution to (A.3) is given by,

$$V(x,Y) = \frac{\gamma}{r}(x - \alpha Y)^2 + \gamma\left[\frac{\alpha\sigma}{r}\right]^2 + \lambda_1 e^{\beta Y} + \lambda_2 e^{-\beta Y}. \tag{A.5}$$

Now (A.2) implies, using (A.6), that

$$\left[\lambda_1'(x) + \beta\lambda_1'(x)\right]e^{\beta\overline{Y}} + \left[\lambda_2'(x) - \beta\lambda_2'(x)\right]e^{-\beta\overline{Y}} = \frac{2\gamma}{r}(1-\alpha)(\alpha\overline{Y} - x).$$

Exploiting the symmetry of the value function we may simplify this expression,

$$[\lambda'(x) + \beta\lambda'(x)]\, e^{\beta\overline{Y}} - [\lambda'(-x) - \beta\lambda'(-x)]\, e^{-\beta\overline{Y}} = \frac{2\gamma}{r}(1-\alpha)(\alpha\overline{Y} - x). \qquad \text{(A.6)}$$

As a result, (A.5) becomes

$$V(x,Y) = \frac{\gamma}{r}(x - \alpha Y)^2 + \gamma\left[\frac{\alpha\sigma}{r}\right]^2 + \lambda(x)e^{\beta Y} + \lambda(-x)e^{-\beta Y}. \qquad \text{(A.7)}$$

Solving (A.6) and using the result in (A.7) finally recovers the value function. The general solution to (A.6) may be written as

$$\lambda(x) = A + Bx + \kappa e^{-\beta x},$$

where $\kappa e^{-\beta x}$ is the solution to the homogenous part of (A.6). It follows that

$$\lambda'(x) = B - \beta\kappa e^{-\beta x},$$

and, by symmetry, that

$$\begin{aligned} \lambda(-x) &= A - Bx + \kappa e^{\beta x}; \\ \lambda'(-x) &= B - \beta\kappa e^{\beta x}. \end{aligned}$$

Hence we have the following system of equations:

$$\begin{aligned} B(e^{\beta\overline{Y}} - e^{-\beta\overline{Y}}) + \beta A(e^{\beta\overline{Y}} - e^{-\beta\overline{Y}}) &= \frac{2\gamma}{r}(1-\alpha)\alpha\overline{Y}; \\ \beta B(e^{-\beta\overline{Y}} - e^{\beta\overline{Y}}) &= -\frac{2\gamma}{r}(1-\alpha)x. \end{aligned}$$

Clearly, the second of these relations implies that

$$B = -\frac{2\gamma}{r\beta}\frac{1-\alpha}{e^{-\beta\overline{Y}} - e^{\beta\overline{Y}}}.$$

It follows that

$$A = \frac{2\gamma(\alpha - 1)}{r(e^{2\beta\overline{Y}} - e^{-2\beta\overline{Y}})}\left\{-\frac{\alpha Y}{\beta}(e^{\beta\overline{Y}} + e^{-\beta\overline{Y}}) - \frac{1}{\beta^2}(e^{\beta\overline{Y}} - e^{-\beta\overline{Y}})\right\}.$$

Hence, recalling the value function expression,

$$V(x,Y) = \frac{\gamma}{r}(x - \alpha Y)^2 + \gamma\left[\frac{\alpha\sigma}{r}\right]^2 + \lambda(x)e^{\beta Y} + \lambda(-x)e^{-\beta Y}.$$

Straightforward substitution yields

$$V(x,Y) = \frac{\gamma}{r}(x-\alpha Y)^2 + \gamma\left[\frac{\alpha\sigma}{r}\right]^2 + \frac{2\gamma}{r}\frac{\alpha-1}{(e^{2\beta\overline{Y}}-e^{-2\beta\overline{Y}})}\times$$
$$\left\{\begin{array}{c} \left(\frac{x-\alpha\overline{Y}}{\beta}-\frac{1}{\beta^2}\right)e^{\beta\overline{Y}}e^{\beta Y} - \left(\frac{x+\alpha\overline{Y}}{\beta}-\frac{1}{\beta^2}\right)e^{-\beta\overline{Y}}e^{\beta Y} \\ -\left(\frac{x+\alpha\overline{Y}}{\beta}+\frac{1}{\beta^2}\right)e^{\beta\overline{Y}}e^{-\beta Y} + \left(\frac{x-\alpha\overline{Y}}{\beta}+\frac{1}{\beta^2}\right)e^{-\beta\overline{Y}}e^{-\beta Y} \end{array}\right\}$$
$$-\kappa(e^{\beta x}e^{-\beta Y}+e^{-\beta x}e^{\beta Y}). \quad \text{(A.8)}$$

This is expression in the main text and is the same as equation (4.1) in Caplin and Leahy (1997).

There are several other optimality conditions related to an equilibrium of the model. First there is the smooth contact conditions,

$$\left.\frac{\partial V(x,Y)}{\partial x}\right|_{Y=\pm\overline{Y}} = -\left.\frac{\partial V(x,Y)}{\partial Y}\right|_{Y=\pm\overline{Y}}. \quad \text{(A.9)}$$

Then there is the value matching condition,

$$V(-S,\overline{Y}) = V(S,\overline{Y}) + c. \quad \text{(A.10)}$$

The optimal choice of target is given by

$$\left.\frac{\partial V(x,Y)}{\partial x}\right|_{x=-S} = -\left.\frac{\partial V(x,Y)}{\partial x}\right|_{x=S}. \quad \text{(A.11)}$$

The smooth pasting condition is

$$\left.\frac{\partial V(x,Y)}{\partial x}\right|_{x=S,Y=\overline{Y}} = 0. \quad \text{(A.12)}$$

These last two relations may be written more compactly as

$$\left.\frac{\partial V(x,\overline{Y})}{\partial x}\right|_{x=-S} = \left.\frac{\partial V(x,\overline{Y})}{\partial x}\right|_{x=-S} \quad \text{(A.13)}$$

It will be convenient to write the value function in a slightly different way.

$$V(x,Y) = \frac{\gamma}{r}(x-\alpha Y)^2 + \gamma\left[\frac{\alpha\sigma}{r}\right]^2 + \frac{2\gamma}{r}\frac{1-\alpha}{(e^{2\beta\overline{Y}}-e^{-2\beta\overline{Y}})}\times$$
$$\left\{\begin{array}{c} \alpha\overline{Y}\left(e^{\beta\overline{Y}}+e^{-\beta\overline{Y}}\right)(e^{\beta Y}+e^{-\beta Y}) + x\left(e^{\beta\overline{Y}}+e^{-\beta\overline{Y}}\right)(e^{\beta Y}+e^{-\beta Y}) \\ +\frac{1}{\beta}\left(e^{\beta\overline{Y}}+e^{-\beta\overline{Y}}\right)(e^{\beta Y}+e^{-\beta Y}) \end{array}\right\}$$
$$-\kappa(e^{\beta(x-Y)}+e^{-\beta(x-Y)}). \quad \text{(A.14)}$$

Using (A.10), implies after some algebra

$$\frac{4S\alpha\overline{Y}\gamma}{r}+\frac{4S\gamma}{r\beta}\frac{1-\alpha}{(e^{2\beta\overline{Y}}-e^{-2\beta\overline{Y}})}\left(e^{\beta\overline{Y}}+e^{-\beta\overline{Y}}\right)^2-\kappa\left[e^{-\beta S}\left(e^{-\beta\overline{Y}}+e^{\beta\overline{Y}}\right)+e^{\beta S}\left(e^{\beta\overline{Y}}+e^{-\beta\overline{Y}}\right)\right]=c$$

Drawing on some trigonometric identities, this can be rewritten as

$$\frac{4S\alpha\overline{Y}\gamma}{r}+\frac{4S\gamma(1-\alpha)}{r\beta}\tanh\beta\overline{Y}-4\kappa\sinh\beta S\sinh\beta\overline{Y}=c. \tag{A.15}$$

Now recall that $V_1(S,\overline{Y})=0=V_1(-S,\overline{Y})$. Hence we have the following pair of relations

$$\begin{aligned}\frac{2\gamma}{r}(S-\alpha\overline{Y})+\frac{2\gamma}{r\beta}(1-\alpha)\tanh\beta\overline{Y} &= \kappa\beta\left[e^{\beta S}e^{-\beta\overline{Y}}-e^{-\beta S}e^{\beta\overline{Y}}\right];\\ -\frac{2\gamma}{r}(S+\alpha\overline{Y})+\frac{2\gamma}{r\beta}(1-\alpha)\tanh\beta\overline{Y} &= \kappa\beta\left[e^{-\beta S}e^{-\beta\overline{Y}}-e^{\beta S}e^{\beta\overline{Y}}\right].\end{aligned}$$

These imply that

$$\kappa=\frac{\gamma S}{r\beta}\frac{1}{\sinh\beta S\cosh\beta\overline{Y}}. \tag{A.16}$$

and

$$\alpha\overline{Y}+\frac{1-\alpha}{\beta}\tanh\beta\overline{Y}=S\coth\beta S\tanh\tanh\beta\overline{Y} \tag{A.17}$$

(A.16) in (A.15) yields

$$S=\frac{r\beta c}{4\alpha\gamma(\beta\overline{Y}-\tanh\beta\overline{Y})} \tag{A.18}$$

(A.17) and (A.18) are two equations in two unknowns, $\overline{Y}$ and S.

Appendix 3: Deriving the Stationary Distribution of Output

Starting from corner B and moving along to A along the edge $\overline{BA}$ we may write the balance equations as follows,

$$\begin{aligned}
\pi_B &= \pi(1); \\
\pi(1) &= \frac{1}{2}\pi(2); \\
\pi(2) &= \frac{1}{2}\pi(1) + \frac{1}{2}\pi(3); \\
&\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

>From the above equations we receive the following recursion:

$$\begin{aligned}
\pi(1) &= \pi_B; \\
\pi(2) &= 2\pi_B; \\
\pi(3) &= 3\pi_B; \\
&\dots\dots\dots\dots\dots \\
\pi(n_1) &= \pi_A = n_1\pi_B.
\end{aligned}$$

Hence $\pi_A = 1/n_2$ and from the last equation it follows that $\pi_B = 1/n_1 n_2$. By symmetry $\pi_D = \pi_B$. So along the edge $\overline{BA}$ joint probabilities are given by:

$$\pi_{BA}(x) = \frac{\zeta}{n_1 n_2}$$

and along the edge CD by

$$\pi_{CD}(x) = \frac{n_1 + 1 - \zeta}{n_1 n_2},$$

where $\zeta = 1, 2, \dots . n_1$. To see where this latter relation comes from, note that as we move along edge $\overline{CD}$ the probabilities are falling, as opposed to rising (along $\overline{BA}$). So, we have that

$$\begin{aligned}
\pi_C &= \pi(1); \\
\pi(1) &= n_1\pi_D; \\
\pi(2) &= (n_1 - 1)\pi_D; \\
&\dots\dots\dots\dots\dots \\
\pi(n_1) &= (n_1 - 1 + 1)\pi_D.
\end{aligned}$$

Proceeding in the same fashion we find that all remaining points in the joint probability equal π_B. Consequently, the balance equations are satisfied for all points. We can verify this by direct calculation. For any points on edge $\overline{BA}$ we have that

$$\pi_{BA}(\zeta) = \frac{1}{2}\pi_{BA}(\zeta-1) + \frac{1}{2}\pi_{BA}(\zeta+1) = \frac{1}{2}\frac{\zeta-1}{n_1 n_2} + \frac{1}{2}\frac{\zeta+1}{n_1 n_2} = \frac{\zeta}{n_1 n_2},$$

while on edge $\overline{CD}$ we have

$$\begin{aligned}
\pi_{CD}(\zeta) &= \frac{1}{2}\pi_{CD}(\zeta-1) + \frac{1}{2}\pi_{CD}(\zeta+1); \\
&= \frac{1}{2}\frac{n_1+1-\zeta+1}{n_1 n_2} + \frac{1}{2}\frac{n_1+1-\zeta-1}{n_1 n_2}; \\
&= \frac{n_1+1-\zeta}{n_1 n_2}.
\end{aligned}$$

At corner A we see that

$$\pi_A = \frac{1}{2}\pi_A + \frac{1}{2}\frac{n_1-1}{n_1 n_2} + \frac{1}{2}\frac{1}{n_1 n_2} = \frac{1}{n_2},$$

and similarly at corner C. For any other point we have the obvious identity $\frac{1}{n_1 n_2} = \frac{1}{2}\frac{1}{n_1 n_2} + \frac{1}{2}\frac{1}{n_1 n_2}$. Finally, we require that the probabilities sum to unity:

$$\begin{aligned}
&\sum_{\zeta}(\pi_{BA}(\zeta) + \pi_{CD}(\zeta)) + \frac{1}{n_1 n_2}(n_1\overline{n}_2 - 2n_1) \\
&= \sum_{x}\frac{n_1+1}{n_1 n_2} + \frac{1}{n_1 n_2}n_1(\overline{n}_2 - 2) \\
&= \frac{n_1(n_1+1)}{n_1 n_2} + \frac{n_1(n_2-n_1-1)}{n_1 n_2} \\
&= 1.
\end{aligned}$$

This completes the solution of equation (4.2).

To further illustrate our solution we rely on the well known fact that $P^k \to P^\star$ in the limit as $k \to \infty$ where $P^\star$ is a stationary Markov matrix whose rows are identical and equal to the stationary eigenvalue vector π^*. We chose $N_1 = 2$, $N_2 =$

4 so that $n_1 = 5$ and $n_2 = 9$ and the dimension of the transition matrix is 25×25 (see the Figure 4.1 in the main text). We enumerate states by index j as follows. The first state, represented by the point A, is the state with $j = 1$ and the last state, represented by C, with $j = 25$.After a sufficiently large number, k, of iterations we obtain $P^\star$ with all rows equal and given by:

$$\pi^* = (0.1111, 0.0889, 0.0667, 0.0444, 0.0222, 0.02222, ...,$$

$$0.0222, 0.02222, 0.0444, 0.0667, 0.0889, 0.1111),$$

from which we read for example $\pi_A = \pi_C = 1/n_2 = 1/9 = 0.1111$, $\pi_B = 1/n_1 n_2 = 1/45 = 0.02222$ and so on.

Appendix 4: Deriving the Stationary Distribution of Output for three sectors

To find the stationary distribution of aggregate output in the case of a three-sector model we proceed in the same fashion as in the case of two sectors. In each sector let output y_i, follow regulated Brownian motions inside their own boundaries $[-\overline{Y}_i, \overline{Y}_i]$ respectively where $i = 1, 2, 3$. As in the case of two sectors we start in a discrete setting constructing the state space the grid $\{n_1, n_2, n_3\}$ where $n_1 < n_2 < n_3$[7]. The same arguments as in the two-sector model apply here: the initial state $\{n_1(0), n_2(0), n_3(0)\}$ can be any point in the parallelepiped $n_1n_2n_3$ but moving through the state space afterward is only in the direction toward interior parallelepiped $ABCD\overline{ABCD}$ as depicted in Figure 6.1. Once the state reaches the interior parallelepiped it stays inside it forever.

In stationary state each sectoral output is uniformly distributed inside its own boundaries and in order to find the stationary distribution for aggregate output defined in the usual way as

$$Y = Y_1 + Y_2 + Y_3$$

we have to find the joint probabilities $\pi(Y_1, Y_2, Y_3)$ which in the stationary state are given by the solution of

$$\pi' = P\pi' \tag{6.1}$$

where P is a Markov transition matrix defined on the state space as follows. To simplify the analysis we move, as before, to an equivalent representation where states are represented by discrete points in a three-dimensional box $n_1 \times (n_2 - n_1) \times (n_3 - n_2)$ as depicted in Figure 6.2 (this is equivalent to moving from the old coordinate system $\{Y_1, Y_2, Y_3\}$ in Figure 6.1 to the new one $\{\xi, \eta, \zeta\}$ in Figure 6.2). The possible transitions are represented by arrows with transition probabilities equal to 1/2. It is clear that the corresponding Markov chain consists of $n_3 - n_2$ connected two-dimensional chains, embedded in $n_1 \times (n_2 - n_1)$ rectangles; this is reminiscent of our two sector analysis, although there are some important differences, which we shall come to presently. The similarity is apparent at the $A\overline{A}$ and $C\overline{C}$ edges; when the state is on the edge $C\overline{C}$ we can either move up to the next rectangle (if money rises) or move back along the line belonging to the plane $CD\overline{CD}$, just as in the two-sector case. However, we see that the difference

[7] $n_i = 2Y_i + 1$ where all numbers are integers. The continous case is obtained by setting $n_i \to \infty$ and $\Delta y_i \to 0$ so that $n_i \Delta y_i \to 2\overline{Y}_i$.

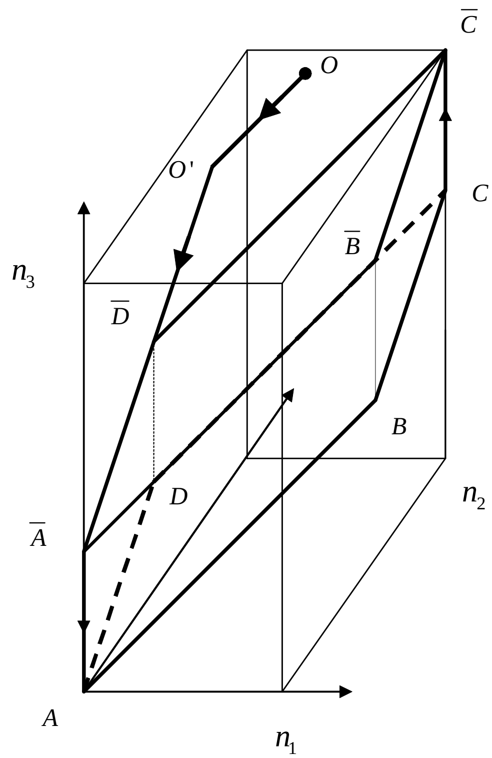

Figure 6.1: Assuming that money falls continuously then starting from the initial point O, we move toward O' along the line $x = y = z = const_1$ and then to the edge $\overline{D}$ along the line $x = 0$, $y = z = const_2$.

between this case and the two-sector case is that the state instead of staying at the ending point (n_1, n_2) with the probability 1/2, can now move upwards. Identical reasoning applies for states on edge $A\overline{A}$, as they can either move back (if money rises) along the line belonging to the plane $AB\overline{AB}$ or move down (if money falls) to the next rectangle. Moving along the first $ABCD$ and the last rectangle $\overline{ABCD}$ is identical except at points A and $\overline{D}$ where with probability 1/2 in the next instant of time states can freeze if money further falls/rises respectively (this situation is represented by the circled arrow). Now it is clear that all states in our Markov chain 'communicate' that is, for any two states i and j we have $P(i \to j) > 0$. In other words there is at least one path leading from i to j. That means there exists an $n \geq 1$ so that all elements of stochastic matrix P^n are strictly positive i.e. $P^n(i, j) > 0 \ \forall (i, j)$. Then P has a unique stationary distribution; that is (6.1) has a unique solution π^*. Moreover $\lim_{t\to\infty} \pi(t) = \pi^*$ independently of the initial distribution $\pi(0)$.The solution of (6.1) in units $1/n_1 n_2 n_3$ is given by:

$$
\begin{array}{ll}
\pi(AB) = \pi(\xi, 1, 1) = n_2(n_1 + 1 - \xi) & \xi = 1, .., n_1 \\
\pi(\overline{DC}) = \pi(\xi, n_2 - n_1 + 1, n_3 - n_2 + 1) = n_2 \xi & \xi = 1, .., n_1 \\
\pi(A\overline{A}) = \pi(1, 1, \zeta) = n_1 & \zeta = 2, .., n_3 - n_2 + 1 \\
\pi(C\overline{C}) = \pi(n_1, n_2 - n_1 + 1, \zeta) = n_1 & \zeta = 1, .., n_3 - n_2 \\
\pi(AB\overline{BA}) = \pi(\xi, 1, \zeta) = n_1 + 1 - \xi & \zeta = 2, .., n_3 - n_2 + 1 \\
 & \xi = 1, .., n_1 \\
\pi(DC\overline{CD}) = \pi(\xi, n_2 - n_1 + 1, \zeta) = \xi & \zeta = 1, .., n_3 - n_2 \\
 & \xi = 1, .., n_1 \\
\pi(ABCD \backslash AB) = \pi(\xi, \eta, 1) = n_2 - n_1 + 1 + \xi - \eta & \eta = 2, .., n_2 - n_1 + 1 \\
 & \xi = 1, .., n_1 \\
\pi(\overline{ABCD} \backslash \overline{DC}) = \pi(\xi, \eta, n_3 - n_2 + 1) = n_1 - \xi + \eta & \eta = 1, ..n_2 - n_1 \\
 & \xi = 1, .., n_1 \\
\pi(\xi, \eta, \zeta) = 1 & \textit{everywhere else}
\end{array}
$$

One can check that the local balance equations are satisfied at each point. The stationary distribution $\pi(\xi, \eta, \zeta)$ represents the occupancy measure of the state (ξ, η, ζ) which is just the joint probability function. Then, in the usual way, we may write the probability distribution function for aggregate output as

$$P(Y \leq \overline{Y}) = \sum_{3\xi + 2\eta + \zeta \leq \overline{Y}} \pi(\xi, \eta, \zeta). \tag{6.2}$$

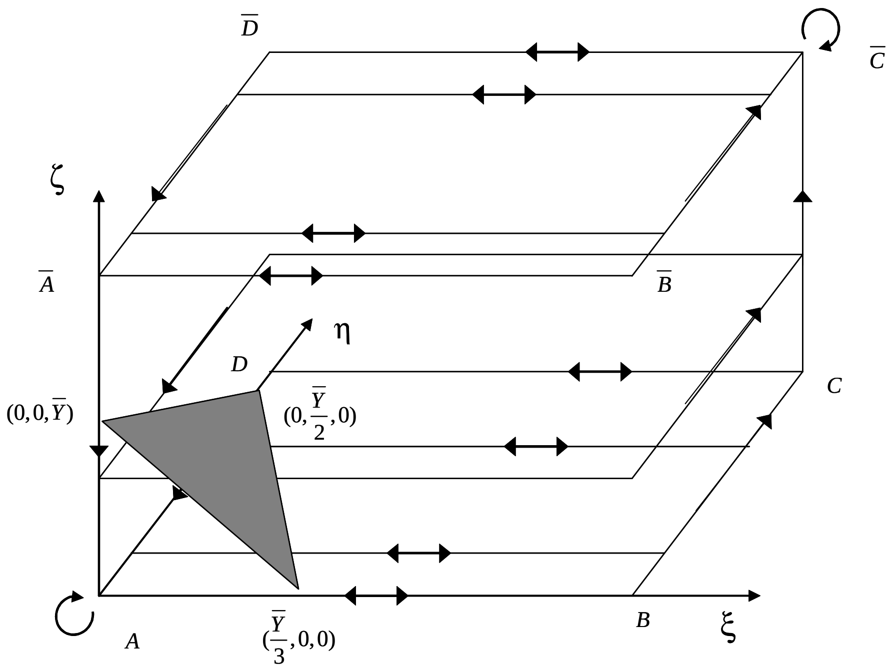

Figure 6.2: The state space in the new coordinate system (ξ, η, ζ) is presented together with the plane $3\xi + 2\eta + \zeta = \overline{Y}$.

In terms of the new coordinates $\{\xi, \eta, \zeta\}$, the condition $Y \leq \overline{Y}$ is equivalent to $3\xi + 2\eta + \zeta \leq \overline{Y}$ which may be interpreted in the following way: The initial problem of finding the probability distribution function of the sum of correlated and uniformly distributed random variables Y_1, Y_2, Y_2 on the intervals n_1, n_2 and n_3 respectively is now transformed, due to the existence of the restricted joint state space, into finding the probability distribution function of a new random variable $z = 3\xi+2\eta+\zeta$ where the probability distribution functions of the random variables ξ, η, ζ are given by:

$$P(\xi \leq \overline{\xi}) = \sum_{\eta,\zeta,\xi\leq\overline{\xi}} \pi(\xi, \eta, \zeta); \tag{6.3}$$

$$P(\eta \leq \overline{\eta}) = \sum_{\xi,\zeta,\eta\leq\overline{\eta}} \pi(\xi, \eta, \zeta); \tag{6.4}$$

$$P(\zeta \leq \overline{\zeta}) = \sum_{\xi,\eta,\zeta\leq\overline{\zeta}} \pi(\xi, \eta, \zeta), \tag{6.5}$$

respectively. We calculate the above probabilities for the continuous case where the sums become Riemann integrals. We perform integration separately over the regions where $\pi \neq 1$ and over interior points where $\pi = 1$. The integrations are trivial so in order to illustrate transition to the continuos case we perform in detail just part of the summation for $P(\xi \leq \overline{\xi})$ along line AB,

$$P_{AB}(\xi \leq \overline{\xi}) = \sum_{\eta=1,\zeta=1,\xi\leq\overline{\xi}} \pi(\xi, 1, 1) = \sum_{\xi=1}^{\overline{\xi}} \frac{n_2(n_1 + 1 - \xi)}{n_1 n_2 n_3}.$$

Now let $n_1, n_2, n_3 \to \infty$ in such a way that $n_1 \Delta n \to \overline{n_1}$, $n_2 \Delta n \to \overline{n_2}$ and $n_3 \Delta n \to \overline{n_3}$ where all $\overline{n}_i$ are continuous and Δn is an infinitesimal increment along the axes. Then, taking limits we have in terms of continuous variables

$$\lim_{n\to\infty} P_{AB}(\xi \leq \overline{\xi}) = \lim_{\Delta n\to 0} \sum_{\xi=\Delta n}^{\overline{\xi}\Delta n} \frac{n_2(n_1 + \Delta n - \xi)}{n_1 n_2 n_3} \Delta n = \frac{1}{n_1 n_2 n_3} \int_0^{\overline{\xi}} n_2(n_1 - \xi) d\xi.$$

In the continuous case, equations (6.3),(6.4) and (6.5) become

$$P(\xi \leq \overline{\xi}) = \frac{\overline{\xi}}{n_1}.$$

The random variable ξ is uniformly distributed on the interval $[0, n_1]$ with density $1/n_1$. For densities for η and ζ are not properly defined because they are discontinuous at the end points

$$P(\eta \leq \overline{\eta}) = \begin{cases} \frac{n_1}{2n_2} & \overline{\eta} = 0 \\ \frac{\overline{\eta}}{n_2} & \overline{\eta} \in (0, n_2 - n_1) \\ \frac{n_1}{2n_2} & \overline{\eta} = n_2 - n_1 \end{cases}$$

and

$$P(\zeta \leq \overline{\zeta}) = \begin{cases} \frac{n_2}{2n_3} & \overline{\eta} = 0 \\ \frac{\overline{\eta}}{n3} & \overline{\eta} \in (0, n_3 - n_2) \\ \frac{n_2}{2n_3} & \overline{\eta} = n_3 - n_2 \end{cases}$$

One can easily check that $P(\xi \leq n_1) = P(\eta \leq n_2 - n_1) = P(\zeta \leq n_3 - n_2) = 1$. Although unconditional probabilities of random variables η and ζ are discontinuous functions it turns out that the joint probability function defined in the usual way as

$$P(\xi \leq \overline{\xi}, \eta \leq \overline{\eta}, \zeta \leq \overline{\zeta}) = \int_0^{\overline{\xi}} \int_0^{\overline{\eta}} \int_0^{\overline{\zeta}} \pi(\xi, \eta, \zeta) d\xi d\eta d\zeta. \tag{6.6}$$

This function is absolutely continuous and an increasing function. In (6.6), $\pi(\xi, \eta, \zeta)$ is now a function of continuous variables defined by:

$$\begin{array}{ll}
\pi(AB) = \pi(\xi, 0, 0) = n_2(n_1 - \xi) & \\
\pi(\overline{DC}) = \pi(\xi, n_2 - n_1, n_3 - n_2) = n_2\xi & \\
\pi(A\overline{A}) = \pi(0, 0, \zeta) = n_1 & \\
\pi(C\overline{C}) = \pi(n_1, n_2 - n_1 0, \zeta) = n_1 & \\
\pi(AB\overline{BA}) = \pi(\xi, 0, \zeta) = n_1 - \xi & \\
\pi(DC\overline{CD}) = \pi(\xi, n_2 - n_1, \zeta) = \xi & \\
\pi(ABCD \backslash AB) = \pi(\xi, \eta, 0) = n_2 - n_1 + \xi - \eta & \\
\pi(\overline{ABCD} \backslash \overline{DC}) = \pi(\xi, \eta, n_3 - n_2) = n_1 - \xi + \eta & \\
\pi(\xi, \eta, \zeta) = 1 & \textit{everywhere else}
\end{array}$$

The previous analysis was necessary to justify why one can calculate first $P(Y \leq \overline{Y})$ and then density $\mu(\overline{Y})$ in the continuous case. It is clear now that (6.2) in the continuous case reads:

$$P(Y \leq \overline{Y}) = P(Y = 3\xi + 2\eta + \zeta \leq \overline{Y}) = \int_{\Omega} \pi(\xi, \eta, \zeta) d\xi d\eta d\zeta \tag{6.7}$$

where the integration region Ω is that sub-space obtained by cutting the state space by the plane $3\xi + 2\eta + \zeta = \overline{Y}$ (Figure 6.2.). To illustrate this we calculate $P(Y \leq \overline{Y})$ for some $\overline{Y} < 3n_1$. First, let $\overline{Y} < 2(n_2 - n_1)$ and $\overline{Y} < n_3 - n_2$ which corresponds to the situation depicted in Figure 6.2. Let $\Omega = \Omega_1 + \Omega_2 + \Omega_3 + \Omega_4$ where Ω_1, Ω_2, Ω_3 and Ω_4 represent the line of integration along the ξ axis, the set in the plane $\eta = 0$, the set in the plane $\zeta = 0$ and the interior points in the integration space respectively. Then (in $1/n_1n_2n_3$ units)

$$
\begin{aligned}
P(Y &= 3\xi + 2\eta + \zeta \leq \overline{Y}) = \int_{\Omega} \pi_{\Omega} d\xi d\eta d\zeta = I_{\Omega_1} + I_{\Omega_2} + I_{\Omega_3} + I_{\Omega_4} = && (6.8)\\
& \int_0^{\overline{Y}/3} n_2(n_1 - \xi)d\xi + \int_0^{\overline{Y}/3} (n_1 - \xi)d\xi \int_0^{\overline{Y}-3\xi} d\zeta + && (6.9)\\
& \int_0^{\overline{Y}/3} d\xi \int_0^{(\overline{Y}-3\xi)/2} (n_2 - n_1 - \eta + \xi)d\eta + \frac{3}{4}\left(\frac{\overline{Y}}{3}\right)^3 && (6.10)\\
&= \frac{n_2 n_1}{3}\overline{Y} + \frac{1}{8}\left(\frac{\overline{Y}}{3}\right)^3 + \frac{n_2 + 3n_1}{4}\left(\frac{\overline{Y}}{3}\right)^2. && (6.11)
\end{aligned}
$$

Now the density is given by

$$
\mu(\overline{Y}) = \frac{dP}{d\overline{Y}} = \frac{n_2 n_1}{3} + \frac{1}{8}\left(\frac{\overline{Y}}{3}\right)^2 + \frac{n_2 + 3n_1}{6}\left(\frac{\overline{Y}}{3}\right).
$$

We see that the density starts from the value $\mu(0) = 1/3n_3$ which is similar to the two dimensional case with the 'dimensionality' factor 1/3. This leads us to conjecture that in the $K-$dimensional case this will read $\mu(K) = 1/Kn_K$. Indeed, note that when $Y = 0$ and $Y = n_1 + n_2 + ... + n_K$ the only states are $(0, 0, ..., 0)$ and $(n_1, n_2, ..., n_K)$, respectively (i.e., the two points in planes $Y_K = 0$ and $Y_K = n_K$) and the joint probabilities are for these two states equal to the unconditional probability, $1/n_K$. However, due to the symmetry in the $K-$dimensional hypercube we will have two lines of discontinuity. One beginning from $(0, 0, ..., 0)$ and terminating on $(n_1, 0, ..., 0)$, and another beginning from $(n_1, n_2, ..., n_K)$ and terminating on $(0, n_2, ..., n_K)$. Along the first one joint probabilities are given by

$$
\pi(\xi, 0, 0, ..., 0) = n_{K-1}n_{K-2} \cdot \cdot n_2(n_1 - \xi), \tag{6.12}
$$

and along the second one by

$$\pi(\xi, n_2, ..., n_K) = n_{K-1}n_{K-2} \cdot \cdot n_2\xi.$$

Then, by defining new random variables ξ_i, $i = 1, 2, ..K$ with corresponding densities $\mu_1 = 1/n_1$ and $\mu_i = 1/(n_i - n_{i-1})$ for $i \neq 1$, expressing Y in the new variables as

$$Y = K\xi_1 + (K-1)\xi_2 + (K-2)\xi_3 + \cdot\cdot + \xi_K,$$

and performing an integration (6.12) in the hypercube along the edge $(\xi_1, 0, ..., 0)$ up to the intersection with hyperplane $Y = K\xi_1 + \sum_{i=2}^{K}(K-i+1)\xi_i$, we get Y/Kn_K in an expression for the probability function. Because the probability function is strictly increasing in Y we always have a simple relation $P(Y) \geq Y/Kn_K$. Due to the symmetry of the probability function we have another relation in the site $(n_1, n_2, ..., n_K)$, $P(Y) \leq Y/Kn_K + 1 - \sum n_i/Kn_K$. The probability function for $n_1 = 0.19$, $n_2 = 0.81$ and $n_3 = 1.77$ is plotted in Figure 6.3. With an increased number of dimensions (sectors), calculating the probability distribution analytically becomes more and more cumbersome. Nevertheless, we still can provide an accurate answer as to how the probability distribution or density of aggregate output behaves when the number of sectors increases. We already know that $Y/Kn_K \leq P(Y) \leq Y/Kn_K + 1 - \sum n_i/Kn_K$ and that the density is shifted 'upwards' by the amount $1/Kn_K$. We saw that it can be interpreted as a weighted uniform measure over the whole interval $[0, \sum n_i]$. What remains can be represented by a weighted mixture of other densities. In other words, when we calculate the probability $P(Y \leq \overline{Y})$ which is simply a sum, we are free to rearrange this sum as we wish. For example in the two dimensional case we rearranged the sum in a specific way over the interior and exterior sets. But in fact we could rearrange the sum in another way considering separately the boundary points and entire interior set (where the joint probabilities are all equal to unity) arriving at the same result. This motivates us to approximate the density function for aggregate output with K sectors as a weighted sum of,

$$\mu(Y, K) = \omega\frac{1}{\sum n_i} + (1-\omega)\overline{\mu}(Y = \sum_{i=1}^{K}(K-i+1)\xi_i) \qquad (6.13)$$

where $\omega = \sum n_i/Kn_K$ and $\overline{\mu}$ is density function of the sum of K independently[8]

[8]By increasing the number of sectors to $K \geq 3$ (dimensions), splitting the density of aggregate

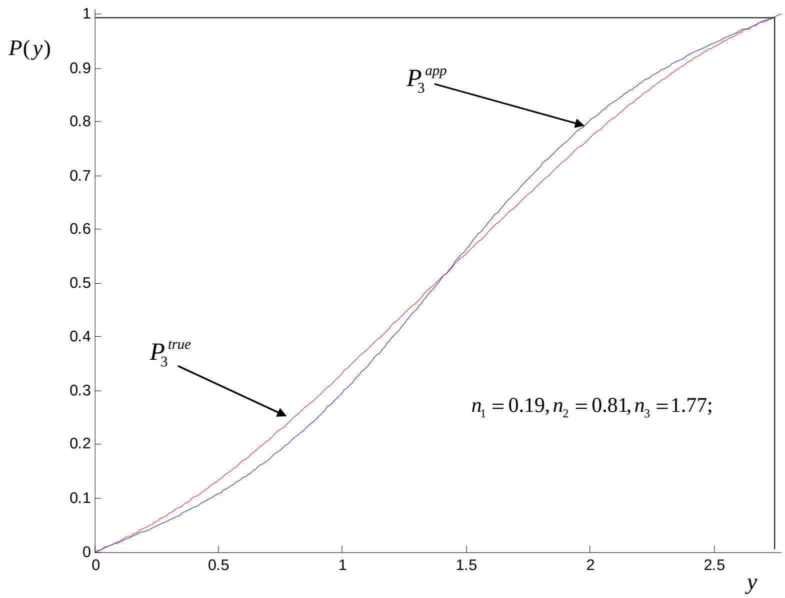

Figure 6.3: Probability functions for aggregate output for three sector model are shown. The exact one P^{true} is obtained by integrating (6.7) over the whole range of Y. The second one P^{app} is obtained by integrating (6.13)

and nonidentically uniformly distributed random variables $0 \leq \xi_i \leq n_i - n_{i-1}$ with densities $\mu(\xi_i) = 1/(n_i - n_{i-1})$, $i = 1, 2, ...K, \;\; and \; n_0 = 0$.

output between a uniform part and the rest is not as simple as in the case of two sectors. Here we assume that what remains after populating the uniform density does not violate "too much" independency.

ABOUT THE CDMA

The **Centre for Dynamic Macroeconomic Analysis** was established by a direct grant from the University of St Andrews in 2003. The Centre funds PhD students and facilitates a programme of research centred on macroeconomic theory and policy. The Centre has research interests in areas such as: characterising the key stylised facts of the business cycle; constructing theoretical models that can match these business cycles; using theoretical models to understand the normative and positive aspects of the macroeconomic policymakers' stabilisation problem, in both open and closed economies; understanding the conduct of monetary/macroeconomic policy in the UK and other countries; analyzing the impact of globalization and policy reform on the macroeconomy; and analyzing the impact of financial factors on the long-run growth of the UK economy, from both an historical and a theoretical perspective. The Centre also has interests in developing numerical techniques for analyzing dynamic stochastic general equilibrium models. Its affiliated members are Faculty members at St Andrews and elsewhere with interests in the broad area of dynamic macroeconomics. Its international Advisory Board comprises a group of leading macroeconomists and, ex officio, the University's Principal.

RECENT WORKING PAPERS FROM THE CENTRE FOR DYNAMIC MACROECONOMIC ANALYSIS

Number	Title	Author(s)
CDMA04/01	Interest Rate Bounds and Fiscal Policy	Jagjit S. Chadha (St Andrews) and Charles Nolan (St Andrews)
CDMA04/02	The Impact of Imperfect Credibility in a Transition to Price Stability	Anamaria Nicolae (Durham) and Charles Nolan (St Andrews)
CDMA04/03	Monetary Policy Rules, Asset Prices and Exchange Rates	Jagjit S. Chadha (St Andrews), Lucio Sarno (Warwick and CEPR) and Giorgio Valente (Warwick)
CDMA04/04	Tax Policy and Irreversible Investment	Sumru Altug (Koç, CEPR and Economic Research Forum), Fanny S. Demers (Carleton and McGill) and Michel Demers (Carleton and McGill)
CDMA04/05	The Role of Preference Shocks and Capital Utilization in the Great Depression	Mark Weder (Humboldt)
CDMA04/06	Optimal Simple Rules for the Conduct of Monetary and Fiscal Policy	Jagjit S. Chadha (St Andrews) and Charles Nolan (St Andrews)
CDMA04/07	Money, Debt and Prices in the UK 1705-1996	Norbert Janssen (Bank of England), Charles Nolan (St Andrews) and Ryland Thomas (Bank of England)
CDMA05/01	Labour Markets and Firm-Specific Capital in New Keynesian General Equilibrium Models	Charles Nolan (St Andrews) and Christoph Thoenissen (St Andrews)
CDMA05/02	The Impact of Simple Fiscal Rules in Growth Models with Public Goods and Congestion	Sugata Ghosh (Cardiff) and Charles Nolan (St Andrews)
CDMA05/03	Inflation Targeting, Committee Decision Making and Uncertainty: The Case of the Bank of England's MPC	Arnab Bhattacharjee (St Andrews) and Sean Holly (Cambridge)
CDMA05/04	How to Compare Taylor and Calvo Contracts: A Comment on Michael Kiley	Huw Dixon (York) and Engin Kara (York)
CDMA05/05	Aggregate Dynamics with Heterogeneous Agents and State-Dependent Pricing	Vladislav Damjanovic (St Andrews) and Charles Nolan (St Andrews)

For information or copies of working papers in this series, or to subscribe to email notification, contact:

Alex Trew
Email: awt2@st-and.ac.uk; Phone: +44 (0)1334 462445; Fax: +44 (0)1334 462444.