Complete and Incomplete Market Models

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1 Introduction

In competitive asset markets, consumers make intertemporal choices in an uncertain environment. Their attitudes toward risk, production opportunities, and the nature of trades that they can enter into determine equilibrium quantities and the prices of assets that are traded. The intertemporal choice problem of a consumer in an uncertain environment yields restrictions for the behavior of individual consumption over time as well as determining the form of the asset pricing function used to price random payoffs.

2 Complete contingent claims

We begin by describing the simplest setup in which consumer choices are made and asset prices determined, namely, a complete contingent claims equilibrium for a pure endowment economy. In such an equilibrium, a consumer can trade claims to contracts with payoffs that depend on the state of the world, for all possible states. We initially consider economies with one date and a finite number of states.

We initially consider economies with one date and a finite number of states. To understand the nature of the trades that take place in a complete contingent claims equilibrium, imagine that all agents get together at time 0 to write contracts that pay off contingent on some state occurring next period. The realization of the states is not known at the time the contracts are written, although agents know the probabilities and the set of all possible states. Once the contracts are signed, the realization of the state is observed by all agents, and the relevant state-dependent trade is carried out.

We assume the following setup:

- There is a set of I consumers, $\{1, 2, \ldots, I\}$
- Each consumer associates the probability π_s^i to state s occurring, where $0 < \pi_s^i < 1$ and

$$\sum_{s=1}^{S} \pi_s^i = 1.$$

- There are M commodities.
- The notation $c_{s,m}^i$ denotes the consumption of agent i in state s of commodity m.
- A consumption vector for agent i is

$$c^i \equiv \{c_{1,1}^i, \ldots, c_{S,1}^i, c_{1,2}^i, \ldots, c_{S,2}^i, c_{1,M}^i, \ldots, c_{S,M}^i\},$$

which is a vector of length $S \times M$. Consumption is always nonnegative and real so that $c^i_{s,m} \in \Re_+$. The *commodity space* is $\Re^{SM}_+$. The commodity space is the space over which consumption choices are made. When there are a finite number of states (or dates) and a finite number of commodities at each state (or date), we say that the commodity space is finite-dimensional.

- The endowment of agent i is a vector of length $S \times M$,

$$\omega^i = \{\omega^i_{1,1}, \ldots, \omega_{S,M}\}$$

The utility of consumer i is a function $u_i : \Re^{SM}_+ \to \Re_+$,

$$u_i(c^i) = \sum_{s=1}^{S} \pi_s U_i(c^i_{s,1}, \ldots, c^i_{s,M}) \tag{2.1}$$

Notice that we assume that utility is additive across states, which is the expected utility assumption.

Here are some definitions.

- An allocation is a vector $(c^1, \ldots, c^I)$
- An allocation is *feasible* is

$$\sum_{i=1}^{I} [c^i_{s,m} - \omega^i_{s,m}] \leq 0 \tag{2.2}$$

  for $s = 1, \ldots, S$ and $m = 1, \ldots, M$. This holds for each commodity and for each state.

- An allocation $(c^1, \ldots, c^I)$ is *Pareto optimal* if there is no other feasible allocation $(\hat{c}^1, \ldots, \hat{c}^I)$ such that

$$u_i(\hat{c}^i) \geq u_i(c^i) \quad \text{for all } i \tag{2.3}$$

  and

$$u_i(\hat{c}^i) > u_i(c^i) \quad \text{for some } i. \tag{2.4}$$

2.1 Contingent claims equilibrium

Imagine now that agents trade contingent claims - which are agreements of the form that, if state s occurs, agent i will transfer a certain amount of his endowment of good m to agent j. Since there are S states and M commodities in each state, a total of $S \times M$ contingent claims will be traded in this economy.

For each state and commodity, let $p_{s,m}$ denote the price of a claim to a unit of consumption of the m'th commodity to be delivered contingent on the s'th state occurring. The set of prices $p \in \Re_+^{SM}$ is a *price system*. The price function p assigns a cost to any consumption c^i and a value to any endowment ω^i; in our application $p : \Re_+^{SM} \to \Re_+$ has an inner product representation:

$$p \cdot c \equiv \sum_{s=1}^{S} \sum_{m=1}^{M} p_{s,m} c_{s,m} = \sum_{s=1}^{S} \left(p_{s,1} c_{s,1} + \ldots + p_{s,M} c_{s,M}\right).$$ [1]

The markets for contingent claims open before the true state of the world is revealed. Afterwards, deliveries of the different commodities are made according to the contracts negotiated before the state is realized and then consumption occurs.

A *complete contingent claims equilibrium (CCE)* is a nonzero price function p on $\Re_+^{SM}$ and a feasible allocation $(c^1, \ldots, c^I)$ such that c^i solves

$$\max_{c^i} u_i(c^i)$$

subject to

$$p \cdot c^i \leq p \cdot \omega^i \tag{2.5}$$

for all i. The complete contingent claims equilibrium allows us to specify a competitive equilibrium under uncertainty by assuming that prices exist for consumption in each possible state of the world.

We can state the following results.

- The First Welfare Theorem: A complete contingent claims equilibrium is Pareto optimal.

- The Second Welfare Theorem: A Pareto optimal allocation can be supported as an equilibrium.

To show the first result, suppose $(c^1, \ldots, c^I, p)$ is an equilibrium, and let $(\hat{c}^1, \ldots, \hat{c}^I, \hat{p})$ be an allocation with $u_i(\hat{c}^i) > u_i(c^i)$ for all i. Then $p \cdot \hat{c}^i > p \cdot c^i$ for all i, which implies that

$$p \cdot \sum_{i=1}^{I} \hat{c}^i > p \cdot \sum_{i=1}^{I} \omega^i.$$

Otherwise, if $\hat{c}^i$ had been affordable, it would have been chosen. But this contradicts the feasibility of $(\hat{c}^1, \ldots, \hat{c}^I)$, namely,

$$\sum_{i=1}^{I} (\hat{c}^i - \omega^i) \leq 0.$$

[1] Notice that $p \cdot (\alpha x + \beta y) = \alpha(p \cdot x) + \beta(p \cdot y)$ for any $\alpha, \beta \in \Re$ and $x, y \in \Re^{SM}$ so that the price function is *linear*.

The existence of equilibrium and the welfare theorems are discussed by Debreu [92], who provides an introduction to competitive equilibrium when the commodity space is finite-dimensional. Early proofs of the existence of a competitive equilibrium are by Arrow and Debreu [20] and McKenzie [272]. Duffie [107, 109] provides a textbook treatment.

2.2 Computing the equilibrium

What is the problem of a consumer in a contingent claims equilibrium? Let $c_s^i = (c_{s,1}^i, \ldots, c_{s,M}^i)'$. The problem in Equation (2.5) can be written as:

$$\max_{\{c_s^i\}_{s=1}^S} \sum_{s=1}^S \pi_s U(c_s^i)$$

subject to

$$\sum_{s=1}^S \sum_{m=1}^M p_{s,m}[\omega_{s,m}^i - c_{s,m}^i] \geq 0.$$

Thus, consumer i chooses a vector of length $S \times M$ to maximize his utility subject to a budget constraint.

To analyze the consumer's problem, we make the following assumption on the utility function $U(c)$.

Assumption 2.1 *Let $U : \Re_+^{S\times M} \to \Re_+$ be concave, increasing, and thrice continuously differentiable with $U''' > 0$, and that*

$$\lim_{c\to 0} U'(c) = +\infty, \quad \lim_{c\to\infty} U'(c) = 0.$$

By the Kuhn-Tucker Theorem, there exists a positive Lagrange multiplier λ^i such that c^i solves the consumer's problem:

$$\max_{c^i\in\Re_+} u_i(c^i) + \lambda^i(p\cdot\omega^i - p\cdot c^i).$$

We can write this equivalently as:

$$\max_{c^i\in\Re_+} \sum_{s=1}^S \pi_s U(c_s^i) + \lambda^i \left[\sum_{s=1}^S \sum_{m=1}^M p_{s,m}\omega_{s,m}^i - p_{s,}c_{s,m}^i\right].$$

Notice that the λ^i for $i = 1, \ldots, I$ are not state dependent. The first-order condition is

$$0 = \pi_s \left(\frac{\partial U(c^i)}{\partial c_{s,m}^i}\right) - \lambda_i p_{s,m} \quad \text{for each } s, m \text{ and } i.$$

For each agent i, this can be written as

$$\frac{\pi_s(\partial U(c^i)/\partial c_{s,m}^i)}{\lambda_i} = p_{s,m} \quad \text{for each } s, m \text{ and } i. \tag{2.6}$$

Define the function

$$g(x) = (U')^{-1}(x),$$

which is well defined since marginal utility is strictly increasing in x. Hence, given $\lambda_i p_{s,m}/\pi_s$, we can define the functions using the implicit function theorem to show there is a solution

$$c^i_{s,m} = g^i(\lambda_i p_{s,m}/\pi_s)$$

for $s = 1, \ldots, S$, $m = 1, \ldots, M$, and $i = 1, \ldots, I$.

How do we solve for the competitive equilibrium? Now go back to the initial budget constraint and substitute in the first-order condition. This yields

$$\sum_{s=1}^{S} \sum_{m=1}^{M} \frac{\pi_s(\partial U(c^i)/\partial g^i(\lambda_i p_{s,m}/\pi_s))}{\lambda_i}[\omega^i_{s,m} - g^i(\lambda_i p_{s,m}/\pi_s)] = 0. \tag{2.7}$$

For each i, this is an equation in the unknown λ_i, given the price system. Once we solve for λ_i for $i = 1, \ldots, I$ from these equations as a function of the prices, p, we can use the market-clearing conditions to solve for the prices.

To illustrate the solution procedure, suppose that the utility function $U^i(c^i)$ has the form:

$$U(c) = \frac{c^{1-\gamma} - 1}{1-\gamma}, \quad \gamma \geq 0. \tag{2.8}$$

Then $U'(c) = c^{-\gamma}$ and $g(x) = x^{-1/\gamma}$. Also specialize to the case of two states, two consumers, and one commodity per state, $S = 2$, $I = 2$ and $M = 1$. Using the four first-order conditions displayed in Equation (2.6), we have

$$c^i_s = (\lambda_i p_s/\pi_s)^{-1/\gamma} \tag{2.9}$$

for $s = 1, 2$ and $i = 1, 2$.

Notice that there are four unknowns – $\lambda_1, \lambda_2, p_1, p_2$ – and four equations – the budget constraints for consumers $i = 1, 2$ and the market-clearing conditions for states $s = 1, 2$. Normalize the price of consumption in state 1 as $p_1 = 1$. We can substitute the expressions for the contingent claims prices and the solutions for c^i_s into Equation (2.7) as:

$$\sum_{s=1}^{2} p_s[\omega^i_s - (\lambda_i p_s/\pi_s)^{-1/\gamma}] = 0 \quad i = 1, \ldots, 2. \tag{2.10}$$

These equations yield the solution for λ_i as

$$\lambda_i = \left\{ \frac{\pi_1^{1/\gamma} + p_2^{(\gamma-1)/\gamma}(1-\pi_1)^{1/\gamma}}{\omega^i_1 + p_2\omega^i_2} \right\}^{\gamma}, \quad i = 1, 2. \tag{2.11}$$

We can substitute these conditions into the market-clearing conditions to solve for the price of consumption in state 2, p_2.

A simpler approach is to use the solutions for c_s^i defined by Equation (2.9) in the market-clearing conditions as:

$$\begin{aligned}(\lambda_1/\pi_1)^{-1/\gamma} + (\lambda_2/\pi_1)^{-1/\gamma} &= \omega_1, \\ (\lambda_1 p_2/(1-\pi_1))^{-1/\gamma} + (\lambda_2 p_2/(1-\pi_1))^{-1/\gamma} &= \omega_2,\end{aligned}$$

where $\omega_s = \omega_s^1 + \omega_s^2$ for $s = 1, 2$. Substituting for λ_i for $i = 1, 2$ using the result in Equation (2.11) and taking the ratio of these conditions yields

$$p_2 = \left(\frac{1-\pi_1}{\pi_1}\right)\left(\frac{\omega_1}{\omega_2}\right)^{\gamma}. \tag{2.12}$$

This says that the price of consumption in state 2 relative to consumption in state 1 is a function of the ratio of the probabilities and endowments across the two states. Notice that p_2 is inversely related to the probability of state 1 and the endowment in state 2. If either π_1 is high or ω_2 is large, then there will be less demand for goods delivered contingent on state 2 occurring, and p_2 will be small.

The approach that we have followed in this section of finding the competitive equilibrium, namely, using the first-order conditions to solve for the consumption allocations as a function of the prices and multipliers and using the budget constraints and market-clearing conditions to solve for the prices and multipliers as a function of the endowments and other exogenous variables, will crop up in other contexts where we solve explicitly for the equilibrium.

2.3 Pareto optimal allocations

In this section, we show the equivalence between the competitive equilibrium and Pareto optimal allocations. In later chapters, we describe how this equivalence can be exploited to characterize competitive equilibrium in a variety of settings.

Assume that U^i are strictly increasing and concave for all i. The social planner assigns weights $\eta_i \in \Re_+$ to each consumer i and chooses allocations $c^i \in \Re^{SM}$ for $i = 1, \ldots, I$ to maximize the weighted sum of individual utilities subject to a set of resource constraints for each state s and each commodity m:

$$\max_{c^1,\ldots,c^I} \sum_{i=1}^{I} \eta_i \sum_{s=1}^{S} \pi_s U(c_s^i) \tag{2.13}$$

subject to

$$\sum_{i=1}^{I} c_{s,m}^i = \sum_{i=1}^{I} \omega_{s,m}^i \quad \text{for all } s, m. \tag{2.14}$$

Let $\mu_{s,m}$ denote the Lagrange multiplier for the resource constraint in each state and for each commodity. The first-order conditions for this problem are

$$\eta_i \pi_s \left(\frac{\partial U_i}{\partial c^i_{s,m}} \right) = \mu_{s,m} \quad \text{for each } s, m \text{ and } i. \tag{2.15}$$

Suppose we set $\eta_i = 1/\lambda_i$ and $p_{s,m} = \mu_{s,m}$, where λ_i are the agent-specific Lagrange multipliers and $p_{s,m}$ are the contingent claims prices. Under these assumptions, the first-order conditions above are identical to those for a complete contingent claims equilibrium. (See the conditions in Equation (2.6).) Since the allocations achieved under the contingent claims equilibrium and the social planning problem both satisfy the resource constraints, they must be equal.

3 Security market equilibrium

In actual asset markets, we observe individuals trading in securities that are claims to random payoffs denominated in units of account, not in commodities. We now describe how to formulate an equilibrium with such securities.

3.1 Definition

The primitives for a security market equilibrium are as follows:

- There are N securities.
- Each security n has the payoffs (denominated in the unit of account, say dollars) $x_{n,s}$ for $s = 1, \ldots, S$.
- We define X as the $N \times S$ matrix of payoffs and $q = (q_1, \ldots, q_N)^T$ in $\Re^N$ denotes the vector of security prices.
- Securities are sold before the state s is realized so that their prices are independent of the realized state.
- After the security markets close, agents trade in spot markets for the M commodities.
- Let $\bar{p}_{s,m}$ denote the unit price of the m'th commodity in state s and define $\bar{p}_s$ in $\Re^M$ as the vector of spot prices in state s.

A *portfolio* is a vector $\theta^i \equiv (\theta^i_1, \ldots, \theta^i_N)$ in $\Re^N$ which has the market value $\theta \cdot q$ and the payoff $X^T\theta$. Some of the elements of θ may be negative. When the total expenditure or portfolio weight on a security is negative ($\theta_k < 0$), we say that security k has been sold short.[2] If the portfolio weights satisfy $\theta_n \geq 0$ for $n = 1, \ldots, N$, then we say that short selling is ruled out. In the security market equilibrium, we will not rule out short sales on any of the securities.

The i'th agent chooses a portfolio $\theta^i \equiv (\theta_1^i, \ldots, \theta_N^i)$ in $\Re^N$ of securities to purchase and a consumption vector c^i in $\Re_+^{SM}$. Given the security and spot prices $(q, \bar{p})$, the i'th agent solves the problem:

$$\max_{c^i, \theta^i} u_i(c^i) = \sum_{s=1}^{S} \pi_s U_i(c_s^i)$$

subject to

$$\theta^i \cdot q \leq 0, \tag{3.16}$$

$$\bar{p}_s \cdot c_s^i \leq \bar{p}_s \cdot \omega_s^i + \theta^i \cdot x_s, \quad s = 1, \ldots, S, \tag{3.17}$$

where $\theta^i \cdot x_s$ is the dollar payoff on the portfolio in state s and c_s^i in $\Re_+^M$ is the consumption vector in state s.

Notice that the consumer faces a separate budget constraint for each state of the world. The consumer's wealth in each realized state s is given by his endowment ω_s^i in that state plus the payoff on his portfolio of assets. Unlike the complete contingent claims equilibrium, the consumer cannot purchase claims to consumption for each possible state subject to a single budget constraint that constrains the value of his consumption to be less than the value of his endowment across all possible states. Instead, his feasible consumption in state s is constrained by his realized wealth at that state. Notice also that the consumers' utility does not depend on consumption in period 0. Hence, without loss of generality, the value of his endowment at date 0 is also taken as zero. This means that purchases of some securities are financed by sales of others. This is known as a *self-financing* portfolio. In Section 1.2.1, we consider a two-period economy in which the consumer receives endowment in both periods of his life.

We define a *security market equilibrium (SME)* as a collection

$$((\theta^1, c^1), \ldots, (\theta^I, c^I), (q, \bar{p}))$$

such that

(i) given prices $(q, \bar{p})$, the allocation (θ^i, c^i) solves the problem for agent i; and

[2] When a security is shorted, the investor receives the current price of the asset in return for a promise to make payments matching those on the security until the short sale is closed by a payment equal to the prevailing price of the asset.

(ii) markets clear:

$$\sum_{i=1}^{I} \theta_n^i = 0, \quad n = 1, \ldots, N, \tag{3.18}$$

$$\sum_{i=1}^{I} (c_{s,m}^i - \omega_{s,m}^i) = 0, \quad s = 1, \ldots, S, \; m = 1, \ldots, M. \tag{3.19}$$

Since spot markets open after securities trading has occurred, we can also refer to this equilibrium as a security-spot market equilibrium.

Note also that the securities are in net zero supply in the economy. We could instead assume that securities are in net positive supply and that consumers have initial holdings of the securities denoted $\bar{\theta}_n^i$ for $n = 1, \ldots, N$. In this case, consumers choose bond holdings to satisfy

$$q \cdot \theta^i \leq q \cdot \bar{\theta}^i$$

and the securities market-clearing condition becomes

$$\sum_{i=1}^{I} \theta^i = \sum_{i=1}^{I} \bar{\theta}^i.$$

3.2 Attaining a CCE by an SME

In the complete contingent claims equilibrium, consumers choose allocations subject to a single budget constraint. In the security market equilibrium, they must choose consumption in each state subject to a budget constraint for that state. This suggests that the consumption allocation in a security market equilibrium may differ from the allocation in a complete contingent claims equilibrium. In a singularly important result, Arrow [18] has shown that if the number of securities equals the number of states, then the allocation in a complete contingent claims equilibrium can be attained in a security market equilibrium.

Denote the allocation and prices in a complete contingent commodity markets equilibrium by $(c^1, \ldots, c^I, p)$. Suppose that $N = S$ and that the columns of the payoff X are linearly independent. Then without loss of generality we can define the dividend or payoff vector as:

$$x_{n,s} = \begin{cases} 1 & \text{if } s = n \\ 0 & \text{otherwise,} \end{cases} \tag{3.20}$$

for $s = 1, \ldots, S$ and $n = 1, \ldots, S$. Define the price of the s'th security such that

$$q_s \bar{p}_{s,m} = p_{s,m} \;\; \forall s, m. \tag{3.21}$$

Then notice that the consumer confronted with these prices has the same range of alternatives that are available under the contingent claims equilibrium. Define the portfolio weights so that the number of units of the s'th security that is held by consumer i is equated to the cost of the net consumption choice by i in state s:

$$\theta_s^i = \bar{p}_s \cdot (c_s^i - \omega_s^i). \tag{3.22}$$

If the prices and allocations defined in this manner constitute a security market equilibrium, then (θ^i, c^i) must be feasible for consumer i in the security market equilibrium. Thus,

$$\begin{aligned} \theta^i \cdot q &= \sum_{s=1}^{S} \sum_{m=1}^{M} \left(\frac{p_{s,m}}{\bar{p}_{s,m}} \right) \bar{p}_{s,m} (c_{s,m}^i - \omega_{s,m}^i) \\ &= \sum_{s=1}^{S} \sum_{m=1}^{M} p_{s,m} (c_{s,m}^i - \omega_{s,m}^i) \leq 0, \end{aligned}$$

since $(p, c^1, \ldots, c^I)$ constitutes a contingent claims equilibrium. Likewise,

$$\bar{p}_s \cdot c_s^i = \bar{p}_s \cdot \omega_s^i + \theta_s^i = \bar{p}_s \cdot \omega_s^i + \theta^i \cdot x_s \quad \text{for } s = 1, \ldots, S$$

since $x_{n,s} = 1$ only if $s = n$. Hence, given the prices $(q, \bar{p})$, (θ^i, c^i) satisfy consumers' budget constraints in the security market equilibrium.

To show that (θ^i, c^i) solves the consumer's problem, assume that $(\phi, \hat{c})$ also satisfies the budget constraints and $u_i(\hat{c}) > u_i(c)$. Since c^i is optimal for consumer i in the complete contingent claims equilibrium, we have that $p \cdot \hat{c} > p \cdot c^i$; otherwise c^i would not have been chosen. If $(\phi, \hat{c})$ satisfies the consumer's budget constraints, then $\phi_s = \phi \cdot x_s \geq p_s \cdot (\hat{c}_s - \omega_s^i)$ and $\phi \cdot q \geq \sum_{s=1}^{S} p_s(\hat{c}_s - \omega_s^i)$ since $q_s \bar{p}_{s,m} = p_{s,m}$. But $p \cdot \hat{c} > p \cdot c^i$ implies that $\phi \cdot q > 0$ which contradicts Equation (3.16). Spot markets clear because $(c^1, \ldots, c^I)$ is feasible. Security markets clear since

$$\begin{aligned} \sum_{i=1}^{I} \theta_s^i &= \sum_{i=1}^{I} \bar{p}_s \cdot (c_s^i - \omega_s^i) \\ &= \bar{p}_s \cdot \sum_{i=1}^{I} (c_s^i - \omega_s^i) = 0, \quad s = 1, \ldots, S. \end{aligned}$$

since $(c^1, \ldots, c^I)$ is feasible for the contingent claims equilibrium.

Recall that we defined each security to have a unit payoff if state $s = n$ occurs and zero otherwise. More generally, a complete contingent claims allocation can be attained in a security market equilibrium with an arbitrary payoff matrix X whose columns are linearly independent. When the columns of X span $\Re^S$, we say that markets are *complete.* With such spanning securities, a security market equilibrium can be converted into a contingent claims equilibrium. Since the complete contingent claims equilibrium allocation is Pareto optimal, the allocation in the security market equilibrium will also be Pareto optimal. In the absence of spanning, markets are *incomplete* and a security market equilibrium may exist but the equilibrium allocation is not necessarily Pareto optimal.

Now suppose there is only one commodity in each state so that $M = 1$. Without loss of generality, assume that the spot price of consumption in each state is unity, $\bar{p}_s = 1$. Notice from Equation (3.17) that $\theta^i \cdot x_s = c_s^i - \omega_s^i$. Suppose we choose the security price q_n as:

$$q_n = \sum_{s=1}^{S} x_{n,s} p_s, \tag{3.23}$$

where p_s is the price of a contingent claim that pays off in state s. It is straightforward to show that (θ^i, c^i) is feasible for consumer i following our earlier argument. Notice that Equation (3.23) yields a simple representation for the price of the n'th security in terms of the price function for the complete contingent claims equilibrium. Specifically, it says that the price of any security that has random payoffs $x_{n,s}$ in state s will be priced such that these payoffs are valued by the contingent claims price in that state.

4 Arbitrage and asset valuation

In the theoretical finance literature, the absence of arbitrage opportunities in securities trading has been exploited by Ross [304], Harrison and Kreps [192], Chamberlain and Rothschild [70], and others to show the existence of a pricing function that is used to value random payoff streams and to characterize its properties. We now illustrate their approach for the simple setup that we have been studying. We assume that there is one date, S states, and one commodity in each state. As a consequence, spot commodity prices can be normalized as unity. We assume that there is one date, S states, and one commodity in each state. As a consequence, spot commodity prices can be normalized as unity. As before, we assume that there are S states of the world, and N securities where $N \leq S$. Hence, the number of securities may be less than the number of states, implying that markets are incomplete.

The $N \times S$ matrix of payoffs on the N securities is given by:

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1S} \\ \vdots & \vdots & & \vdots \\ x_{N1} & x_{N2} & \cdots & x_{NS} \end{bmatrix}. \tag{4.24}$$

Recall that θ_n is the units purchased on security n, and q_n is the price of security n.

Example 2.1. Consider an economy with 1 date and 3 states. Suppose there are 3 securities. Security 1 pays 1 in state 1 and zero otherwise. Security 2 pays 1 in state 2 and zero otherwise. Suppose that the third security is a risk-free bond. What is the payoff matrix?

$$X_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}. \tag{4.25}$$

Notice that markets are complete since the rank of X is 3. Hence, consumers can attain any pattern of state-contingent consumption by trading in the 3 securities. In fact, if they just hold the risk-free bond, they can guarantee a consumption of 1 for each state.

Now suppose that there are only 2 states. In this case, one of the securities is a *redundant* security, that is, a security whose payoff structure can be constructed by using the other securities in the market. In this case, the payoff matrix becomes

$$X_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}. \tag{4.26}$$

The consumer can attain a consumption of 1 in each state by i) by holding 1 unit each in the first 2 securities or (ii) by holding a unit of the risk-free bond. In the first case, the portfolio satisfies $\theta^1 = (1, 1, 0)'$ whereas in the latter case it is given by $\theta^2 = (0, 0, 1)'$. A redundant security is thus a security whose payoffs can be generated by holding a portfolio of the other securities. In this case, holding the first portfolio yields exactly the same payoff as holding the second portfolio.

We can also consider the case with 3 states and 2 securities. Suppose, in particular, that only the 2 risky security are available. The payoff matrix is

$$X_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{4.27}$$

Now there is no way for consumers to trade in securities to obtain consumption in state 3. Hence, markets are incomplete as the number of securities is less than the number of states.

To characterize behavior in securities markets, we need to make further assumptions regarding the price of a portfolio and its payoff. These issues are discussed next.

4.1 The law of one price

Recall that for any vector of security prices q in $\Re^N$, a portfolio θ has market value $q \cdot \theta$ and payoff $X^T\theta$. We now make some additional assumptions. The first has to do with the notion of a *payoff space*. For the case of complete markets with S states, the payoff space consists of $\Re^S$. In this case, $N = S$ and investors can attain any payoff in $\Re^S$ by forming portfolios based on the S linearly independent columns of X. When markets are incomplete, there are only $N < S$ linearly independent columns of X and the set of payoffs that investors can obtain is $\mathcal{X} \subset \Re^S$.

We now make two additional assumptions regarding portfolio payoffs and their prices.

Assumption 4.2 *(Free Portfolio Formation)*

$$X_1^T\theta \in \mathcal{X}, X_2^T\phi \in \mathcal{X} \Rightarrow X_1^T\theta + X_2^T\phi \in \mathcal{X}.$$

This means that if investors can attain the payoff $X_1^T\theta \in \mathcal{X}$ and the payoff $X_2^T\phi \in \mathcal{X}$, then they can also attain the payoff $X_1^T\theta + X_2^T\phi \in \mathcal{X}$. This assumption rules out short sales constraints and transactions costs of various types. In later chapters we discuss the role of such frictions in determining equilibrium asset returns and allocations.

Assumption 4.3 *(Law of One Price)*

$$(a\theta + b\phi) \cdot q = a(\theta \cdot q) + b(\phi \cdot q).$$

This says that the value of the new portfolio $a\theta + b\phi$, where a and b are constants, must equal the value of its parts. Notice that this assumptions rules out arbitrage opportunities of various types. For example, it says that an investor who purchases a claim to the NYSE Composite Index at the price q_1 and sells it at the same price cannot make an instantaneous profit by buying 2 units of the claim and selling those two units for a higher price. Thus, if $\theta = (-1, 0, \ldots, 0)$ and $\phi = (1, 0, \ldots, 0)$, then the value of $2\theta + 2\phi$ cannot be greater than $-2q_1 + 2q_1$. (Notice we have allowed for short sales in this instance. Thus, we also have free portfolio formation.)

Assumptions 2.1 and 2.2 also imply that the zero payoff must be available and must have zero price. Otherwise, any payoff could be obtained at any price.

Suppose an equilibrium exists. Notice that if $S = N$ so that markets are complete, the price of any security n is expressed as the discounted value of its payoffs using the contingent claims prices p_s. Hence, if markets are complete, then the law of one price holds. The price of any security is expressed as a linear function of its payoffs using the state-contingent prices:

$$\begin{aligned} q = \sum_{s=1}^{S} p_s x_s &= \sum_{s=1}^{s} \pi_s m_s x_s \\ &= E[mx], \end{aligned} \tag{4.28}$$

where $m_s = p_s/\pi_s$. We have that the stochastic discount factor is strictly positive, $m > 0$. Take any payoff $x = y + z \in \mathcal{X}$. Then its price is given by

$$q = E[xm] = E[(y + z)m].$$

Even if markets are not complete, the linearity of the pricing function implies that the law of one price holds. In this case, there may not exist a unique discount factor that satisfies Equation (4.28). Nevertheless, the equilibrium pricing function still exhibits the linearity property.

It is also possible to prove the converse of this result, namely, if the law of one price holds, then there is a discount factor that can be used to price payoffs. We have already demonstrated that the pricing function is linear. The existence of a discount factor based on the law of one price derives from the fact that any linear function on a payoff space $\mathcal{X}$ can be represented as an inner product. In other words, the pricing function that is used to value any payoff $x \in \mathcal{X}$ is a linear function and has the representation $q = E[x^\star x]$ for $x^\star \in \mathcal{X}$. The existence of the discount factor $x^\star$ is guaranteed by means of a projection argument. Following a more general approach described by Hansen and Richard [187], Hansen and Jagannathan [186] consider a payoff space $\mathcal{X}$ that can be expressed in terms of N basis payoffs. Organize the basis payoffs into the vector $x = (x_1, x_2, \ldots, x_N)$ and also their prices. They show that the discount factor $x^\star \in \mathcal{X}$ that satisfies the law of one price has the form:

$$x^\star = q' E(xx')^{-1} x, \tag{4.29}$$

where q denotes the price of the basis payoffs under $x^\star$, that is, $q = E[x^\star x]$.

To show this, notice that the payoff space is generated as $\mathcal{X} = \{c'x\}$. The discount factor that we are seeking must be in the payoff space. Hence, $x^\star = c'x$. Define c so that $x^\star = c'x$ prices the basis payoffs. Hence, we require that $q = E(x^\star x) = E(xx'c)$. Hence, $c = E(xx')^{-1}q$ provided the inverse $E(xx')^{-1}$ exists. But this is guaranteed by the law of one price. Hence, the discount factor is defined as in equation (4.29). The discount factor is a linear combination of x so it is in the payoff space $\mathcal{X}$. By construction it prices the basis payoffs and it also prices any $x \in \mathcal{X}$:

$$E[x^\star(x'c)] = E\left[q' E(xx')^{-1} xx'c\right] = q'c.$$

Finally by linearity we have that $q(c'x) = c'q(x)$.

Note, however, that the discount factor is not guaranteed to be strictly positive. For that we need to define stronger forms of the notion of absence of arbitrage.

4.2 Arbitrage opportunities

Stronger forms of arbitrage opportunities can also be defined. We have two definitions.

An *arbitrage of the first kind* is a portfolio θ in $\Re^N$ with

$$q \cdot \theta \leq 0 \quad \text{and} \quad X^T\theta > 0. \tag{4.30}$$

An *arbitrage of the second kind* is a portfolio θ in $\Re^N$ with

$$q \cdot \theta < 0 \quad \text{and} \quad X^T\theta \geq 0. \tag{4.31}$$

An economy with an arbitrage opportunity of the first type is given by the following example. Suppose there are three states of the world, $S = 3$, and two securities, $N = 2$. Let the prices of the securities be $q = (1, 1)'$. Then

$$X = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}. \tag{4.32}$$

Notice that for any vector (θ_1, θ_2) where $\theta_1 > 0$, the payoff on the portfolio is given by

$$X^T\theta = \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \theta_1 \\ -\theta_1 - \theta_2 \\ \theta_1 + \theta_2 \end{bmatrix}. \tag{4.33}$$

Thus, $X^T\theta \geq 0$ only if $\theta_1 + \theta_2 = 0$, which is just the definition of an arbitrage opportunity of the first type. Hence, choosing $\theta_1 > 0$ and $\theta_1 = -\theta_2$, the initial outlay for the portfolio is zero but it yields positive profits in some state of nature.

An arbitrage opportunity of the second type is as follows. Suppose that there are two states of the world, $S = 2$, and two securities, $N = 2$. Let the matrix of payoffs be equal to

$$X = \begin{bmatrix} 4 & -4 \\ 1 & -1 \end{bmatrix}. \tag{4.34}$$

Suppose that the vector of total purchases on the two assets is given by $\theta = (1, -4)$. Then

$$X^T\theta = \begin{bmatrix} 4 & 1 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.35}$$

Notice that the payoff on the portfolio is zero in all states of nature. Let the price of security 1 be equal to 1. As long as the price of security 2 is greater than 1/4, the initial outlay is negative. For example, if $q_2 = 1$, then,

$$q \cdot \theta = (1 \;\; 1) \begin{bmatrix} 1 \\ -4 \end{bmatrix} = -3,$$

which means that the investor receives funds to hold the portfolio.

Clearly, the existence of such arbitrage opportunities cannot be consistent with a security market equilibrium. From the definition of the agent's problem in the security market equilibrium, we see that an arbitrage of the first kind implies that a zero endowment yields a nonnegative, nonzero consumption allocation. Consider the problem in Section 1.2.1 where the consumer maximizes date 1 utility by choosing a portfolio of assets subject to the budget constraints in (3.16) and (3.17). If

$$q \cdot \theta \leq 0,$$

then

$$\bar{p}_s \cdot c_s \leq \theta \cdot x_s$$

is satisfied with $c_s > 0$ for some s if $X^T\theta > 0$, which is equivalent to $\theta \cdot x_s > 0$ for some s.

Likewise, for the problem in Section 1.2.2 where the consumer derives utility from consumption in date 0 and date 1, if

$$q \cdot \theta < 0 \quad \text{and} \quad X^T\theta \geq 0,$$

then the agent can guarantee a strictly positive consumption at date 0:

$$c_0 = -q \cdot \theta > 0.$$

Thus, we can show that if utility functions are strictly increasing (so that there is at least one non-satiated consumer), an arbitrage does not exist in a competitive equilibrium. Recall that in a security market equilibrium, security prices can be expressed as the discounted value of their state-contingent payoffs, where the discounting is done using a strictly positive discount factor. In the next section, we examine the converse statement, namely, in the absence of arbitrage there exists a strictly positive state-price vector.

4.3 Existence of a state-price vector

The proof that there exists a strictly positive price vector that can be used to price the payoffs on any security based solely on the absence of arbitrage arguments is due to Ross [302]. In this section, we provide a simple proof for an economy with a single date and a finite number of states.[3]

Define a *state-price vector* as a vector ψ in $\Re^S_{++}$ with $q = X\psi$. Notice that it is a strictly positive price vector that is used to assign a price to the random payoffs paid by each security n. If a complete contingent claims equilibrium exists, then the state-price vector is defined as the price function p. Clearly, if a state-price vector exists, then there is no arbitrage. We can also prove the converse.

[3] See Kreps [234] for a proof with continuous time and a continuum of states.

Theorem 4.1 *There is no arbitrage if and only if there is a state-price vector.*

PROOF.
Partition the matrix X as:

$$X = \begin{bmatrix} X_1 \\ \cdots \\ X_2 \end{bmatrix} \tag{4.36}$$

such that the payoff of the first N_1 securities are linearly independent with payoff matrix X_1 and the payoff on the other $N_2 = N - N_1$ securities are linear combinations of the first N_1 securities with payoff matrix X_2. Then there exists an $N_2 \times N_1$ matrix K such that $X_2 = KX_1$. Thus, we can decompose the payoff matrix X as

$$X = (X_1^T, X_1^T K^T)^T.$$

Let q_1 be the price vector associated with the first N_1 securities and q_2 the price vector associated with the other N_2 securities. Then

$$q = (q_1^T, q_2^T)^T.$$

First, we prove that if there is no arbitrage, then there must be a state-price vector $\psi \in \Re^S_{++}$ such that $q_1 = X_1\psi$. Suppose not. We define

$$A \equiv \{X_1\psi : \psi \in \Re^S_{++}\} \tag{4.37}$$

and

$$B \equiv \{\lambda q_1 : \lambda \in \Re_+\}, \tag{4.38}$$

then $A \cap B$ is empty. It follows from the Separating Hyperplane Theorem,[4] that there exists a nonzero $\theta_1 \in \Re^{N_1}$ such that

$$\lambda \theta_1^T q_1 \leq \theta_1^T X_1 \psi \ \ \forall \lambda \in \Re_+, \ \psi \in \Re^S_{++}. \tag{4.39}$$

[4] We state a version that is applicable in $\Re^n$. Define a *linear functional* F on $\mathbf{R}^n$ as a function $F : \mathbf{R}^n \to \mathbf{R}$ satisfying

$$F(\alpha x + \beta y) = \alpha F(x) + \beta F(y), \quad x, y \in \mathbf{R}^n, \ \ \alpha, \beta \in \mathbf{R}.$$

We have the following theorem.

Theorem 4.2 *(Separating Hyperplane Theorem) Suppose A and B are convex, disjoint sets in $\mathbf{R}^n$. There is some linear functional F such that $F(x) \leq F(y)$ for each x in A and y in B. Moreover, if x is in the interior of A or y is in the interior of B, then $F(x) < F(y)$.*

For a more general version that is applicable in arbitrary vector spaces, see Luenberger [253, p.133].

This implies that $\theta_1^T q_1 \leq 0$ and $\theta_1^T X_1 \geq 0$. Since X_1 has full rank, there is no $\theta_1 \neq 0$ such that $\theta_1^T X_1 = 0$ and thus we must have $\theta_1^T X_1 > 0$. Let $\theta^T = (\theta_1^T, 0)^T$, then

$$\theta^T q = \theta_1^T q_1 \leq 0 \text{ and } \theta^T X = \theta_1^T X_1 > 0, \tag{4.40}$$

which implies that θ is an arbitrage.

Second, we prove that $q_2 = X_2\psi$, where ψ was proven to exist in the previous paragraph. Suppose $q_2 \neq X_2\psi$, then there exists a $\theta_2 \in \Re^{N_2}$ such that $\theta_2^T(q_2 - X_2\psi) < 0$. Let $\theta_1 = -K^T\theta_2$ and $\theta^T = (\theta_1^T, \theta_2^T)$, then

$$\theta^T X = \theta_1^T X_1 + \theta_2^T K X_1 = 0 \text{ and } \theta^T q = \theta_2^T(q_2 - X_2\psi) < 0. \tag{4.41}$$

Thus, θ is an arbitrage.

Therefore, we have that $q = X\psi$ with $\psi \in \Re^S_{++}$. Finally it is easy to check that if a state-price vector exists, then there is no arbitrage. ∎

This theorem shows the existence of a strictly positive state-price vector ψ.[5] However, it says nothing about uniqueness. If $N = S$ so that markets are complete, then we know that the state-price vector is unique, and satisfies the equation

$$q = X\psi = Xp,$$

where $p = (p_1 \ldots p_S)'$ is the vector of contingent claims prices. Hence, the state-prices ψ_s may be obtained as the (unique) solution to a set S linear equations in S unknowns. However, if $N < S$, there may be many solutions for ψ satisfying $q = X\psi$. Some of these solutions may not be strictly positive. In the case of solutions for ψ which are are note strictly positive, that s, which have some elements that are zero or negative, then we know that there will exist arbitrage.

In general, depending on the nature of market incompleteness characterizing the underlying economy, the prices attached to payoffs in different states of the world may differ from the state prices in a complete contingent claims equilibrium. In later chapters, we discuss economies with alternative forms of market incompleteness and study the nature of the equilibrium prices and allocations that arise in them.

4.4 Risk-free asset

Suppose that a risk-free asset exists, say asset 0. Such an asset has the same pay-off in all states of the world, i.e., $x_{0,s}$ is independent of s, or $x_{0,s} = x \; \forall s$.

[5] An alternative proof of this result relies on Stiemke's Lemma. See LeRoy and Werner [244] or Duffie [109].

Hence, the price of such an asset today is given by

$$
\begin{aligned}
p^f &= \sum_{s=1}^{S} \psi_s x \\
&= x \sum_{s=1}^{S} \psi_s. \qquad (4.42)
\end{aligned}
$$

Otherwise, there would exist an arbitrage opportunity. For $x = 1$, the price of an asset today that pays off 1 unit in all states of nature next period is given by

$$
p^f = \sum_{s=1}^{S} \psi_s. \qquad (4.43)
$$

Suppose $p^f > \sum \psi_s$. Then an investor could sell the security at date 0 and receive p^f. At date 1, s/he would deliver 1 regardless of the state s. Since the state prices ψ_s are strictly positive, the investor would make a profit of $p^f - 1 > 0$. Conversely, if $p^f < \sum_s \psi_s$, investors could make a profit by buying the security at date zero for p^f and obtaining the strictly positive payoff $1 - p^f$ in period 1. A similar argument implies that the risk-free rate of return r^f is given by

$$
\begin{aligned}
r^f &= \frac{1}{p^f} \\
&= \frac{1}{\sum_{s=1}^{S} \psi_s}. \qquad (4.44)
\end{aligned}
$$

4.5 Risk neutral pricing

Notice that we can use the state-price vector to derive an alternative representation of security prices. Given a state-price vector ψ_s, we can define the *risk-adjusted probability* of state s as

$$
\begin{aligned}
\pi_s^{\star} &= \frac{1}{\sum_{s=1}^{S} \psi_s} \pi_s \left(\frac{\psi_s}{\pi_s} \right) \\
&= \frac{\psi_s}{\sum_{s=1}^{S} \psi_s}. \qquad (4.45)
\end{aligned}
$$

where $\pi_s > 0$ is the probability of state s occurring. Then we can express the price of security n as

$$
q_n = \sum_{s=1}^{S} x_{n,s} \psi_s
$$

$$= \frac{1}{r^f} \sum_{s=1}^{S} r^f \pi_s \left(\frac{\psi_s}{\pi_s} \right) x_{n,s}$$

$$= \frac{1}{r^f} \sum_{s=1}^{S} \pi_s^\star x_{n,s}. \tag{4.46}$$

Using the definition of the risk-adjusted probabilities, we can write this last relation as

$$q_n = \frac{1}{r^f} E^\star(x_n), \tag{4.47}$$

where the expectation $E^\star(\cdot)$ is taken with respect to $\pi_s^\star$ for $s = 1, \ldots, S$. This asset valuation formula says that security prices are determined as the expected discounted value of future payoffs, with the risk-adjusted probabilities being used to evaluate the expectation of the random payoffs for each security $n = 1, \ldots, N$. The relation in (4.47) can be interpreted as a *certainty equivalent* approach to asset pricing. In other words, $E^*(x_n)$ computes the market-adjusted certainty equivalent of the payoff x_n. In general, different utility functions will assign different certainty equivalents to the same risk embodied in the random set of payoffs X. Since the price of the asset is determined as a certainty equivalent, the time discounting is done using the risk-free rate. This type of valuation is known as *risk neutral valuation*, and it provides a convenient approach for pricing assets without having to specify some underlying structural model of consumption and asset allocation.

Example 2.2 Suppose there 2 securities and 2 states of the economy. Security 1 pays off 2 in state 1 and 1 in state 2 while security 2 pays off 1 in state 1 and 2 in state 2. For simplicity, suppose that the prices are unity for securities 1 and 2, that is, $q_i = 1$ for $i = 1, 2$. The security prices and payoffs satisfy the relation

$$q = X\psi \quad \Leftrightarrow \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}. \tag{4.48}$$

Since the number of securities equals the number of states, we can solve uniquely for the state prices as

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = X^{-1} q = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}. \tag{4.49}$$

Given the state prices, the risk-adjusted probabilities are defined as

$$\pi_1^\star = \frac{\psi_1}{\sum_s \psi_s} = \frac{1/3}{2/3} = \frac{1}{2} \text{ and } \pi_2^\star = \frac{1}{2}. \tag{4.50}$$

We can also compute the risk-free interest rate for this economy as

$$r^f = \frac{1}{p^f} = \frac{1}{\psi_1 + \psi_2} = \frac{3}{2}, \tag{4.51}$$

which implies that the (net) real interest rate is $r^f - 1 = 1.5 - 1 = 0.5$.

Suppose instead that there are 3 states of the economy. Suppose that security 1 pays off 1 in state 3 while security 2 pays off 2. Now the payoff matrix becomes:

$$X = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}. \tag{4.52}$$

The state prices satisfy the set of equations:

$$1 = 2\psi_1 + \psi_2 + \psi_3 \tag{4.53}$$

$$1 = \psi_1 + 2\psi_2 + 2\psi_3. \tag{4.54}$$

Since the number of states exceeds the number of securities, there exist multiple solutions for the state prices. Solving for $\psi_1 = 1 - 2\psi_2 - 2\psi_3$ from the second equation and substituting the result into the first shows that any set of ψ_2 and ψ_3 that satisfies the relation $3\psi_2 + 3\psi_3 = 1$ is a solution for the state prices. One set of strictly positive state prices that satisfy these conditions is given by $\psi_1 = \frac{1}{3}$, $\psi_2 = \frac{1}{6}$ and $\psi_3 = \frac{1}{6}$. Associated with these prices are the risk-adjusted probabilities:

$$\pi_1^\star = \frac{\psi_1}{\sum_s \psi_s} = \frac{1/3}{4/6} = \frac{1}{2} \tag{4.55}$$

$$\pi_2^\star = \frac{\psi_2}{\sum_s \psi_s} = \frac{1/6}{4/6} = \frac{1}{4} \tag{4.56}$$

$$\pi_3^\star = \frac{\psi_3}{\sum_s \psi_s} = \frac{1/6}{4/6} = \frac{1}{4}. \tag{4.57}$$

The risk-free for this economy is given by:

$$r^f = (p^f)^{-1} = \left(\sum_s \psi_s\right)^{-1} = 1.5 \tag{4.58}$$

as before. However, this is not the unique solution. Another set of state prices is obtained by setting $\psi_3 = 0$ and $\psi_1 = \psi_2 = \frac{1}{3}$. However, we know by Theorem 4.1 that such a set of state prices would admit arbitrage.

4.6 The discount factor

Returning to the results of the previous chapter, we can use the existence of the state-price vector to give an alternative representation for security prices. For this purpose, define:

$$m_s = \frac{\psi_s}{\pi_s}, \tag{4.59}$$

where π_s is the probability of state s. (Notice that this definition is consistent with the existence of a complete contingent claims equilibrium in which the contingent claims prices are equal to the probability-weighted intertemporal MRS for any consumer i.)

Example 2.3 Consider the economy with 2 dates and 2 states in Example 2.2. For the security prices and payoff matrix is given in equation (4.48), the state prices are given by $\psi_1 = \psi_2 = \frac{1}{3}$. Suppose the objective probabilities of state 1 and state 2 are given by $\pi_1 = 0.1$ and $\pi_2 = 0.9$. Then the unique (strictly positive) stochastic discount factor is given by

$$m_1 = \frac{\psi_1}{\pi_1} = \frac{1/3}{1/10} = \frac{10}{3} \quad (4.60)$$

$$m_2 = \frac{\psi_2}{\pi_2} = \frac{1/3}{9/10} = \frac{10}{27}. \quad (4.61)$$

If the state prices are not uniquely defined, there also exist multiple solutions for the stochastic discount factor and some of them may not be strictly positive.

Using the definition for m_s, we can derive an expression for security prices as

$$\begin{aligned} q_n &= \sum_{s=1}^{S} \psi_s x_{n,s} \\ &= \sum_{s=1}^{S} \pi_s m_s x_{n,s} \\ &= E[m x_n]. \end{aligned} \quad (4.62)$$

Notice that m_s is the ratio of the state price of state s to the probability of state s; hence, it is positive because state prices and probabilities are both positive. As before, we refer to m_s as the stochastic discount factor for state s. We note that if m_s is small, then state s is “cheap” in the sense that investors are unwilling to pay a high price to receive wealth in that state. If we define the risk-neutral probabilities, then the price of the asset is also proportional to the expected value of the random payoffs. However, we note that the existence of the stochastic discount factor as defined in Equation (4.59) does not depend on the existence of any specific asset pricing model but only on the absence of arbitrage opportunities.

5 Asset pricing relations

Consider a two-period version of the model under uncertainty with one commodity in which the state is revealed in the second period. The commodity space is $\Re_+^{S+1}$. Let $c^i = (c_0^i, c_1^i, \ldots, c_S^i)$ represent c_0 units of consumption in the first period and c_s represent units of consumption contingent on the state s occurring, $s \in \{1, \ldots, S\}$. Suppose preferences satisfy expected utility:

$$u_i(c^i) = U_i(c_0^i) + \sum_{s=1}^{S} \pi_s^i V_i(c_s^i), \tag{5.63}$$

where $\pi_s^i > 0$ denotes the probability of state s occurring and U_i and V_i are strictly increasing, strictly concave, and differentiable functions.

5.1 Contingent claims equilibrium

Suppose $(c^1, \ldots, c^I, p)$ where $p \in \Re_+^{S+1}$ is a complete contingent claims equilibrium. In this case, the consumer maximizes (5.63) subject to the budget constraint:

$$p_0 c_0^i + \sum_{s=1}^{S} p_s c_s^i \leq p_0 \omega_0^i + \sum_{s=1}^{S} p_s \omega_s^i. \tag{5.64}$$

Without loss of generality, let us normalize $p_0 = 1$. The first-order necessary and sufficient conditions for c^i to be an optimal plan are:

$$\frac{\partial U_i(c^i)}{\partial c_0^i} = \lambda_i, \tag{5.65}$$

$$\frac{\partial \pi_s V_i(c^i)}{\partial c_s^i} = \lambda_i p_s, \quad s = 1, \ldots, S, \tag{5.66}$$

where λ_i is a Lagrange multiplier. For any two states k and l, a consumer i chooses consumption satisfying:

$$\frac{\pi_k^i V_i'(c_k^i)}{\pi_l^i V_i'(c_l^i)} = \frac{p_k}{p_l}. \tag{5.67}$$

This condition says that the ratio of the contingent claims prices for any two states is equal to the ratio of the marginal utilities of consumption in those states weighted by the probability of occurrence of those states. We can also derive an expression for the price of a contingent claims contract that pays off in state s as:

$$p_s = \frac{\pi_s^i V_i'(c_s^i)}{U_i'(c_0^i)}. \tag{5.68}$$

5.2 Security market equilibrium

Now consider the security market equilibrium. Suppose that there are $N \geq S$ securities with payoffs $x_{n,s}$ for $n = 1, \ldots, N$ and $s = 1, \ldots, S$. Without loss of generality, assume that the spot price of consumption is unity for each

$s \in \{0, 1, \ldots, S\}$. The budget constraints for the consumer's problem are given by:

$$c_0^i + \theta^i \cdot q \leq \omega_0^i, \tag{5.69}$$

$$c_s^i \leq \omega_s^i + \theta^i \cdot x_s, \quad 1 \leq s \leq S. \tag{5.70}$$

Let μ_0^i and μ_s^i, $s = 1, \ldots, S$, denote the Lagrange multipliers for the consumer's budget constraints. The first-order conditions with respect c_0^i, c_s^i, and θ_n^i are:

$$U'(c_0^i) = \mu_0^i, \tag{5.71}$$

$$\pi_s^i V'(c_s^i) = \mu_s^i, \quad s = 1, \ldots, S, \tag{5.72}$$

$$\mu_0^i q_n = \sum_{s=1}^{S} \mu_s^i x_{n,s}, \quad n = 1, \ldots, N. \tag{5.73}$$

Assume that $\pi_s^i = \pi_s$ for $s = 1, \ldots, S$. Then, using these conditions, we can express the price of the n'th security as:

$$\begin{aligned} q_n &= \sum_{s=1}^{S} \frac{\pi_s V'(c_s^i)}{U'(c_0^i)} x_{n,s} \\ &= E\left[\frac{V_i'(c_s^i) x_{n,s}}{U_i'(c_0^i)}\right]. \end{aligned} \tag{5.74}$$

5.3 The stochastic discount factor

In Section 1.2.2, we showed that a complete contingent claims equilibrium allocation can be attained in a security market equilibrium in which the spot price of consumption is normalized as one and the price of the n'th security is defined in Equation (3.23). Equation (5.74) shows that the relation in equation (3.23) holds provided the contingent claims prices p_s are defined as the probability-weighted intertemporal marginal rate of substitution (MRS) in consumption for any consumer i. Define the quantity:

$$m_s \equiv \frac{V'(c_s^i)}{U'(c_0^i)} \ \forall i \tag{5.75}$$

as the common intertemporal MRS in state s. Then, using the result in (5.74), we can express the price of any security n as

$$\begin{aligned} q_n &= \sum_{s=1}^{S} \pi_s m_s x_{n,s} \\ &= E\left[m x_n\right]. \end{aligned} \tag{5.76}$$

The relation in (5.76) shows that the price of a security is determined as the expected discounted value of its future payoff, the discount factor defined as the intertemporal marginal rate of substitution (MRS) in consumption. Since the individual MRSs are equated across consumers in a complete contingent claims equilibrium, the discount factor used to price random payoffs in Equation (5.74) or (5.76) is evaluated using the MRS for *any* consumer i. Based on the representation in Equation (5.76), the intertemporal MRS in consumption is sometimes called the *stochastic discount factor.* As we examine alternative dynamic equilibrium frameworks and derive restrictions for individual choices, we will also derive restrictions for the stochastic discount factor.

6 Multi-period contingent claims

Up to this point, we have considered simple one- or two-period economies with a discrete number of states and commodities. We now consider complete contingent claims equilibrium in economies with an infinite number of dates. We describe how to price claims that have payoffs for all possible events that can occur and discuss the implications of perfect risk sharing for such economies.

Assume agents live an infinite number of periods and that there is only one commodity or $M = 1$. Uncertainty is defined in terms of a random variable s_t that can take on S possible values in the set $\bar{S}$. Thus, $s_t \in \bar{S}$ at each period. The state of the economy is given by the history s^t of realizations of the random variable s_r for $r \leq t$, defined as

$$s^t \equiv (s_1, \ldots, s_t) = (s^{t-1}, s_t).$$

Let $\pi_t(s^t) \in [0,1]$ denote the probability that $s^t \in \bar{S}^t$ occurs. Notice that

$$\sum_{t=1}^{\infty} \sum_{s^t \in \bar{S}^t} \pi_t(s^t) = 1.$$

We assume that s_t follows a first-order Markov process so that

$$\pi(s_j \mid s_i) = \text{Prob}(s_{t+1} = s_j \mid s_t = s_i). \tag{6.77}$$

denotes the probability of moving from state s_i to state s_j in one period. Then

$$\pi_t(s^t) = \pi(s_t \mid s_{t-1})\pi(s_{t-1} \mid s_{t-2}) \ldots \pi(s_2 \mid s_1) \tag{6.78}$$

is the probability of a time path $s^t = (s_1, \ldots, s_t)$.

The endowment good is non-storable. Let $\omega^i(s_t)$ denote the endowment of agent i in state s_t, where the process is stationary and $\omega(s_t)$ is nonnegative (so it might equal 0 in some periods). The consumption of agent i at time t with history s^t is denoted $c_t^i(s^t)$. The t subscript is added to show that

s^t is a t-dimensional vector. Let $p_t(s^t)$ denote the time 0 price of a unit of consumption at time t in history s^t. Agent $i \in I$ maximizes

$$\sum_{t=1}^{\infty} \sum_{s^t \in S^t} \beta^t U(c_t^i(s^t)) \tag{6.79}$$

subject to

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} p_t(s^t)[\omega^i(s^t) - c^i(s^t)] = 0. \tag{6.80}$$

Notice that the budget constraint requires that the expected discounted present value of lifetime expenditures must equal the expected discounted present value of lifetime endowment. It does not require that the agent's budget must be balanced for a particular history $\{s^t\}_{t=0}^{\infty}$. The Lagrange multiplier for the budget constraint λ^i is not indexed by time or state. The first-order condition is

$$\frac{\beta^t \pi_t(s^t) U'(c_t^i(s^t))}{\lambda^i} = p_t(s^t). \tag{6.81}$$

6.1 Implications of risk sharing

In the previous sections, we described some of the implications of the complete contingent claims equilibrium for asset pricing. We now describe its implications for the behavior of individual consumption in a heterogeneous population. If there are 2 types of agents, A and B, we can show that

$$\frac{p_t(s^t)}{\beta^t \pi_t(s^t)} = \frac{U'(c_t^a(s^t))}{\lambda^a} = \frac{U'(c_t^b(s^t))}{\lambda^b}. \tag{6.82}$$

This condition says that the weighted marginal utility of consumption is equated across consumers. We refer to this feature of the complete contingent claims equilibrium as *complete risk-sharing* or *full insurance*. Wilson [373] was one of the first to note that optimal sharing of risk by members of a risk-averse group is equivalent to the existence of a set of individual-specific weights such that individuals' marginal utility satisfies a version of Equation (6.82).[6]

When markets are incomplete, there do not exist claims that pay off for each possible state of the world. In this case, consumers face a set of budget constraints, one for each state (and date) of the world. As a result, marginal rates of substitution in consumption across different states are not equated across consumers and full insurance does not occur. Private information considerations are typically given as the rationale for market incompleteness and

[6] See Wilson [373, pp. 123-24].

the absence of full insurance opportunities. Individuals may have private information about their characteristics, which can give rise to adverse selection due to asymmetric information. Alternatively, the private actions of individuals may affect the probability distribution of future outcomes. This is known as moral hazard arising from the unobservability of actions and it is commonly assumed to characterize delegated decision making, labor contracting, and insurance markets. In this case, it is not possible to issue claims that pay off in each possible state of the world because such payments cannot be enforced. Thus, markets are incomplete and the optimal risk-sharing rule defined by Equation (6.82) no longer holds. Typically, the second-best risk-sharing rule implies that some risk must be imposed on the privately informed party to ensure that she will undertake the right action. We examine these issues in later chapters.

The last equation can be rewritten to express the ratio of marginal utilities as a function of the ratio of Lagrange multipliers:

$$\frac{U'(c_t^a(s^t))}{U'(c_t^b(s^t))} = \frac{\lambda^a}{\lambda^b}. \tag{6.83}$$

This condition illustrates a property of efficient risk sharing that the marginal rate of substitution across agents is equal to a constant across states.

To solve for the individual allocations, we can substitute in the goods market-clearing conditions and apply the inverse function theorem to find consumption as a function of the total endowment and the ratio of multipliers. The unknown multipliers can then be found by substitution into the budget constraint. If the endowment process is stationary, say follows a stationary first-order Markov process, then the consumption of each type of agent is stationary so that consumption of type A is a function $c^a(s_t)$.

6.1.1 Idiosyncratic risk

Some of the implications of the model are now examined. Suppose that there are only 2 types of agents, A and B, and that the endowment always satisfies

$$\omega = \omega^a(s^t) + \omega^b(s^t), \tag{6.84}$$

where ω is constant. Hence, there is no aggregate risk in this economy, which would occur if the aggregate endowment $\omega(s^t)$ were to vary as a functon of the history s^t. Let $\pi(s^t)$, the probability of the history s^t equal the share of output that goes to agent A, so that

$$\omega^a(s^t) = \pi(s^t) \tag{6.85}$$

and so $\omega^b(s^t) = 1 - \pi(s^t)$. For simplicity, we are assuming that the probability is i.i.d., that is, $\pi(s^t|s^{t-1}) = \pi(s^t)$. However, there is no difficulty keeping it

Markov. Hence the only uncertainty in this economy is how the total endowment is split between the 2 types of agents.

The first-order condition is

$$\frac{\beta^t \pi(s^t) U'(c_t^a(s^t))}{\lambda^a} = p_t(s^t) = \frac{\beta^t \pi(s^t) U'(c_t^b(s^t))}{\lambda^b}. \tag{6.86}$$

The goods market-clearing condition is

$$\omega = c_t^a + c_t^b. \tag{6.87}$$

Using this condition in the first-order condition and rewriting,

$$\frac{U'(c^a(s^t))}{U'(\omega - c^a(s^t))} = \frac{\lambda^a}{\lambda_b}. \tag{6.88}$$

Notice that the right side is constant over time and states. Also, since the total endowment is fixed, the solution to this equation, if λ^a/λ_b were known, has constant consumption c^a. We can solve for the constant consumption using the inverse function theorem. Suppose that the utility function for each agent has the form in Equation (2.8). Then we can solve for the consumption of agent A for history s^t as

$$c^a(s^t) = \bar{c}^a = \left[1 + \left(\frac{\lambda^a}{\lambda^b}\right)^{-1/\gamma}\right] \left(\frac{\lambda^a}{\lambda^b}\right)^{-1/\gamma} \omega.$$

It follows that the consumption of type B is also constant. Hence, if the only risk is idiosyncratic and if markets are complete, then agents can completely insure against endowment risk.

6.1.2 Aggregate risk

Of course the aggregate economy cannot diversify away aggregate risk. The best that can be done is to shift risk to those best able to bear it. Our model predicts that the marginal rate of substitution across agents at each point in time and in each state will be equal to a constant. This is demonstrated in Equation (6.88).

Suppose that the aggregate endowment fluctuates randomly but that type A's fraction of output is constant. In particular, assume that

$$\omega^a(s^t) = \delta\omega(s^t), \quad 0 < \delta < 1.$$

Assume again that the utility function for each agent has the form in Equation (2.8). Then using Equation (6.88), we can solve for the consumption of agent A for any given history of the shocks s^t as

$$c^a(s^t) = \left[1 + \left(\frac{\lambda^a}{\lambda^b}\right)^{-1/\gamma}\right] \left(\frac{\lambda^a}{\lambda^b}\right)^{-1/\gamma} \omega(s^t).$$

Hence, we observe that the aggegate economy is affected. Even with complete risk sharing, the allocations of individual agents vary with fluctuations in the aggregate economy. If, as we have assumed, output is nonstorable, the economy cannot insure against aggregate fluctuations. The best it can do is share the risk efficiently, which is achieved when the marginal rate of substitution across agents is equal to a constant for all states.

The perfect risk sharing hypothesis has been used in a variety of applications in the recent macroeconomics literature. The assumption of complete markets has been used to examine the implications of intertemporal models of consumption and labor supply. It also underlies many of the tests of asset pricing relations based on the Euler equations for a representative consumer's problem. Other papers have examined issues of international risk sharing, and the interaction of risk sharing with specialization. We discuss this literature in later chapters.

6.2 Sequential trading

The discussion above assumes that all of the contracts are negotiated at time 0. Once time progresses and the economy moves along a sample path s^t, there will be no need to renegotiate a contract and the state-contingent trades are carried out. Suppose instead that consumers can trade one-period contingent claims at each date. One could ask whether the resulting equilibrium is identical to the one that is obtained with a time 0 trading scheme. To answer this question, we examine the case with sequential trading.

The aggregate endowment fluctuates randomly and the uncertainty follows a Markov process, as described in (6.77). We define $q(s_{t+1}, s_t)$ as the time t price in state s_t of a unit of consumption to be delivered in state s_{t+1} at time $t+1$. Hence, $q(s_{t+1}, s_t)$ denotes the price of one-period contingent claim. Likewise, let $z_i(s_{t+1}, s_t)$ denote the number of units of the consumption good that agent i receives (if $z_i > 0$) or pays out ($z_i < 0$) if state s_{t+1} occurs. This has been written as a function of both s_{t+1} and s_t. However, to carry out the state contingent trades, it will suffice to know the trade (amount and the sign - plus or minus) that must be transferred in each state. Hence, a vector $\{z_i(1), \ldots, z_i(S)\}$ summarizes the trades. Notice that this vector is independent of the state which occured last period. So henceforth, we drop the additional state variable. Agent i has a budget constraint in period t:

$$z_i(s_t) + \omega^i(s_t) \geq c_t^i + \sum_{s_{t+1} \in \bar{S}} q(s_{t+1}, s_t) z_i(s_{t+1}). \tag{6.89}$$

Let z_t^i denote the S-dimensional vector of contingent claims. Let $\lambda^i(s_t)$ denote the Lagrange multiplier. The first-order conditions are

$$U'(c_t^i) = \lambda^i(s_t) \tag{6.90}$$

$$\lambda^i(s_t) q(s_{t+1}, s_t) = \beta \pi(s_{t+1} \mid s_t) \lambda^i(s_{t+1}). \tag{6.91}$$

Notice that we are picking state contingent trades, so that there is no expectations operator in the second equation. Eliminating the Lagrange multiplier, we have

$$U'(c^i(s_t))q(s_{t+1}, s_t) = \beta\pi(s_{t+1} \mid s_t)U'(c^i(s_{t+1})). \tag{6.92}$$

Rewriting, observe that

$$q(s_{t+1}, s_t) = \frac{\beta\pi(s_{t+1} \mid s_t)U'(c^i(s_{t+1})}{U'(c^i(s_t))} \tag{6.93}$$

for all agents.

How is this price, which is the price of a unit of consumption to be delivered at time $t+1$ in state s_{t+1} determined in time t, conditional on the state s_t occurring, related to the time 0 price in (6.81)? To answer this, solve (6.81) for $U'(c_t^i(s^t))$ and substitute into (6.93) and then update by 1 time period and substitute for $U'(c_{t+1}^i(s^{t+1}))$ to obtain

$$\begin{aligned} q(s_{t+1}, s_t) &= \left[\frac{\beta\pi(s_{t+1} \mid s_t)p_{t+1}(s^{t+1})\lambda^i}{\beta^{t+1}\pi_{t+1}(s^{t+1})}\right]\left[\frac{p_t(s^t)\lambda^i}{\beta^t\pi_t(s^t)}\right]^{-1} \\ &= \frac{p_{t+1}(s^{t+1})}{p_t(s^t)}. \end{aligned} \tag{6.94}$$

Hence, we find that the one-period contingent claims price for state s_{t+1} conditional on s_t occurring in period t is a ratio of the contingent claims prices that negotiated at time zero for the histories s^{t+1} and s^t. If we assume that there are 2 types of agents, A and B as before, it is straightforward to show that the solution to the simple 2 agent problem at time 0 is also the solution to the sequential trading problem. (To show this, we can use the relation in Equation (6.94) to write the sequence of budget constraints as a lifetime budget constraint. Thus, allocations which satisfy the first-order conditions and sequence of budget contraints for the sequential problem also satisfy the first-order conditions and budget constraint for the time zero problem.) An important implication is that, after the contingent trades have been made at time 0, there is no interest in renegotiating any trades. This is despite the fact that the economy is moving along a particular sample path, whereas in the time 0 trading model, the lifetime budget constraint holds in expected value over all sample paths.

6.3 Implications for pricing assets

Why do we care? We don't see trades like the contingent claims trades - or while we do see some assets with contingencies such as options, there aren't many. So why is the contingent claims price system of interest? The reason is that all asset prices will be a bundle of contingent claims prices.

Consider the price of a claim to one unit of the consumption with certainty next period. Let b denote the amount of the asset you buy and let $Q(s_t)$ denote the price. The budget constraint is

$$b_{t-1} + \omega_t \geq c_t + Q(s_t) b_t.$$

The first-order condition is

$$U'(c_t) Q(s_t) = \beta \sum_{s_{t+1} \in \bar{S}} \pi(s_{t+1} \mid s_t) U'(c_{t+1}).$$

Using the results of this section, we can express the price of the asset as

$$Q(s_t) = \sum_{s_{t+1} \in \bar{S}} q(s_{t+1}, s_t). \tag{6.95}$$

Hence, once we have the contingent claims prices, we can construct any kind of security price.

7 A Model with Idiosyncratic Risk

In this chapter and the next, we will discuss models in which there exist market incompleteness and other market frictions. There are several reasons for examining such models. We have already seen that different versions of the representative consumer model have been unable to rationalize such asset pricing anomalies as the equity premium puzzle, the average real risk-free rate puzzle, and the behavior of the term premiums. In a representative agent model, all asset returns are driven by a common stochastic discount factor which suggests that, to some extent, stocks and bonds should tend to move together.[7] Yet the empirical evidence appears to be at odds with this requirement. By introducing market incompleteness, borrowing constraints, and other sorts of frictions, this close link can be broken.

The first topic we consider in this chapter is analyzing the equilibrium allocations and prices in an economy with idiosyncratic risk. We examine two cases: one in which markets are complete and the other in which borrowing constraints with asymmetric information so that markets are incomplete. Our discussion follows Scheinkman and Weiss [325]. We examine the complete markets case to highlight the role that *ex ante* heterogeneity plays in the economy. In the borrowing constraint model, idiosyncratic income risk is nondiversifiable because these shocks are not publicly observed. Hence agents have limited opportunities to borrow against future income and cannot totally insure against all types of risks. Using this framework, we characterize the equilibrium in a model with heterogeneous consumers and borrowing constraints.

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[7] This point has been explored by Barsky [27], among others.

This model also has implications for the behavior of individual consumption and leisure/labor supply allocations. Even in the absence of aggregate shocks, the model generates random fluctuations in aggregate output, the labor input, and the relative price of the asset that is traded in equilibrium. Another implication is that the cross-sectional distribution of nonhuman wealth is an important determinant of aggregate economic activity. By contrast, representative consumer models imply that fluctuations in aggregate unemployment arise solely from the intertemporal substitution of labor and have been rejected in alternative tests based on aggregate data. (See, for example, Mankiw, Rotemberg and Summers [265], Eichenbaum, Hansen and Singleton [119], and Altug [10], among others.)

7.1 The model

We start our discussion with the paper by Scheinkman and Weiss [325]. For consistency with the remainder of the material, we present a discrete-time version of their model. Because we wish to appeal to the law of large numbers in describing the aggregate properties of individual risk, we will introduce individual risk based on Feldman and Gilles [133].

Assume that there is a countable infinity of agents and that the index set of agents is $A = \{1, 2, \ldots\}$. Let ν denote the probability measure defined over A such that $\nu(A) = 1$. Let $B \subset A$ be an infinite subset consisting of type 1 agents[8] and assume that $\nu(B) = \alpha$ where α is the proportion of agents that are type 1. Let B^c be the infinite subset of agents that are type 2 such that $\nu(B^c) = 1 - \alpha$.

Uncertainty is introduced through a productivity shock. A stochastic process is a collection of random variables $\{s(t, \omega), t \in T\}$ defined on the same probability space $(\Omega, \mathcal{F}, P)$, where $T = \{0, 1, 2, \ldots\}$ and $s : T \times \Omega \to S$. For a fixed $\omega \in \Omega$, $s(\cdot, \omega)$ is the sample path or the realization. For a fixed $t \in T$, $s(t, \cdot)$ is a random variable. Let F be the transition probability function which is assumed to have the Feller property. Let the state space be $S = \{1, 2\}$. To conserve on notation, we let $s_t(\omega) = s(t, \omega)$.

We now relate the aggregate uncertainty and individual risk through the productivity shock. Each agent, whether a type 1 or type 2, has time-additive preferences over consumption and leisure streams. An agent can produce one unit of the consumption good per one unit of labor when he is productive, but he may suffer random spells of nonproductivity. In the absence of *ex ante* heterogeneity, agents become differentiated from each other as a result of their histories of productivity. Define a function $\theta : S \to \{0, 1\}$ indexed by $a \in A$.

[8] Examples of infinite subsets are the set of even numbers $\{2, 4, 6, \ldots\}$ or a Fibonocci sequence $\{1, 2, 3, 5, \ldots\}$ in which the (i+1) element equals the sum of elements i and i-1.

Assume that if $s_t(\omega) = 1$, then

$$\theta_{a,t}(s) = \begin{cases} 1 & \text{if } a \in B \\ 0 & \text{otherwise,} \end{cases} \tag{7.96}$$

while if $s_t(\omega) = 2$, then

$$\theta_{a,t}(s) = \begin{cases} 1 & \text{if } a \in B^c \\ 0 & \text{otherwise.} \end{cases} \tag{7.97}$$

The production function for an agent of type i is

$$y_i = \theta_i \ell_i, \tag{7.98}$$

where ℓ_i is the labor supply.

A typical type i consumer has preferences over stochastic sequences $\{c_{i,t}, \ell_{i,t}\}$ of the form

$$U^i = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t [U(c_{i,t}) - W(\ell_{i,t})] \right\}. \tag{7.99}$$

We make the following assumptions on preferences.

Assumption 7.4 *(i) Let* $U : \Re_+ \to \Re_+$ *be concave, increasing, and thrice continuously differentiable with* $U''' > 0$*, and that*

$$\lim_{c \to 0} U'(c) = +\infty, \quad \lim_{c \to \infty} U'(c) = 0.$$

Also assume that $-U''(c)c/U'(c) \leq 1$ *and* $-U'''/U''$ *is decreasing.*
(ii) Let $W : [0, L)+ \to \Re_+$ *be continuous, strictly increasing, thrice continuously differentiable, and convex with* $W(0) = 0$, $W'(0) = 0$ *and* $\lim_{\ell \to L} W'(\ell) = \infty$.

The assumption that $-U''(c)c/U'(c) \leq 1$ ensures that the substitution effect is greater than the income effect. The assumption that $-U'''/U''$ is decreasing implies that the utility function U displays decreasing absolute risk prudence. The concept of absolute risk prudence is studied by Kimball [230]. Decreasing absolute risk prudence implies increasing absolute risk aversion when agents are strictly risk averse.

When a type i agent is productive, he chooses the labor input to maximize his expected discounted present value of utility, trading the gain in the utility of consumption made available by producing against the disutility of labor. An unproductive agent chooses $\ell = 0$ to avoid any disutility from labor.

7.2 Complete contingent claims equilibrium

We will discuss the complete markets case in two ways. First, we describe the equilibrium allocations under the assumption that all trades are made at time zero. In the second case, we provide a sequential definition of the equilibrium. For the sequential equilibrium, we describe several schemes. The first is to allow borrowing while the second is to treat each type of individual as a firm issuing claims on its output stream.

In our discussion below, we assume that $W(\ell) = \ell$. In equilibrium, all agents of the same type will be identical and so we will describe the behavior of the representative type i agent. All variables are expressed as per-capita.

At time zero, all agents trade in the market for claims to consumption and labor supply contingent on state ω at time t. In other words, define $\{\mathcal{F}_t\}_{t=1}^{\infty}$ as an increasing sequence of σ–algebras generated by $\{s_s(\omega), s \leq t\}$. Agents trade for claims contingent on a given history $\{s_s(\omega), s \leq t\}$ for all $w \in \Omega$. Here $(\Omega, \mathcal{F}, \mathcal{P})$, Ω is the set of sample points, $\mathcal{F} \supset \mathcal{F}_t$ for all t is the set of possible events, which are subsets of Ω. $\mathcal{F}$ is closed under the taking of complements and countable unions.

Define $p_t(\omega)$ as the price of a right to delivery of 1 unit of consumption in state ω at time t for all t and for $\omega \in \Omega$. In a complete markets equilibrium, a representative type i consumer maximizes

$$E[U^i|F_0] = \sum_{t=0}^{\infty} \int_{\Omega} \beta^t [U(c_{i,t}(\omega)) - \ell_{i,t}(\omega)] \mathcal{P}(d\omega) \tag{7.100}$$

subject to the single budget constraint

$$\sum_{t=0}^{\infty} \int_{\Omega} [p_t(\omega)(c_{i,t}(\omega) - \theta_{i,t}(\omega)\ell_{i,t}(\omega))] \mathcal{P}(d\omega) \leq 0, \tag{7.101}$$

and the constraints that $c_{i,t}(\omega) \geq 0$ and $ell_{i,t}(\omega) \geq 0$. Notice that in (7.100) and (7.101), we integrate over all possible realizations ω and sum over time.

The equilibrium allocations must also satisfy the market-clearing conditions, which indicate that the consumption of the two types of individuals at any given date and in a given state must equal the total amount produced at that date and in that state. More precisely,

$$\alpha c_{1,t}(\omega) + (1-\alpha) c_{2,t}(\omega) = \\ \alpha \theta_{1,t}(\omega)\ell_{1,t}(\omega) + (1-\alpha)\theta_{2,t}(\omega)\ell_{2,t}(\omega). \tag{7.102}$$

Now let us characterize the complete markets equilibrium. Let λ_i denote the Lagrange multiplier associated with the budget constraint for a type i agent. The first-order conditions with respect to $c_{i,t}(\omega)$ and $\ell_{i,t}(\omega)$ for an agent of type i are

$$\beta^t U'(c_{i,t}(\omega)) = \lambda_i p_t(\omega), \tag{7.103}$$

$$\beta^t = \lambda_i \theta_{i,t}(\omega) p_t(\omega). \tag{7.104}$$

When $\theta_{i,t}(\omega) = 1$, the first-order conditions for agent i imply that $\lambda_i = \beta^t/p_t(\omega)$, and $U'(c_{i,t}(\omega)) = 1$. Define the function g as

$$g(x) = (U')^{-1}(x),$$

which is well-defined because marginal utility is strictly concave. Define $\bar{c}$ as $\bar{c} \equiv g(1)$. The consumption of an agent i with $\theta_{i,t}(\omega) = 1$ (a productive agent) equals $\bar{c}$. When $s_t(\omega) = i$, the price satisfies $p_t(\omega) = \beta^t/\lambda_i$. Notice that λ_i does not vary over time or over realizations ω.

An unproductive agent (one with $\theta_{j,t}(\omega) = 0$) chooses $\ell_{j,t}(\omega) = 0$ and sets consumption to satisfy $\beta^t U'(c_{j,t}(\omega)) = \lambda_j p_t(\omega)$. Substituting in the price $p_t(\omega)$, which we related earlier to the multiplier for the productive agent λ_i, we have

$$\beta^t U'(c_{j,t}(\omega)) = \beta^t \lambda_j/\lambda_i, \tag{7.105}$$

so that the consumption of the unproductive agent satisfies

$$\hat{c}_{j,t} = g(\lambda_j/\lambda_i). \tag{7.106}$$

We now look at the market–clearing conditions. For any t, ω such that $\theta_{1,t}(\omega) = 1$,

$$\begin{aligned} \alpha c_{1,t}(\omega) + (1-\alpha)c_{2,t}(\omega) &= \alpha g(1) + (1-\alpha)g(\lambda_2/\lambda_1) \\ &= \alpha \ell_{1,t}(\omega). \end{aligned} \tag{7.107}$$

For any t, ω such that $\theta_{2,t}(\omega) = 2$, market-clearing requires

$$\begin{aligned} \alpha c_{1,t}(\omega) + (1-\alpha)c_{2,t}(\omega) &= \alpha g(\lambda_1/\lambda_2) + (1-\alpha)g(1) \\ &= (1-\alpha)\ell_{2,t}(\omega). \end{aligned} \tag{7.108}$$

To proceed further, we make a simplifying assumption. Assume that the random variable $s_t(\omega)$ is i.i.d and that the probability that $s_t(\omega) = 1$ is π so that the probability that $s_t(\omega) = 2$ is $1 - \pi$. The expected present value of lifetime earnings of a type 1 agent are

$$\sum_{t=0}^{\infty} \int_{\Omega} p_t(\omega)\theta_{1,t}(\omega)\ell_{1,t}(\omega) = \frac{1}{1-\beta}\frac{\pi \ell_1}{\lambda_1},$$

where we have substituted $p_t(\omega) = \beta^t/\lambda_1$ for $t \geq 0$. The expected present value of the type 1 agent's consumption stream is

$$\sum_{t=0}^{\infty} \int_{\Omega} p_t(\omega)c_{1,t}(\omega) = \frac{1}{1-\beta}\frac{\pi g(1)}{\lambda_1} + \frac{1}{1-\beta}\left[\frac{(1-\pi)}{\lambda_2}g(\lambda_1/\lambda_2)\right].$$

Equating the two expressions and using the market-clearing condition (7.107) to solve for $\ell_1 - g(1)$, we have

$$\left(\frac{1-\alpha}{\alpha}\right) g\left(\frac{\lambda_2}{\lambda_1}\right) = \left(\frac{1-\pi}{\pi}\right)\left(\frac{\lambda_1}{\lambda_2}\right) g\left(\frac{\lambda_1}{\lambda_2}\right). \tag{7.109}$$

We can repeat the same steps for a type 2 agent; this results in

$$\left(\frac{\alpha}{1-\alpha}\right) g\left(\frac{\lambda_1}{\lambda_2}\right) = \left(\frac{\pi}{1-\pi}\right)\frac{\lambda_2}{\lambda_1} g\left(\frac{\lambda_2}{\lambda_1}\right).$$

Define $x \equiv \lambda_1/\lambda_2$; then the equilibrium condition (7.109) becomes

$$\left(\frac{1-\alpha}{\alpha}\right) g\left(\frac{1}{x}\right) = \left(\frac{1-\pi}{\pi}\right) xg(x). \tag{7.110}$$

We now consider an important case corresponding to the complete market equilibrium studied in Scheinkman and Weiss. Suppose that $\pi = 1/2$ and that $\alpha = 1/2$. Then each period half of the agents are productive and each type of agent expects to be productive with the same probability as any other agent. Under these assumptions, a stationary solution is $\lambda_1 = \lambda_2 = 1$. In this case, individual $a \in A$ consumes a constant amount equal to $\bar{c}$ at all dates and in all states. Output is constant and equal to $2\bar{c}$. Prices are also constant and the real interest rate r satisfies

$$E_t\left(\frac{p_{t+1}}{p_t}\right) = \frac{1}{1+r} = \beta.$$

This is the case of complete insurance in which the opportunities to pool risks enable all agents to consume a fixed amount regardless of the particular realization ω which determines their earnings stream.

Suppose now that $\pi = 2/3$ but retain the assumption that $\alpha = 1/2$. Then each period, one half of the agents are productive just as before. But now notice that the expected present value of the lifetime earnings for a type 1 agent is greater than that of a type 2 agent. Equation (7.110) now becomes

$$\frac{1}{2} g(x)x = g\left(\frac{1}{x}\right). \tag{7.111}$$

Suppose utility displays constant relative risk aversion so that $U'(c) = c^{-\gamma}$. Then the solution is $x = (\frac{1}{2})^{\frac{\gamma}{2-\gamma}}$. The real interest rate r_1 when type 1 agents are productive ($s_t(\omega) = 1$) is

$$\frac{1}{1+r_1} = \pi\beta + \beta(1-\pi)x,$$

and the real interest rate r_2 when type 2 agents are productive ($s_t(\omega) = 2$) is

$$\frac{1}{1+r_2} = \pi x\beta + \beta(1-\pi).$$

Hence both agents experience fluctuations in consumption over time, depending on the realization of the random variable. The economy experiences aggregate fluctuations in output, prices and real interest rates because agents are no longer indentical in expected present value of expected lifetime earnings. There is no market incompleteness here and risks are pooled.

When agents have the same discounted present value of labor income, then they can borrow and lend to smooth consumption to an extent that the agent's consumption is no longer dependent on the particular time path of his wealth. When agents are no longer identical in expected present value, the ability of each agent to smooth consumption is affected. When $\pi = 2/3$, agent 1 is better off with fluctuating consumption than in the case where consumption is constant at $\bar{c}$. The same result would hold if $\pi = 1/2$ and $\alpha \neq 1/2$. In that case, although any individual agent expects to be productive with the same probability as any other agent, the proportion of agents that are productive varies so that aggregate output fluctuates because of the concentration of the productivity shock.

7.3 Sequential equilibrium

Instead of assuming that all trades take place at time 0 in terms of contingent claims, imagine instead that agents make their consumption and labor decisions sequentially. We discuss two cases. First we permit borrowing and lending. This can take place through a financial intermediary or by transactions among individual agents. We then look at the case where a household issues an equity share which is a claim to some portion of its earnings. When the household is productive, it pays a dividend to shareholders while when it is not productive, no dividend is paid.

In all cases, we search for a stationary equilibrium.

Borrowing and Lending

Suppose that borrowing and lending are permitted. Assume that there is a durable and nondepletable asset that is fixed in per-capita supply at one unit. The asset is bought and sold at a real price q_t at time t. Let $z_{a,t}$, where $a \in A$, denote the asset holdings of agent a at time t. If $a \in B$ then agent a is a type 1 and a type 2 otherwise. Let $x_{i,t}$ denote the holdings of the asset by the representative type i agent at the beginning of time t.

The supply of the asset is fixed at unity. Thus, market-clearing requires that

$$\alpha x_{1,t} + (1-\alpha) x_{2,t} = 1 \tag{7.112}$$

We can determine x_2 if we know x_1 so that we need only keep track of the per-capita asset holdings of one type of agent. We will find it convenient later on to let x be the vector (x_1, x_2) and to let the state of the system be described by the pair (x, s).

The representative type i agent, for $i = 1, 2$ chooses stochastic sequences $\{c_{i,t}, \ell_{i,t}\}$ to maximize

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t [U(c_{i,t}) - \ell_{i,t}] \right\}, \tag{7.113}$$

subject to the set of constraints

$$z_{i,t+1} - z_{i,t} = (y_{i,t} - c_{i,t})/q_t, \tag{7.114}$$

$$y_{i,t} = \theta_i(s_t)\ell_{i,t}, \tag{7.115}$$

$$\ell_{i,t} \geq 0,\ c_{i,t} \geq 0, \tag{7.116}$$

and given the initial distribution of the asset, which satisfies

$$1 = \alpha x_{1,0} + (1-\alpha)x_{2,0}.$$

We will assume that all agents of the same type are identical so that in equilibrium $z_a = x_1$ if $a \in B$ and $z_a = x_2$ otherwise. We allow z and x to be negative; this can be interpreted as a debt (or borrowing). Let $\xi_{i,t}$ be the multiplier associated with the constraint. The state variables clearly consist of s and the vector x. The state of an individual agent depends on his asset holdings z_a and the system state variables, (x, s).

We now set this up as a dynamic programming problem. The problem is

$$V_i(z, x, s) = \max_{\{c,\ell,z'\}} [U(c) - \ell + \beta \int_S V_i(z', x', s')F(s, ds')] \tag{7.117}$$

subject to the constraints (7.114)–(7.116) and the law of motion for x. We assume that $z \in \mathcal{Z} = [-\bar{z}, \bar{z}]$ where $1 < \bar{z} < \infty$, $\ell \in [0, L]$ where $L < \infty$, and $c \in [0, \bar{Y}]$ where $\bar{Y} < \infty$. The upper bound $\bar{Y}$ can be justified by setting $\bar{Y} = 2L$ which is the maximum possible output that could be attained. Also assume that $x_i \in \mathcal{Z}$ for $i = 1, 2$. Define $\mathcal{S} \equiv \mathcal{Z} \times S$. Let Q be the set of functions $q : \mathcal{S} \rightarrow \Re_+$ such that $\{q : 0 < q(x, s) < \infty, (x, s) \in \mathcal{S}$.

Notice that if $q(x, s)$ is strictly positive, then the set of values $\{c, \ell, z'\}$ satisfying (7.114 – 7.116) can be denoted $\phi(z, x, s)$; this set is compact and convexed-valued. If q is continuous, then under assumption (7.4), ϕ is continuous in s. Let $\mathcal{V}$ be the space of bounded, continuous, real-valued functions $V_i(z, x, s)$ on $\mathcal{Z} \times \mathcal{S}$ with the norm $\|V_i\| = \sup |V_i(z, x, s)|$. Given any continuous, strictly positive price, it is straightforward to show that there exists a unique value function satisfying (7.117). This summarizes the information we need for the individual agent $a \in A$. Let us define an equilibrium for this economy.

Definition 7.1 *A stationary equilibrium is a set of functions* $\hat{q} : \mathcal{Z} \times \mathcal{S} \rightarrow \Re_+$, $\hat{c}_i(z, x, s)$, $\hat{\ell}_i(z, x, s)$, *and* $\hat{z}_i(z, x, s)$, *defined on* $\mathcal{Z} \times \mathcal{S}$ *and measurable with respect to* $\mathcal{F}_t$, *such that*
(i) $\hat{c}_i(z, x, s)$, $\hat{\ell}_i(z, x, s)$, *and* $\hat{z}_i(z, x, s)$ *solve (7.117) subject to the constraints (7.114), (7.115) and (7.116);*
(ii) markets clear:

$$1 = \alpha \hat{z}_1(z_1, x, s) + (1-\alpha)\hat{z}_2(z_2, x, s), \tag{7.118}$$

and

$$\alpha\hat{y}_1 + (1-\alpha)\hat{y}_2 = \alpha\hat{c}_1(z_1, x, s) + (1-\alpha)\hat{c}_2(z_2, x, s), \tag{7.119}$$

where $\hat{y}_i = \theta_i(s)\hat{\ell}_i(z_i, x, s)$, *and*
(iii) the laws of motion for the system variables x *evolve as*

$$x_i' = \hat{z}_i(x_i, x, s), \quad i = 1, 2. \tag{7.120}$$

Recall that if we know x_1, then we can determine x_2 from the market–clearing condition (7.112). Without any loss of information, we can define $\xi_i(x_i, s)$ as the equilibrium multiplier on the constraint (7.114) for the representative type i agent when the average holdings of the durable asset by the type i agents at the beginning of the period is x_i. The pair (x_i, s) completely describes the state of the system. Similarly, let $c_i(x_i, s)$ denote the equilibrium consumption for the representative type i agent, and let $\ell_i(x_i, s)$ denote the equilibrium labor supply. Finally, let φ_i denote the partial derivative of the value function with respect to its first argument, or $\varphi_i(x_i, s) = V_i'(x_i, x, s)$.

The equilibrium first-order conditions for the representative type i agent are

$$U'(c_i(x, s)) = \frac{\xi_i(x_i, s)}{q(x, s)} \tag{7.121}$$

$$1 = \frac{\theta_i(s)\xi_i(x_i, s)}{q(x, s)} \tag{7.122}$$

$$\xi_i(x_i, s) = \beta E_s[\varphi_i(x_i', s')]. \tag{7.123}$$

The envelope condition is $\varphi_i(x_i, s) = U'(c_i(x_i, s))q(x, s)$.

If $\theta_i(s) = 1$, then $\xi_i(x_i, s)/q(x, s) = 1$ and $c_i(x_i, s) = \bar{c}$, where $\bar{c}$ was defined earlier. Suppose that agents of type i are productive while type j agents are not. Then the first-order condition for the representative type j agent is

$$U'(c_j(x_j, s)) = \xi_j(x_j, s)/q(x, s) = \frac{\xi_j(x_j, s)}{\xi_i(x_i, s)},$$

where we have substituted $q(x, s) = \xi_i(x_i, s)$. It becomes apparent when we compare the first-order conditions for the sequential equilibrium with the contingent claims equilibrium, that $\xi_i = \lambda_i$ and that the price of the asset equals the Lagrange multiplier for the productive agent, which is independent over dates and alternative states of the economy.

In the discussion above, the borrowing and lending takes place between individuals of different types. Instead of borrowing and lending, suppose that agents act as if they were firms and issue equities shares that are claims to their earnings stream. We will show that the allocation is the same as that under the borrowing and lending.

Equities trading
We assume that there is one outstanding claim to each earnings stream. Let $z_i^j(s_t)$ denote the shares to the jth earnings stream in state s_t held by agent i. The share sells at the price Q_t^j. The sum of shares satisfies

$$\alpha x_{i,t}^j + (1-\alpha)x_{j,t}^j = 1, \quad i,j = 1,2. \tag{7.124}$$

An agent of type i buys shares of the equity issued by agents of type j. The type i agent also issues shares and pays dividends. Agent i maximizes (7.113) subject to the current period budget constraint

$$\begin{aligned} c_{i,t} + Q_t^i x_{i,t+1}^i + Q_t^j x_{i,t+1}^j &\leq (Q_t^i + d_{i,t})x_{i,t}^i \\ &+ (Q_t^j + d_{j,t})x_{i,t}^j + \theta_{i,t}\ell_{i,t} - d_{i,t}(\alpha x_{i,t}^i + (1-\alpha)x_{j,t}^i). \end{aligned} \tag{7.125}$$

We have written this constraint assuming that the representative type i also buys and sells claims to his own earnings stream. To be consistent, we also assume that dividends equal output, or $d_{i,t} = \theta_{i,t}\ell_{i,t}$. We will see shortly that many of these terms will drop out. To start, we look for a stationary equilibrium in which $x_{i,t}^j = x_{i,t+1}^j$ and (7.124) is satisfied. There may be other stationary solutions, but we examine only this one because, as we will show it has a natural interpretation. Under these assumptions, the budget constraint becomes

$$c_{i,t} \leq d_{i,t}x_{i,t}^i + d_{j,t}x_{i,t}^j. \tag{7.126}$$

Let $\xi_{i,t}$ denote the multiplier associated with the constraint (7.125) for consumer i. If $s_t = i$, then $\theta_j(s_t) = 0$ for $j \neq i$, and no output is produced by type j agents so that $d_{j,t} = 0$. The first-order conditions imply that when $s_t = 1$, type 1 agents consume $\bar{c}$, and type 2 agents consume $g(\xi_{2,t})$. Recall that in the contingent claims equilibrium, the share of output consumed by a type 1 agent when $\theta_1 = 1$ was $\bar{c}/\theta_1\ell_1$ and the share consumed by type 2 agents was

$$\frac{1-\alpha}{\alpha\theta_1\ell_1} g(\lambda_2/\lambda_1).$$

When $\theta_2 = 1$, the share of output consumed by a type 2 agent was $\bar{c}/\theta_2\ell_2$ and the share consumed by type 1 agents was

$$\frac{\alpha}{1-\alpha\theta_2\ell_2} g(\lambda_1/\lambda_2).$$

Setting the equity shares equal to the consumption shares,

$$\begin{aligned} x_1^1 &= \frac{g(1)}{\theta_1\ell_1}, \quad x_2^1 = \frac{1-\alpha}{\alpha}\frac{g(\lambda_2/\lambda_1)}{\theta_1\ell_1}, \\ x_1^2 &= \frac{g(1)}{\theta_2\ell_2}, \quad x_2^2 = \frac{\alpha}{1-\alpha}\frac{g(\lambda_1/\lambda_2)}{\theta_2\ell_2}. \end{aligned}$$

One can verify that this allocation is market-clearing and satisfies the first-order conditions. To show that these distribution of shares, together with the implied consumption and labor supply allocations, can be used to replicate the complete markets equilibrium, merely set $\xi_{i,t}(\omega) = \lambda_i p_t(\omega)$ for each $\omega \in \Omega$.

The first-order conditions can be used to find expressions for equity prices as[9]

$$Q_t^j = E_t \left[\frac{\beta U'(c_{i,t+1})}{U'(c_{i,t})} (Q_{t+1}^j + d_{j,t+1}) \right], \quad i, j = 1, 2. \tag{7.127}$$

Under the complete markets assumption, consumers set their intertemporal MRS's equal to the common ration $p_{t+1}(\omega)/p_t(\omega)$. Since this ratio varies with ω, the price of a claim to type i's earning stream is the expected discounted value of that stream. We could introduce aggregate uncertainty into this setup, and allow for a production technology that yields an exogenous output stream $\{d_t\}$. Then the price of a claim to this output stream would be determined as in (7.127), with the common intertemporal MRS used as the stochastic discount factor.

We already studied the pricing of such claims in the representative consumer pure exchange economy of Chapter 2. There the stochastic discount factor is equal to the random intrtemporal MRS of the representative consumer and can be evaluated using a parametric specification of preferences and aggregate or per-capita consumption data.[10] With incomplete markets, there is in general no common stochastic discount factor, and asset pricing relations based on an intertemporal MRS evaluated with aggregate or per-capita consumption data are not valid. Likewise, market frictions such as short sales constraints and bid-ask spreads will alter the relationship between individual intertemporal MRS's and the common stochastic discount factor used to value random payoffs. The results of Luttmer [254] and He and Modest [198] suggest that we can construct volatility bounds for stochastic discount factors in the presence of various forms of market frictions, provided there exist a complete set of contingent claims. We describe their methods in a later section. For the incomplete markets case, we address the issue of characterizing the stochastic discount factor by analyzing a specific model with frictions.

7.4 Borrowing constraints

From our review of the empirical evidence based on representative consumer models, we know that such models fail to account for the temporal behavior of asset returns due to the lack of correlation of aggregate consumption growth

[9] In this case, we use the budget constraint (7.125) with the last two terms equal to each other, and differentiate with respect to $x_{i,t}^j$ for $i, j = 1, 2$.

[10] The Euler equation tests that we described in Chapter 4 typically assume *ex ante* heterogeneity among consumers but rely on the aggregation conditions specified by Rubenstein [309] in order to derive representations involving aggregate or per-capita consumption.

with asset returns. Thus, explaining the empirical facts requires that the link between the intertemporal MRS and asset returns be loosened. As we noted before, market incompleteness and borrowing constraints are frictions that can potentially accomplish this. In a later section, we discuss other types of frictions, such as transactions costs, that may have the same effect.

Several authors, including Bewley [34] and by Mankiw [262], have noted that introducing market frictions can help to explain the equity premium. Mankiw uses a two-period model in which the risk-free rate is fixed and shows how the concentration of idiosyncratic shocks throughout the population affects the equity premium. Under certain circumstances, an econometrician who uses per-capita consumption series and a representative agent framework will overpredict the degree of risk aversion required to generate an equity premium of the magnitude observed. Hence, a potential explanation of the equity premium puzzle is that it is an artifact of the representative agent model. Problem 3 below is based on the Mankiw paper. This point has been further studied by Weil [365] who also models the risk-free rate. Kahn [227] also develops a two-period model with moral hazard and imperfect risk sharing.

There is a recent literature studying these issues using infinite-lived agent models. While the results are very preliminary, there appear to be some general conclusions. In a model with no aggregate uncertainty and with i.i.d. shocks for individuals, Aiyagari and Gertler [7] have found in simulations that the borrowing constraints did not generate enough volatility of asset returns. To improve their results, they also included transactions costs. This is similar to the results of the papers by Heaton and Lucas [199], who work with a three-period model and incorporate transactions costs, short sales constraints and borrowing constraints. Telmer [354] develops a model in which there is both aggregate and individual uncertainty. While he is unable to prove formally existence and uniqueness of equilibrium, he does have a computational algorithm which allows him to simulate the model. He finds that introducing a risk-free asset allows the agents to do a great deal of consumption smoothing.[11] Constantinides and Duffie [80] have pointed out that in most of these models, the idiosyncratic labor income shocks are i.i.d. and hence, transient so that the permanent income of agents is almost equal across agents despite imperfect risk sharing. Hence, the consumption smoothing opportunities afforded by a risk-free bond are almost enough to allow risk sharing and that this is the reason transactions costs and short sales constraint are needed.

We now study equilibrium with incomplete markets and borrowing constraints. We first describe how to prove existence and uniqueness of the competitive equilibrium for this model, and then study some of its implications. To construct the equilibrium, we start by fixing the marginal valuation function for the asset, which is equal to the Lagrange multiplier on the budget

[11] Other related papers are by Brown [56] and Danthine, Donaldson, and Mehra [88].

constraint. We then determine the price that clears the market, holding the Lagrange multiplier fixed. The market-clearing price that results is then held fixed as we solve for the marginal value function. The method of proof in this step follows that of Deaton and Laroque [90]. We then show that the marginal value functions are increasing and concave in the market-clearing price. In the final step, we show that there exists a unique price function that clears the market that is also used to construct the marginal valuation function.

All variables are measured as per-capita. We retain the assumption that there is one unit of the durable asset. We will search for a stationary equilibrium. The state of the system at time t is described by the Markov process s_t and the distribution of the durable asset across the type 1 and 2 agents. As before, we assume that agents within a class – type 1 or type 2 – are identical. The proportion of each type of agent in the population is fixed. Let x_i denote the amount of the asset held by the average type i agent and let x be the vector (x_1, x_2). At the beginning of the period, the distribution of the durable asset across agents satisfies

$$1 = \alpha x_1 + (1 - \alpha) x_2. \tag{7.128}$$

The state of the economy is summarized by (x, s).

The representative type i agent, for $i = 1, 2$, chooses stochastic sequences $\{c_{i,t}, \ell_{i,t}\}$ to maximize

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t [U(c_{i,t}) - W(\ell_{i,t})] \right\}, \tag{7.129}$$

subject to the set of constraints

$$z_{i,t+1} - z_{i,t} = (y_{i,t} - c_{i,t})/q_t, \tag{7.130}$$

$$y_{i,t} = \theta_i(s_t)\ell_{i,t}, \tag{7.131}$$

the nonnegativity constraints

$$z_{i,t+1} \geq 0, \ \ell_{i,t} \geq 0, \ c_{i,t} \geq 0, \tag{7.132}$$

and initial conditions $x_{i,0} = z_{i,0}$ with $1 = \alpha x_{1,0} + (1 - \alpha) x_{2,0}$.

There are two features worth noting about this problem. First, it rules out complete insurance of idiosyncratic risk by ruling out the existence of prices to consumption contingent on any possible history $\{s_s(\omega), s < t\}$. This can be described in terms of the underlying probability space. One possible reason for idiosyncratic risk to be uninsurable may be that the shocks to individuals' productivity are not publicly observable. Second, the above problem assumes that individuals in this economy face borrowing constraints. The borrowing constraints are introduced through the constraint that the asset holdings of the consumer must be nonnegative at all dates and all states; i.e., $z_{i,t+1} \geq 0$.

Notice that if the agent is productive, he chooses both consumption and labor supply. Otherwise, he chooses only consumption. Also, when the agent is unproductive, he is able to consume a positive amount by running down his asset holdings.

We now study the consumer's problem as a dynamic programming problem. The average type i agent who begins the period with asset holdings z solve

$$V_i(z, x, s) = \max_{\{c, \ell, z'\}} \left[U(c) - W(\ell) + \beta \int_S V_i(z', x', s') F(s, ds') \right] \tag{7.133}$$

subject to the constraints (7.130)– (7.132) and the law of motion for x. We assume that $z \in \mathcal{Z} = [0, \bar{z}]$ where $1 < \bar{z} < \infty$, $\ell \in [0, L]$ where $L < \infty$, and $c \in [0, \bar{Y}]$ where $\bar{Y} < \infty$. Also assume that $x_i \in \mathcal{Z}$ for $i = 1, 2$.

The equilibrium price is a function $q : \mathcal{S} \to \Re_+$ such that $\{q : 0 < q(x, s) < \infty, (x, s) \in \mathcal{S}\}$. Notice that if q is strictly positive, then the set of values $\{c, \ell, z'\}$ satisfying (7.130 – 7.132), denoted $\phi(z, x, s)$, is compact and convex-valued. If q is continuous, then under Assumption 7.4, ϕ is continuous in s. Let $\mathcal{V}$ be the space of bounded, continuous, real-valued functions $V_i(z, x, s)$ on $\mathcal{Z} \times \mathcal{S}$ with the norm $\|V_i\| = \sup |V_i(z, x, s)|$. Given any continuous, strictly positive price q, it is straightforward to show that there exists a unique value function satisfying (7.133). This summarizes the information we need for the individual agent $a \in A$. Let us define an equilibrium for this economy.

Definition 7.2 *A stationary equilibrium is a set of functions* $\hat{q} : \mathcal{Z} \times \mathcal{S} \to \Re_+$, $\hat{c}_i(z, x, s)$, $\hat{\ell}_i(z, x, s)$, *and* $\hat{z}_i(z, x, s)$, *defined on* $\mathcal{Z} \times \mathcal{S}$ *and measurable with respect to* $\mathcal{F}_t$, *such that*
(i) $\hat{c}_i(z, x, s)$, $\hat{\ell}_i(z, x, s)$, *and* $\hat{z}_i(z, x, s)$ *solve (7.133) subject to the constraints (7.130)–(7.132);*
(ii) markets clear:

$$1 = \alpha \hat{z}_1(z_1, x, s) + (1 - \alpha) \hat{z}_2(z_2, x, s), \tag{7.134}$$

and

$$\alpha \hat{y}_1 + (1 - \alpha) \hat{y}_2 = \alpha \hat{c}_1(z_1, x, s) + (1 - \alpha) \hat{c}_2(z_2, x, s), \tag{7.135}$$

where $\hat{y}_i = \theta_i(s) \hat{\ell}_i(z_i, x, s)$, *and*
(iii) the laws of motion for the system variables x *evolve as*

$$x_i' = \hat{z}_i(x_i, x, s), \quad i = 1, 2. \tag{7.136}$$

Recall that if we know x_1, then we can determine x_2 from the market-clearing condition (7.128). Without any loss of information, we can define $\xi_i(x_i, s)$

as the equilibrium multiplier on the constraint (7.130) for the representative type i agent when the average holdings of the durable asset by the type i agents at the beginning of the period is x_i. The pair (x_i, s) completely describes the state of the system. Similarly, let $c_i(x_i, s) = \hat{c}_i(x_i, x, s)$ denote the equilibrium consumption for the representative type i agent, and let $\ell_i(x_i, s) = \hat{\ell}_i(x_i, x, s)$ denote the equilibrium labor supply. Finally, let φ_i denote the partial derivative of the value function with respect to its first argument, or $\varphi_i(x_i, s) = V_i'(x_i, x, s)$.

The equilibrium first-order conditions for the representative type i agent are

$$U'(c_i(x_i, s)) = \frac{\xi_i(x_i, s)}{q(x, s)}, \tag{7.137}$$

$$W'(\ell_i(x_i, s)) = \frac{\theta_i(s)\xi_i(x_i, s)}{q(x, s)}, \tag{7.138}$$

$$\xi_i(x_i, s) = \beta E_s[\varphi_i(x_i', s')] + \mu_i, \tag{7.139}$$

where μ is the multiplier on the nonnegativity constraint for $z_{i,t+1}$ so that $\mu_i = 0$ only if $z_{i,t+1} > 0$. The envelope condition is $\varphi_i(x_i, s) = U'(c_i(x_i, s))q(x, s)$. When $\theta_i = 1$, the agent always produces enough so that $z_{i,t+1} > 0$ and $\mu = 0$. When $\theta_i = 0$, the maximum that the agent can consume is $c_i = z_i q$. Hence the multiplier ξ_i obeys

$$\xi_i(x_i, s) = \max\left[U'(x_i q(x, s))q(x, s), \beta E_s \xi_i(x_i', s')\right]. \tag{7.140}$$

Define the function h by

$$h(k) \equiv (W')^{-1}(k)$$

for $k \geq 0$ so that $h : \Re_+ \to [0, L]$. Recall that the definition of the function g is $g \equiv (U')^{-1}$so that $g : \Re_+ \to [0, \bar{Y}]$. Given (x, s), for fixed $\xi_i \geq 0$ and $q > 0$, equations (7.137–7.138) are four equations ($i = 1, 2$) in four unknowns $(c_1, c_2, \ell_1, \ell_2)$ which are the value of the functions $c_i(x_i, s)$ and $\ell_i(x_i, s)$ when $\xi_i = \xi_i(x_i, s)$ and $q = q(x, s)$. The values (c_i, ℓ_i) satisfy $c_i = g(\xi_i/q)$ and $\ell_i = h(\theta\xi_i/q)$. For notational convenience, define the function H as

$$H(k, \theta) \equiv \theta h(\theta k) - g(k).$$

Proposition 7.1 *Under Assumption 7.4, $H_1 > 0$ and $H_{1,1} < 0$, where H_i denotes the partial derivative with respect to the ith argument. Also,* $\lim_{k\to 0} H(k, \theta) = -B = -\infty$ *and* $\lim_{k\to\infty} H(k, \theta) = L$.

It is straightforward to verify these properties under Assumption 7.4. Notice that g is a function satisfying $U'(g(k)) = k$ such that $g'(k) = (U'')^{-1} < 0$ and $g''(k) = -U'''/(U'')^2 < 0$. The function h satisfies $W'(h(k)) = k$. The solution (c_i, ℓ_i) for $i = 1, 2$ to equations (7.137–7.138) can be used in the budget constraints (7.130) to solve for the average asset holdings next period x_i' of type i agents.

So far we have established that for fixed (x, s) and given $q > 0$ and $\xi_i \geq 0$, equations (7.130) and (7.137–7.138) form a system of six equations in six unknowns. We now fix only the functions ξ_i and determine the value of the price q such that markets clear; essentially we are adding one more equation and one unknown. Substituting for x_i' into (7.134)and using $\ell_i = \ell_i(x_i, s)$, $c_i = c_i(x_i, s)$ and $\theta_i = \theta_i(s)$, the market-clearing price satisfies

$$\begin{aligned} 1 &= \alpha \left[\frac{\theta_1 \ell_1 - c_1}{q} + x_1 \right] + (1-\alpha) \left[\frac{\theta_2 \ell_2 - c_2}{q} + x_2 \right] \\ &= 1 + \alpha[\theta_1 \ell_1 - c_1] + (1-\alpha)[\theta_2 \ell_2 - c_2] \end{aligned}$$

Subtracting 1 from both sides and substituting H and taking as given the values $\xi_i = \xi_i(x_i, s)$ for fixed x_i, s, the market-clearing price $q > 0$ satisfies

$$\alpha H(\xi_1/q, \theta_1) = -(1-\alpha) H(\xi_2/q, \theta_2). \tag{7.141}$$

We have the following result.

Theorem 7.3 *Under Assumption 7.4, for fixed $x_i \in \mathcal{Z}$ and $s \in S$ and given $\xi_i = \xi_i(x_i, s)$ such that $\xi_i > 0$, there exists a unique solution $\hat{q} : \mathcal{Z} \times S \times \Re_+ \times \Re_+ \rightarrow \Re_+$ that is strictly positive and continuous.*

Proof.
Under Assumption 7.4, $\lim_{q \rightarrow 0} H(\xi/q, \theta) = L$ and $\lim_{q \rightarrow \infty} H(\xi/q, \theta) = -B$. From Proposition 7.1, it follows that the left-side of (7.141) is decreasing in q and the right-side is increasing. At $q = 0$, $\lim_{\psi \rightarrow \infty} h(\psi) = L$ and $\lim_{\psi \rightarrow \infty} g(\psi) = 0$ so that $\lim_{\psi \rightarrow \infty} h(\psi) - g(\psi) = L$. As q increases, $h - g$ decreases. For each $s \in S$, either $\theta_1 = 0$ or $\theta_2 = 0$ so that the labor supply is constant at zero for one type of agent. Hence, there exists a unique q solving (7.141). Because H is continuous in ξ, it follows that q is continuous in ξ. ∎

It is straightforward to show by differentiation that $\hat{q}$ is increasing in both ξ_1 and ξ_2. Given $\xi_i(x_i, s)$, define $q(x, s) \equiv \hat{q}(x, s, \xi_1(x_1, s), \xi_1(x_1, s))$.

Next we hold q fixed such that $0 < q \leq \infty$. In our discussion below, we drop the index of the agent type for convenience. Let $\mathcal{S} \equiv \mathcal{Z} \times S$ and define $\mathcal{C}(\mathcal{S})$ denote the space of continuous bounded functions defined on the state space and let $\mathcal{D}(\mathcal{S}) \in \mathcal{C}(\mathcal{S})$ be the subspace of continuous functions that are nonincreasing in their first argument. The space $\mathcal{D}(\mathcal{S})$ is a Banach space.

We have the following lemma.

Lemma 7.1 *Let $\xi \in \mathcal{D}(\mathcal{S})$ so that ξ is nonincreasing in its first argument. Fix $q \in \Re_+$ such that $0 < q < \infty$. Let $\psi \in \Re_+$ and let the function G :*

$\Re_+ \times \mathcal{S} \to \Re_+$ *be defined by*

$$G(\psi, x, s) \equiv \max\left[U'(xq)q, \beta \int_S \xi\left(x + \frac{1}{q}H(\psi/q,\theta), s'\right) F(s, ds')\right]. \qquad (7.142)$$

Then G *is nonincreasing in* ψ *and* x*. Furthermore,* $\lim_{\psi\to 0} G(\psi, x, s) = \infty$ *and* $\lim_{\psi\to\infty} G(\psi, x, s) = \underline{G} \geq \bar{U} > 0$.

Proof.
The assumptions on ξ and q ensure that G is nonincreasing in its first argument. An increase in ψ increases $H(\psi/q, \theta)$ which decreases ξ, hence G is nonincreasing in its second argument. As $\psi \to 0$, $H(\psi, \theta) = -B$ and as $\lim_{\psi\to\infty} H(\psi, \theta) = L$. As $\psi \to \infty$, $H(\psi, \theta) = L$. Hence,

$$\underline{G}(x, s) \equiv \lim_{\psi\to\infty} G(\psi, x, s) = \beta \int_S \xi(x + L/q, s')F(s, ds') \geq 0.$$

∎

Let q be fixed as before and let f be a solution to

$$\begin{aligned} f(x, s) &= G(f(x, s), x, s) \qquad (7.143)\\ &= \max\left[U'(xq)q, \beta \int_S \xi\left(x + \frac{1}{q}H(f(x, s)/q, \theta), s'\right) F(s, ds')\right] \end{aligned}$$

Define T_1 as the operator that assigns the solution f to the function G so that $f = T_1 G$. We have the following lemma.

Lemma 7.2 *Assume that* $\xi \in \mathcal{D}(\mathcal{S})$ *so that* G *as defined in (7.142) satisfies the conditions of Lemma 7.1. For fixed* q *such that* $0 < q < \infty$*, let* $f : \mathcal{S} \to \Re_+$ *be the solution to (7.143). Then:*
i) There exists a unique $f^\star \in \mathcal{C}(\mathcal{S})$ *satisfying (7.143).*
ii) The solution function satisfies $f^\star \in \mathcal{D}(\mathcal{S})$ *so that* $T_1 : \mathcal{D}(\mathcal{S}) \to \mathcal{D}(\mathcal{S})$.
iii) If $G_1 \geq G_2$ *for all* (ψ, x, s)*, then* $T_1 G_1 \geq T_1 G_2$.

Proof.
Under the conditions of Lemma 7.1, $G(\psi, x, s) - \psi$ is continuous and strictly decreasing in ψ. For a fixed (x, s), $H(\psi/q, \theta)$ is increasing in ψ. For $G \in \mathcal{D}(\mathcal{S})$, let ψ satisfy

$$\begin{aligned} \psi &= G(\psi, x, s)\\ &= \max\left[U'(xq)q, \beta \int_S \xi\left(z + \frac{1}{q}H(\psi/q, \theta), s'\right) F(s, ds')\right]. \qquad (7.144) \end{aligned}$$

Clearly the left-side is increasing in ψ which, under the assumption that $G \in \mathcal{D}(\mathcal{S})$, implies that the right-side is decreasing in ψ. As $\psi \to 0$, the left-side tends to 0 and the right-side tends to ∞. As ψ increases the left-side increases and the right-side tends to $\underline{G}$. Hence, there exists a unique ψ that satisfies (7.144).

It also follows that

$$\max\left[U'(xq)q, \beta\int_S \xi\left(z+\frac{1}{q}H(\psi/q,\theta),s'\right)F(s,ds')\right]-\psi$$

is continuous and strictly decreasing in ψ. Therefore, $f^\star$ is continuous and $f^\star(x,s)$ is decreasing in its first argument.

Suppose that $G_1 \in \mathcal{D}(\mathcal{S})$ and $G_2 \in \mathcal{D}(\mathcal{S})$ and that $G_1 > G_2$ where

$$G_i(\psi,x,s) \equiv$$
$$\max\left[U'(xq)q, \beta\int_S \xi_i\left(z+\frac{1}{q}H(\psi/q,\theta),s'\right)F(s,ds')\right],$$

for $i=1,2$.

Let ψ_1 be the solution to the equation $[0 = G_1(\psi_1,x,s)-\psi_1]$ and let ψ_2 be the solution to $[0 = G_2(\psi_2,x,s)-\psi_2]$. It follows that

$$G_1(\psi_2,x,s)-\psi_2 \geq G_2(\psi_2,x,s)-\psi_2 = 0$$

so that

$$\psi_1-\psi_2 \geq G_1(\psi_1,x,s)-G_2(\psi_2,x,s).$$

Therefore, $f_1^\star = TG_1 \geq TG_2 = f_2^\star$. ∎

For a fixed $0<q<\infty$, let the operator $T_2 : \mathcal{D}(\mathcal{S}) \to \mathcal{D}(\mathcal{S})$ be defined by

$$\begin{aligned}\xi^{n+1}(x,s) &= (T_2\xi^n)(x,s) \qquad (7.145)\\ &= \max\left[U'(xq)q, \beta\int_S \xi^n\left(x+\frac{1}{q}H\left(\frac{T_2\xi^n}{q},\theta\right),s'\right)F(s,ds')\right].\end{aligned}$$

We have the following theorem.

Theorem 7.4 *Let $0<q<\infty$ be fixed and let $T_2 : \mathcal{D}(\mathcal{S}) \to \mathcal{D}(\mathcal{S})$ be defined by (7.145). Under Assumption 7.4, T_2 is a contraction.*

Proof.

For an initial guess $\xi^0 \in \mathcal{D}(\mathcal{S})$, it is clear that

$$\beta\int_S \xi^0\left(x+\frac{1}{q}H(\psi/q,\theta),s'\right)F(s,ds')$$

is an element of $\mathcal{D}(\mathcal{S})$ for fixed ψ such that $0 \leq \psi < \infty$. Under the conditions of Lemma 7.2, the solution

$$f^\star(x,s) =$$
$$\max\left[U'(xq)q, \beta\int_S \xi^0\left(x+\frac{1}{q}H(f^\star(x,s)/q,\theta),s'\right)F(s,ds')\right]$$

is an element of $\mathcal{D}(\mathcal{S})$.

Let $\xi_1, \xi_2 \in \mathcal{D}(\mathcal{S})$ and assume that $\xi_1 > \xi_2$. For fixed ψ, it follows that

$$\max\left[U'(xq)q, \beta\int_S \xi_1\left(x+\frac{1}{q}H(\psi/q,\theta),s'\right)F(s,ds')\right] \geq$$
$$\max\left[U'(xq)q, \beta\int_S \xi_2\left(x+\frac{1}{q}H(\psi/q,\theta),s'\right)F(s,ds')\right].$$

Under the conditions of Lemma 7.2, it follows that

$$\begin{aligned} &T_2\xi_1(x,s) = \\ &\quad \max\left[U'(xq)q, \beta\int_S \xi_1\left(x+\frac{1}{q}H(T_2\xi_1/q,\theta),s'\right)F(s,ds')\right] \\ &\quad \geq \max\left[U'(xq)q, \beta\int_S \xi_2\left(x+\frac{1}{q}H(T_2\xi_2/q,\theta),s'\right)F(s,ds')\right]. \end{aligned}$$

Hence, $T_2\xi_1 \geq T_2\xi_2$ so that T_2 is monotone.

Let $0 < a < \infty$. To show that H has the discounting property, notice that

$$\begin{aligned} &\beta\int_S (\xi+a)\left(x+\frac{1}{q}H((\xi+a)(x,s)/q,\theta),s'\right)F(s,ds') \\ &\qquad \leq \beta\int_S (\xi+a)\left(x+\frac{1}{q}H(\xi(x,s)/q,\theta),s'\right)F(s,ds') \\ &\quad \leq \quad \beta\int_S \xi\left(x+\frac{1}{q}H(\xi(x,s)/q,\theta),s'\right)F(s,ds') + \beta a, \end{aligned}$$

so that T_2 has the discounting property. Hence, T_2 has a unique fixed point $\xi^\star$. ∎

To find the equilibrium, the first step is to study the behavior of $\xi^\star$ as a function of q. Before we can do this, we show that if U' is convex and W' is concave, then $\xi^\star$ is convex in x. The argument is basically as follows. We have established that there exists a unique fixed point in the space $\mathcal{D}(\mathcal{S})$. If we start in the subspace of $\mathcal{D}(\mathcal{S})$ consisting of nonincreasing convex functions and show that the operator T_2 maps those functions into other functions in the same subspace, then, because of uniqueness, we know that the fixed point is a function that is convex in x.

Proposition 7.2 *If $U''' > 0$ and $W''' < 0$, then $\xi^\star$ is convex in x.*

PROOF.
Fix q such that $0 < q < \infty$. Suppose that $\xi \in \mathcal{D}(\mathcal{S})$ and that ξ is convex in its first argument. Then the function G defined in (7.142)

$$G(\psi,x,s) = \max\left[U'(xq)q, \beta\int_S \xi\left(x+\frac{1}{q}H(\psi/q,\theta),s'\right)F(s,ds')\right]$$

is convex in (ψ, x). For fixed (ψ, s), it is straightforward to verify convexity in x. To show that G is convex in ψ, notice that

$$\frac{\beta}{q^2}\int_S\left[\xi_{11}(x',s')H'(\psi/q,\theta)+\xi_1(x',s')\frac{1}{q}H''(\psi/q,\theta)\right]F(s,ds')>0,$$

where the convexity of U' and concavity of W' are used to show that $H''<0$. Because G is convex in (ψ, x), the solution to $[G(\psi,s)-\psi=0]$ is also convex. Recall that $T_2\xi$ is the solution ψ to the equation $G(\psi,x,s)-\psi=0$. Let $\lambda\in[0,1]$ and let $x_1,x_2\in X$. Define $\psi_1=T_2\xi(x_1,s)$ and $\psi_2=T_2\xi(x_2,s)$. Then $G(\psi_1,x_1,s)-\psi_1=G(\psi_2,x_2,s)-\psi_2=0$. Because G is convex,

$$\begin{aligned}&G(\lambda T_2\xi(x_1,s)+(1-\lambda)T_2\xi(x_2,s),\lambda x_1+(1-\lambda)x_2)\\&\qquad-[\lambda T_2\xi(x_1,s)+(1-\lambda)T_2\xi(x_2,s)]\leq 0.\end{aligned}$$

Since G is decreasing in its first argument, it follows that $T_2\xi$ is convex in x if ξ is convex in x. ∎

For fixed x, s, notice that $x'=x+(1/q)H(\xi^\star/q)$ is a function of q; for notational convenience, we write $x'=a(x,q,s)$. It follows that the fixed point $\xi^\star$ is also a function of q; to emphasize this, we will write $\xi^\star=\Omega(q)$. If we can show how x' changes as q varies, we can determine how Ω changes as q varies. The results are summarized in the next proposition.

Proposition 7.3 *Let $0<q<\infty$ and let $\Omega(q)=\xi^\star_q$ be a fixed point of T_2 where T_2 is defined in (7.145). Then $\xi^\star_q$ is continuous, increasing and concave in q.*

Proof.
For a fixed $0<\psi<\infty$, notice that

$$\frac{\partial a(x,q,s)}{\partial q}=-\frac{1}{q^2}\left[H(\psi/q)+\frac{\psi}{q}H(\psi/q)\right]<0.$$

For this to be true for the unproductive agent, we have used the condition $-U''c/U'<1$ in Assumption 7.4. It follows that $\xi^\star$ is increasing in q.

Next,

$$\frac{\partial^2 a(x,q,s)}{\partial q^2}=\frac{1}{q^3}\left[\frac{\psi}{q}\left(H'+\frac{H''\psi}{q}\right)+H+\frac{2H'\psi}{q}\right]>0$$

because, under Assumption 7.4,

$$H'+\frac{H''\psi}{q}=\frac{W'}{(W'')^2}\left[\frac{W''}{W'}-\frac{W'''}{W''}\right]-\frac{U'}{(U'')^2}\left[\frac{U''}{U'}-\frac{U'''}{U''}\right]>0.$$

Hence x' is decreasing and convex in q. Because $\xi^\star$ is nonincreasing and convex in x' and

$$\frac{\partial}{\partial q}\left[\frac{\partial\xi^\star_1(x',s)}{\partial x'}\frac{\partial x'}{\partial q}\right]=\xi^\star_{11}a_1(q,x,s)+\xi^\star_1 a_{11}(q,x,s),$$

it follows that $\xi^\star$ is nonincreasing and concave in q; or $\Omega' > 0$ and $\Omega'' < 0$. ∎

Although we have found a fixed point $\xi^\star_q$ for a given q and determined the market clearing price $\hat{q}$ for given ξ_1 and ξ_2, we have not shown that $q = \hat{q}$. In fact, we must address the issues of whether a solution exists and if it exists, whether it is unique. Fix (x, s) and define the function $\Omega_i(q) = \xi^\star_{i,q}$, which expresses the fixed point $\xi^\star$ as a function of q. Define the function $\nu : [0, \infty) \to [0, \infty)$ as the solution to

$$\alpha \left[H \left(\frac{\Omega_1(q)}{\nu(q)}, \theta_1 \right) \right] + (1 - \alpha) \left[H \left(\frac{\Omega_2(q)}{(q)}, \theta_2 \right) \right] = 0. \tag{7.146}$$

For notational convenience, let H'_i denote the partial derivative with respect to the first argument of H for an agent of type i. We have the following proposition.

Proposition 7.4 *Let ν be as defined in (7.146). Under Assumption 7.4, ν is increasing and concave.*

Proof.
Differentiating (7.146) with respect to q and solving for ν', we have

$$\nu'(q) = \left[\alpha H'_1 \frac{\Omega_1(q)}{\nu(q)} + (1 - \alpha) H'_2 \frac{\Omega_2(q)}{\nu(q)} \right]^{-1} \times$$
$$\alpha H'_1 \Omega'_1(q) + (1 - \alpha) H'_2 \Omega'_2(q) > 0. \tag{7.147}$$

For notational convenience, define $A_i \equiv \Omega'_i(q) - \Omega_i(q)\nu'(q)/\nu(q)$. Then (7.147) can be written as $\alpha H'_1 A_1 + (1 - \alpha) H'_2 A_2 = 0$. Differentiating the preceding equation with respect to q and simplifying, we have

$$\alpha \left[H''_1 (A_1)^2 + H'_1 \Omega''_1 \right] + (1 - \alpha) \left[H''_2 (A_2)^2 + H'_2 \Omega''_2 \right] =$$
$$\left[\alpha H'_1 \frac{\Omega_1}{\nu(q)} + (1 - \alpha) H'_2 \frac{\Omega_2}{\nu(q)} \right] \nu''(q).$$

Because $H''_i < 0$ and $\Omega''_i < 0$, the left-side is negative. The coefficient on ν'' is positive. Hence $\nu'' < 0$. ∎

We have the following proposition.

Proposition 7.5 *Under Assumption 7.4, there is a unique fixed point $q = \nu(q)$.*

Proof.
Notice that if

$$\alpha \left[H \left(\frac{\Omega_1(q)}{q}, \theta_1 \right) \right] + (1 - \alpha) \left[H \left(\frac{\Omega_2(q)}{q}, \theta_2 \right) \right] > 0,$$

then $\nu(q) < q$, while if

$$\alpha\left[H\left(\frac{\Omega_1(q)}{q},\theta_1\right)\right] + (1-\alpha)\left[H\left(\frac{\Omega_2(q)}{q},\theta_2\right)\right] < 0,$$

then $\nu(q) > q$. Recall that $U'(xq)q$ is increasing in q and that

$$(1/q)H(\psi/q)$$

is decreasing in q. Because U' is unbounded, we can show that for all $c \leq c^\star$, $cU'(c) \geq \underline{U}$ and $\underline{U} > 0$. Notice that ξ is bounded below by 0 and bounded above by some $\bar{\xi}$. Hence as $q \to 0$, $\xi^\star \to \underline{U}$. Then, for some $0 < \epsilon < \infty$, as $\lim_{q\to 0}$, we have

$$\begin{aligned}
&\lim_{q\to 0}[\alpha\left[H\left(\frac{\Omega_1(q)}{q},\theta_1\right)\right] + (1-\alpha)\left[H\left(\frac{\Omega_2(q)}{q},\theta_2\right)\right] \\
&= [\alpha\left[H\left(\underline{U}/0,\theta_1\right)\right] + (1-\alpha)\left[H\left(\underline{U}/0,\theta_2\right)\right] = \alpha L > 0.
\end{aligned}$$

Hence $\nu(0) > 0$. As $q \to \infty$, $\xi^\star$ is bounded and $\lim_{q\to\infty} U'(xq)q = \infty$. Recall that as $\nu \to \infty$, $H(\nu) \to L$. We show that

$$\begin{aligned}
&\lim_{q\to \infty}[\alpha\left[H\left(\frac{\Omega_1(q)}{q},\theta_1\right)\right] + (1-\alpha)\left[H\left(\frac{\Omega_2(q)}{q},\theta_2\right)\right] \\
&= [\alpha\left[H\left(\frac{L}{\infty},\theta_1\right)\right] + (1-\alpha)\left[H\left(\frac{L}{\infty},\theta_2\right)\right] = -2B < 0.
\end{aligned}$$

Hence there is some Q such that for $q \geq Q$,

$$\alpha\left[H\left(\frac{\Omega_1(q)}{q},\theta_1\right)\right] + (1-\alpha)\left[H\left(\frac{\Omega_2(q)}{q},\theta_2\right)\right] < 0.$$

Hence, $\nu(q) < q$ for $q \geq Q$. Because $\nu(0) > 0$, $\nu' > 0$, $\nu'' < 0$ and there exists some Q such that $\nu(Q) < Q$, a fixed point exists and is unique. ∎

The fixed point $\nu(q) = q$ was constructed holding the state (x, s) fixed, so that we can define the function $q(x, s)$. This function has the properties that markets clear and the fixed point of the marginal valuation function was constructed holding the function q fixed. This is the unique stationary equilibrium for which we have been searching.

We now wish to study some of the implications of the equilibrium. Notice that in equilibrium, the first-order condition of a type i agent with respect to asset holdings can be written

$$1 = \beta E_s\left[\frac{U'(c_i(x',s'))}{U'(c_i(x,s))}\frac{q(x',s')}{q(x,s)}\right],$$

which can be rewritten as

$$\begin{aligned}
\beta E_s\left[\frac{U'(c_i(x',s'))}{U'(c_i(x,s))}\right] &= \frac{q(x,s)}{E_s(q(x',s'))}\times \\
&\left[1 - \text{Cov}_s\left(\beta\frac{U'(c_i(x',s'))}{U'(c_i(x,s))}, \frac{q(x',s')}{q(x,s)}\right)\right].
\end{aligned}$$

The covariance of the asset return with an agent's intertemporal marginal rate of substitution will depend on what type the agent is, and there is no reason to believe that agents will set the *ex ante* intertemporal MRS equal. This occurs because we have not introduced a risk-free asset into the model. As Scheinkman and Weiss note, introducing additional assets into the model may change this result. It is still the case, however, that the *ex post* MRS will be different across agents because of the limits to pooling risk resulting from the borrowing constraints.

Another feature of the equilibrium is that the borrowing constraints do not bind in equilibrium. The argument goes as follows. Recall that we assumed $\lim_{c\to 0} U'(c) = \infty$. For a fixed price $0 < q < \infty$, suppose that the constraint was binding for some agent so that $\xi(x,s) = U'(xq)q$, implying that $c = xq$. Then the multiplier next period when there is no savings and the agent is unproductive, which is always a possibility, is equal to

$$\xi(0,s') = \max[U'(0)q', \beta E_{s'}\xi(0,s'')] = \infty.$$

In that case, $\beta E_s \xi(0,s') = \infty$ so that $\xi(x,s) = \infty$. If $\xi(x,s) = \infty$, then for the first–order conditions to hold,

$$U'(c) = \frac{\infty}{q}$$

so that $c = 0$, which is a contradiction. Thus, the solution to equation (7.139) is satisfied with $\mu_i = 0$ and

$$\xi_i(x,s) = \beta E_s[U'(c_i(x_i', s'))q(x', s')]. \tag{7.148}$$

This feature also arises in the models considered by Bewley [33] and Deaton [89]. Nevertheless, the nonnegativity constraint on the accumulation of non-human wealth alters the ensuing equilibrium because, as we showed in Section 7.1.2, allowing unrestricted borrowing yields allocations that are identical to the complete markets equilibrium.

The intertemporal MRS of both agents are used to price the asset in equilibrium. An econometrician using aggregated consumption data would not be able to evaluate the equilibrium Euler equation. One of the model's implications is that the distribution of asset holdings across consumers affects the asset price and output. Using this implication may provide a way of testing the model's restrictions. We postpone discussion of some of these issues until the next chapter.

8 Volatility Bounds for MRSs

In the previous sections, we derived restrictions from the optimal portfolio choice problem of some representative or average consumer and showed how such restrictions could be tested by making parametric assumptions about the form of the utility function. We now describe how to derive restrictions for the unconditional mean and variance of the intertemporal MRS without making such parametric assumptions. Such unconditional moment restrictions are described in terms of the mean-standard deviation frontier for intertemporal MRSs, which is related to the mean-standard deviation frontier for asset returns derived by Chamberlain and Rothschild [70]. Here we describe a similar approach that Hansen and Jagannathan [186] develop. We derive volatility bounds assuming that there are no short sales constraints, transactions costs, and other frictions. Volatility bounds with such frictions are studied by Luttmer [254] and He and Modest [198].

We describe the construction of the volatility bounds using a setup that is based on Hansen and Jagannathan, although we have described versions of this framework in Chapters 1 and 2. There are multiple consumers who trade in securities markets and who may differ in terms of their preferences and information sets. Trading takes place at time 0 and the assets or securities pay off in date τ. We let I^j denote the information set of consumer j at time 0 and $I = \cap I^j$, where the intersection is taken over the consumers in the economy who trade securities. We assume that consumers can trade in assets at time zero that have payoffs p denominated in the numeraire good at time τ. Let P denote the set of portfolio payoffs that are traded by consumers. The first-order condition for the portfolio choice problem of consumer j is given by:

$$\pi_I(p) = E(p\mathcal{M}^j | I^j), \quad p \in P. \tag{8.149}$$

In this expression, $\mathcal{M}^j$ is the intertemporal marginal rate of substitution in consumption (MRS) for consumer j and $\pi_I(p)$ is the price at time 0 of a portfolio that pays p units of the numeraire good at date τ. Since consumers know the prices of securities that are traded at time zero, prices are represented as a function $\pi_I(p)$ mapping P into I. We can also define a functional π that maps portfolio payoffs into the expected value of prices; that is, $\pi(p) = E[\pi_I(p)]$.

It follows from an iterated expectation in Equation (8.149) that

$$\pi_I(p) = E\left[E(p\mathcal{M}^j | I^j) | I\right] = E(p\mathcal{M}^j | I), \quad p \in P, \tag{8.150}$$

since security prices are observed by all consumers. Notice that this relationship holds for all consumers j provided they engage in securities trading. Hence, we will drop the j superscript and refer to the intertemporal MRS as $\mathcal{M}$.

We can study the implications of the asset pricing relation in Equation (8.150) by assuming that the payoff space P is defined from a subset of the asset payoffs that are traded by consumers. Suppose first that the asset payoffs at date τ consist of an n-dimensional vector $\boldsymbol{x}$. The time zero prices of these n

payoffs is represented with an n-dimensional vector $\boldsymbol{q}$. In this case, the asset pricing relation in Equation (8.150) can be represented as:

$$\boldsymbol{q} = E(\boldsymbol{x}\mathcal{M}|I). \tag{8.151}$$

We can also define the payoff space $P \equiv \{\boldsymbol{c}\cdot\boldsymbol{x} : \boldsymbol{c} \in \mathbf{R}^n\}$ by taking all possible independent linear combinations of the vector $\boldsymbol{x}$. The prices of payoffs in P are given by the corresponding linear combinations of the prices $\boldsymbol{q}$; that is,

$$\pi_I(\boldsymbol{c}\cdot\boldsymbol{x}) \equiv \boldsymbol{c}\cdot\pi_I(\boldsymbol{x}) = \boldsymbol{c}\cdot\boldsymbol{q}.$$

Since $\boldsymbol{q}$ is an element of the common information set I, π_I maps P into I while π maps P into the real line $\mathbf{R}$. Later we also consider payoffs which are found by taking nonlinear combinations of the payoffs in P.

We can express the restrictions in Equation (8.151) by using an iterated expectation argument as follows:

$$E(\boldsymbol{q}) = E(\boldsymbol{x}\mathcal{M}). \tag{8.152}$$

This is just the unconditional counterpart of Equation (8.151). If consumers are nonsatiated in the numeraire good at time τ, then it must be the case that

$$\mathcal{M} > 0. \tag{8.153}$$

Notice that this condition is sufficient to imply the absence of arbitrage opportunities: if $\mathcal{M} > 0$, then nonnegative payoffs that are strictly positive with positive probability conditioned on I have positive prices.

We first derive the implications of the condition in Equation (8.152) for the behavior of intertemporal MRSs denoted $\mathcal{M}$. We begin by constructing random variables in P that have the same mean as $\mathcal{M}$ and that have the smallest variance among all random variables satisfying the unconditional moment restriction in Equation (8.152). We denote such random variables by $\mathcal{M}^\star$. Since P is assumed to contain a finite vector of asset payoffs, we can write $\mathcal{M}^\star$ as $\mathcal{M}^* = \boldsymbol{x'}\boldsymbol{\alpha_0}$ where $\boldsymbol{\alpha_0}$ is in $\mathbf{R}^n$. Using Equation (8.152), we solve for $\boldsymbol{\alpha_0}$

$$\boldsymbol{\alpha_0} = E(\boldsymbol{x}\boldsymbol{x'})^{-1}E(\boldsymbol{q}), \tag{8.154}$$

since $E(\boldsymbol{x}\boldsymbol{x'})$ is nonsingular. Thus, given asset market data on the payoffs $\boldsymbol{x}$ and prices $\boldsymbol{q}$ of n assets, we can construct the random variable $\mathcal{M}^*$.

We now consider two cases, when there exists a riskless payoff p in P such that $p = 1$ and when there does not. In the former case, Equation (8.152) implies that $E(\mathcal{M}^*) = \pi(1) = E(\mathcal{M})$, where $\pi(1)$ is the price of the sure payoff. Since both $\mathcal{M}$ and $\mathcal{M}^*$ satisfy Equation (8.152), we have that $E[\boldsymbol{x}(\mathcal{M}-\mathcal{M}^*)] = 0$, which means that the discrepancy between $\mathcal{M}$ and $\mathcal{M}^*$ is orthogonal to the random vector $\boldsymbol{x}$. Thus, $\mathcal{M}^*$ is the least-squares projection of $\mathcal{M}$ onto P, which implies that

$$\sigma^2(\mathcal{M}) = \sigma^2(\mathcal{M}^*) + \sigma(\mathcal{M} - \mathcal{M}^*)^2.$$

This yields the first volatility result, which can be expressed as:

$$\sigma^2(\mathcal{M}) \geq \sigma^2(\mathcal{M}^*), \quad E(\mathcal{M}^*) = E(\mathcal{M}). \tag{8.155}$$

Now suppose there does not exist a riskless payoff in P. Let $\boldsymbol{x^a}$ denote the $(n+1)$-dimensional random vector formed by augmenting $\boldsymbol{x}$ with a unit payoff. Since $E(\boldsymbol{xx'})$ is nonsingular, $E(\boldsymbol{x^a x^{a\prime}})$ is also nonsingular. Define the augmented payoff space P^a to contain a unit payoff by using $\boldsymbol{x^a}$ in place of $\boldsymbol{x}$. To proceed as before, we must assign a number ν to the expected price of the hypothetical unit payoff; that is, to $\pi(1)$. Such price data may not be available. Thus, we have to examine the implications of Equation (8.152) for an array of such prices. Let ν be a candidate for $\pi(1)$ and π_ν the expected pricing function that corresponds to P^a. Let us construct a random variable $\mathcal{M}_\nu$ in P^a such that $E(\boldsymbol{x}\mathcal{M}_\nu) = E(\boldsymbol{q})$ implies $E(\mathcal{M}_\nu) = \nu$. We get this result because $\mathcal{M}_\nu$ satisfies Equation (8.152) with the unit payoff. Consider any random variable $\mathcal{M}$ that has mean ν and satisfies Equation (8.152). As before, we can show that $\mathcal{M}_\nu$ is the least-squares projection of $\mathcal{M}$ on P^a so that:

$$\sigma^2(\mathcal{M}) = \sigma^2(\mathcal{M}_\nu) + \sigma(\mathcal{M} - \mathcal{M}_\nu)^2,$$

which implies that

$$\sigma^2(\mathcal{M}) \geq \sigma^2(\mathcal{M}_\nu). \tag{8.156}$$

We can replicate the above analysis to construct a set of random variables $\mathcal{M}_\nu$ that satisfy Equation (8.152) and have means ν for all real numbers ν. Define the region:

$$S = \{(\nu, w) \in \mathbf{R}^2 : w \geq \sigma(\mathcal{M}_\nu)\}. \tag{8.157}$$

This set is the set of random variables that have mean ν and variance at least as great as $\mathcal{M}_\nu$. Since this set contains the mean and standard deviation for intertemporal MRSs that satisfy the condition in Equation (8.152), its boundary is called the *mean-standard deviation frontier* for intertemporal MRSs.

Let us now derive an expression for $\mathcal{M}_\nu$. From Equation (8.152), we know that $E(\boldsymbol{x}\mathcal{M}_\nu) = E\left[(\boldsymbol{x} - E\boldsymbol{x})(\mathcal{M}_\nu - \nu)\right] + \nu E(\boldsymbol{x}) = E(\boldsymbol{q})$. By definition, $\mathcal{M}_\nu$ is a linear combination of a unit payoff and the entries of $\boldsymbol{x}$ and $E(\mathcal{M}_\nu)$ is ν so that $\mathcal{M}_\nu = (\boldsymbol{x} - E\boldsymbol{x})'\beta_\nu + \nu$ for some β_ν in $\mathbf{R}^n$. Substituting this expression above yields $\beta_\nu = \Sigma^{-1}[E(\boldsymbol{q}) - \nu E(\boldsymbol{x})]$ where $\Sigma = E[(\boldsymbol{x} - E\boldsymbol{x})(\boldsymbol{x} - E\boldsymbol{x})']$. The standard deviation of $\mathcal{M}_\nu$ is given by:

$$\begin{aligned} \sigma(\mathcal{M}_\nu) &= E\left[(\mathcal{M}_\nu - \nu)^2\right]^{1/2} \\ &= \left[(E\boldsymbol{q} - \nu E\boldsymbol{x})'\Sigma^{-1}(E\boldsymbol{q} - \nu E\boldsymbol{x})\right]^{1/2}. \end{aligned} \tag{8.158}$$

For a given ν, $\sigma(\mathcal{M}_\nu)$ depends only on the means of $\boldsymbol{x}$ and $\boldsymbol{q}$ and the covariance matrix of $\boldsymbol{x}$ and can be computed from asset market data. Suppose $\mathcal{M}$ is equal to the value of ν in all states of the world so that there is *risk-neutral pricing*. Then Equation (8.152) implies that expected prices are proportional to expected payoffs, $E(\boldsymbol{q}) = \nu E(\boldsymbol{x})$. The result in Equation (8.158) can be interpreted as saying that for a fixed Σ, the bounds on the variability of $\mathcal{M}$ will be larger for economies with more risk averse agents.

We can extend this analysis to construct minimum variance random variables among the class of nonnegative random variables satisfying the positivity constraint in Equation (8.153) by considering random variables that can be interpreted as either European call or put options on payoffs in P. Recall that when the payoff on the underlying portfolio is p and the strike price is k, a European call option entitles an investor to the payoff $\max\{p-k, 0\}$ and a put option to the payoff $\max\{k-p, 0\}$. These payoffs are nonnegative but they may be nonlinear functions of the payoff vector $\boldsymbol{x}$. There are again two cases, the case when a riskless payoff exists and when it does not. In the former case, finding the minimum variance nonnegative random variable $\tilde{\mathcal{M}}$ satisfying Equation (8.152) is equivalent to finding a vector $\boldsymbol{\alpha_0}$ in $\mathbf{R}^n$ such that

$$E\left[\boldsymbol{x}(\boldsymbol{x'}\boldsymbol{\alpha_0})^+\right] = E(\boldsymbol{q}), \tag{8.159}$$

where $\tilde{\mathcal{M}} = (\boldsymbol{x'}\boldsymbol{\alpha_0})^+$. If the constructed $\tilde{\mathcal{M}}$ is positive with probability one, then the volatility bound cannot be sharpened further. This occurs only if $\tilde{\mathcal{M}}$ coincides with $\mathcal{M}^\star$.

Define $R = \{p \in P : \pi(p) = 1\}$. When $\boldsymbol{q}$ is nonrandom, R is the collection of (gross) returns on portfolios in P. We cannot solve Equation (8.159) to obtain $\boldsymbol{\alpha_0}$ because it is a nonlinear function of $\boldsymbol{\alpha_0}$. However, we can solve the problem of finding a payoff r in R whose (positive) truncation has the smallest second moment:

$$\min_{r \in R} \|r^+\|^2, \tag{8.160}$$

where $\|r\| \equiv [E(r^2)]^{1/2}$. We scale $\tilde{r}^+$ as $\tilde{\mathcal{M}} = \tilde{r}^+/\|\tilde{r}^+\|$. It can be shown that $\|\tilde{r}^+\|$ must be strictly positive as long as there is one random variable $\mathcal{M}$ satisfying Equations (8.152) and (8.153). It is easy to show that $\tilde{\mathcal{M}}$ satisfies Equation (8.152).

In the case when there does not exist a unit payoff in the payoff space P, the payoff space must be augmented with a unit payoff to form the augmented payoff space, P^a. As before, assign alternative, strictly positive numbers ν to $\pi(1)$ and define R_ν as the set of payoffs with expected prices equal to one. The counterpart to Equation (8.160) is:

$$\delta_\nu \equiv \inf_{r \in R_\nu} \|r^+\|^2, \tag{8.161}$$

If this problem has a solution with $\delta > 0$, denote the minimum variance, nonnegative random variable with mean ν that satisfies Equation (8.152) by $\tilde{\mathcal{M}}_\nu$. The counterpart to the region S given in Equation (8.157) is:

$$S^+ \equiv \left\{(\nu, w) : \delta_\nu > 0 \text{ and } w \geq \sigma(\tilde{\mathcal{M}}_\nu)\right\}. \tag{8.162}$$

Figure 3.1 plots the mean-standard deviation frontiers S and S^+ using data on all the firms in the New York Stock Exchange and the American Exchange for which we have monthly returns between January 1926 and December 1990. Notice that the impact of imposing the strict positivity constraint on the MRS is to reduce the admissible mean-standard deviation region.

The mean-standard deviation region can be used as a diagnostic tool by plotting the unconditional means and standard deviations for the intertemporal MRS implied by alternative asset pricing models and examining whether these means and standard deviations fall within the admissible region implied by data on asset returns. As an example, the intertemporal MRS for the model with time-additive, CRR preferences is given by:

$$\mathcal{M}_t \equiv \beta \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}, \quad \gamma \geq 0.$$

Suppose we are given a sample of observations on consumption $\{c_t\}_{t=1}^T$. For any value of the unknown parameters entering the relevant expression for $\mathcal{M}_t$, we can calculate the sample means and sample standard deviations of $\mathcal{M}_t$ to determine the set of parameters of the underlying economic model that are consistent with data on asset returns. Hansen and Jagannathan [186] and Cochrane and Hansen [74] calculate the mean-standard deviation region for intertemporal MRSs as a way of examining the implications of alternative asset pricing models.

9 Volatility Bounds with Frictions

In Chapter 4, we described how to derive the mean-standard deviation region for intertemporal MRS's that are used to price random payoffs in dynamic asset pricing models. We now extend this discussion to account for short sales constraints, transaction costs and borrowing constraints. As in in our earlier discussion, the volatility bounds we derive here can be used as a diagnostic tool for determining the class of asset pricing models that are consistent with asset market data.

We derive restrictions for intertemporal MRS's with various forms of frictions by using a sequential interpretation of the complete contingent claims equilibrium that we described in Section 7.1. Let's define $\tilde{q}_{t+1}(\omega)\tilde{z}_{t+1}(\omega)$ as the payoff on securities purchased at time t that is realized at time $t+1$. We assume consumers can purchase securities that pay off for each possible realization of the economy. Portfolios with such payoffs can be purchased at the price $p_{t+1}(\omega)/p_t(\omega)$ in period t. Using this notation, notice that the single budget constraint facing agent i can be written in terms of a sequence of one-period constraints:

$$c_{i,t}(\omega) + E_t\left[\frac{p_{t+1}(\omega)}{p_t(\omega)}\tilde{q}_{t+1}(\omega)\tilde{z}_{t+1}(\omega)\right] \leq \ell_{i,t}(\omega) + \tilde{q}_t(\omega)\tilde{z}_t(\omega), \qquad (9.163)$$

for $t \geq 0$ where $E_t(\cdot)$ denotes expectation conditional on the history of shocks, $\{s_s(\omega), s \leq t\}$. We obtain the single budget constraint (7.101) by solving (9.163) forward, where we implicitly impose a condition that the value of limiting portfolio payoff goes to zero.

Volatility bounds with frictions have been derived by Luttmer [254] and He and Modest [198] who consider different types of constraints. Luttmer considers a *solvency* constraint of the form:

$$\tilde{q}_{t+1}(\omega)\tilde{z}_{i,t+1}(\omega) \geq 0. \qquad (9.164)$$

According to this constraint, any contingent contract that allows debt in some state of the world is prohibited. A weaker version of the constraint is employed by He and Modest who require that

$$E_t\left[\frac{p_{t+1}(\omega)}{p_t(\omega)}\tilde{q}_{t+1}(\omega)\tilde{z}_{i,t+1}(\omega)\right] > 0. \qquad (9.165)$$

This states that the value of the portfolio payoff today must be nonnegative. It does not preclude $\tilde{q}_{t+1}(\omega)\tilde{z}_{i,t+1}$ from being negative in some states of the world. We refer to it as the *market-wealth* constraint. We can show that the borrowing constraint in Scheinkman and Weiss is a market-wealth constraint. Consider the Kuhn-Tucker condition for the nonnegativity constraint, $z_{i,t+1} \geq 0$ given by $\mu_{i,t}z_{i,t+1} = 0$. Using the first-order conditions (7.137 -7.139), this can be expressed as

$$E_t[\beta U'(c_{i,t+1})q_{t+1} - U'(c_{i,t})q_t]z_{i,t+1} = 0.$$

Using the above result, we can substitute for $q_t z_{i,t+1}$ in the budget constraint $z_{i,t+1} - z_{i,t} = (\theta_{i,t}\ell_{i,t} - c_{i,t})/q_t$ as

$$c_t + E_t\left[\frac{\beta U'(c_{i,t+1})}{U'(c_{i,t})}q_{t+1}z_{i,t+1}\right] = \theta_{i,t}\ell_{i,t} + q_t z_t.$$

Since q_t is strictly positive, the requirement that $z_{i,t+1} \geq 0$ is equivalent to the form of the market-wealth constraint (9.165) postulated by He and Modest.

Now we analyze the implications of these constraints for individuals intertemporal MRS's with complete markets. Let $\xi_{i,t}(\omega)$ denote the multiplier on the single-period budget constraints (9.163). Using the same preferences as in Section 10.1.1, the first-order condition with respect to the portfolio weights $z_{i,t+1}(\omega)$ imply that

$$E_t\left[-\frac{p_{t+1}(\omega)}{p_t(\omega)}q_{t+1}(\omega)\xi_{i,t}(\omega)-\tilde{q}_{t+1}(\omega)\xi_{i,t+1}(\omega)-\mu_{i,t}(\omega)\right]=0,$$

where the elements of the vector $\mu_{i,t}(\omega)$ equal zero if and only if the corresponding elements of $\tilde{z}_{i,t+1}(\omega)$ are strictly positive. Substituting for $\xi_{i,t}(\omega)$,

$$E_t\left[\frac{\beta U'(c_{i,t+1}(\omega))}{U'(c_{i,t}(\omega))}\tilde{q}_{t+1}(\omega)\right]\leq E_t\left[\frac{p_{t+1}(\omega)}{p_t(\omega)}\tilde{q}_{t+1}(\omega)\right]. \tag{9.166}$$

Let $\mathcal{M}^i$ denote the individual intertemporal MRS in the above expression and $\wp$ the ratio of the contingent claims prices. Since we assumed complete markets in the construction of the payoffs of the traded securities, and given that both $\mathcal{M}^i$ and $\wp$ are nonnegative, we also have that

$$\mathcal{M}^i\leq\wp. \tag{9.167}$$

Thus, with solvency constraints, the individual intertemporal MRS is downward biased relative to the market-determined stochastic discount factor that is used to value payoffs on one-period securities. For certain classes of utility functions (including exponential and power utility functions), we can show that the intertemporal MRS evaluated with per-capita consumption data also inherits this downward bias:

$$\mathcal{M}^a\leq\wp \tag{9.168}$$

where $\mathcal{M}^a\equiv\beta U'(\bar{c}_{t+1}(\omega))/U'(\bar{c}_t(\omega))$, and U is a function of the average subsistence levels, $\bar{\gamma}$, and per-capita consumption, $\bar{c}_t$. (See Problem 7.5 at the end of the chapter.)

Now let us consider the implications of the less restrictive market-wealth constraint. Consumers can now form portfolios in addition to those described above. Let Z denote the set of one-period security payoffs with zero market prices, or equivalently, the set of excess returns. Any payoff in Z satisfies the market-wealth constraint. Furthermore,

$$E_t[\mathcal{M}^i z]=E_t[\wp z]\quad\text{for } z\in Z. \tag{9.169}$$

The payoff $\mathcal{M}^i-\wp E_t(\wp\mathcal{M}^i)/E_t(\wp^2)$ has a zero market price, that is, $E_t[\wp(\mathcal{M}^i-\wp E_t(\wp\mathcal{M}^i)/E_t(\wp^2))]=0$. Using this payoff for z in relation (9.169), we have

$$\mathcal{M}^i=\psi^i\wp\ \text{ for }\psi^i=E_t(\wp\mathcal{M}^i)/E_t(\wp^2). \tag{9.170}$$

Furthermore, (9.166) implies that $0<\psi^i\leq 1$. For the the power utility function, we can show that

$$\mathcal{M}^a=\psi^a\wp, \tag{9.171}$$

where $0 < \psi^a \leq 1$. (See Problem 10.5.) Recall that the market-wealth constraint is less restrictive than the solvency constraint. As the above results demonstrate, the less restrictive constraint imposes the more stringent proportionality requirement on the aggregate intertemporal MRS.

Cochrane and Hansen [74] describe in detail how to compute the boundary of the mean-standard deviation region for intertemporal MRS's or stochastic discount factors that satisfy (9.168) and (9.171) in the case of two limited liability securities. They consider quarterly value-weighted stock returns on the NYSE and T-bill returns. Let $\boldsymbol{x}$ denote a random vector formed by stacking these two returns, and let P^+ denote the cone of random variables or limited-liability payoffs that can be constructed from constant-weighted portfolios of these returns:

$$P^+ \equiv \{p : p = \boldsymbol{c} \cdot \boldsymbol{x} \quad \text{for } \boldsymbol{c} \in \Re^2, p \geq 0\}.$$

Define the region of stochastic discount factors that satisfy (9.168) by B^+. It turns out the region B^+ is an expanded version of the region S^+ that we calculated in Chapter 4 for any intertemporal MRS or stochastic discount factor satisfying the the asset pricing relation without a solvency constraint, or short sales constraints. We can also construct a region, denoted W^+ for the set of random random variables $\mathcal{M}^a$ such that the proportionality restriction in (9.171) holds. W^+ is also an expanded version of S^+, but it is smaller than B^+.

The mean-standard region for random variables or stochastic discount factors that satisfy (9.168) can be constructed by using two so-called edge portfolios, denoted p_1 and p_2. Any other payoff in P^+ is a convex combination of these edges with nonnegative portfolio weights. Since the original two returns have nonnegative payoffs, each edge has a positive portfolio weight on one of the securities and a nonpositive weight on the other. Let us normalize these edge payoffs so that their price is one, that is, $E(\wp p_1) = E(\wp p_2) = 1$, and order them so that $E(p_1) \geq E(p_2)$.

The boundary of B^+ has three segments. To see how the first segment arises, notice that for any constant discount factor $\mathcal{M}^a$ such that $0 \leq \mathcal{M}^a \leq 1/E(p_1)$, the inequalities in (9.168) are satisfied, that is,

$$E(\mathcal{M}^a p_i) \leq E(\wp p_i) = 1, \quad i = 1, 2,$$

Let $\sigma(\mathcal{M})$ denote the standard deviation of some stochastic discount factor $\mathcal{M}$. Then there is a horizontal segment at $\sigma(\mathcal{M}^a) = 0$ from $E(\mathcal{M}^a) = 0$ to $E(\mathcal{M}^a) = 1/E(p_1)$. Furthermore, as long as the constant discount factor $\mathcal{M}^a$ is strictly less than $1/E(p_1)$, the inequalities in (9.168) will be strict. When $\mathcal{M}^a = 1/E(p_1)$, (9.168) will hold with equality for p_1. The second segment begins at this point.

Now consider the mean-standard region region for the set of (strictly positive) stochastic discount factors that correctly price p_1. Following the notation of Chapter 4, we denote this region S_1^+. Notice that the point $(1/E(p_1), 0)$ is on the boundary of this set because the constant discount factor $1/E(p_1)$ prices p_1 correctly. Also, it is easy to see that any other frontier random variable for S_1^+ will also be on the boundary of B^+ provided (9.168) is satisfied for p_2. In other words, any $\mathcal{M}^1$ on the boundary of S_1^+ such that $1 = E(\mathcal{M}^1 p_1)$ will also be on the boundary of B^+ provided $E(\mathcal{M}^1 p_2) \leq E(\wp p_2)$. Thus, we follow the right boundary of S_1^+ until we find a frontier discount factor that prices p_2 correctly.

The third segment of the boundary for B^+ coincides with the mean-standard deviation region of stochastic discount factors, S^+, which correctly price both p_1 and p_2. Thus, we can find the minimum variance random variables $\mathcal{M}$ such that $E(\mathcal{M}p_i) = 1$ for $i = 1, 2$. For these random, (9.168) will hold with equality for all possible payoffs in P^+.

To construct the boundary of W^+, multiply both sides of (9.171) by some payoff x. Taking expectations (first conditional on the information set at time t, and then unconditionally), we obtain $E(\mathcal{M}^a x) = E(\psi^a)\tilde{Q}$ where $0 < E(\psi^a) \leq 1$ and $\tilde{Q}$ is the price of the random payoff x. Thus, for any $\mathcal{M}^a$ satisfying (9.171), we can find a stochastic discount factor $\mathcal{M}^a/E(\psi^a)$ that prices the payoffs x correctly. Since the mean and standard deviation of random variables that are scale multiples the same scaling, we can construct the stochastic discount factors in W^+ by scaling the discount factors in S^+ by arbitrary numbers between zero and one.

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