02/05/2006, Dr Marcus Rossberg, mr30@st-and.ac.uk
office hours: Thursdays, 10-12 (*Arché* building, College St, 19.4)

PY1003

Lecture 20

Revision II

(1) Many-place predicates

All predicates have a certain number of *places*. This is equal to the number of names by which they need to be followed in order to create a grammatical atomic sentence. The order of the names (or variables) following a many-place predicate is important.

<u>A two-place predicate:</u>
Hxy : x hates y

d : Dave
a : Arnold

Dave hates Arnold	Hda
Arnold hates Dave	Had
Everyone hates Arnold	$\forall xHxa$
Arnold hates everyone	$\forall xHax$

<u>A three-place predicate:</u>
$Pxyz$: x prefers y to z

c : Carrie

Dave prefers Carrie to Arnold	$Pdca$
Dave prefers Arnold to Carrie	$Pdac$

Remember: a predicate letter always has a fixed number of places.

<u>Example:</u> Mary is an aunt and Emma is John's aunt.

You *cannot* formalise this as: $Aa \wedge Abc$ ✗

a : Mary
b : Emma
c : John
A : ??

Rather, use an existential quantifier to help you out:

Key: *a*: Mary
b: Emma
c: John
Axy : *x* is an aunt of *y*

Mary is an aunt and Emma is John's aunt: $\exists x Aax \wedge Abc$ ✓

Resist the temptation to introduce two different predicate letter, a one-place (monadic) predicate letter for '*x* is an aunt', and a two-place (binary) predicate letter for '*x* is an aunt of *y*'. Otherwise, you will not be able to show that

Mary is John's aunt. Therefore, Mary is an aunt.

is valid – which is blatantly is.

(2) Multiple generality

When two or more quantifiers have overlapping scope, we must use different variable letters with each quantifier:

✗ $\exists x \forall x Lxx$

✓ $\exists x \forall y Lxy$

The order of the quantifiers is also important:

There is someone who hates everyone $\exists x \forall y Hxy$

Everyone is hated by someone or other $\forall x \exists y Hxy$

(With '*Hxy*' for '*x* hates *y*'; domain: people.) It does not make a bit of a difference whether you use '*x*' or '*y*' or '*z*', however, *as long as you keep them apart*. Variables are mere place-holder. The place they mark is the only important thing, what letter they are does not matter. All three following sentences say exactly the same:

$$\forall x \exists y Hyx$$

$$\forall y \exists x Hxy$$

$$\forall z \exists y Hyz$$

namely, that everybody is hated by somebody (and not all necessarily by the same person).

Using multiple generality we can translate some quite complex phrases, e.g.:

Some student is taller than some professor $\exists x \exists y[(Sx \wedge Py) \wedge Txy]$

Sx : x is a student
Px : x is a professor
Txy : x is taller than y

Some boy hates every girl $\exists x[\mathrm{B}x \wedge \forall y(Gy \rightarrow Hxy)]$

Bx : x is a boy
Gx : x is a girl
Hxy : x hates y

Remember:

I. *Only apply the tree rule for a quantifier (or negated quantifier) if it has the widest scope*. You *cannot* apply it to an embedded quantifier.

Exercises: What is the correct rule to apply to the following sentences?

1. $\exists x \forall y Rxy$
2. $\forall x \forall y(Sx \wedge Py)$
3. $\neg \forall x(Fx \vee \exists y Lxy)$
4. $\neg(\forall x Fx \vee \exists y Gy)$
5. $\neg \exists x \neg Px$

II. *Never apply two quantifier rules at once.*

✗

Fa
$\exists x \forall y Rxy$ ✓
|
Rbc

✓

Fb
$\exists x \forall y Rxy$ ✓
|
$\forall y \mathrm{R}ay$ ✓a✓b
|
Raa
Rab

(3) Infinite Trees

Infinite trees can occur in predicate Logic. If an argument is valid or a set of sentences is inconsistent, your tree will close (provided you apply the rules correctly). If, however, the argument is invalid or the set is consistent, it may have infinitely long open branches.

Sometimes it may look like you have an infinite branch, but in fact the branch can be closed if you apply the correct rule. You must apply the rule that will close the branch in these cases.

A simple example:

✗ $\forall x \exists y Rxy$ ✓a✓b
Fa
$Fa \rightarrow \neg Fa$
$\exists y Ray$ ✓
Rab
$\exists y Rby$ ✓
Rbc
⋮

✓ $\forall x \exists y Rxy$
Fa
$Fa \rightarrow \neg Fa$ ✓
/ \
$\underline{\neg Fa}$ $\underline{\neg Fa}$

Models for infinite trees become difficult, because many sentences will appear on an infinite branch (and hence cannot be assigned arbitrary truth-values) although you have not actually written them down.

We have covered this briefly but don't worry: you will not be assessed on models for infinite trees. ☺

(4) Identity

Tree Rules for Identity

Identity Rule 1:

For any constants *a* and *b*, if a sentence of the form '*a* = *b*' appears on a branch of a tree, we may substitute '*a*' for '*b*' (and *vice versa*) in any sentence that occurs *on that branch* (not on any other branch). The identity statement is never ticked off.

Fa
$a = b$
|
Fb

Identity Rule 2:

For any constant 'a', if a statement of the form '¬ *a* = *a*' occurs, close that branch.

You will not be assessed on models or counterexamples for sentences or arguments involving identity. ☺

Note that this does not mean that every tree with identity statements that you are asked to do will close!

(5) Only, at least *n*, at most *n*, exactly *n*

You're the only one I love $Lab \wedge \forall y(Lay \rightarrow y=b)$

a : me
b : you
Lxy : x loves y

Only Lister and Frankenstein survived $[(Sa \wedge Sb) \wedge \forall y(Sy \rightarrow (y=a \vee y=b))]$

a : Lister
b : Frankenstein
Sx : x survived

There was at least one survivor $\exists xSx$

There were at least two survivors $\exists x\exists y[(Sx \wedge Sy) \wedge \neg x=y]$

There was at most one survivor $\forall x\forall y[(Sx \wedge Sy) \rightarrow x=y]$

There were at most two survivors $\forall x\forall y\forall z[((Sx \wedge Sy) \wedge Sz) \rightarrow (x=y \vee (x=z \vee y=z))]$

There was exactly one survivor $\exists x(Sx \wedge \forall y(Sy \rightarrow y=x))$

(=There was at least one survivor and any survivor is identical to her)

There were exactly two survivors $\exists x\exists y[((Sx \wedge Sy) \wedge \neg x=y) \wedge \forall z(Sz \rightarrow (z=x \vee z=y))]$

(=There were at least two survivors and any survivor is identical to one of those two)

(6) Definite Descriptions

The lecturer is not dressed properly $\exists x[(Lx \wedge \forall y(Ly \rightarrow y=x)) \wedge \neg Dx]$

Lx : x is a lecturer
Dx : x is dressed properly

Andy's only sister is psychic $\exists x[(Sxa \wedge \forall y(Sya \rightarrow y=x)) \wedge Px]$

a : Andy
Sxy : x is a sister of y
Px : x is psychic

The Head of School is the man

$\exists x([Hx \wedge \forall y(Hy \rightarrow x=y)] \wedge \exists z[Mz \wedge \forall w(Mw \rightarrow w=z)] \wedge x=z)$

Hx : x is a Head of School
Mx : x is a man