

# Magnetic ordering of nuclear spins in an interacting two-dimensional electron gas

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We investigate the magnetic behavior of nuclear spins embedded in a two-dimensional (2D) interacting electron gas using a Kondo lattice model description. We derive an effective magnetic Hamiltonian for the nuclear spins, which is of the Rudermann-Kittel-Kasuya-Yosida type and where the interactions between the nuclear spins are strongly modified by the electron-electron interactions. We show that the nuclear magnetic ordering at finite temperature relies on the (anomalous) behavior of the 2D static electron spin susceptibility and thus provides a connection between low-dimensional magnetism and nonanalyticities in interacting 2D electron systems. Using various perturbative and nonperturbative approximation schemes in order to establish the general shape of the electron spin susceptibility as a function of its wave vector, we show that the nuclear spins locally order ferromagnetically and that this ordering can become global in certain regimes of interest. We demonstrate that the associated Curie temperature for the nuclear system increases with the electron-electron interactions up to the millikelvin range.

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## I. INTRODUCTION

In the past decade, the field of spintronics has seen remarkable progress.<sup>1–3</sup> Among them, the possibility of confining electron spins in quantum dots opens the door to quantum spintronics. This is based on the possibility of controlling and manipulating single electron spins in order to build devices able to achieve operations for quantum information processing. The most promising and challenging idea is the use of spins of confined electrons in quantum dots to realize quantum bits.<sup>4</sup> Within the last years, all the necessary requirements for spin-based quantum computation have been realized experimentally, going from the coherent exchange of two electron spins in a double dot<sup>5</sup> to the coherent control of a single electron spin, including the observation of Rabi oscillations.<sup>6</sup> These achievements have become possible because electron spins in semiconductor quantum dots are relatively weakly coupled to their environment and are therefore long lived quantities, quite robust against decay. Indeed, longitudinal relaxation times in these systems have been measured to be of the order of 1 s.<sup>7–9</sup> A lower bound on the spin decoherence time for an ensemble of electron spins in GaAs quantum dots has been measured to be typically larger than 100 ns,<sup>10</sup> while a coherence time in a single quantum dot exceeding 1  $\mu$ s has been recently achieved using spin-echo techniques.<sup>5</sup> It is by now well established that one of the major sources of decoherence for a single electron spin confined in a quantum dot is the contact hyperfine interaction with the surrounding lattice nuclear spins.<sup>11</sup>

One possibility to lift this source of decoherence is the development of quantum control techniques, which effectively lessen or even suppress the nuclear spin coupling to the electron spin.<sup>5,12,13</sup> Another possibility is to narrow the nuclear spin distribution<sup>14–16</sup> or dynamically polarize the nuclear spins.<sup>11,14,17–19</sup> However, in order to extend the spin decay time by 1 order of magnitude through polarization of the nuclear spins, a polarization of above 99% is required,<sup>14</sup>

quite far from the best result so far reached in quantum dots, which is around 60%.<sup>19</sup> A common point to the aforementioned approaches is their aim at mitigating nuclear spin fluctuations by external actions. Recently, the possibility was raised of an intrinsic polarization of nuclear spins at finite but low temperature in the two-dimensional electron gas (2DEG) confined by the GaAs heterostructure.<sup>20</sup>

The nuclear spins within the 2DEG interact mainly via the Rudermann-Kittel-Kasuya-Yosida (RKKY) interaction,<sup>21</sup> which is mediated by the conduction electrons (the direct dipolar interactions between the nuclear spins are much weaker, see below). An intrinsic nuclear spin polarization relies on the existence of a temperature dependent magnetic phase transition, at which a ferromagnetic ordering sets in, thus defining a nuclear spin Curie temperature.

The first estimate of such a Curie temperature was obtained for three-dimensional (3D) metallic samples, using a Weiss mean field treatment by Fröhlich and Nabarro more than 60 years ago.<sup>22</sup> They determined the nuclear spin Curie temperature to be in the microkelvin range or less for 3D metals. A Weiss mean field treatment also gives a nuclear spin Curie temperature  $T_C$  in the microkelvin range for a typical 2DEG made from GaAs heterostructures;<sup>20</sup> yet, a more detailed analysis is desirable for at least two reasons. First, a Weiss mean field analysis does not properly take into account the dimensionality of the system. Second, it ignores electron-electron (e-e) interactions. In two dimensions, the Mermin-Wagner theorem<sup>23</sup> states that there is no phase transition at finite temperature for spin systems with Heisenberg (isotropic) interactions, provided that the interactions are short ranged enough. However, RKKY interactions are long ranged and, strictly speaking, the Mermin-Wagner theorem does not apply, although a conjecture extending the Mermin-Wagner theorem for RKKY interactions due to noninteracting electron systems has been recently formulated (and proven in some particular cases).<sup>24</sup>

In Ref. 20, we started from a Kondo lattice description for the system composed of nuclear spins and electrons, then

derived a rather general effective Hamiltonian for nuclear spins after integrating out electron degrees of freedom, and finally performed a spin wave analysis around a ferromagnetic ground state (which we assumed to be the lowest energy state). We indeed showed that  $T_C=0$  for noninteracting electrons in agreement with the latter conjecture. However, taking into account e-e interactions changed drastically this conclusion. It turns out that e-e interactions modify the long-range nature of the two-dimensional (2D) RKKY interactions (which are directly related here to the static electron spin susceptibility) and thereby allow some ordering of the nuclear spins at finite temperature.<sup>20</sup> Furthermore, we showed that the temperature scale at which this ordering takes place is enhanced by e-e interactions.<sup>20</sup>

The study of thermodynamic quantities in *interacting* electron liquids (especially in two dimensions) has attracted some theoretical<sup>25–34</sup> and experimental<sup>35</sup> interest recently with the goal to find deviations from the standard Landau-Fermi liquid behavior. It is therefore quite remarkable that the macroscopic magnetic properties of nuclear spins in a 2DEG, and thus their finite temperature ordering, are directly related to the corrections to the static electron spin susceptibility induced by e-e interactions. They may therefore be associated with an indirect signature of Fermi liquid nonanalyticities. Nevertheless, it turns out that the temperature dependence of the electron spin susceptibility  $\chi_s(T)$  is rather intricate. On the one hand, from perturbative calculations in second order in the short-ranged interaction strength, one obtains that  $|\chi_s(T)|$  increases with temperature.<sup>29–31</sup> A recently developed effective supersymmetric theory, which takes into account renormalization effects, reproduces this behavior and also allows us to determine subleading logarithmic corrections.<sup>32</sup> These subleading logarithmic corrections have also been found by a direct diagrammatic approach, taking into account renormalization effects.<sup>33,34</sup> When these corrections are important, they lead to a non-monotonic behavior of  $|\chi_s(T)|$ , which first decreases with temperature.<sup>33,34</sup> This latter behavior is in agreement with recent experiments on 2DEGs.<sup>35</sup>

In view of these recent results, we want to reconsider the question of a finite temperature ordering of nuclear spins by taking into account renormalization effects of the static spin susceptibility  $\chi_s(q)$ , where  $q$  is the wave vector, and therefore going beyond Ref. 20. It turns out that, *a priori*, different nuclear spin orderings can occur, depending on temperature and other sample parameters such as the interaction strength, measured by the dimensionless parameter  $r_s$  (essentially, the ratio between Coulomb and kinetic energies of the electrons). We consider at least two possible ordered phases in the nuclear system: a ferromagnetic ordering<sup>36</sup> and a helical spin ordering where the nuclear spins align ferromagnetically at the scale of the nuclear lattice constant but point in opposite directions at the scale of the Fermi wavelength (roughly 2 orders of magnitude larger than the nuclear lattice spacing at small  $r_s \leq 1$ ). Depending on the general nonperturbative shape of  $\chi_s(q)$  (which may have a complex dependence on  $T$  and  $r_s$ ), we discuss the possible ordered phases and their associated magnetic properties.

The outline of the paper is as follows: In Sec. II, we formulate a Kondo lattice description of our problem where

the nuclear spins are playing a role analogous to magnetic impurities embedded in an electron liquid. We then derive a general effective magnetic Hamiltonian for nuclear spins where the interaction is controlled by the electron spin susceptibility in two dimensions. In Sec. III, we calculate the electron spin susceptibility in an interacting 2DEG using various approximation schemes for both short-ranged and long-ranged interactions. Particular attention is paid to renormalization effects in the Cooper channel, which turn out to be important. Section IV is devoted to the magnetic properties of the nuclear spins depending on the general wave-vector dependence of the electron spin susceptibility. We discuss two different phases: a ferromagnetic phase and a helical phase with a period of the order of the electron Fermi wavelength. Finally, Sec. V contains a summary of our main results and also perspectives. The Appendix contains some details of the derivation of the effective nuclear spin Hamiltonian and of the reduction to a strictly 2D problem.

## II. MODEL HAMILTONIAN

### A. Kondo lattice description

In order to study an interacting electron gas coupled to nuclear spins within the 2DEG, we adopt a tight-binding representation in which each lattice site contains a single nuclear spin and electrons can hop between neighboring sites. A general Hamiltonian describing such a system reads

$$\begin{aligned} H &= H_0 + \frac{1}{2} \sum_{j=1}^{N_l} A_j c_{j\sigma}^\dagger \boldsymbol{\tau}_{\sigma\sigma'} c_{j\sigma'} \cdot \mathbf{I}_j + \sum_{ij} v_{ij}^{\alpha\beta} I_i^\alpha I_j^\beta \\ &= H_0 + H_n + H_{dd}, \end{aligned} \quad (1)$$

where  $H_0$  denotes the conduction electron Hamiltonian,  $H_n$  the electron-nuclear spin hyperfine interaction, and  $H_{dd}$  the general dipolar interaction between the nuclear spins.  $H_0$  can be rather general and includes electron-electron (e-e) interactions. In Eq. (1),  $c_{j\sigma}^\dagger$  creates an electron at the lattice site  $\mathbf{r}_j$  with spin  $\sigma = \uparrow, \downarrow$ , and  $\boldsymbol{\tau}$  represents the Pauli matrices. We have also introduced  $\mathbf{I}_j = (I_j^x, I_j^y, I_j^z)$ , the nuclear spin located at the lattice site  $\mathbf{r}_j$ , and  $A_j$ , the hyperfine coupling constant between the electron and the nuclear spin at site  $\mathbf{r}_j$ . Summation over the spin components  $\alpha, \beta = x, y, z$  is implied. The electron spin operator at site  $\mathbf{r}_j$  is defined by  $\mathbf{S}_j = \hbar c_{j\sigma}^\dagger \boldsymbol{\tau}_{\sigma\sigma'} c_{j\sigma'}$  (for convenience, we normalize the spin operator here to 1).  $N_l$  denotes the total number of lattice sites. From here on, we assume that  $A_j = A > 0$ , which means we assume that the hyperfine interaction is antiferromagnetic and the same for all atoms that constitute the heterostructures (typically Ga and As and their isotopes).

The nuclear spins are also coupled via the dipolar interaction to other nuclear spins, which are not embedded in the 2DEG. Taking into account this interaction as well makes the problem of the magnetism of nuclear spins in GaAs heterostructures an *a priori* 3D tremendously complicated one. Nevertheless, it turns out that the dipolar interaction energy scale  $E_{dd}$  is the smallest one. It has been estimated to be  $E_{dd} \approx 100$  nK.<sup>37</sup> In particular,  $k_B T \gg E_{dd}$ , where  $T$  is the temperature of a typical experiment. In the rest of the paper, we

neglect all direct dipolar interactions between the nuclear spins, which are in general smaller than the indirect interaction, as we will see. Therefore, we assume that  $v_{ij}^{\alpha\beta} \approx 0$  in Eq. (1). This assumption is important since it allows us to focus only on those nuclear spins that lie within the support of the electron envelope wave function (in growth direction).

The general Hamiltonian in Eq. (1) is the well-known Kondo lattice Hamiltonian (KLH), though  $H_0$  also contains e-e interactions. The KLH is one of the most studied models in condensed matter theory due to its large variety of applications. The KLH has been used to describe the properties of transition metal oxides,<sup>38</sup> heavy fermion compounds,<sup>39,40</sup> more recently, also magnetic semiconductors (or semimetals) in the series of rare earth substances,<sup>41</sup> and diluted magnetic semiconductors such as  $\text{Ga}_{1-x}\text{Mn}_x\text{As}$ ,<sup>42,43</sup> to list only a few. The nuclear spins play a role analogous to magnetic impurities in the Kondo lattice problem. The regime in which we are interested corresponds to the weak Kondo coupling regime in the sense that  $A \ll E_F$ , where  $E_F$  is the Fermi energy. Furthermore, the nuclear spin density  $n_s$  is far larger than the electron density  $n_e$ . It is worth noticing that the single nuclear spin Kondo temperature  $T_K \approx D \exp(-E_F/A)$  (with  $D$  being the electron bandwidth) is extremely small compared to all other energy scales. We are therefore far away from the so-called controversial exhaustion regime<sup>44</sup> where the individual screening of the impurity competes with an indirect magnetic exchange between the nuclear spins.

In this low electron density regime, the ground state of the magnetic system (here, the nuclear spins) has been shown to be ordered ferromagnetically in three dimensions using various treatments that go beyond mean field theory and which notably include spin wave modes (but neglect e-e interactions).<sup>39</sup>

### B. Derivation of an effective magnetic Hamiltonian

We first go to Fourier space and rewrite  $H_n$  in Eq. (1) as

$$H_n = \frac{A}{2N_l} \sum_{\mathbf{q}} \mathbf{S}_{\mathbf{q}} \cdot \mathbf{I}_{\mathbf{q}}, \quad (2)$$

where  $\mathbf{I}_{\mathbf{q}} = \sum_j e^{i\mathbf{q} \cdot \mathbf{r}_j} \mathbf{I}_j$  and  $\mathbf{S}_{\mathbf{q}} = \sum_j e^{-i\mathbf{q} \cdot \mathbf{r}_j} \mathbf{S}_j$  are the Fourier transforms of  $\mathbf{I}_j$  and  $\mathbf{S}_j$ , respectively. (From now on, we set  $\hbar = 1$ .) Since  $A$  is a small energy scale in our case, we can perform a Schrieffer-Wolff (SW) transformation in order to eliminate terms linear in  $A$ , followed by integrating out the electron degrees of freedom. Furthermore, we can reduce our initial 3D model to a genuine 2D problem. The main steps of these calculations are given in the Appendix. We are left with an effective Hamiltonian  $H_{\text{eff}}$  for the nuclear spins in a 2D plane,

$$H_{\text{eff}} = \frac{A^2}{8n_s N} \sum_{\mathbf{q}} J_{\mathbf{q}}^{\alpha} \chi_{\alpha\beta}(q) I_{-\mathbf{q}}^{\beta}, \quad (3)$$

where

$$\chi_{\alpha\beta}(\mathbf{q}, \omega) = -\frac{i}{Na^2} \int_0^{\infty} dt e^{-i\omega t - \eta t} \langle [S_{\mathbf{q}}^{\alpha}(t), S_{-\mathbf{q}}^{\beta}] \rangle \quad (4)$$

is a general 2D electron spin susceptibility tensor,  $\chi_{\alpha\beta}(q) = \chi_{\alpha\beta}(\mathbf{q}, \omega=0)$  and  $q = |\mathbf{q}|$ .  $N$  is the number of lattice sites in the 2D plane and  $a$  denotes the lattice spacing for nuclear spins. Note that  $\langle \dots \rangle$  means average over electron degrees of freedom only. We have normalized  $\chi$  such that it coincides with the density-density Lindhard function (see below) in the isotropic and noninteracting limit.

The only assumptions we make are time reversal symmetry of  $H_0$ , as well as translational and rotational invariance. The effective Hamiltonian in Eq. (3) is therefore quite general and does not depend on the dimensionality of the system. Note that Eq. (3) is also valid when electron-electron interactions are taken into account. It is worth emphasizing that the SW transformation neglects retardation effects. This is appropriate since the nuclear spin dynamics is slow compared to the electron one (in terms of energy scales, this is related to the fact that  $A \ll E_F$ ). Therefore, electrons see an almost static nuclear spin background, and the adiabatic approximation (for the conduction electrons) is well justified. In the case of ferromagnetic semiconductors, such an approximation breaks down and retardation effects must be taken into account.<sup>45</sup> If we also assume spin isotropy in the 2DEG, then  $\chi_{\alpha\beta}(q, \omega \rightarrow 0) = \delta_{\alpha\beta} \chi_s(q)$ , where  $\chi_s(q)$  is the isotropic electron spin susceptibility in the static limit.

In real space, the effective nuclear spin Hamiltonian reads

$$H_{\text{eff}} = -\frac{1}{2} \sum_{\mathbf{r}, \mathbf{r}'} J_{\mathbf{r}-\mathbf{r}'}^{\alpha\beta} \mathbf{I}_{\mathbf{r}}^{\alpha} \mathbf{I}_{\mathbf{r}'}^{\beta}, \quad (5)$$

where

$$J_{|\mathbf{r}|}^{\alpha\beta} = -(A^2/4n_s) \chi_{\alpha\beta}(|\mathbf{r}|) \quad (6)$$

is the effective exchange coupling. The nuclear spins  $\mathbf{I}_{\mathbf{r}}$  are therefore interacting with each other, this interaction being mediated by the conduction electrons. This is just the standard RKKY interaction,<sup>21</sup> which, however, as we shall see, can be substantially modified by electron-electron interactions compared to the free electron case.

### III. ELECTRON SPIN SUSCEPTIBILITY IN A TWO-DIMENSIONAL INTERACTING ELECTRON GAS

The main result of the previous section is that the magnetic exchange interaction between the nuclear spins is mediated by the electron gas. Therefore, the key quantity governing the magnetic properties of the nuclear spins is the electron spin susceptibility  $\chi_s(\mathbf{q})$  in two dimensions. The calculation of this quantity in an interacting 2DEG has been the subject of intense efforts in the past decade in connection with nonanalyticities in the Fermi liquid theory.<sup>25-34</sup> On a more fundamental level, incorporating e-e interactions in the calculations of thermodynamic quantities has been an important area of condensed matter theory over the past 50 years. In particular, the study of nonanalytic behavior of thermodynamic quantities and susceptibilities in electron liquids has

attracted recent interest, especially in two dimensions.<sup>25–34</sup> Of particular importance for the following is the recent findings by Chubukov and Maslov,<sup>29</sup> namely, that the static non-uniform spin susceptibility  $\chi_s(q)$  depends *linearly* on the wave vector modulus  $q=|\mathbf{q}|$  for  $q \ll k_F$  in two dimensions (while it is  $q^2$  in three dimensions), with  $k_F$  the Fermi momentum. This nonanalyticity arises from the long-range correlations between quasiparticles mediated by virtual particle-hole pairs, despite the fact that e-e interactions was assumed to be short ranged.

Let us first recall the case of noninteracting electrons. In this case,  $\chi_s$  coincides with the usual density-density (or Lindhard) response function  $\chi_L$ ,<sup>46</sup>

$$\chi_L(q) = \frac{1}{Na^2} \sum_{\mathbf{k}, \sigma} \frac{n_{\mathbf{k}, \sigma} - n_{\mathbf{k}+\mathbf{q}, \sigma}}{\epsilon_{\mathbf{k}, \sigma} - \epsilon_{\mathbf{k}+\mathbf{q}, \sigma} + i\eta}, \quad (7)$$

where  $n_{\mathbf{k}}$  is the electron number operator,  $\epsilon_{\mathbf{k}}$  the dispersion relation, and  $\eta > 0$  an infinitesimal regularization parameter. This Lindhard function can be evaluated exactly and reads as follows in two dimensions:<sup>46</sup>

$$\chi_L(q) = -N_e \left( 1 - \Theta(q - 2k_F) \frac{\sqrt{q^2 - 4k_F^2}}{q} \right), \quad (8)$$

where  $N_e = n_e / E_F$  is the electron density of states (per spin). Note that  $N_e = m^* / \pi$ , where  $m^*$  is the effective electron mass in a 2DEG. It follows from Eq. (8) that

$$\delta\chi_L(q) \equiv \chi_L(q) - \chi_L(0) = 0 \quad \text{for } q \leq 2k_F. \quad (9)$$

Let us now include electron-electron interactions. It is convenient to introduce a relativistic notation, with  $\bar{p} \equiv (p_0, \mathbf{p})$  being the  $(D+1)$  momentum where  $p_0$  denotes the frequency and  $\mathbf{p}$  the  $D$ -dimensional wave vector (here  $D=2$ ). In a zero-temperature formalism, the susceptibility can be written diagrammatically<sup>46</sup> as

$$\chi_s(\bar{q}) = -\frac{i}{L^D} \sum_{\bar{p}_1, \sigma, \sigma'} \sigma \sigma' G_\sigma(\bar{p}_1 - \bar{q}/2) G_{\sigma'}(\bar{p}_1 + \bar{q}/2) \Lambda_{\bar{p}_1, \sigma \sigma'}(\bar{q}), \quad (10)$$

with  $\sigma, \sigma' = \pm$ , and where  $L = aN^{1/D}$  is the system length,  $G_\sigma(\bar{p}_1)$  is the exact single-particle Green's function, and  $\Lambda(\bar{q})$  is the exact vertex function, which can be expressed in terms of the exact scattering amplitude  $\Gamma(\bar{q})$  as follows:<sup>46</sup>

$$\begin{aligned} \Lambda_{\bar{p}_1, \sigma \sigma'}(q) &= \delta_{\sigma \sigma'} \\ &- \frac{i}{L^D} \sum_{\bar{p}_2} \Gamma_{\bar{p}_1, \bar{p}_2}^{\sigma \sigma \sigma' \sigma'}(\bar{q}) G_{\sigma'}(\bar{p}_2 + \bar{q}/2) G_{\sigma'}(\bar{p}_2 - \bar{q}/2). \end{aligned} \quad (11)$$

This scattering amplitude plays a crucial role, as we will see next.  $\Gamma$  is, for a general scattering event, a function of four spin variables:  $\sigma_1, \sigma'_1, \sigma_2,$  and  $\sigma'_2$ . Nevertheless, one can use a convenient parametrization that ensures rotational spin invariance,<sup>46</sup>

$$\Gamma_{\bar{p}_1, \bar{p}_2}^{\sigma_1 \sigma'_1 \sigma_2 \sigma'_2}(\bar{q}) = \Gamma_{\bar{p}_1, \bar{p}_2}^+ \delta_{\sigma_1 \sigma'_1} \delta_{\sigma_2 \sigma'_2} + \Gamma_{\bar{p}_1, \bar{p}_2}^- \boldsymbol{\tau}_{\sigma_1 \sigma'_1} \cdot \boldsymbol{\tau}_{\sigma_2 \sigma'_2}, \quad (12)$$

where  $\boldsymbol{\tau}$  is a vector whose components are the Pauli matrices  $(\tau^x, \tau^y, \tau^z)$  and  $\boldsymbol{\tau}_{\sigma_1 \sigma'_1} \cdot \boldsymbol{\tau}_{\sigma_2 \sigma'_2} = \sum_{a=x,y,z} \tau_{\sigma_1 \sigma'_1}^a \tau_{\sigma_2 \sigma'_2}^a$ . Note that  $\Gamma^\pm$  is spin independent and corresponds to the charge and spin channels, respectively. Following Ref. 46, we next write the Bethe-Salpeter (BS) equation for  $\Gamma^-$  (corresponding to the spin channel) as follows:

$$\Gamma_{\bar{p}_1, \bar{p}_2}^-(\bar{q}) = (\Gamma_{\text{irr}}^-)_{\bar{p}_1, \bar{p}_2}(\bar{q}) + \frac{1}{L^D} \sum_{\bar{p}''} (\Gamma_{\text{irr}}^-)_{\bar{p}_1, \bar{p}''}(\bar{q}) R_{\bar{p}''}(\bar{q}) \Gamma_{\bar{p}'', \bar{p}_2}^-(\bar{q}), \quad (13)$$

where  $(\Gamma_{\text{irr}}^-)_{\bar{p}\bar{p}'}(\bar{q})$  is the exact irreducible electron-hole scattering amplitude in the spin channel, and

$$R_{\bar{p}}(\bar{q}) = -2iG(\bar{p} + \bar{q}/2)G(\bar{p} - \bar{q}/2) \quad (14)$$

is the electron-hole bubble.<sup>47</sup>

One can exactly solve, at least formally, the BS equation [Eq. (13)] using a matrix notation where the matrix indices run over  $\bar{p}$ . Within this notation,  $R$  is a diagonal matrix. We find that

$$\Gamma_{\bar{p}_1, \bar{p}_2}^- = \sum_{\bar{p}''} (\Gamma_{\text{irr}}^-)_{\bar{p}_1, \bar{p}''} \left( \frac{1}{1 - \Gamma_{\text{irr}}^-(\bar{q})R(\bar{q})} \right)_{\bar{p}'' \bar{p}_2}. \quad (15)$$

This enables us to derive an exact and closed expression for the spin susceptibility, given by

$$\chi_s(\bar{q}) = \frac{1}{L^D} \sum_{\bar{p}, \bar{p}'} \left( R(\bar{q}) \frac{1}{1 - \Gamma_{\text{irr}}^-(\bar{q})R(\bar{q})} \right)_{\bar{p}\bar{p}'}. \quad (16)$$

In general,  $\Gamma_{\text{irr}}^-$  cannot be calculated exactly, and some approximations are required. The approximation we use in the following consists in replacing the exact irreducible electron-hole scattering amplitude  $(\Gamma_{\text{irr}}^-)_{\bar{p}, \bar{p}'}$  by an averaged value calculated with respect to all possible values of  $p$  and  $p'$  near the Fermi surface. This is equivalent to the following approximation:

$$(\Gamma_{\text{irr}}^-)_{\bar{p}, \bar{p}'}(\bar{q}) \approx \Gamma_{\text{irr}}^-(\bar{q}) \quad \forall p, p'. \quad (17)$$

We now assume  $q_0=0$  and suppress the  $q_0$  argument in what follows since we are interested in the static properties of the spin susceptibility.

### A. Short-ranged interaction

In this section, we consider a  $q$ -independent short-ranged interaction potential, which corresponds within our notations to  $\Gamma_{\text{irr}}^-(q) = -U$ . This approximation considerably simplifies the BS equation [Eq. (13)] and the formal expression of  $\chi_s$  in Eq. (16). The derivative of  $\chi_s(q)$  with respect to  $q$  can be expressed in a simple compact form,

$$\frac{\partial \chi_s(q)}{\partial q} = \frac{\partial \Pi(q)}{\partial q} \frac{1}{[1 + U\Pi(q)]^2}, \quad (18)$$

where  $\Pi(q) = \sum_{\vec{p}} R_{\vec{p}}(q)/L^D$ . In the low  $q \ll k_F$  limit, one can approximate the term  $\Pi(q)$  in the denominator of Eq. (18) by its noninteracting value  $\chi_L(0) = -N_e$ .

The multiplicative factor  $1/(1 - UN_e)^2$  in Eq. (18) signals the onset of the ferromagnetic Stoner instability when  $UN_e$  approaches unity. The Stoner instability is supposed to occur for very large  $r_s \sim 20$  according to Monte Carlo results.<sup>48</sup> For smaller  $r_s \leq 10$ , we are still far from the Stoner instability. Though this multiplicative term does not play a significant role at small  $r_s$ , it increases with  $r_s$ , showing the tendency.

### 1. Perturbative calculation

The corrections to the polarization bubble  $\Pi(q)$  are dominated by the first bubble correction to the self-energy of  $G_\sigma(p)$ .<sup>29</sup> These corrections have been calculated in second order in  $U$  in the small  $q$  limit in Ref. 29, with the result

$$\delta\Pi(q) = \Pi(q) - \Pi(0) \approx -4q\chi_s \frac{|\Gamma(\pi)|^2}{3\pi k_F}, \quad (19)$$

where  $\chi_s = |\chi_s(0)|$  and  $\Gamma(\pi) \sim -Um^*/4\pi$  is the  $(2k_F)$  backscattering amplitude. When  $UN_e \ll 1$ , we recover from Eq. (18) the known result  $\delta\chi_s(q) = \delta\Pi(q)$ .<sup>29</sup> This perturbative calculation therefore gives  $\delta\chi_s(q) = \alpha q$ , with  $\alpha < 0$ , or, equivalently,  $|\chi_s(q)|$  increases with  $q$  at low  $q$ .<sup>36</sup> Finite temperature calculations along the same line imply that  $\delta\chi_s(T) = \gamma T$ , with  $\gamma < 0$ .<sup>30,31</sup> This result has been confirmed by using a supersymmetric effective theory of interacting spin excitations.<sup>32</sup>

On the other hand, recent experiments on a low density interacting electron gas in silicon metal oxide semiconductor field effect transistors (MOSFETs) have found that  $|\chi_s(T)|$  decreases at low temperature,<sup>35</sup> in apparent contradiction with perturbative calculations. This rather puzzling situation demands a nonperturbative approach.

### 2. Beyond lowest order perturbation theory: Renormalization effects

The previous calculations give a spin susceptibility, which is quadratic in the backscattering amplitude. However, it is known that at low enough energy, the backscattering amplitude becomes renormalized. One should therefore instead consider some type of renormalized perturbation theory approximation (RPTA). Shekhter and Finkel'stein<sup>33</sup> argued recently that the strong renormalization of the scattering amplitude in the Cooper channel may explain the sign of  $\delta\chi_s(T)$  in the experiment by Prus *et al.*<sup>35</sup>

Let us introduce  $\Gamma(\theta, T)$ , the two-particle scattering amplitude at a particular scattering angle  $\theta$  and temperature  $T$ . The Cooper channel corresponds to  $\theta = \pi$ , and we denote by  $\Gamma_c(T) = \Gamma(\pi, T)$  the corresponding two-particle scattering amplitude. In the Cooper channel, the two particles that scatter have exactly opposite momenta. We first expand  $\Gamma_c$  in Fourier harmonics such that

$$\Gamma_c(T) = \sum_n (-1)^n \Gamma_{c,n}(T), \quad (20)$$

where  $n$  is an integer representing the angular momentum quantum.

In order to analyze the temperature dependence of  $\Gamma_{c,n}(T)$ , it is enough to write the BS equation for  $\Gamma_c$  in a way fully analogous to Eq. (13). In the Cooper channel, due to the fact that the momentum of the center of motion of the two scattering particles is zero, the integration over the electron-hole bubble gives rise to logarithmic infrared (IR) divergences. These divergences are due to particle-hole excitations around the Fermi sea and already occur in the ladder approximation (i.e., by considering only the bare short-ranged interaction potential in the BS equation). We refer the reader to Refs. 49 and 33 for more details. The logarithmic IR divergences can be absorbed by a rescaling of the scattering amplitudes in the Cooper channel such that

$$\Gamma_{c,n}(T) = \frac{\Gamma_{c,n}}{1 + \Gamma_{c,n} \ln(E_F/k_B T)}, \quad (21)$$

where  $\Gamma_{c,n} = \Gamma_{c,n}(E_F)$  is the bare value. The logarithmic factors give rise to the well known Cooper instability.<sup>50,51</sup> They are just the 2D equivalent of the one found in the discussion of Kohn-Luttinger superconductivity.<sup>52</sup> In Eq. (21), the logarithmic divergence is cut off by the temperature at low energy.

Another way of recovering Eq. (21) is to regard the backscattering amplitudes  $\Gamma_{c,n}$  as some energy scale dependent coupling constants  $\Gamma_{c,n}(\Lambda)$  of the 2DEG [ $\Lambda$  is some running energy scale, the equivalent of  $T$  in Eq. (21)]. The next step is to write renormalization group (RG) equations for these couplings. These couplings are marginal, and the RG equations at the lowest order read

$$\frac{d\Gamma_{c,n}}{d \ln(\Lambda/E_F)} = -(\Gamma_{c,n})^2. \quad (22)$$

The solutions of the RG equations for  $\Gamma_{c,n}(\Lambda)$  are given by Eq. (21), where, essentially,  $T$  is replaced by  $\Lambda$ .<sup>33</sup>

The low-energy behavior of the  $\Gamma_{c,n}(T)$  strongly depends on the bare scattering amplitudes being repulsive ( $\Gamma_{c,n} > 0$ ) or attractive ( $\Gamma_{c,n} < 0$ ). From Eq. (21), we immediately infer that when  $\Gamma_{c,n} > 0$ ,  $\Gamma_{c,n}(T) \rightarrow 0$  at low energy/temperature. On the other hand, when  $\Gamma_{c,n} < 0$ ,  $\Gamma_{c,n}(T)$  renormalizes to the strong coupling regime. Assuming that there exists at least one harmonic  $n_0$  such that  $\Gamma_{c,n_0} < 0$  implies that there exists a temperature below which the scattering in the Cooper channel is entirely dominated by  $\Gamma_{c,n_0}$ .<sup>33</sup>

This reasoning relies on the fact that at least one bare scattering amplitude in the  $n$ th harmonic is negative. This assumption can be further substantiated with some explicit perturbative calculations of the irreducible scattering amplitudes.<sup>52-54</sup> By computing the lowest order corrections contributing to the irreducible scattering amplitude  $\Gamma_{c,irr}$  [therefore going beyond the ladder approximation in the BS equation, which leads to Eq. (21)], one can actually prove that there exist higher harmonics such that  $\Gamma_{c,n} < 0$ .<sup>49</sup>

Note that these calculations imply that a superconducting instability should develop at a temperature  $T \lesssim T_L = E_F \exp(-1/|\Gamma_{c,n_0}|/k_B)$ . This is entirely analogous to the Kohn-Luttinger mechanism for superconductivity.<sup>52,55</sup> Nevertheless, in a typical 2DEG, disorder or some other mechanism may provide a natural infrared cutoff preventing this superconducting instability to be reached. Let us call  $\Delta > T_L$  this infrared cutoff. It is worth noting that  $\Delta$  should not be too large either in order to let  $\Gamma_{c,n_0}$  flow to relative large values of order 1 for this mechanism to be relevant.

As shown in Ref. 33, this mechanism, if it takes place, leads to a nonmonotonic behavior of the temperature dependence of the electron spin susceptibility and more specifically to the existence of a scale  $T_0$ , below which  $\frac{d\chi_s}{dT} > 0$  [note that we use a different sign convention for  $\chi_s(0)$  compared to Ref. 33]. By fitting the experimental data of Prus *et al.* with such a theory, the estimate  $\Gamma_{c,n_0} \sim 0.25-0.3$  was obtained in Ref. 33, which implies  $T_0 \sim 10$  K, a surprisingly large temperature scale of the order of  $E_F \sim 40$  K in this experiment. It is worth emphasizing that such a scale, being dependent on the bare value of  $\Gamma_{c,n_0}$ , is nonuniversal and therefore sample dependent.

This RPTA also raises a similar issue concerning the behavior of the electrostatic spin susceptibility  $\chi_s(q)$  and, more specifically, of  $\Gamma_c(q)$ . Following Ref. 49, one can write a BS equation at zero temperature for  $\Gamma_c(\xi)$ , where  $\xi$  is an energy scale in the vicinity of the Fermi energy  $E_F$ . By linearizing  $\xi$  around  $E_F$  such that  $\xi = v_F q$ , one immediately infers from Eq. (21) that

$$\Gamma_{c,n}(q) = \frac{\Gamma_{c,n}}{1 + \Gamma_{c,n} \ln(E_F/v_F q)} \approx \frac{\Gamma_{c,n}}{1 + \Gamma_{c,n} \ln(k_F/2q)}. \quad (23)$$

Note that at finite temperature and for a realistic system, the infrared cutoff  $\Lambda$  is replaced by  $\max\{v_F q, k_B T, \Delta\}$ . This result is also obtained from a renormalization group approach by working directly in momentum space at low energy where the energy can be linearized around  $k_F$ . At low enough momentum,  $\Gamma_c(\theta, q)$  is dominated by the harmonic  $n_0$  such that

$$\Gamma_c(q) \approx (-1)^{n_0} \frac{-|\Gamma_{c,n_0}|}{1 - |\Gamma_{c,n_0}| \ln(k_F/2q)}. \quad (24)$$

By replacing the bare value of the scattering amplitude in Eq. (19) by the renormalized one, we find that  $\delta\chi_s(q) \approx -4qN_e \frac{|\Gamma(\pi, q)|^2}{3\pi k_F}$  also acquires a nontrivial  $q$  dependence. In particular, using Eq. (24), we obtain that

$$\frac{d\chi_s}{dq} \approx -\frac{4N_e}{3\pi k_F} (\Gamma_c^2 - 2|\Gamma_c^3|), \quad (25)$$

which is positive when  $|\Gamma_c(q)| > 1/2$ .

Let us assume that  $\Delta$ , the aforementioned infrared cutoff, is close to the Cooper instability, i.e.,  $k_B T_L \lesssim \Delta$ . In such a case, the flow of  $\Gamma_c(q)$  in Eq. (24) is cut off by  $\Delta$ . Using  $1/\Gamma_{c,n_0} = \ln(E_F/k_B T_L)$ , one obtains that

$$\Gamma_c \approx \frac{(-1)^{n_0+1}}{\ln\left(\frac{\Delta}{k_B T_L}\right)} \quad \text{for } q \lesssim \Delta/v_F \quad (26)$$

is renormalized toward large values. This implies that there exists a  $q_0 \sim 10q_L$  (where  $q_L = k_B T_L/v_F$ ) such that  $\frac{d}{dq} \delta\chi_s(q) > 0$ ,  $\forall q < q_0$ , and  $\frac{d}{dq} \delta\chi_s(q) < 0$  when  $q \geq q_0$ .

We must point out that the RG equation in Eq. (25) assumes that all other scattering amplitudes  $\Gamma_n$  for  $n \neq n_0$  can be neglected below some low energy. In order to go beyond this approximation, one has to solve (numerically) a set of RG differential equations instead.

This scenario relies on a fine tuning of  $\Delta$  compared to  $k_B T_L$  (typically, this demands  $T_L \lesssim \Delta/k_B < 10T_L$ ). If this condition is not met, one may then expect  $\delta\chi_s(q) < 0$  in accordance with lowest order perturbative calculations. Nevertheless, one should mention that an alternative theory, also giving  $\delta\chi_s(T) > 0$  at low  $T$ , has been put forward.<sup>34</sup> This alternative scenario applies for vanishing Cooper amplitudes. In such a case, the anomalous temperature dependence of the spin susceptibility is dominated by nonanalytic contributions from particle-hole rescattering with small momentum transfer.<sup>34</sup> Whether such a scenario, not considered here, also implies that  $\delta\chi_s(q) > 0$  is an interesting but open question.

The aforementioned considerations immediately raise the issue about the amplitude of  $q_0$  in a typical interacting 2DEG. In order to describe the temperature dependence of  $\chi_s(T)$ , Shekhter and Finkel'stein determined  $\Gamma_{n_0}$  such that the experimental behavior of Ref. 35 is reproduced. This fixes the value of this parameter and also the scale  $k_B T_0$  to the order of  $E_F$ . One may therefore expect  $q_0$  to be of the order of  $k_F$  for a similar 2DEG. Another independent way of substantiating these estimates is to go beyond the ladder approximation in the BS equation.<sup>49</sup> First, as we mentioned before, this allows one to show that there exists a value  $n_0$  at which  $\Gamma_{c,n_0} < 0$ . Second, this can give us an estimate for  $T_L$  and therefore for  $q_0$ . Indeed, in Ref. 49 the Cooper instability has been estimated to set in at the temperature  $k_B T_L \sim E_F e^{-16E_F/(v_s^3 \xi)}$ , where  $\xi \sim \Delta$  plays the role of the infrared cutoff in Ref. 49, and the condition  $\xi/E_F \ll 1$  was assumed. Consistency requires that  $T_L \sim \Delta \sim E_F$  at large  $r_s$ , implying  $q_0 \sim k_F$ . This is in agreement with our previous estimate.

The previous calculations also give us information about the possible shapes for  $\chi_s(q)$ . When renormalization in the Cooper channel is important, we obtain at least one extremum around some wave vector  $q_0 \sim O(k_F)$ . Furthermore, at large  $q$ , we should recover the noninteracting behavior and therefore  $\chi_s(q) \sim \chi_L(q) \rightarrow 0$  for  $q \rightarrow \infty$ . Because  $\chi_s(q_0) < 0$ , we expect another extremum around a value  $q_1 > q_0$ . Since the noninteracting behavior is recovered for  $q \gg 2k_F$ , one may suspect  $q_1 \sim O(2k_F)$ . From the previous considerations, we therefore conclude that there exist (at least) two extrema for the electron spin susceptibility  $\chi_s(q)$ . It is worth emphasizing that this double-extremum structure is a direct consequence of the nontrivial renormalization of the scattering amplitude in the Cooper channel. We have schematically drawn in Fig. 1 the possible qualitative shapes denoted by (a) and (b) of  $\bar{\chi}_s(q) = \chi_s(q)/|\chi_s(0)|$  as a function of  $q/k_F$  and

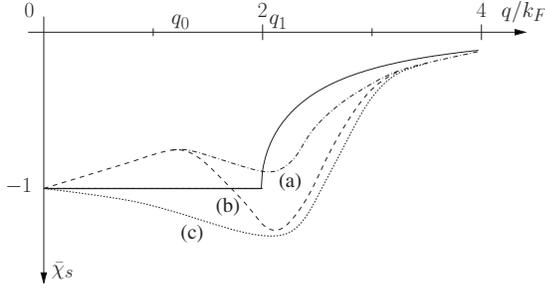


FIG. 1. Three possible qualitative shapes denoted by (a), (b), and (c) for the normalized susceptibility  $\bar{\chi}_s(q) = \chi_s(q)/|\chi_s(0)|$  as a function of  $q/k_F$  (dashed lines) compared to the noninteracting value (thin full line). Here,  $q_0$  and  $q_1$  are the positions of the extrema for curves (a) and (b). In contrast, the local field factor approximation discussed in Sec. III B results in a monotonic increase of  $\bar{\chi}_s(q)$  (not sketched in the figure), being always larger than the noninteracting value.

compared it to the (normalized) noninteracting  $\chi_L(q)/N_e$  at  $T=0$ . In the case denoted by (a), we choose  $\chi_s(q_2) > \chi_s(0)$ , whereas in the case denoted by (b), the absolute value of the susceptibility at  $q_2 \approx 2k_F$  is chosen to exceed the static value, i.e.,  $\chi_s(q_2) < \chi_s(0)$ . The previous considerations do not allow us to discriminate between these two possible shapes of  $\chi_s(q)$ . Furthermore, by increasing  $r_s$ ,  $\chi_s$  can evolve from one shape to another.

On the other hand, if the renormalization in the Cooper channel does not take place, e.g., when it is cut off by disorder, then the perturbative calculations at the lowest order apply and give instead  $\delta\chi_s(q) < 0$  at low  $q$ .<sup>29-31</sup> A possible shape for  $\chi_s(q)$ , consistent with these calculations, has been drawn in Fig. 1 and corresponds to label (c). We should note, however, that the effect of rescattering of a pair of quasiparticles in all different channels<sup>34</sup> should be carefully examined and may still lead to shapes (a) or (b) in Fig. 1.

## B. Long-ranged Coulomb interactions

In the preceding section, we replaced  $\Gamma_{\text{irr}}^-(q)$  by an almost  $q$ -independent constant operator, assuming that the Coulomb interaction was screened and, therefore, short ranged. Let us consider in this section the bare 2D Coulomb interaction,  $V(q) = 2\pi e^2/q$ , where  $e$  is the electron charge.

### 1. Local field factor approximation

One of the most successful approximations for the calculation of electron response functions is the local field factor approximation (LFFA). It improves the random phase approximation<sup>46</sup> for which the effective field seen by an electron is the field that would be seen by a classical test charge embedded in the electron gas. The idea of the LFFA to correct the random phase approximation and to better account for the correlations existing in the electron gas is to replace the average electrostatic potential by a local field effective potential seen by an electron with spin  $\sigma$ , which is part of the 2DEG. (We refer to Ref. 46 for a review). The local field factor  $G_-$  can be defined as follows:

$$G_-(q)V(q) = \chi_s^{-1}(q) - \chi_L^{-1}(q). \quad (27)$$

Equivalently, the static spin susceptibility  $\chi_s$  can be written as

$$\chi_s(q) = \frac{\chi_L(q)}{1 + V(q)G_-(q)\chi_L(q)}. \quad (28)$$

The precise determination of  $G_-(q)$  for all  $q$  is an open problem. However, the asymptotic regimes, particularly the  $q \rightarrow 0$  limit, are quite well established because they are strongly constrained by sum rules.<sup>46</sup> In this work, we use a semiphenomenological interpolation formula given in Ref. 46

$$G_-(q) \approx g_0 \frac{q}{q + g_0(1 - \chi_P/\chi_S)^{-1}\kappa_2}. \quad (29)$$

Here,  $(g\mu_B)^{-2}\chi_P$  is the Pauli susceptibility ( $\mu_B$  the Bohr magneton and  $\chi_P > 0$ ),  $\chi_S = |\chi_s(0)|$  is the renormalized value of the spin susceptibility at  $q=0$ ,  $\kappa_2 = k_F r_s \sqrt{2}$  is the 2D Thomas-Fermi wave vector, and  $g_0$  is the pair-correlation function at  $r=0$ , describing the probability of finding two electrons (of opposite spins) at the same position in the electron gas. This phenomenological form for  $G_-$  has been modified from the one originally proposed by Hubbard<sup>56</sup> in order to satisfy exactly the compressibility sum rule.<sup>46,57</sup> The main weakness of this approach is the arbitrariness of the chosen form for  $G_-$ . For noninteracting electrons,  $\chi_P/\chi_S = 1$ . An approximate form for  $g_0$ , giving a good agreement with quantum Monte Carlo (QMC) calculations, has been proposed recently by Gori-Giorgi *et al.* and reads<sup>58</sup>

$$g_0(r_s) \approx (1 + Ar_s + Br_s^2 + Cr_s^3)e^{-Dr_s/2}. \quad (30)$$

The parameters  $A=0.088$ ,  $B=0.258$ ,  $C=0.00037$ , and  $D=1.46$  are fitting parameters reproducing QMC results for  $g_0$  in a 2DEG.<sup>58</sup> This approximation yields

$$\chi_s(q) \approx -N_e \frac{q + g_0\kappa_2\alpha}{q + g_0\kappa_2(\alpha - 1)} \quad \text{for } q < 2q_F, \quad (31)$$

where  $\alpha = \chi_S/(\chi_S - \chi_P)$  can be regarded as a Fermi liquid parameter. The low- $q$  semiphenomenological approximation for the electron spin susceptibility given in Eq. (31) results in  $\frac{d\chi_s(q)}{dq} > 0$ ,  $\forall q$ , in contrast to the lowest order perturbative calculations.<sup>29,30</sup>

Note that a direct estimate of  $G_-(q)$  by recent QMC in a 2DEG gives an almost linear in  $q$  behavior up to rather large values of  $q \lesssim 2k_F$ , followed by a more complex nonmonotonic behavior around  $2k_F$ , and finally diverges in the large- $q$  limit.<sup>46,59,60</sup> This large- $q$  limit is not reproduced by Eq. (29). This is not a serious drawback since most quantities of interest are dominated by the low- $q$  regime.

However, the scale  $q^* = g_0\kappa_2(\alpha - 1)$  decreases exponentially with  $r_s$  according to Eq. (30). This would imply an almost constant behavior for  $G_-(q)$ , except at low  $q$ . When we compare this behavior with available QMC data,<sup>46,59</sup> we find that there is a manifest contradiction. Therefore, this raises some doubt about the presence of  $g_0$  (a short distance quantity) in Eq. (29).

## 2. Modified local field factor approximation

If we instead replace  $g_0$  in Eq. (29) by a parameter  $g_1$ , such that  $g_1\kappa_2(\alpha-1)=g_1r_s\sqrt{2}(\alpha-1)k_F\gg 2k_F$ , we have checked that the QMC data for  $G_-(q)$  are much better reproduced for  $q$ 's up to  $2k_F$  than by the expression given in Eq. (29). In such a modified local field factor approximation (MLFFA),  $g_1$  is approximately given by

$$g_1 \approx \frac{\zeta_1}{r_s(\alpha-1)} = \frac{\zeta_1}{r_s} \left( \frac{\chi_S}{\chi_P} - 1 \right), \quad (32)$$

with  $\zeta_1$  as some numerical constant of order 1.

We should mention that some other more complicated analytical fits of the QMC data have been obtained in Ref. 61. Nevertheless, we note that the fits used in that paper lead to  $\delta\chi_s(q) \sim q^2$  for two-dimensions, which is in contradiction with all previous approximations. It seems desirable to test Eq. (32) with more detailed QMC calculations.

## C. Comparison of the various approximation schemes

If we summarize the various approximation schemes presented in the previous sections, which are perturbative or semiphenomenological, we can clearly ascertain that  $\delta\chi_s(q) \propto q$  for  $q \ll q_F$ . Nevertheless, the sign of the proportionality constant depends on the approximation scheme we used.

Lowest order perturbation theory in the interaction strength leads to  $d\chi_s(q)/dq < 0$  at low  $q$ .<sup>29,30</sup> However, within the RPTA, renormalization effects in the Cooper channel<sup>33,49</sup> are important and change the picture given by lowest order perturbation theory. In this latter case, the RPTA yields an opposite sign for  $d\chi_s(q)/dq$  below some wave vector  $q_0$ . The LFFA we used implies that  $d\chi_s(q)/dq > 0$  for all  $q$  and, therefore, a monotonic behavior (not shown in Fig. 1), whereas the RPTA leads to a nonmonotonic behavior (see Fig. 1). Establishing a microscopic connection between these two different approaches is obviously a rather difficult and open issue.

The LFFA is a semiphenomenological approximation in which an analytical expression for the local field factor  $G_-(q)$  is ‘‘guessed’’ with the constraints that the asymptotic behavior should reproduce some known results inferred from exact sum rules. The unknown parameters of  $G_-(q)$  are fixed from a fit to the QMC data.<sup>61</sup> One may wonder whether one can extract some information about the possible shapes of  $\chi_s(q)$  directly from the original QMC data. The QMC data shows a rather complicated structure with two extrema for  $G_-(q)$  around  $2k_F$  [see Ref. 59 or Ref. 46 (p. 244)]. Though it might be tempting to relate the double-extremum structure obtained by the RPTA to the QMC results, it turns out to be impossible to extract the behavior of  $\chi_s(q)$  from available QMC data for  $G_-(q)$ . New QMC calculations directly computing  $\chi_s(q)$  instead of  $G_-(q)$  are thus highly desirable.<sup>62</sup>

Finally, we have seen that the low  $q$  dependence of  $\delta\chi_s(q)$  mimics the temperature dependence of  $\delta\chi_s(T)$ , which is in agreement with the experiment by Prus *et al.* at low  $T$ .<sup>35</sup> These experimental features may provide another, though indirect, consistency check of the RPTA.

## IV. MAGNETIC PROPERTIES OF THE NUCLEAR SPINS

We assume in this section that some nuclear spin ordering actually takes place at low enough temperature and analyze how this ordering is destroyed when the temperature is raised.

### A. Mean field approximation

Since the interaction between nuclear spins is of RKKY type, the interaction is ferromagnetic at short distances  $d \ll k_F^{-1}$  [the large  $q$  behavior of  $\chi_s(q)$  is only weakly modified by e-e interactions]. Furthermore, many mean field calculations performed for the 3D Kondo lattice at low electron density (neglecting e-e interactions though) predict a ferromagnetic ordering.<sup>39,64</sup> Therefore, assuming a low temperature ferromagnetic ordering of the nuclear spins seems a reasonable assumption.

We first recall the mean field results for completeness.<sup>20</sup> The Weiss mean field theory gives a Curie temperature,

$$T_C^{MF} = -\frac{I(I+1)}{3k_B} \frac{A^2}{4n_s} \chi_L(q=0), \quad (33)$$

where  $I$  is the nuclear spin value.

In two dimensions, this mean field theory yields for  $T_C^{MF}$  a dependence on the ratio  $n_e/n_s$ . For a metal with about one conduction electron per nuclear spin, the ratio  $n_e/n_s \sim 1$ , and we recover the result derived more than 60 years ago by Fröhlich and Nabarro for a 3D bulk metal.<sup>22</sup> For a 2D metal, the Weiss mean field theory then gives  $k_B T_C = I(I+1)A^2/12E_F$ . For a 2D semiconductor, however, the smaller Fermi energy is compensated by the smaller ratio  $n_e/n_s \ll 1$ . With typical values for GaAs heterostructures,  $I=3/2$ ,  $A \sim 90 \mu\text{eV}$ , and  $a \sim 2 \text{ \AA}$ ,<sup>14</sup> we estimate  $T_C \sim 1 \mu\text{K}$ , which is very low. (For such low  $T_C$ 's, ignoring nuclear dipole-dipole interactions from the start would not be legitimate.) However, this estimate is just based on the simplest mean field theory and, moreover, does not include the effect of e-e interactions. It still leads to a finite  $T_C$  under which the nuclear spins order ferromagnetically.

### B. Spin wave analysis around a ferromagnetic ground state

We shall now go beyond the above mean field approximation and perform a spin wave analysis. The collective low-energy excitations in a ferromagnet are then given by spin waves (magnon excitations).

#### 1. Magnetization and Curie temperature of the nuclear spins

From the standard spin wave analysis,<sup>65</sup> the dispersion relation of the spin waves in the ferromagnet simply reads

$$\omega_q = I(J_0 - J_q) = I \frac{A^2}{4} a^2 [\chi_s(q) - \chi_s(0)], \quad (34)$$

where  $J_q$  is the Fourier transform of  $J_r$  defined in Eq. (6).

At this stage, we already see that the stability of the ferromagnetic ground state demands that  $\delta\chi_s(q) = \chi_s(q) - \chi_s(0) > 0$ . We can therefore conclude that the second order calculation implies that the ferromagnetic ground state is always

unstable.<sup>36</sup> On the other hand, the renormalized perturbation theory approximation developed in Sec. III A 2 shows that it is necessary to go beyond the lowest order perturbation theory.

When renormalization effects are not important in the RPTA, the lowest order perturbative results are recovered, and the ferromagnetic ground state seems unstable (though renormalization effects in all channels must be carefully taken into account as described in Ref. 34). When renormalization effects in the Cooper channel are important, we expect the two possible shapes denoted by (a) and (b) for the static spin susceptibility  $\chi_s(q)$  at  $T=0$  (see Fig. 1). If the case (b) is favored, then there exists a value of  $q$  at which  $\omega_q < 0$ , signaling an instability of the ferromagnetic ground state. From this perspective, cases (b) and (c) are similar. Another ground state must then be assumed, and a subsequent analysis is required. This will be detailed in Sec. IV D.

On the other hand, if the shape of the susceptibility denoted by (a) is favored, the ferromagnetic assumption is self-consistent. The RPTA predicts that there exists a temperature  $T_0$  above which  $d\chi_s/dq < 0$  at low  $q$ . This implies that there exists another temperature  $T_1 \leq T_0$  at which the minimum in  $q_1$  touches the horizontal axis, signaling an instability. If the Curie temperature  $T_C$  is larger than  $T_1$ , then there exists a temperature regime (typically for  $T > T_1$ ) where the ferromagnetic ground state becomes unstable, and a different ordering may be favored. This case will be analyzed in Sec. IV D. On the other hand, if the Curie temperature  $T_C$  is smaller than  $T_1$ , the ferromagnetic ground state is self-consistent below  $T_C$ .<sup>63</sup> Such a scenario is in accordance with the one obtained from LFFA. Let us therefore analyze this latter case.

The magnetization  $m$  per site for a ferromagnet at finite  $T$  is defined by

$$m(T) = I - \frac{1}{N} \sum_{\mathbf{q}} n_{\mathbf{q}} = I - \frac{1}{N} \sum_{\mathbf{q}} \frac{1}{e^{\beta\omega_{\mathbf{q}}} - 1}, \quad (35)$$

where  $n_{\mathbf{q}}$  is the magnon occupation number and the summation is over the first Brillouin zone of nuclear spins. In the continuum limit, this becomes

$$m(T) = I - a^2 \int \frac{d\mathbf{q}}{(2\pi)^2} \frac{1}{e^{\beta\omega_{\mathbf{q}}} - 1}. \quad (36)$$

We define the Curie temperature  $T_C$  as the temperature at which the magnetic order is destroyed by those spin waves. This procedure is equivalent to the decoupling scheme of Bogolyubov and Tyablikov.<sup>66</sup> Another way of determining the Curie temperature is to analyze at which temperature  $T_C$  the spin wave analysis breaks down. The Curie temperature  $T_C$  may then be defined by  $m(T_C)=0$ , which can be written as

$$1 = \frac{a^2}{I} \int \frac{d\mathbf{q}}{(2\pi)^2} \frac{1}{e^{\omega_{\mathbf{q}}/k_B T_C} - 1}. \quad (37)$$

For noninteracting electrons in two dimensions,  $\chi_s(q) - \chi_s(0) = 0$  for  $q < 2k_F$ ,<sup>46</sup> where  $k_F$  is the Fermi wave vector. The spin wave analysis, therefore, gives  $T_C = 0$ . This is in

agreement with a recent conjecture extending the Mermin-Wagner theorem<sup>23</sup> for RKKY interactions to a noninteracting 2D system.<sup>24</sup> For interacting electrons, however, the long-range decay of the RKKY interactions can be altered substantially, and no conclusion can be drawn from the Mermin-Wagner theorem or its extensions.

Let us now include electron-electron interactions (obtained by either the RPTA or the LFFA). All approximations imply that the magnon dispersion is linear in  $q$  at low  $q$ , i.e., that  $\omega_q \approx cq$ , where

$$c = I \frac{A^2}{4} a^2 \left. \frac{\partial \chi_s(q)}{\partial q} \right|_{q \rightarrow 0} \quad (38)$$

can be regarded as the spin wave velocity.<sup>67</sup> Such linear spin wave behavior is usually associated with antiferromagnets, while one would expect a quadratic dispersion for ferromagnetically ordered states like those considered here. This somewhat unexpected linear dispersion comes purely from electron-electron interactions.

The perturbative calculations or their extensions to include the Cooper pair instability allow us to extract only the low  $q$  asymptotic behavior of  $\delta\chi_s(q)$ . Monte Carlo results, however, seem to indicate that the local field factor  $G_-(q)$  is almost linear in  $q$  up to  $q \sim O(2k_F)$ .<sup>46</sup> We will therefore assume that  $\omega_q \approx cq$  for  $q < q^* \sim O(k_F)$ .

This implies that for  $T < T^*$ , where

$$T^* = cq^*/k_B, \quad (39)$$

the integral determining  $m$  in Eq. (36) is entirely dominated by the linear dispersion behavior. Since fast modes corresponding to  $q \gg q^*$  are exponentially suppressed, we can easily compute it, assuming  $\omega_q$  linear in  $q$  for the whole  $q$  range (extending the upper integration limit to infinity). We obtain

$$m(T) = I[1 - (T/T_C)^2] \quad \text{for } T < T^*, \quad (40)$$

where

$$T_C = \frac{2c}{k_B a} \sqrt{\frac{3I}{\pi}} = \frac{A^2 I}{2k_B} \sqrt{\frac{3I}{\pi n_s}} \left. \frac{\partial \chi_s(q)}{\partial q} \right|_{q \rightarrow 0} \quad (41)$$

is the Curie temperature. Note that with these estimates, one has

$$\frac{T^*}{T_C} = \frac{aq^*}{2\sqrt{3I/\pi}} \leq 1. \quad (42)$$

Such a definition of  $T_C$  has been obtained assuming  $\omega_q \approx cq$  for all  $q$ . This approximation has two major aspects: First, it regularizes naturally the integral in Eq. (37) in the UV limit. Second, only the low-energy dependence of  $\omega_q$  is taken into account, which is consistent with a spin wave approximation.

## 2. Alternative UV regularization schemes

In the previous section, we have assumed that  $\omega_q \approx cq$  for all  $q$ . On the other hand, one can assume that we know explicitly  $\omega_q$  for all  $q$  in the first Brillouin zone, i.e., for  $q < \pi/a$  despite only the asymptotic limits of  $\delta\chi_s(q)$  (and therefore of  $\omega_q$ ) are well established. At large  $q$ , we expect

electron-electron interactions to play a minor role and the electron spin susceptibility to be well approximated by its noninteracting value, which decreases as  $1/q^2$  [see Eq. (8)]. This implies that  $\delta\chi_s(q) \approx -\chi_s(0) = \chi_S$  at large  $q$  and that the integral in Eq. (36) or (37) are actually diverging when  $a \rightarrow 0$ . If we adopt such a procedure, the integral in Eq. (37) is fully dominated by the short-distance modes, i.e., by the UV cutoff (and therefore independent of any e-e interactions). Such a regularization scheme is not very satisfying and, furthermore, even inconsistent for a spin-wave approximation, which relies on the long-ranged modes. Note that the  $T_C$  we obtain with such procedure is similar (up to a prefactor of order unity) to the Curie temperature obtained within the mean field theory in Eq. (33).

Another regularization scheme consists in cutting off the integral in Eq. (37) to  $q < 2\zeta_F k_F$ , with  $\zeta_F$  as a constant larger than 1. This can be justified by integrating out fast modes directly at the Hamiltonian level in Eq. (3) since  $\chi(q)$  decreases as  $1/q^2$  for  $q \gg k_F$ . Such a reasoning is equivalent in real space to a decimation procedure in which a square plaquette containing  $(\zeta_F a k_F)^{-1} \times (\zeta_F a k_F)^{-1}$  nuclear spins is replaced by another plaquette with a single average spin. Since at short distance, the RKKY interaction is mainly ferromagnetic, this is equivalent to a mean field procedure. The long distance interaction is not substantially modified. The main effect of this integration over fast modes is that the UV cutoff in Eq. (37) is now of order  $2k_F$  instead of  $\pi/a$ . Although such a procedure does not allow for an exact calculation of  $T_C$  since  $\omega_q$  is not known around  $2k_F$ , this considerably boosts the Curie temperature by orders of magnitudes compared to the previous regularization scheme and in the same range as the Curie temperature determined from Eq. (41).

In the following, we will therefore use Eq. (41) as our definition of  $T_C$ . This has the advantage of providing us with a simple closed formula. Furthermore, such a  $T_C$  is consistent with the long-range approximation for spin waves.

### 3. Numerical estimate of the Curie temperature

Equation (41) gives us an estimate of the Curie temperature as a function of the derivative of the electron spin susceptibility. We have computed these quantities in Sec. III for various approximate schemes. We are therefore now ready to give estimates for the Curie temperature  $T_C$ . Let us start with the RPTA. Assuming  $\Gamma_c \sim O(1)$  in Eq. (26), we obtain a Curie temperature  $T_C \sim 20 \mu\text{K}$  for typical GaAs 2DEG parameters with  $r_s \sim 1$ . Setting  $q^* \sim 2k_F$  in Eq. (42) and using the same parameters, one obtains  $T^* \sim T_C/50$ .

At larger  $r_s$ ,  $T_C$  is enhanced for two reasons: First,  $k_F^{-1}$  increases linearly with  $r_s$ ; second, the value of the spin susceptibility at  $q=0$ ,  $\chi_S = |\chi_S(0)|$ , which is essentially the Pauli susceptibility at small  $r_s$ , increases linearly with  $r_s$ .<sup>46</sup> An approximate value of  $\chi_S$  can be extracted from QMC calculations.<sup>46</sup> One obtains, for example,  $T_C \sim 0.3 \text{ mK}$  for  $r_s=5$  and  $T_C \sim 0.7 \text{ mK}$  for  $r_s=8$ . One may even obtain larger values of  $T_C$  in the millikelvin range for larger  $r_s$  since  $T_C$  increases quadratically with  $r_s$ . Furthermore, when  $UN_e$  is no longer negligible compared to 1,  $T_C$  ( $T^*$ ) is even further enhanced by an additional factor of  $1/(1-UN_e)^2$  [see Eq. (18)].

Close to the ferromagnetic Stoner instability of the electron system, reached when  $UN_e \sim 1$ , the Curie temperature  $T_C$  for the nuclear system is strongly enhanced as could have been anticipated.

On the other hand, if we use the local field factor approximations developed in Sec. III B, we can determine  $T_C$  by inserting  $\chi_s(q)$  obtained from the LFFA into Eq. (41),

$$T_C = \frac{IA}{2k_B} \sqrt{\frac{3I}{\pi} \frac{A}{(\alpha-1)^2 g_i V(a)}}, \quad (43)$$

where  $g_i = g_0$  [Eq. (30)] or  $g_i = g_1$  [Eq. (32)]. The energy scale  $(\alpha-1)^2 g_i V(a)$  can be interpreted as a renormalized screened potential due to collective interaction effects that are incorporated in the LFFA.

If we use  $g_i = g_0$ , the LFFA gives an exponential enhancement of  $T_C$  with the increasing interaction parameter  $r_s$ . Yet, as we have discussed in Sec. III B, this form for the local field factor  $G_-$  cannot really be trusted at large  $r_s > 1$  when compared to QMC data. If we use the MLFFA instead, which seems to be in better agreement with QMC data, then  $g_i = g_1$ , and we obtain  $T_C \sim 0.4 \text{ mK}$  for  $r_s \sim 5$  and  $T_C \sim 1 \text{ mK}$  for  $r_s \sim 8$ . These values are consistent with the ones found using renormalized perturbation theory.

Note that the ratio  $A/(\alpha-1)^2 g_0 V(a)$  can be regarded as the small parameter of our theory. For some large value of  $r_s$ , the dimensionless parameter  $A/(\alpha-1)^2 g V(a)$  may approach unity, and the truncation of the Schrieffer-Wolff transformation at the lowest order becomes unjustified.

### C. Self-consistent calculation of the Curie temperature

If we assume a ferromagnetic ordering of the nuclear spins, this generates some rather large nuclear Overhauser magnetic field  $B_{\text{eff}}(m) \sim O(1 \text{ T})$  in a GaAs 2DEG for  $m \leq I$  that we have neglected before. Therefore, the spin degeneracy of the conduction electrons will be lifted by the Zeeman energy  $g\mu_B B_{\text{eff}}$ , which should have some effects on the determination of  $T_C$ , too. A self-consistent procedure including feedback effects is therefore required. This is the purpose of this section.

The electron spin susceptibility is no longer isotropic. In the following, we will assume that it is still diagonal but has now a longitudinal component  $\chi_s^z$  (in the direction of the magnetic field produced by the nuclear spins) and a transverse component  $\chi_s^\perp$ . Taking into account such an anisotropy, our SW transformation is still valid, and we obtain an effective spin anisotropic 2D long-ranged Hamiltonian, which replaces Eq. (5),

$$H_{\text{eff}} = -\frac{1}{2} \sum_{\mathbf{r}, \mathbf{r}'} \left[ J_{|\mathbf{r}-\mathbf{r}'|}^z I_{\mathbf{r}}^z I_{\mathbf{r}'}^z + \frac{J_{|\mathbf{r}-\mathbf{r}'|}^\perp}{2} (I_{\mathbf{r}}^+ I_{\mathbf{r}'}^- + I_{\mathbf{r}}^- I_{\mathbf{r}'}^+) \right], \quad (44)$$

where, in  $q$  space,  $J_q^{z/\perp} = -(A^2/4n_s) \chi^{z/\perp}(q)$ . The dispersion relation for magnons now becomes

$$\omega_q = I(J_0^z - J_q^\perp), \quad (45)$$

and therefore generically acquires a gap for  $B_{\text{eff}} \neq 0$ . Let us denote by  $\Delta(B_{\text{eff}}) = \Delta(m)$  such a gap. The expansion of the magnon dispersion for small  $q$  leads to

$$\omega_q \approx I\Delta + Iq \left. \frac{\partial J_q^\perp}{\partial q} \right|_{q=0} = I\Delta + c'q, \quad (46)$$

where we have assumed a linear in  $q$  behavior for  $\chi_s^\perp(q)$  and defined  $c' = I \frac{\partial J_q^\perp}{\partial q}(q=0)$ . We assume ferromagnetic order, and so  $c' > 0$ . In the following, we develop a simple approach of the Landau type. Equation (37), which determines the Curie temperature, is still valid and now reads

$$1 = \frac{a^2}{I} \int \frac{d\mathbf{q}}{(2\pi)^2} \frac{1}{e^{\beta I\Delta + \beta c'|\mathbf{q}|} - 1}, \quad (47)$$

where  $\beta = 1/k_B T$ . As before, we assume the linear dispersion to be valid for all  $q$ , which naturally regularizes the integral in the UV limit. The latter integral equation can be rewritten in a more compact form as

$$1 = \frac{a^2}{2\pi I (\beta c')^2} \text{Li}_2(e^{-\beta I\Delta[m(T)]}), \quad (48)$$

where  $\text{Li}_2$  is the dilogarithm function. Let us first focus on the magnetization  $m$ . From Eq. (36), we have

$$m(T) = I - \frac{a^2}{2\pi (\beta c')^2} \text{Li}_2(e^{-\beta I\Delta[m(T)]}). \quad (49)$$

When  $T \rightarrow T_C$ ,  $m \rightarrow 0$  and, hence,  $\Delta(m) \rightarrow 0$ . Let us denote by  $T_C^0$  the Curie temperature obtained from Eq. (37) by neglecting  $B_{\text{eff}}$ . We assume that the temperature is below but close to  $T_C$ . We can therefore expand the gap as  $\Delta = \Delta' m + \dots$ , with  $\Delta' = \frac{d\Delta}{dm}(m=0)$ . The dilogarithm around  $\Delta=0$  then expands to

$$\begin{aligned} \text{Li}_2(e^{-\beta I\Delta' m}) &= \frac{\pi^2}{6} + g(\beta I\Delta' m) - \frac{(\beta I\Delta')^2}{4} m^2 + \frac{(\beta I\Delta')^3}{72} m^3 \\ &+ \dots, \end{aligned} \quad (50)$$

where  $g(x) = x(\ln x - 1)$ . Introducing the dimensionless parameters  $t = T/T_C - 1$ ,  $t_0 = T/T_C^0 - 1$ , and  $b = I\Delta'/k_B T_C$ , the self-consistent equation [Eq. (49)] for  $m(T)$  becomes

$$m(t) = -2t_0 + \frac{6}{\pi^2} b m [1 - \ln(bm)] + \frac{3b^2}{2\pi^2} m^2 - \frac{b^3}{12\pi^2} m^3 + \dots \quad (51)$$

This equation can be easily solved numerically. Note that by integrating this equation with respect to  $m$ , one obtains the Landau functional.

In Fig. 2, we plotted the magnetization as a function of  $t_0$  for various values of the dimensionless parameters  $b$ . Two interesting features can be noticed: First,  $T_C$  increases compared to  $T_C^0$  by increasing  $b$ . Second, the magnetization rises sharply as  $T$  is lowered through  $T_C$ , which is reminiscent of a first order phase transition for a system in the thermodynamic limit. This effect becomes pronounced at values of  $b \sim 0.15$ .

Let us now estimate the value of  $b$  in our system. To this end, we need to estimate  $\Delta'$ , which can be regarded as the susceptibility of the susceptibility. Since the limits  $T \rightarrow 0$  and  $B_{\text{eff}} \rightarrow 0$  for the susceptibility  $\chi_s(T, B_{\text{eff}})$  do not commute,<sup>34</sup> a little care is required. When the nuclear spins in the GaAs heterostructure are polarized, they generate a rather large ef-

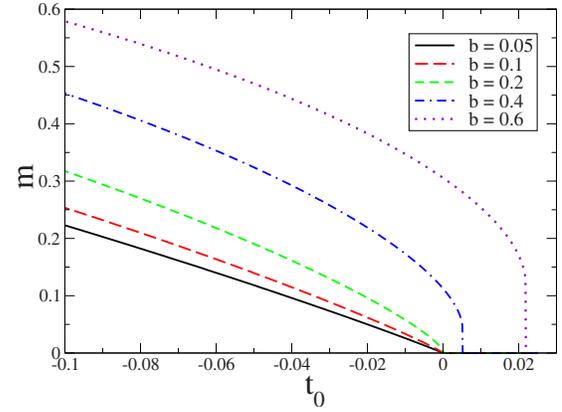


FIG. 2. (Color online) Magnetization as a function of  $t_0 = (T/T_C^0) - 1$  for various values of the dimensionless parameter  $b$ . For small values of  $b$ , the behavior described by Eq. (40) is recovered.

fective magnetic field  $B_{\text{eff}}$ , which gives an electronic Zeeman energy scale  $\Delta(m=1)$  of the order  $IA \sim O(1 \text{ K})$ . This scale is much larger than the typical Curie temperature we estimated before. In the ferromagnetic phase, we are therefore in a regime in which  $\Delta(m=1) \gg k_B T$ , where  $T$  is the temperature.

In this regime, one can use Ref. 34 to estimate  $\Delta' \sim A^3/E_F^2 (ak_F)^2 \Gamma_c^2$ , where  $\Gamma_c$  is the renormalized scattering amplitude defined in Eq. (24). Using Eq. (37) for the Curie temperature  $T_C$ , we can estimate  $b \sim (ak_F)(A/E_F) \ll 1$ . Our problem thus corresponds to a regime in which feedback effects are negligible, which justifies the previous isotropic approximation for the electron spin susceptibility.

#### D. Spin wave analysis around a helical ground state

We have assumed so far that the ground state is ferromagnetic. This assumption obviously depends on the sign of  $\delta\chi_s(q)$ . We have seen that the RPTA predicts two possible scenarios: Either  $\delta\chi_s(q) < 0$  in some range of  $q$  [case (b) in Fig. 1] or  $\delta\chi_s(q)$  becomes negative above some temperature  $T_1$ . A positive sign, on the other hand, is always obtained from LFFA. In contrast, second order perturbation theory gives a negative sign for  $\delta\chi_s(q)$ , independent of the magnitude of  $q$ . If we assume that there is a temperature range where  $\delta\chi_s(q) < 0$ , this clearly implies that the assumption of ferromagnetic order is invalid since the magnon spectrum has  $\omega_q < 0$ . This corresponds to the situations (b) or (c) in Fig. 1, which may depend on the Fermi liquid parameters or arise when  $T$  exceeds some temperature  $T_1$ .

##### 1. A new ground state

A different ground state for the nuclear spins thus has to be assumed. If the minimum of  $\delta\chi_s(q)$  is reached at some finite wave vector  $q_1 \sim k_F$ , a reasonable ground state of such a system could be an incommensurate helical state, or a spiral state, with a wavelength of the helix of the order of  $\lambda_1 = 2\pi/q_1$ .

We stress that *on the scale of the lattice spacing*, the RKKY interaction remains ferromagnetic and the neighbor-

ing nuclear spins are still ferromagnetically aligned. On the scale of the Fermi wavelength, however, an opposite spin alignment is favored. A gain in energy may be obtained by a slow variation of the nuclear spin alignment, i.e., by helical order. If such a nuclear spin arrangement is reached for some temperature range, it implies that the total magnetization vanishes over distances much longer than  $\lambda_F$ .

We assume that the ground state of the nuclear spins is described by a local magnetization  $\mathbf{m}_i = \langle \mathbf{I}_i \rangle$  that is confined to the spin  $(x, y)$  plane and that rotates in spin space around the  $z$  direction. A full rotation is described by the wave vector  $\mathbf{q}_1$ .

These  $\mathbf{m}_i$  play the role of the order parameter in the system. In order to compare with a ferromagnet, we can characterize the order by locally rotating the  $\mathbf{m}_i$  so that they all map onto the same  $\mathbf{m}_0$ ,

$$\mathbf{m}_i = R_i \mathbf{m}_0 = \begin{pmatrix} \cos(\mathbf{r}_i \cdot \mathbf{q}_1) & \sin(\mathbf{r}_i \cdot \mathbf{q}_1) & 0 \\ -\sin(\mathbf{r}_i \cdot \mathbf{q}_1) & \cos(\mathbf{r}_i \cdot \mathbf{q}_1) & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{m}_0, \quad (52)$$

where  $\mathbf{m}_0 = \mathbf{m}(\mathbf{r}_i = 0)$  is the magnetization at site  $\mathbf{r}_i = 0$ . Note that  $\mathbf{m}_0$  has the same interpretation as the average magnetization of a ferromagnet.

The helical ground state breaks the translational as well as the rotational symmetry of the Hamiltonian in Eq. (3). In particular, the ground state is degenerate with respect to a change of the direction of  $\mathbf{q}_1$  in the 2DEG plane. With the choice of a specific direction  $\mathbf{q}_1$ , this symmetry is broken. There exists, therefore, a Goldstone mode that tends to restore this rotational symmetry by inducing changes on the direction of  $\mathbf{q}_1$ . Obviously, such a mode would destroy the assumed order immediately. Realistically, however, the specific helical state can be pinned by the system, such as by a disorder lattice configuration or by some Dzyaloshinskii-Moryia interactions, and the Goldstone mode becomes massive.<sup>68</sup> In the following, we will assume that this is the case. Concretely, this means that excitations with small  $\mathbf{q}$  in the perpendicular direction of  $\mathbf{q}_1$  are strongly suppressed.

## 2. Spin wave analysis

Our starting point is the effective spin Hamiltonian of Eq. (5) with an exchange interaction matrix  $J_{ij}^{\alpha\beta} = J^{\alpha\beta}(|\mathbf{r}_i - \mathbf{r}_j|)$  defined in Eq. (6) that can be off-diagonal in general. Since  $J_{ij}^{\alpha\beta}$  is proportional to the spin susceptibility, as can be seen from Eq. (A6), we can assume, however, that  $J_{ij}^{xx} = J_{ij}^{yy}$  and that  $J_{ij}^{\alpha\beta} = -J_{ij}^{\beta\alpha}$  for  $\alpha \neq \beta$ . These assumptions are related to the conservation of the total spin of the electron system.<sup>69</sup>

A convenient way to perform the spin wave analysis is to first perform a local rotation of each nuclear spin  $I_j$  as in Eq. (52) such that they all become parallel to each other like in a ferromagnet. We thus define a local (right-handed) set of axes described by the unit vectors  $\mathbf{e}_i^1$ ,  $\mathbf{e}_i^2$ , and  $\mathbf{e}_i^3$ , with  $\mathbf{e}_i^1$  being parallel to  $\mathbf{m}_i$ .<sup>70</sup> In principle,  $\mathbf{e}_i^2$  and  $\mathbf{e}_i^3$  can be chosen arbitrarily. It is convenient though to choose  $\mathbf{e}_i^3$  parallel to the spin rotation axis  $z$ . Then, we can write  $\mathbf{I}_i = \tilde{I}_i^1 \mathbf{e}_i^1 + \tilde{I}_i^2 \mathbf{e}_i^2 + \tilde{I}_i^3 \mathbf{e}_i^3$ . These new components  $\tilde{I}_i^\alpha$  are connected to the original components  $I_i^\alpha$  through the matrices  $R_i$  as  $(\tilde{I}_i^1, \tilde{I}_i^2, \tilde{I}_i^3)^T$

$= R_i^\dagger (I_i^x, I_i^y, I_i^z)^T$ , where  $T$  denotes here the transposition. Let  $\mathbf{J}_{ij}$  be the  $3 \times 3$  matrix associated with  $J_{ij}^{\alpha\beta}$ ; then, the Hamiltonian for the nuclear spins can be written as

$$H_{\text{eff}} = -\frac{1}{2} \sum_{ij} (\tilde{I}_i^1, \tilde{I}_i^2, \tilde{I}_i^3) R_i \mathbf{J}_{ij} R_j^\dagger \begin{pmatrix} \tilde{I}_j^1 \\ \tilde{I}_j^2 \\ \tilde{I}_j^3 \end{pmatrix}. \quad (53)$$

In this new basis, the spin-wave analysis is analogous to the ferromagnetic case and is rather standard (see, e.g., Ref. 71).

The ground state energy of the helimagnet then becomes

$$E_0 = -\frac{I^2 N}{2} J_{q_1}^{xx}. \quad (54)$$

Let us compare Eq. (54) to the ground state energy of a ferromagnet. If all spins are aligned along the  $x$  direction, the ground state energy of the ferromagnet is

$$E_0^{\text{FM}} = -\frac{I^2}{2} \sum_{ij} J_{ij}^{xx} = -\frac{I^2 N}{2} J_{q=0}^{xx}. \quad (55)$$

The energy of the helical state is thus lower than that of the ferromagnet if  $J_{q_1}^{xx} > J_{q=0}^{xx}$ . Furthermore, the helical state has the lowest energy at the wave vector  $\mathbf{q}_1$ , where  $J_{q_1}^{xx} = J_q^{xx}$  has its maximum.

The low energy excitations above the ground state can also be obtained in a straightforward manner from Eq. (53). We find the following two branches of the spin wave spectrum:

$$\omega_q^{(1)} = \frac{I}{2} (J_{q_1}^{xx} - J_{|\mathbf{q}_1 + \mathbf{q}|}^{xx}), \quad (56)$$

$$\omega_q^{(2)} = \frac{I}{2} (J_{q_1}^{xx} - J_q^{zz}). \quad (57)$$

Clearly,  $\omega_q^{(1)} = 0$  at  $\mathbf{q} = 0$ . On the other hand, we see that the helical ground state can only be stable if  $\omega_q^{(2)} \geq 0$ , which means that  $J_{|\mathbf{q}|}^{zz}$  must not exceed  $J_{q_1}^{xx}$  in the vicinity of  $\mathbf{q} \approx 0$ . If  $J_q^{zz} = J_q^{xx}$ , this is indeed the case. The second branch,  $\omega_q^{(2)}$ , then has a gap.

## 3. Effect of gapless modes

We see that there is a gapless mode  $\omega_{\mathbf{q}}$  in the system, given by Eq. (56). If  $\mathbf{q}$  is such that  $|\mathbf{q}_1 + \mathbf{q}| = q_1$ , then  $\omega_{\mathbf{q}}$  remains strictly zero. This is the aforementioned Goldstone mode. Such fluctuations, therefore, are assumed to acquire a mass, associated with an energy scale  $\Delta_G$ . In the longitudinal direction, however, where  $\mathbf{q}$  is parallel to  $\mathbf{q}_1$ ,  $\omega_{\mathbf{q}}$  grows for  $|\mathbf{q}_1 + \mathbf{q}| > q_1$  and  $|\mathbf{q}_1 + \mathbf{q}| < q_1$  proportionally to the increase of  $\chi_s(|\mathbf{q}_1 + \mathbf{q}|)$  with respect to its minimum.

The minima of the spin wave spectrum are pushed to finite momenta  $q$ , notably to the minimum of the spin susceptibility  $\chi_s(q)$  at finite  $q = q_1$ . Provided that  $\chi_s(q)$  remains analytic around the minimum at  $q_1$ , it no longer grows linearly but, in general, as  $(q - q_1)^2$ . The system, therefore, can no longer benefit from a linear  $\chi_s(q)$  to stabilize the ferro-

magnetic order, and different, nonuniversal, energy and length scales will affect the Curie temperature  $T_C$  (see below).

We have seen above that the quantity  $m_0 = |\mathbf{m}_0|$  can be used as the order parameter of the helical state. Every spin wave reduces  $m_0$ , and we can thus use again Eq. (36) in order to express how the average local magnetization is reduced by the spin excitations,

$$m_0 = I - a^2 \int \frac{d\mathbf{q}}{(2\pi)^2} \frac{1}{e^{\beta\omega_{\mathbf{q}}} - 1}. \quad (58)$$

For  $q \rightarrow 0$ , the integrand becomes singular as  $1/q^2$ .<sup>72</sup> The singularity cannot be compensated by the factor  $q$  of the spherical integration measure  $q dq$ , as in the previously discussed ferromagnetic case. This is the same situation that is met when calculating spin wave excitations for systems without long-ranged interactions. There, the singularity is directly linked to the Mermin-Wagner theorem.<sup>23</sup> In the present case, however, the small  $q$  values are cut off at some finite inverse length scale  $\pi/L$ , associated with the energy  $\Delta$ , the infrared cutoff frequency introduced in Sec. III.

The scale  $L$  lifts the singularity in the integral [Eq. (58)] by cutting off the momentum at  $q \approx \pi/L$ , and so by effectively opening a gap for the excitations. The spin system can maintain a quasicrystal over the length  $L$ . An additional reduction of the singularity also arises from the fact that the system is not truly 2D but a layer with a finite width  $w_z$ , containing about 50–100 planes of nuclear spins (see also the Appendix). This length scale, however, must be compared with the typically much longer wavelengths of the spin waves, and thus can only account for a partial regularization. The singularity is dominated by the minimal curvature of  $\omega_{\mathbf{q}}$  in all directions of  $\mathbf{q}$ . In the present case, there are two main directions, the longitudinal one parallel to  $\mathbf{q}_1$ , where  $\omega_{\mathbf{q}} = Cq^2$ , with  $C = -\partial^2 J_q / \partial q^2|_{q=q_1}$ , and the transverse one perpendicular to  $\mathbf{q}_1$ , where the curvature is imposed by  $\Delta_G$ . For the stability of the ground state, we must assume that the pinning strength  $\Delta_G$  is large compared to the energies imposed by the  $J_q^{\alpha\beta}$ . The singularity in Eq. (58) is thus dominated by  $1/\beta C q^2$ .

Let us assume  $L \sim 10 \mu\text{m}$ . Cutting off the upper integration limit by  $k_F$ , the singular part of Eq. (58) becomes

$$\int_{\pi/L}^{k_F} \frac{dq}{C\beta q} = \frac{1}{C\beta} \ln(k_F L / \pi). \quad (59)$$

The logarithm yields a factor exceeding 1. The helical order cannot be stable if the expression in Eq. (59) becomes larger than 1. This allows us to define a temperature

$$k_B T^G = \frac{C}{I a^2 \ln(k_F L / \pi)}, \quad (60)$$

above which the gapless mode definitely destroys the helical order.

We see that much of the stability depends on the value of  $C$ , which means on the curvature of  $\chi_s(q)$  around its minimum at  $q_1$ . We can very roughly estimate  $C/a^2$

$\sim J_{q_1} / (a k_F)^2$  and see that this temperature  $T^G$  can actually be quite high. Let us now write the integral in Eq. (58) in the form

$$\int \frac{d\mathbf{q}}{(2\pi)^2} \frac{1}{e^{\beta\omega_{\mathbf{q}}} - 1} = \frac{\ln(k_F L / \pi)}{\beta C} + \int_{|\mathbf{q}| > k_F} \frac{d\mathbf{q}}{(2\pi)^2} \frac{1}{e^{\beta\omega_{\mathbf{q}}} - 1}. \quad (61)$$

Let us now further introduce a temperature  $T^{**}$ , similar to the temperature  $T^*$  for the ferromagnetic case, below which the integral determining  $m_0$  is entirely dominated by the quadratic dispersion behavior. This means that we assume that  $\omega_{\mathbf{q}} \approx Cq^2$  (in the direction parallel to  $\mathbf{q}_1$ ) up to a  $q^{**} < q_1, k_F$ . We can set

$$T^{**} = C(q^{**})^2 / k_B. \quad (62)$$

For  $T_1 < T < T^{**}$ , therefore, Eq. (61) is controlled entirely by the first logarithmic part. We thus obtain

$$m_0 \approx I - a^2 \frac{\ln(k_F L / \pi)}{C} k_B T = I(1 - T/T^G), \quad (63)$$

valid for  $T < T^{**}$ .

The temperature  $T^G$  can be seen as a generalization of  $T_C$  for the helical case. It differs from the ferromagnetic case through its dependence on external, nonuniversal cutoff scales. This loss of universality is an essential difference compared to the previously studied ferromagnetic order. This also indicates that the helical order is much more fragile with respect to external conditions than the ferromagnetic order.

At higher temperatures, the formation of defects and magnetic domains will further tend to destabilize the order as well. It is therefore possible that the helical order is destroyed well below  $T^G$ .

## V. CONCLUSION AND DISCUSSION

*Summary.* In this paper, we have examined the interplay between an interacting electron liquid in two dimensions with magnetic order in a lattice of nuclear spins. We have based our investigation on a Kondo lattice model, in which the electrons couple weakly to the nuclear spins through the hyperfine interactions. In this way, an effective coupling of the RKKY type is induced between the nuclear spins which, as obtained through a Schrieffer-Wolff transformation, is expressed in terms of the static electron spin susceptibility  $\chi_s(q)$ .

Electron-electron interactions in two dimensions can substantially modify the shape of  $\chi_s(q)$  and therefore profoundly affect the magnetic properties of the nuclear spin system. A magnetic order can arise because the conditions for the Mermin-Wagner theorem are not met due to the long-range character of  $\chi_s(r)$ . Much depends thus on the precise shape of  $\chi_s(q)$ . Based on a renormalized perturbation theory, we argued that for short-ranged interactions,  $\chi_s(q)$  should have the forms sketched in Fig. 1. When renormalization of the scattering amplitudes is important,  $\chi_s(q)$  has two extrema at values  $q_0, q_1 \sim k_F$ , which lead to the two generic situations labeled by (a) and (b) in Fig. 1. If renormalization effects are

unimportant, this leads to form (c), as sketched in Fig. 1. The distinction between (a), (b), and (c) is nonuniversal and, presumably, depends on the sample disorder, the interactions, and the temperature.

For long-ranged Coulomb interactions, on the other hand, a calculation based on a local field factor approximation produces a monotonic increase of  $\chi_s(q)$  (not sketched in Fig. 1).

Such a monotonic increase or case (a) stabilizes a nuclear ferromagnet. In case (b), the ferromagnetic order has an instability at the wave vector  $q_1 \sim k_F$ , which corresponds to the absolute minimum of  $\chi_s(q)$ , but ferromagnetic coupling is maintained at short distances (as compared to the Fermi wavelength). A similar behavior emerges also for case (c). We argued that the nuclear ground state then has a quasi-order, which is (nearly) ferromagnetic on short distances but rotates the local magnetization on a scale of  $1/q_1$ , thus providing a helical order.

*Experimental implications.* The transition temperature  $T_C$  describes the temperature above which the magnetic order breaks down. While the mean field approximation predicts a very low transition temperature of the order of  $\sim 1 \mu\text{K}$ , we have seen that electron-electron interactions can considerably increase the value of  $T_C$ . From our various approximation schemes we obtain a  $T_C$  in the millikelvin range for both short-ranged and long-ranged interactions for  $r_s = 5-10$  (see Sec. IV B 3).

If temperature is decreased below  $T_C$ , the nuclear spins order and generate an effective magnetic field  $B_{\text{eff}}$ , which can be very large in GaAs 2DEGs, this being in contrast to Si MOSFETs, which have a much smaller Overhauser field. This internal magnetic field has important consequences for the thermodynamic behavior of the electron spin susceptibility, which can be derived from the nonanalytic dependence of  $\chi_s(T, B_{\text{eff}})$  on temperature and magnetic field following Ref. 34 (the limits  $T \rightarrow 0$  and  $B_{\text{eff}} \rightarrow 0$  do not commute). In a Si MOSFET 2DEG, we expect  $g\mu_B B_{\text{eff}} \ll k_B T_C$ ; therefore, the calculated and observed<sup>35</sup> linear in  $T$  behavior of  $\chi_s(T)$  is also valid below  $T_C$ . However, in GaAs 2DEGs, one has clearly  $g\mu_B B_{\text{eff}} \gg k_B T_C$ , and one may therefore expect  $[\chi_s(T) - \chi_s(0)] \propto T$  above  $T_C$  and  $[\chi_s(T) - \chi_s(0)] \propto m \propto [1 - (T/T_C)^2]$  below  $T_C$ , in contrast to Si MOSFETs. This implies an upturn of  $\chi_s(T)$  around  $T = T_C$  in a GaAs 2DEG.

*Open questions.* There remain many open questions. Mainly, a detailed study of  $\chi_s(q)$  at values  $q \sim 2k_F$  could clarify the scenarios qualitatively sketched in Fig. 1. Monte Carlo simulations and experiments may provide further insights here. It would be desirable to establish a general magnetic phase diagram for the nuclear spins as a function of  $r_s$  and  $T$ . Possible new phases such as a nuclear spin glass phase are likely due to the complexity and richness of the problem. Finally, disorder may play an important role by providing a further cutoff length, and the interplay with electron-electron interactions requires a separate investigation.

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## APPENDIX: DERIVATION OF THE EFFECTIVE MAGNETIC LOW-ENERGY HAMILTONIAN

### 1. Effective Hamiltonian

We start from the general Kondo lattice Hamiltonian in Eq. (1) with  $H_{dd}=0$ . Since  $A$  is a small energy scale in our case, we can perform a Schrieffer-Wolff transformation in order to eliminate terms linear in  $A$ , followed by “integrating out” the electronic degrees of freedom. Keeping the lowest order terms in  $A^2$  of the SW transformation, we are left with an effective interaction Hamiltonian  $H_{SW}$  that reads

$$H_{SW} = H_0 - \frac{1}{2}[S, [S, H_0]]. \quad (\text{A1})$$

$S$  is defined by  $H_n + [S, H_0] = 0$ , which is solved as  $S = L_0^{-1} H_n$ , where  $L_0$  is the Liouvillian superoperator.

Let us define

$$U = \frac{1}{2}[S, [S, H_0]], \quad (\text{A2})$$

which can be rewritten as  $U = \frac{1}{2}[L_0^{-1} H_n, H_n]$ . Using an integral representation for  $L_0^{-1} = -i \int_0^\infty dt e^{i(L_0 + i\eta)t}$ , one obtains

$$U = -\frac{i}{2} \int_0^\infty dt e^{-\eta t} [H_n(t), H_n], \quad (\text{A3})$$

where  $H_n(t) = e^{iL_0 t} H_n = e^{iH_0 t} H_n e^{-iH_0 t}$ , and  $\eta \rightarrow 0^+$  ensures the convergence of the time integration. Using the definition of  $H_n$  given in Eq. (2), we are left in  $q$  space with

$$\begin{aligned} U &= -\frac{iA^2}{8N_l^2} \sum_{\mathbf{q}, \mathbf{q}'} \int_0^\infty dt e^{-\eta t} [\mathbf{I}_{\mathbf{q}} \cdot \mathbf{S}_{\mathbf{q}}(t), \mathbf{I}_{\mathbf{q}'} \cdot \mathbf{S}_{\mathbf{q}'}] \\ &= -\frac{iA^2}{8N_l^2} \sum_{\mathbf{q}, \mathbf{q}'} \int_0^\infty dt e^{-\eta t} \{ I_{\mathbf{q}}^\alpha I_{\mathbf{q}'}^\beta [S_{\mathbf{q}}^\alpha(t), S_{\mathbf{q}'}^\beta] + [I_{\mathbf{q}}^\alpha, I_{\mathbf{q}'}^\beta] S_{\mathbf{q}}^\beta S_{\mathbf{q}'}^\alpha(t) \}. \end{aligned} \quad (\text{A4})$$

Summation over Greek indices is implied. The first commutator enters the definition of the susceptibility in Eq. (A6) below. The second commutator can be straightforwardly computed by going back to real space. We obtain  $[I_{\mathbf{q}}^\alpha, I_{\mathbf{q}'}^\beta] = i\epsilon^{\alpha\beta\gamma} I_{\mathbf{q}+\mathbf{q}'}^\gamma$ , where  $\epsilon^{\alpha\beta\gamma}$  is the fully antisymmetric tensor.

We next take the equilibrium expectation value over electronic degrees of freedom only, denoted by  $\langle \dots \rangle$ . Furthermore, we assume translational invariance in the 2DEG, which implies  $\langle O_{\mathbf{q}} O_{\mathbf{q}'} \rangle = N \langle O_{\mathbf{q}} O_{-\mathbf{q}} \rangle \delta_{\mathbf{q}+\mathbf{q}', 0}$ , with  $N$  being the number of sites in a 2D lattice. Since  $H_0$  has time-reversal symmetry, the term proportional to  $\epsilon^{\alpha\beta\gamma} I_{\mathbf{q}+\mathbf{q}'}^\gamma$  in Eq. (A4) drops out. Together with the reduction to a 2D problem (discussed below), this allows us to bring  $\langle U \rangle$  to a much simpler form,

$$\langle U \rangle = \frac{A^2}{8N} a^2 \sum_{\mathbf{q}} I_{\mathbf{q}}^\alpha \chi_{\alpha\beta}(q) I_{-\mathbf{q}}^\beta, \quad (\text{A5})$$

where  $n_s = a^{-2}$  is the 2D nuclear spin density. The quantity

$$\chi_{\alpha\beta}(q, \omega) = -\frac{i}{Na^2} \int_0^\infty dt e^{-i\omega t - \eta t} \langle [S_{\mathbf{q}}^\alpha(t), S_{-\mathbf{q}}^\beta] \rangle \quad (\text{A6})$$

is the 2D electron spin susceptibility, which includes electron-electron interactions.

## 2. Reduction to two dimensions

As already mentioned, the electron gas is not strictly 2D but has a finite thickness, typically of the order of  $w_z \sim 5$  nm and therefore contains several layers of 2D nuclear spin planes. Since the electron wave function is confined in the third dimension ( $z$  direction), we can reduce the original 3D system to an effective 2D one. Let us suppose that a lattice site is labeled by  $\mathbf{r}_j = (\mathbf{r}_{j\parallel}, z_j)$ , where  $\mathbf{r}_{j\parallel}$  is the planar coordinate and  $z_j$  the position in the perpendicular direction. We consider  $N_z \sim 50$  layers such that  $N_l = N \times N_z$ .

If we assume that  $w_z$  is sufficiently small so that the electrons are confined in a single mode  $\phi(z)$  in the  $z$  direction, all nuclear spins in a column along the  $z$  direction couple to the same electron wave function. Fluctuations of the nuclear spins along this direction are expected to be weak. In a mean-field-like description, we may thus replace the column by a single 2D nuclear spin as follows. Let us separate out the perpendicular mode from the electron spin operator as

$$\mathbf{S}_i(\mathbf{r}_{i\parallel}, z_i) \rightarrow \frac{|\phi(z_i)|^2}{w_z} \mathbf{S}_i(\mathbf{r}_{i\parallel}), \quad (\text{A7})$$

with  $\mathbf{S}_i(\mathbf{r}_{i\parallel})$  being a 2D electron spin operator. The mode  $\phi(z_i)$  can then be used as an envelope function for the nuclear spins,

$$\mathbf{I}_i(\mathbf{r}_{i\parallel}) = \frac{1}{w_z} \int dz_i |\phi(z_i)|^2 \mathbf{I}_i(\mathbf{r}_{i\parallel}, z_i). \quad (\text{A8})$$

Since  $w_z$  is determined by  $w_z = \int dz_i |\phi(z_i)|^2$ , these new operators  $\mathbf{I}_i(\mathbf{r}_{i\parallel})$  satisfy the standard spin commutation relations. The remaining Hamiltonian is now strictly 2D.

Alternatively, we can argue as follows. Since  $k_F^{-1} \gg w_z$ , we can approximate  $|\mathbf{r}_i - \mathbf{r}_j| \approx |\mathbf{r}_{i\parallel} - \mathbf{r}_{j\parallel}|$  and therefore

$$\begin{aligned} \frac{1}{N_z} \sum_{z_i, z_j=1}^{N_z} J^{\alpha\beta}(|\mathbf{r}_i - \mathbf{r}_j|) I_i^\alpha I_j^\beta &\approx \left( \frac{1}{\sqrt{N_z}} \sum_{z_i} I_i^\alpha(\mathbf{r}_{i\parallel}, z_i) \right) J^{\alpha\beta}(|\mathbf{r}_{i\parallel} - \mathbf{r}_{j\parallel}|) \\ &\times \left( \frac{1}{\sqrt{N_z}} \sum_{z_j} I_j^\beta(\mathbf{r}_{j\parallel}, z_j) \right). \end{aligned} \quad (\text{A9})$$

This amounts to replacing a single nuclear spin at position  $(\mathbf{r}_{j\parallel}, z_j)$  by an average nuclear spin  $\frac{1}{\sqrt{N_z}} \sum_{z_j} \mathbf{I}_j(\mathbf{r}_{j\parallel}, z_j)$ . This is fully consistent with the fact that the RKKY interaction is almost constant and ferromagnetic in the  $z$  direction perpendicular to the 2DEG (this is at least the case for the RKKY interaction obtained by neglecting electron-electron interactions, but we expect that the interactions do not modify significantly the RKKY interactions in the  $z$  direction).

Our problem has now been reduced to a 2D system consisting of  $N = N_l / N_z$  nuclear spins interacting with long-ranged interactions. The effective nuclear spin Hamiltonian  $H_{\text{eff}} = \langle U \rangle$  is finally given by Eq. (3).

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