14th April 2006 Dr Philip Ebert
pae1@st-andrews.ac.uk

Lecture 15

Multiple generality

1. Recap

a) Introduced n-placed predicates, e.g. the following (two-place) relation:

Rab

a relates to *b* in a certain way, specified by the two-place (or *binary*) predicate *R*.

Simon loves Charis
Ssm

s = Simon, m = Charis, Sxy = x loves y

We noted that the *order of the names* is important. This is also important once we generalise on relations by using quantifiers:

Simon loves everyone -- Everyone loves Simon
$\forall$x(Ssx) $\forall$x(Sxs)

b) Tree-rule

The usual tree rule apply and there are no new rules for n-place predicates

c) Models and countermodels

The only complication here, compared to one-place predicates is that we have to respect *the order of the names*! A n-place predicate has as *extension* a set of n-tuples.

The topic for this lecture is to introduce multiple generality involving n-place predicates.

2. Quantifiers and scope

In earlier lectures we discussed sentences which contain more than just one quantifier:

If there is something that is F, then everything is G
$\exists xFx \rightarrow \forall yGy$

In this case, the *main* connective is the $\rightarrow$ which combines two quantified sentences. The *scope* of the second quantifier is distinct from the scope of the first. The sentence can also (correctly) formalised as:

$$\exists x\ Fx \rightarrow \forall x\ Gx$$

Using n-place predicates we need to be careful about the *scope* of the quantifier. For instance consider

Everyone loves someone
For all x, there is a y such that x loves y
$\forall x\ \exists y\ Lxy$

The *existential* quantifier is in the *scope* of the *universal* quantifier. The "main connective" is the universal quantifier.

Contrast this with

Someone loves everyone
There is someone such that he/she loves everyone
There is an x, such that for all y, x loves y
$\exists x\ \forall y\ Lxy$

Here the "main connective" is *existential* quantifier. The *universal* quantifier is in the scope of the *existential* quantifier.

So, if there are quantifiers in the scope of other quantifiers – as often happen with two(or more)-placed predicates – we need to be careful about the choice of variable-letters. Always use *different variable-letters* for the embedded quantifiers.

It is easy to see that

$$\exists x\ \forall x\ Axx$$

is not just confusing but it is crucially unclear what this statement says:

There is an x such that for all x, x admires x
⇒ This will be marked as a mistake!

Let me give you some more examples. Take the sentence

Everyone is the daughter/son of someone
$\forall x\ \exists y\ Dxy$
(Dxy: x is the daughter/son of y)

In contrast,

$$\exists x\ \forall y\ Dxy$$

would say something rather different:
There is an x, such that for all y, x is the daughter/son of y
There is someone that is the daughter/son of everyone.
⇒ *Be carefully about the order of the quantifier*

3. **Multiple Generality involving one- and two-place predicates.**

Remember last week Marcus, in full respect of his well-known humbleness used the following sentence:

Every student loves Marcus

Which he formalised as

$$\forall x(Sx \rightarrow Lxm)$$

now here we can also quantify into 'Marcus' and write

$$\forall x(Sx \rightarrow \exists y\,(Lxy))$$

here the *existential quantifier* is in the scope of the universal one and we need to choose a different variable letter.

Let us consider the following similar examples

Some boy loves some girl
$\exists x(Bx \wedge \exists y(Gy \wedge Lxy))$

Some boy loves every girl
$\exists x(Bx \wedge \forall y(Gy \rightarrow Lxy))$

Every boy loves some girl
$\forall x(Bx \rightarrow \exists y(Gy \wedge Lxy))$

Every boy loves every girl
$\forall x(Bx \rightarrow \forall y(Gy \rightarrow Lxy))$

4. **More complicated examples.**

There is someone who is admired by everyone (s)he meets:
$\exists x \forall y(Mxy \rightarrow Ayx)$
Mxy: x meets y, Axy: x admires y

Nobody loves anybody:
$\neg \exists x \exists y Lxy$

Everyone is loved by themself if they are loved by anyone:
$\forall x(\exists y Lyx \rightarrow Lxx)$

5. **Applying tree rules to sentences involving multiple generality**

We do not need any new tree-rules. We only need to apply the already-known rules for the (negated) universal and (negated) existential quantifiers. The rule for the *outermost* quantifier is to be applied first

∀x∃yRxy	Apply the rule for ∀ first
¬∀x∃yRxy	Apply the rule for ¬∀ first
∃x∀y(Fx→Fy)	Apply the rule for ∃ first
¬∃x∃y(Fx→Fy)	Apply the rule for ¬∃ first

Some examples:

∃x∀y(Lxy) |- ∃x (Lxm)

∃x∀y(Lxy) √[a]
¬∃x (Lxm) √
|
∀x¬(Lxm) √[a]
|
∀y(Lay) √[a,m]
|
Laa
Lam
|
¬Lam
CLOSED

This is a fairly easy example.

Rules of Thumb:

a) Do not rush – apply every rule individually.

b) Be careful with the following:

∀x∃yRxy

since the *universal quantifier* rule is not ticked, there is *potential* for an infinite tree.

∀x∃yLxy √[a] √[b] √[c]
|
∃yLay √[b]
|
Lab
|
∃yLby √[c]
|
Lbc
|
∃yLcy

This leads us to discussing some *meta-theoretical* issues about first-order logic in the next section.

But before some other examples:

Faa, $\forall x \forall y\ (Fxy \rightarrow Gxy)\ \vdash \exists x \exists y\ (Gxy)$

6. Metatheory: First-order logic is semi-decidable. The Limits of the tree method.

Once we introduced multiple generality and so arrived at *full* first-order predicate logic, there will be

some invalid arguments such that you cannot use trees to prove that they are invalid.

However,

if an argument is valid, then there is a tree that closes.

This *restriction* of what can be proven is not simply due to the proof procedure – the tree method – that we are using. Rather it generalises to any effective decision procedure.

Sentential logic has the property of being *decidable*. That is given a valid argument there is a finite proof of its validity and given an invalid argument there is a finite proof that it is invalid.

In contrast, full first-order predicate logic is only *semi-decidable*. If an argument is valid, then there is a finite proof (e.g. a tree that closes). Yet, this does not hold for an invalid argument.