A Simple Proof of Lorenz Dominance Criterion

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Abstract

This article provides a simple proof of the Lorenz dominance criterion for two non-decreasing income transformations. The criterion is extended the most general case, with only very mild restrictions on the form of initial income distribution or the properties of the income transformations.

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1 Introduction

To investigate the causes and consequences of inequality and also to assess government policy targeting disparity, we need to measure and compare inequality associated with different income transformations. The most popular ordering of inequality is Lorenz dominance introduced in Atkinson (1970). Atkinson demonstrated that if the Lorenz curve (which shows the proportion of total income received by the poorest $t\%$ of the population) for one distribution lies below the Lorenz curve associated with another, then inequality in the first case is higher for a wide class of inequality measures.

However, it is somewhat cumbersome to construct the Lorenz curve every time we need to compare the inequality resulting from different income transformations. In some circumstances, for example when a new tax code is designed, we may not even know the true income distribution. In such a situation we need to know when one income transformation leads to higher inequality than another for any initial income distribution. The Lorenz Dominance criterion helps us to answer this question. In particular, it states that for two non-decreasing strictly positive income transformations the necessary and sufficient condition for Lorenz dominance is that the ratio of two transformations is a monotonic function.

Jakobsson (1976) and Kakwani (1977) first formulated and proved the Lorenz dominance criterion for income transformations. However, the formulations provided in Jakobsson (1976) and Kakwani (1977) were rather complicated and the criterion was proved only for continuous and differentiable income distribution functions and for differentiable income transformations. Keen, Papapanagos and Shorrocks (2000) have proved a similar criterion for discrete income distributions. Le Breton, Moyes and Trannoy (1996) provided the proof for discrete distributions. Although this proof, without loss of generality, could be extended to the case of general income distributions, this work has not been completed. Moreover, the proof is likely to be rather complicated. The present paper provides a much simpler proof and extends the class of income transformations. In particular, we do not restrict the domain of income to be finite or positive and we do not require the income transformations to be non-negative.

The aim of this article is to provide a simple proof of the Lorenz dominance criterion for two income transformations in the case of the most general form of initial income distribution and non-decreasing income transformations.

2 Main statement and proof

Let x denote initial income and $F(X)$ be its cumulative distribution function, i.e. $F(X) = P(x \leqslant X)$, where P is the probability measure. Let $L(t)$ denote a corresponding Lorenz curve, which shows the proportion of total income received by the poorest $t\%$ of the population. Then, following Le Breton, Moyes and Trannoy (1996), we can construct the implicit expression for the calculation of the Lorenz curve:

$$L(t) = \frac{\int_{-\infty}^{x} y dF(y)}{\int_{-\infty}^{+\infty} y dF(y)}, \quad where \;\; x = \inf \{x, F(x) \geqslant t\},$$

where $\int_{-\infty}^{x} y dF(y)$ calculates the total income possessed by the $t\%$ poorest fraction of the population.

Let $g(x)$ and $h(x)$ be two non-decreasing income transformations; $F_g(x)$ and $F_h(x)$ are cumulative distribution functions for income transformed by $g(x)$

and $h(x)$ correspondingly; $L_g(t)$ and $L_h(t)$ are Lorenz curves for income distributions $F_g(x)$ and $F_h(x)$, respectively.

Definition 1 *Following Atkinson (1970) we will say that cumulative distribution F_h weakly Lorenz dominates cumulative distribution F_g iff $L_g(t) \leqslant L_h(t)$ for any $t \in [0,1]$.*

Proposition 1 provides the sufficient condition for weak Lorenz domination of $F_g(x)$ over $F_h(x)$.

Proposition 1 *Let $g(x)$ and $h(x)$ be non-decreasing functions such that after each transformations the income possessed by the total population is positive. Then $L_g(t) \leqslant L_h(t)$ for any initial distribution of income $F(x)$ if*

for any y, z such that $y < z$, $g(y)h(z) - g(z)h(y) \geqslant 0$.

Proof. *First we will describe the Lorenz curve for transformed income $g(x)$. The total income possessed by poorest $t * 100\%$ fraction of total population is equal to $\int_{-\infty}^{x} g(z)dF(z)$, $x = \inf\{x, F(x) \geqslant t\}$. Therefore a new Lorenz curve could be expressed as follows,*

$$L_g(t) = \frac{\int_{-\infty}^{x} g(z)dF(z)}{\int_{-\infty}^{+\infty} g(z)dF(z)} \quad where \quad x = \inf\{x, F(x) \geqslant t\}. \tag{1}$$

Similarly, for transformation $h(x)$ we have,

$$L_h(t) = \frac{\int_{-\infty}^{x} h(z)dF(z)}{\int_{-\infty}^{+\infty} h(z)dF(z)}. \tag{2}$$

Using equations (1) and (2) we can formulate the Lorenz dominance criterion,

$$L_g(t) - L_h(t) = \frac{\int_{-\infty}^{x} g(z)dF(z)}{\int_{-\infty}^{+\infty} g(z)dF(z)} - \frac{\int_{-\infty}^{x} h(z)dF(z)}{\int_{-\infty}^{+\infty} h(z)dF(z)} \geqslant 0. \tag{3}$$

Due to the assumption that the transformed incomes by the total population is positive, equation (3) is equivalent to (4)

$$\int_{-\infty}^{x} g(y)dF(y) \int_{-\infty}^{+\infty} h(z)dF(z) \geqslant \int_{-\infty}^{x} h(z)dF(z) \int_{-\infty}^{+\infty} g(y)dF(y). \qquad (4)$$

Subtracting the common term $\int_{-\infty}^{x} g(y)dF(y) \int_{-\infty}^{x} h(z)dF(z)$ *from the both sides of inequality (4) we get (5)*

$$\int_{-\infty}^{x} g(y)dF(y) \int_{x}^{+\infty} h(z)dF(z) \geq \int_{-\infty}^{x} h(y)dF(y) \int_{x}^{+\infty} g(z)dF(z), \qquad (5)$$

which is similar to inequality (6)

$$\iint_{y<x;x<z} \left[g(y)h(z) - g(z)h(y)\right] dF(z)dF(y) \geq 0. \qquad (6)$$

Since $y \leqslant x \leqslant z$*, inequality* $g(y)h(z) - g(z)h(y) \geq 0$ *is true, which implies that inequality (6) is true for any cumulative distribution function* $F(x)$. ■

Conclusion 1 *Proposition 1 implies that if* $h(x)$ *and* $g(x)$ *are positive, then a sufficient condition for weak Lorenz dominance of* $F_g(x)$ *over* $F_h(x)$ *is that the ratio* $\frac{g(x)}{h(x)}$ *is non-increasing on the set where* $h(x) \neq 0$.

In general, inequality $g(y)h(z) - g(z)h(y) \geq 0$ implies $\frac{g(y)}{h(y)} \leq \frac{g(z)}{h(z)}$ only if $h(y) < 0$ and $h(z) > 0$.

3 Conclusion

This paper provides a simple proof of the Lorenz dominance criterion for income transformations for a given general form of income distribution function. I believe this work provides a useful reference for the researcher investigating the impact of disparity on different economic outcomes.

References

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