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PY1003

Lecture 17

Identity, Number, and Definite Descriptions

Identity

Recall that we introduced '=' into our formal language in order to be able to show that arguments like the following are valid:

Cicero is a Roman orator.
Cicero is identical to Tully.
____
Therefore, Tully is a Roman orator.

We formalised this argument as:

Key: c : Cicero
t : Tully
Rx : x is a Roman orator

$Rc,\ c = t \vdash Rt$

Recap: Tree Rules for Identity

First rule for '=':

For any constants a and b, if a sentence of the form '$a = b$' appears on a branch of a tree, we may substitute 'a' for 'b' (and *vice versa*) in any sentence that occurs *on that branch* (not on any other branch).
The identity statement is *never* ticked off.

$$\begin{array}{c} Fa \\ a = b \\ | \\ Fb \end{array}$$

Note: You can also substitute into identity statements.

Second rule for '=':

For any constant 'a', if a statement of the form '$\neg\ a = a$' occurs, close that branch.

$$\begin{array}{c} \vdots \\ \neg\ a = a \\ \hline \end{array}$$

Examples:

$\forall y(Fy \to Gy), Fa, \neg Gb \vdash \neg a{=}b$

$$\forall y(Fy \to Gy) \checkmark a$$
$$Fa$$
$$\neg Gb$$
$$\neg\neg a{=}b \checkmark$$
$$a{=}b$$
$$Fa \to Ga \checkmark$$

$\neg Fa$	Ga
———	Gb
	———

The argument is valid.

$\exists x\, x{=}a \vee \forall x\, x{=}a \vdash a{=}b$

$$\exists x\, x{=}a \vee \forall x\, x{=}a \checkmark$$
$$\neg a{=}b$$

$\exists x\, x{=}a \checkmark$	$\forall x\, x{=}a \checkmark b$
$c{=}a$	$b{=}a$
$\vdots$	$\neg b{=}b$
	———

The left branch remains open: the argument is not valid.

Properties of Identity

Towards the end of the last lecture we proved that

$$\forall x\forall y(x = y \to y = x)$$

This property of the identity relation is called **symmetry**: whenever a certain *x* is identical to a certain *y* then *y* is also identical to *x*.

This is of course not surprising: that *x* is identical to *y* means that they are the *very same* thing.

Other relations have that property, too. … *is married to…*, for example, is symmetric, and so is … *is a sibling of…* .

Identity has to other important properties, namely **reflexivity** and **transitivity**, as they are defined below. Whenever a relation has *all* of those three properties, we say that this relation is an **equivalence relation**.

In general, these properties are defined like this:

Reflexivity: $\forall x Rxx$

Symmetry: $\forall x\forall y(Rxy \to Ryx)$

Transitivity: $\forall x\forall y\forall z((Rxy \wedge Ryz) \to Rxz)$

Identity fulfils all three conditions, and is hence an equivalence relation. (The proofs that identity is reflexive and transitive is left as an exercise.)

… *is a sibling of*… is transitive, too, but it is not reflexive: no-one is her or his own sibling. Reflexive relations are, for instance, … *is as tall as* …, or … *has the same colour as* … .

Number

With the help of identity, we are now also in the position to express and reason about statements that are more precise than just saying that something, everything, or nothing is such-and-such.

Recall that '$\exists x\ Fx$' is read as 'there is something that is F', in the sense of 'there is *at least one* thing that is F'.

How can we say that there are *at least two* things that are F?
We can just say that there is something that is F, and again something that is F, and those two are not identical:

$$\exists x \exists y((Fx \wedge Fy) \wedge \neg x{=}y)$$

This obviously extends to any number of objects:

There are at least three things:

$$\exists x \exists y \exists z[((Fx \wedge Fy) \wedge Fz) \\ \wedge ((\neg x{=}y \wedge \neg x{=}z) \wedge \neg y{=}z)]$$

There are at least four things:

$$\exists x \exists y \exists z \exists v[(((Fx \wedge Fy) \wedge Fz) \wedge Fv) \\ \wedge (((((\neg x{=}y \wedge \neg x{=}z) \wedge \neg x{=}v) \\ \wedge \neg y{=}z) \wedge \neg y{=}v) \wedge \neg z{=}v)]$$

and so on.

What if we wanted to say that there is *exactly* one F? We can paraphrase that as 'there is an F, and nothing except for that thing is F'. If we now realise that this means that there is an F, and everything that is F is identical to that thing, we can express that in our formal language of predicate logic with identity:

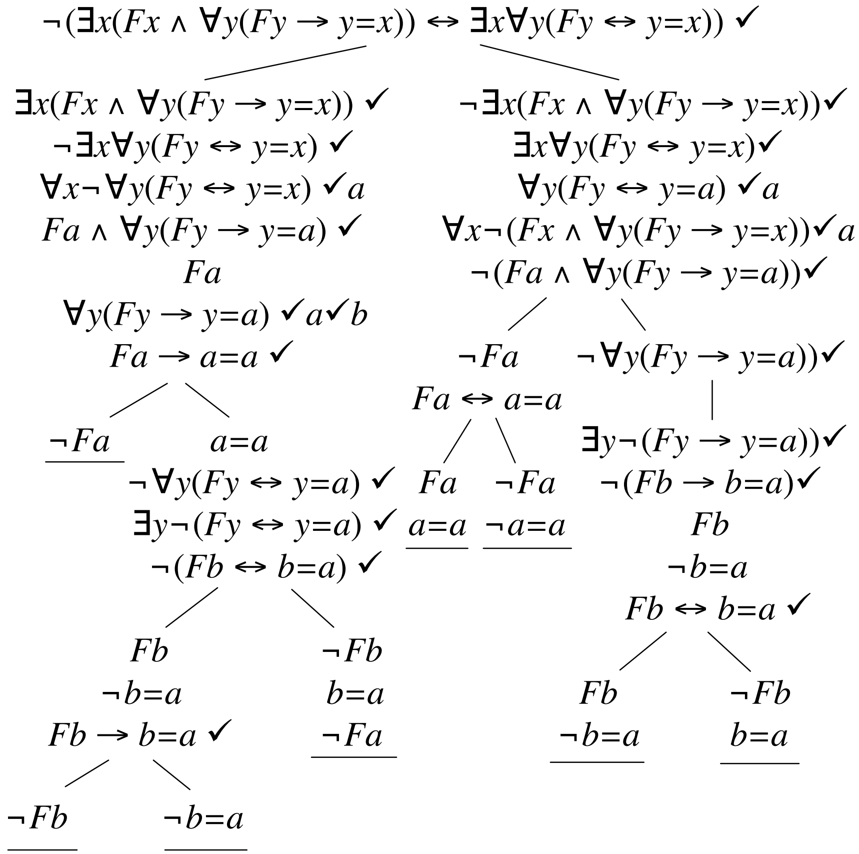
There is exactly one F :

$$\exists x(Fx \wedge \forall y(Fy \rightarrow y{=}x))$$

Alternatively, we could also write:

$$\exists x \forall y(Fy \leftrightarrow y{=}x)$$

A tree shows that the two formalisations of ‘There is exactly one *F*’ are equivalent:



The tree closes. The equivalence holds.

Taking these two insights together, we can now express that there are exactly two things that are *F*:

$$\exists x\exists y[((Fx \wedge Fy) \wedge \neg x{=}y) \wedge \forall z(Fz \rightarrow (z{=}x \vee z{=}y))]$$

Or shorter: $\exists x\exists y\forall z[(Fz \leftrightarrow (z{=}y \vee z{=}x)) \wedge \neg x{=}y)]$

There are exactly three things (short version only):

$$\exists x\exists y\exists w\forall z[(Fz \leftrightarrow ((z{=}y \vee z{=}x) \vee z{=}w)) \wedge ((\neg x{=}y \wedge \neg x{=}w) \wedge \neg y{=}w)]$$

Definite Descriptions

By using the method of saying that there is exactly one thing that is such-and-such, Russell manage to solve the problem of the present King of France. ‘The present King of France is bald’ is false, since there is no King of France presently – for the same reason, though, ‘the present King of France is not bald’ is false, too. But is the second of those sentences not the negation of the first and should thus be true if the first is false?

Russell’s solution is that both sentences actually claim that *there is exactly one* present King of France, and then make differing claims about his hair. So, the second is, despite appearance, not the negation of the first. The sentences should thus be formalised as (‘*Kx*’ for ‘*x* is presently King of France’, ‘*B*’ for ‘*x* is bald’):

The present King of France is bald:

$$\exists x[(Kx \wedge \forall y(Ky \rightarrow y{=}x)) \wedge Bx]$$

The present King of France is not bald:

$$\exists x[(Kx \wedge \forall y(Ky \rightarrow y{=}x)) \wedge \neg Bx]$$